Project of Distributed Systems: Paradigms and Models
Iterative Jacobi Method

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Contents

1 Analysis of the problem

The pseudo-code to execute the algorithm sequentially for a fixed amount of iteration l, is shown in 1. Note that at each iteration we're forced to use an auxiliary vector x' to compute the updated elements of x for the current iteration, and then, before starting the following iteration, we copy the elements of x' in x. This is necessary because if at iteration k, we compute $x_1^{(k)}$ modifying x in-place, then when we compute $x_2^{(k+1)}$ instead of using $x_1^{(k)}$, we'll use $x_1^{(k+1)}$.
The algorithm is composed by three nested loops, where the first iterates l times, and the other two iterate n times, so the total time complexity is $O(ln^2)$. The swap function used in the pseudo-code to copy the elements of x' in x, can be implemented with $O(1)$ cost, by simply swapping the pointers of the two array (this works in C++ where we see as vectors).

Algorithm 1 Sequential code for Jacobi method

Require: A matrix, b and x vectors, l positive integer
1: **for** $k \leftarrow 1$ to *limit* **do**
2: **for** $i \leftarrow 1$ to n **do**
3: $val \leftarrow 0$
4: **for** $j \leftarrow 1$ to n **do**
5: **if** $j \neq i$ **then**
6: $val \leftarrow val + A[i, j] * x[j]$
7: **end if**
8: **end for**
9: $x'[i] \leftarrow \frac{1}{A[i,i]}(b[i] - val)$
10: **end for**
11: swap(x, x')
12: **end for**

Let's now analyze how we can exploit the properties of the problems in order to parallelize the algorithm. The first important thing we notice is that in order to compute the updated values for the vector x at iteration k, we need the full updated vector x at iteration $k-1$, therefore the iterations of the algorithm needs to be strictly sequential. So let's focus on understanding what we can parallelize inside a single iteration. As we said in the description of the problem, at each iteration k every component of the new vector $x^{(k+1)}$ can be computed independently from the others components using the formula in equation **??**. So this is a data parallel computation where (for the reasons described before) we cannot modify the vector x in place. So we can parallelize the code by assigning to each worker a different partition of components of x.
The usage of multiple workers introduces the following synchronization problems:

1. Start iteration $k+1$ only after all the workers finished their computations for iteration k, since as we said at the beginning of the analysis iterations must be strictly sequential

2. Executing the swap of the main and auxiliary vectors only after all the workers finished their computations, otherwise it might happen that the vectors are swapped while other vectors have still to finish their computation

3. Keep track of the number of iteration so that all the workers will stop after *limit* iterations.

The first problem can be solved simply by putting a barrier after the execution of the map, so that all the worker will be forced to wait the others before starting the next iteration. In this way we can also solve the third problem by selecting one of the worker to count the number of iterations of the algorithm executed so far. The designated worker can simply use

a shared variable among the workers to increase by one at the end of every iteration, before the barrier (since the variable is modified by just one worker it doesn't need synchronization mechanism to be accessed). This solution guarantees that before starting iteration k the shared variable will have value k for all the workers, so every worker can individually check if $k = limit$ and stop its execution. The second problem can be solved simply by adding a second barrier, which will be placed between the end of the map and the next iteration barrier, and designate one of the worker to execute the swap of the vectors after the barrier (it can be the same worker which updates the iterations counter). In this way the swap will happen only after every worker finished its computations for the current iteration.

Algorithm 2 Worker pseudo-code

Require: A matrix, b and x vectors, l positive integer
1: **while** $k < l$ **do**
2: **for** $i \leftarrow start$ to end **do**
3: $val \leftarrow 0$
4: **for** $j \leftarrow 1$ to n **do**
5: **if** $j \neq i$ **then**
6: $val \leftarrow val + A[i, j] * x[j]$
7: **end if**
8: **end for**
9: $x'[i] \leftarrow \frac{1}{A[i,i]}(b[i] - val)$
10: **end for**
11: **Vectors swap barrier**
12: **if** start $= 0$ **then**
13: swap(x, x')
14: $k \leftarrow k + 1$
15: **end if**
16: **Next iteration barrier**
17: **end while**

The variables *start* and *end* represent respectively the first and the last index of x assigned to the worker to compute. Note that both of the barrier are needed, because if we only use the copy barrier, it might happen that one of the worker might start the next iteration before the designated worker could swap the vectors and update the iteration counter, so both barriers are needed.

2 Time analysis

In the sequential case as we said the complexity of the algorithm is:

$$T_{seq} = T_{init} + k(n^2 t_{\oplus} + t_{swap})$$

as we already said the swap can be implemented just by swapping the pointer and the, so the time is approximately:

$$T_{seq} \approx kn^2 t_{\oplus}$$

In the parallel case in the second innermost loop, instead of iterating n times, we will iterate n/nw times. The overhead introduced is given by the forking and joining of the threads (which can be done just once) and by the time required by the synchronization with the two barriers.

$$T_{par}(nw) = T_{init} + T_{fork} + k(\frac{n}{nw} n t_{\oplus} + T_{synch} + T_{swap} + T_{synch}) + T_{join}$$

So the speedup is:

$$Speedup(nw) = \frac{T_{init} + k(n^2 t_{\oplus} + t_{swap})}{T_{init} + T_{fork} + k(\frac{n}{nw} n t_{\oplus} + T_{synch} + T_{swap} + T_{synch}) + T_{join}}$$

If we consider negligible the time for the preparations of the data and the management of the workers then the speedup is more or less:

$$Speedup(nw) \approx \frac{k n^2 t_{\oplus}}{k \frac{n}{nw} n t_{\oplus}} = nw$$

which is the ideal speedup which we can theoretically obtain with a map pattern. Of course to have a better estimate of the speedup we should consider the serial fraction of the program. We can make a rough estimate by considering that the total number of operations done by the algorithm is more or less in the order of kn^2, since we have to do some fixed amount of iterations inside the three loops. Ideally, with n workers, every worker could compute its $x_i^{(k)}$, reducing the number of operation to kn, which are the number of operations that we have to do sequentially in the best case, so a rough estimation of the serial fraction f is:

$$f = \frac{kn}{kn^2} = \frac{1}{n}$$

if we now consider the speedup given by the Ahamdal's law, we get that in the best case the maximum speedup is:

$$Speedup < \frac{1}{f} = \frac{1}{\frac{1}{n}} = n$$

So the speedup is upper-bounded by the size of the matrix. This makes sense because in the best case we have a thread for each component x_i of the vector, so at most n. So we should expect to have a better speedup as the size of n increases.

3 Implementation

The parallel program has bee implemented in three versions:

- The first using C++ threads
- The second using FastFlow
- The third using Open MP

3.1 C++ thread

In this implementation we use the fork-join mechanism to spawn nw threads, each of them is assign to a chunk of x of size $\lfloor \frac{n}{nw} \rfloor$, and each of them execute code in 2.

3.2 FastFlow

For the fast flow implementation we simply use the `ParalleFor` class of FastFlow to parallelize the for loop which iterates over

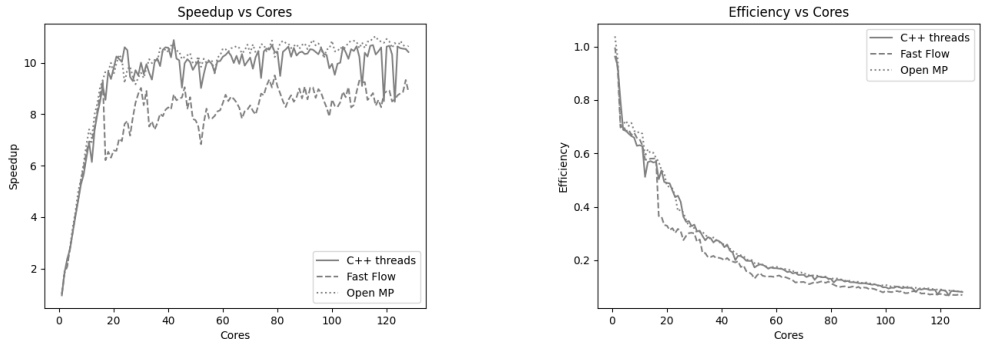
3.3 Open MP

4 Experiments

The goals of the experiments that we will execute are the following:

- Comparing the performances obtained by the three implementations
- Check how the performances change with respect to the size of the matrix

To compare the performances of the three implementations we will compute the speedup obtained when executed with nw workers, varying nw from 1 to 128, and showing the results in a plot. To see if the size of the matrix impact on the speedup, we'll repeat the procedure for matrices of different sizes (100, 500, 1000). Note: in order to avoid good and bad case, to take the time obtain by one implementation with a certain number of worker, we execute the program 5 times and take the average time.
After that we'll select the number of workers for which we obtain the best speedup and we'll execute the algorithms with that number of workers for matrices of different sizes (10, 20, 100, 200, 500, 1000, 2000, 5000, 10000) to better understand the impact that the size of the matrix has on the performances. Even in this case each time is the mean of five executions.



(a) label 1 (b) label 2

Figure 1: 2 Figures side by side

5 Manual

To compile the code is sufficient to execute the command `make` in the `src` directory of the project. The compilation will produce the executable `main` which can be called to launch the program providing the following arguments:

- **n**: and integer which represent the number of row and columns of the matrix
- **nw**: the number of workers that will be used in the parallel versions of the algorithm
- **iterations**: the number of iterations of the algorithm

Once execute the program will execute the sequential algorithm and the three parallel implementations and report the execution times in nanoseconds. The execution of the experiment is handled by two bash shell scripts.