Utilice el método del ejemplo 6 para encontrar $\frac{\partial w}{\partial x}$ y $\frac{\partial w}{\partial y}$ como funciones de x y y

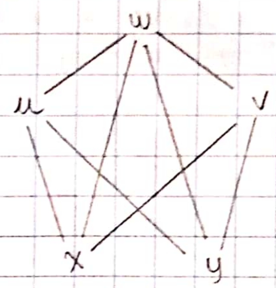
25. $w = u^2 + v^2 + x^2 + y^2$; $u = x - y$, $v = x + y$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}$$

$$= 2u(1) + 2v(1) + 2x$$

$$= 2u + 2v + 2x \Rightarrow 2(x-y) + 2(x+y) + 2x$$

$$= 2x - \cancel{2y} + 2x + \cancel{2y} + 2x \Rightarrow 6x$$



$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial y}$$

$$= 2u(-1) + 2v(1) + 2y$$

$$= -2u + 2v + 2y \Rightarrow -2(x-y) + 2(x+y) + 2y$$

$$= -\cancel{2x} + 2y + \cancel{2x} + 2y + 2y \Rightarrow 6y$$

26. $w = uv - xy$; $u = \frac{x}{x^2+y^2}$, $v = \frac{y}{x^2+y^2}$

$$w = f(u, v, x, y)$$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}$$

$$= v\left[\frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right] + u\left[-y\,(x^2+y^2)^{-1-1}(2x)\right] + (-y)$$

$$= v\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] + u\left[-y\,(x^2+y^2)^{-2}(2x)\right] - y$$

$$= v\left[\frac{-x^2+y^2}{(x^2+y^2)^2}\right] + u\left[\frac{-2xy}{(x^2+y^2)^2}\right] - y$$

$$= \frac{y}{x^2+y^2}\left[\frac{-x^2+y^2}{(x^2+y^2)^2}\right] + \frac{x}{x^2+y^2}\left[\frac{-2xy}{(x^2+y^2)^2}\right] - y$$

$$= -\frac{x^2y + y^3}{(x^2+y^2)^3} - \frac{2x^2y}{(x^2+y^2)^3} - y$$

$$= -\frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^3} - y$$

$$= -\frac{3x^2y + y^3}{(x^2+y^2)^3} - y$$

$$= -\frac{3x^2y + y^3 - y(x^2+y^2)^3}{(x^2+y^2)^3}$$

$$= -\frac{3x^2y + y^3 - y(x^6+3x^4y^2+3x^2y^4+y^6)}{(x^2+y^2)^3}$$

$$= -\frac{3x^2y + y^3 - x^6y - 3x^4y^3 - 3x^2y^5 - y^7}{(x^2+y^2)^3}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial y}$$

$$= v\cdot\left[x\,\frac{1}{(x^2+y^2)}\right] + u\left[\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right] + (-x)$$

$$= v\left[-x\,(x^2+y^2)^{-1-1}\right] + u\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] - x$$

$$= v\left[-x\,(x^2+y^2)^{-2}\,2y\right] + u\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] - x$$

$$= v\left[\frac{-2xy}{(x^2+y^2)^2}\right] + u\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] - x$$

$$= \frac{y}{x^2+y^2}\left[\frac{-2xy}{(x^2+y^2)^2}\right] + \frac{x}{x^2+y^2}\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] - x$$

$$= -\frac{2xy^2}{(x^2+y^2)^3} + \frac{x^3-xy^2}{(x^2+y^2)^3} - x$$

$$= \frac{-xy^2+x^3}{(x^2+y^2)^3} - x$$

$$= -\frac{x\cancel{(y^2+x^2)}}{(x^2+y^2)^{\cancel{3}}} - x$$

$$= -\frac{x}{(x^2+y^2)^2} - x$$

$$= -\frac{x + x^5 + 2x^3y^2 + xy^4}{(x^2+y^2)^2}$$

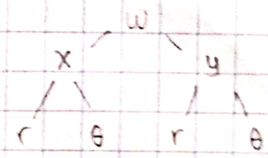
//

40. Suponga que $w = f(x,y)$, $x = r\cos\theta$ y que $y = r\,\text{sen}\,\theta$. Demuestre que

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\,\text{sen}\,\theta$$

$$\frac{\partial y}{\partial r} = \text{sen}\,\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\cdot\text{sen}\,\theta \Rightarrow \cos\theta\frac{\partial w}{\partial x} + \text{sen}\,\theta\frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial w}{\partial x}(-r\,\text{sen}\,\theta) + \frac{\partial w}{\partial y}\cdot r\cos\theta \Rightarrow r\cos\theta\frac{\partial w}{\partial y} - r\,\text{sen}\,\theta\frac{\partial w}{\partial x}$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\cos\theta\frac{\partial w}{\partial x} + \text{sen}\,\theta\frac{\partial w}{\partial y}\right)^2 + \frac{1}{r^2}\left(r\cos\theta\frac{\partial w}{\partial y} - r\,\text{sen}\,\theta\frac{\partial w}{\partial x}\right)^2$$

$$= \cos^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \cancel{2\left(\cos\theta\frac{\partial w}{\partial x}\right)\left(\text{sen}\,\theta\frac{\partial w}{\partial y}\right)} + \text{sen}^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{\cancel{r^2}}\cancel{r^2}\left[\cos^2\theta\left(\frac{\partial w}{\partial y}\right)^2\right.$$

$$\left. \cancel{-2\left(\cos\theta\frac{\partial w}{\partial y}\right)\left(\text{sen}\frac{\partial w}{\partial x}\right)} + \text{sen}^2\theta\left(\frac{\partial w}{\partial x}\right)^2\right]$$

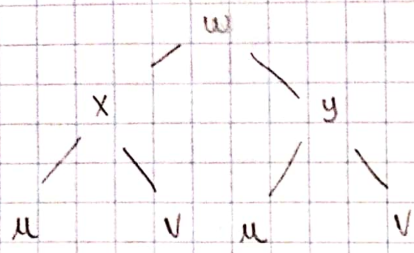
$$= \cos^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \text{sen}^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + \cos^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + \text{sen}^2\theta\left(\frac{\partial w}{\partial x}\right)^2$$

$$= \left(\frac{\partial w}{\partial x}\right)^2\underbrace{(\cos^2\theta + \text{sen}^2\theta)}_{1} + \left(\frac{\partial w}{\partial y}\right)^2\underbrace{(\text{sen}^2\theta + \cos^2\theta)}_{1}$$

$$= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

43. Suponga que $w = f(x,y)$, donde $x = u + v$ y $y = u - v$. Demuestre que

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial u \partial v}$$



$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = -1 \qquad \frac{\partial y}{\partial v} = -1$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial w}{\partial x} \cdot 1 + \frac{\partial w}{\partial y} \cdot 1 \Rightarrow \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$

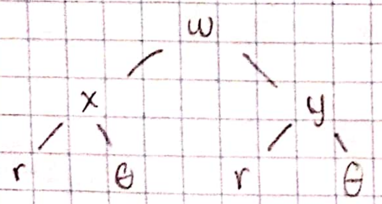
$$= \frac{\partial w}{\partial x} \cdot 1 + \frac{\partial w}{\partial y} \cdot (-1) \Rightarrow \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial^2 w}{\partial u \partial v} \Rightarrow \frac{\partial w}{\partial u} \cdot \frac{\partial w}{\partial v} = \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)$$

$$= \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right)_{//}$$

45. Suponga que $w = f(x,y)$, $x = r\cos\theta$ y $y = r\,\text{sen}\,\theta$. Demuestre que

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$



$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\,\text{sen}\,\theta$$

$$\frac{\partial y}{\partial r} = \text{sen}\,\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} \qquad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial w}{\partial x} \cdot \cos\theta + \frac{\partial w}{\partial y} \cdot \text{sen}\,\theta \qquad = \frac{\partial w}{\partial x}(-r\,\text{sen}\,\theta) + \frac{\partial w}{\partial y} \cdot r\cos\theta$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) \Rightarrow \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial w}{\partial x} + \operatorname{sen}\theta \frac{\partial w}{\partial y}\right)$$

$$= \cos \frac{\partial w_x}{\partial r} + \operatorname{sen} \frac{\partial w_y}{\partial r} \qquad \boxed{w_x = \frac{\partial w}{\partial x} \qquad w_y = \frac{\partial w}{\partial y}}$$

$$= \cos\theta\left(\frac{\partial w_x}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w_x}{\partial y}\cdot\frac{\partial y}{\partial r}\right) + \operatorname{sen}\theta\left(\frac{\partial w_y}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w_y}{\partial y}\cdot\frac{\partial y}{\partial r}\right)$$

$$= \cos\theta\left(\frac{\partial^2 w}{\partial x^2}\cos\theta + \frac{\partial^2 w}{\partial x \partial y}\operatorname{sen}\theta\right) + \operatorname{sen}\theta\left(\frac{\partial^2 w}{\partial x \partial y}\cdot\cos\theta + \frac{\partial^2 w}{\partial y^2}\operatorname{sen}\theta\right)$$

$$= \cos^2\theta \frac{\partial^2 w}{\partial x^2} + \cos\theta\operatorname{sen}\theta \frac{\partial^2 w}{\partial x \partial y} + \cos\theta\operatorname{sen}\theta\frac{\partial^2 w}{\partial x \partial y} + \operatorname{sen}^2\theta \frac{\partial^2 w}{\partial y^2}$$

$$\frac{\partial^2 w}{\partial r^2} = \cos^2\theta \frac{\partial^2 w}{\partial x^2} + 2\cos\theta\operatorname{sen}\theta \frac{\partial^2 w}{\partial x \partial y} + \operatorname{sen}^2\theta \frac{\partial^2 w}{\partial y^2}$$

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial \theta}\right) \Rightarrow \frac{\partial}{\partial \theta}\left(r\cos\theta \frac{\partial w}{\partial y} - r\operatorname{sen}\theta \frac{\partial w}{\partial x}\right)$$

$$= r\cos\theta \frac{\partial w_x}{\partial \theta} - r\operatorname{sen}\theta \frac{\partial w_y}{\partial \theta}$$

$$= r\cos\theta\left(\frac{\partial w_x}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w_x}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right) - r\operatorname{sen}\theta\left(\frac{\partial w_y}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w_y}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right)$$

$$= r\cos\theta\left(\frac{\partial^2 w}{\partial x^2}(-r\operatorname{sen}\theta) + \frac{\partial^2 w}{\partial x \partial y} r\cos\theta\right) - r\operatorname{sen}\theta\left(\frac{\partial^2 w}{\partial x \partial y}\cdot(-r\operatorname{sen}\theta) + \frac{\partial^2 w}{\partial y^2}\right.$$

$$\left.\cdot\, r\cos\theta\right)$$

$$= -r^2\cos\theta\operatorname{sen}\theta \frac{\partial^2 w}{\partial x^2} + r^2\cos^2\theta \frac{\partial^2 w}{\partial x \partial y} + r^2\operatorname{sen}^2\theta \frac{\partial^2 w}{\partial x \partial y}$$

$$- r^2\cos\theta\operatorname{sen}\theta \frac{\partial^2 w}{\partial y^2}.$$

$$= r^2\left(\cos\theta\operatorname{sen}\theta \frac{\partial^2 w}{\partial x^2} + \cos^2\theta \frac{\partial^2 w}{\partial x \partial y} + \operatorname{sen}^2\theta \frac{\partial^2 w}{\partial x \partial y}\right.$$

$$\left.- \cos\theta\operatorname{sen}\theta \frac{\partial^2 w}{\partial y^2}\right)$$

$$= \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2}$$

$$= \cos^2\theta \frac{\partial^2 w}{\partial x^2} + 2\left(\cos\theta \operatorname{sen}\theta \frac{\partial^2 w}{\partial x \partial y}\right) + \operatorname{sen}^2\theta \frac{\partial^2 w}{\partial y^2} + \frac{1}{r}\left(\cos\theta \frac{\partial w}{\partial x} + \operatorname{sen}\theta \frac{\partial w}{\partial y}\right)$$

$$+ \frac{1}{\cancel{r^2}} \cancel{r^2}\left(\cos\theta \operatorname{sen}\theta \frac{\partial^2 w}{\partial x^2} + \cos^2\theta \frac{\partial^2 w}{\partial x \partial y} + \operatorname{sen}^2\theta \frac{\partial^2 w}{\partial x \partial y} - \cos\theta \operatorname{sen}\theta \frac{\partial^2 w}{\partial y^2}\right)$$

$$= \cos^2\theta \frac{\partial^2 w}{\partial x^2} + \sin(2\theta) \frac{\partial^2 w}{\partial x \partial y} + \operatorname{sen}^2\theta \frac{\partial^2 w}{\partial y^2}$$