# Nanoscale hydrodynamics near solids

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## Motivation

- Great interest in the study of fluids in contact with solids in the nanoscale.
- Density layering $\rightarrow$ DFT for equilibrium situations $\rightarrow$ DDFT for the study of the dynamic behaviour of the fluid.
- Slip boundary condition

$$\delta \frac{\partial v}{\partial z} = v_{\mathrm{slip}}, \qquad\qquad \delta = \frac{\eta}{\gamma}$$

- Bocquet and Barrat [**Bocquet1993**]

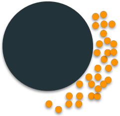
$$\gamma = \frac{1}{S k_B T} \int_0^{\tau} dt \left\langle \hat{F}^x(t) \hat{F}^x \right\rangle$$

- Petravic and Harrowell [**Petravic2007**] disagree with the expression obtained by Bocquet and Barrat.
- The expression for $\gamma$ suffers from the plateau problem.

## Roadmap

**BEHAVIOUR**
**FLUID - SOLID**
**IN THE NANOSCALE**

KAWASAKI-GUNTON PROJECTION OPERATOR + MARKOVIAN APPROXIMATION

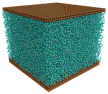


$$
\begin{aligned}
\partial_t \rho(\mathbf{r}) &= -\boldsymbol{\nabla}\cdot\mathbf{g}(\mathbf{r})\\
\partial_t \mathbf{g}(\mathbf{r}) &= -\boldsymbol{\nabla}\cdot(\mathbf{g}(\mathbf{r})\mathbf{v}(\mathbf{r})) - \rho(\mathbf{r})\boldsymbol{\nabla}\frac{\delta\mathcal{F}}{\delta\rho(\mathbf{r})}[\rho,\mathbf{R}]\\
&\quad + \boldsymbol{\nabla}\cdot\boldsymbol{\Sigma}(\mathbf{r}) + \boldsymbol{\mathcal{S}}(\mathbf{r})\\
\dot{\mathbf{R}} &= \frac{\mathbf{P}}{M}\\
\dot{\mathbf{P}} &= -\frac{\partial\mathcal{F}}{\partial\mathbf{R}} - \int d\mathbf{r}\,\boldsymbol{\mathcal{S}}(\mathbf{r})
\end{aligned}
$$

MORI THEORY

$$C(t) = \exp\{-\Lambda^* t\}\cdot C(0)$$

MD SIMULATIONS



SLIP BOUNDARY CONDITION

# Hydrodynamics theory for liquids near solids

## The system and the relevant variables

- We study a fluid with $N$ particles in contact with a solid sphere of $N'$ particles.
- $z = (\mathbf{q}_i, \mathbf{p}_i)$ and $z' = (\mathbf{q}_{i'}, \mathbf{p}_{i'})$
- The relevant variables

$$\hat{\rho}_{\mathbf{r}}(z) = \sum_i^N m\delta(\mathbf{r} - \mathbf{q}_i) \qquad \hat{\mathbf{R}}(z) = \frac{1}{N'}\sum_{i'}^{N'} \mathbf{q}_{i'}$$

$$\hat{\mathbf{g}}_{\mathbf{r}}(z) = \sum_i^N \mathbf{p}_i\delta(\mathbf{r} - \mathbf{q}_i) \qquad \hat{\mathbf{P}}(z) = \sum_{i'}^{N'} \mathbf{p}_{i'}$$

- The derivatives of the relevant variables

$$i\mathcal{L}\hat{\rho}_{\mathbf{r}}(z) = -\boldsymbol{\nabla}\cdot\hat{\mathbf{g}}_{\mathbf{r}}(z) \qquad i\mathcal{L}\hat{\mathbf{R}}(z) = \frac{\hat{\mathbf{P}}(z)}{M}$$

$$i\mathcal{L}\hat{\mathbf{g}}_{\mathbf{r}}(z) = -\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}}_{\mathbf{r}}(z) + \hat{\mathbf{F}}^{\mathrm{s}\to\mathrm{l}}_{\mathbf{r}}(z) \qquad i\mathcal{L}\hat{\mathbf{P}}(z) = -\int d\mathbf{r}\hat{\mathbf{F}}^{\mathrm{s}\to\mathrm{l}}_{\mathbf{r}}(z)$$

## Kawasaki-Gunton projection operator

- Clear separation of timescales between the evolution of the averages and the decay of the memory kernel

$$\frac{\partial}{\partial t} a_i(t) = v_i(t) + \sum_j D_{ij}(t)\lambda_j(t)$$

- Reversible term: $v_i = \mathrm{Tr}[\overline{\rho}_t i\mathcal{L}\hat{A}_i]$
- The relevant ensemble:

$$\overline{\rho}(z) = \frac{1}{Z[\lambda]}\rho_0 \exp\{-\lambda \cdot \hat{A}(z)\}$$

- The dissipative matrix is given by the Green-Kubo formula

$$D_{ij}(t) = \int_0^{\Delta t} dt' \left\langle \mathcal{Q}_t i\mathcal{L}\hat{A}_j \exp\{i\mathcal{L}t'\}\mathcal{Q}_t i\mathcal{L}\hat{A}_i \right\rangle^{\lambda(t)}$$

- The Kawasaki-Gunton projection operator is given by

$$\mathcal{Q}_{t'}\hat{F}(z) = \hat{F}(z) - \mathrm{Tr}[\overline{\rho}_{t'}\hat{F}] - \sum_i (\hat{A}_i(z) - a_i(t'))\frac{\partial}{\partial a_i(t')}\mathrm{Tr}[\overline{\rho}_{t'}\hat{F}]$$

# Equations of nanohydrodynamics

$$
\begin{aligned}
\partial_t \rho(\mathbf{r}) &= -\boldsymbol{\nabla} \cdot \mathbf{g}(\mathbf{r}) \\
\partial_t \mathbf{g}(\mathbf{r}) &= -\boldsymbol{\nabla} \cdot (\mathbf{g}(\mathbf{r})\mathbf{v}(\mathbf{r})) - \rho(\mathbf{r}) \boldsymbol{\nabla} \frac{\delta \mathcal{F}}{\delta \rho(\mathbf{r})}[\rho, \mathbf{R}] + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}(\mathbf{r}) + \boldsymbol{\mathcal{S}}(\mathbf{r}) \\
\dot{\mathbf{R}} &= \frac{\mathbf{P}}{M} \\
\dot{\mathbf{P}} &= -\frac{\partial \mathcal{F}}{\partial \mathbf{R}} - \int d\mathbf{r} \boldsymbol{\mathcal{S}}(\mathbf{r})
\end{aligned}
$$

- $\mathcal{F}[\rho, \mathbf{R}]$: free energy density functional of a fluid in the presence of a solid sphere.
- $\Sigma(\mathbf{r})$: fluid stress tensor.
- $\mathcal{S}(\mathbf{r})$: irreversible surface force density on the fluid.

# The transport kernels

- The fluid stress tensor $\boldsymbol{\Sigma}(\mathbf{r})$ is given by

$$\boldsymbol{\Sigma}^{\alpha\beta}(\mathbf{r}) = \int d\mathbf{r}' \boldsymbol{\eta}_{\mathbf{r}\mathbf{r}'}^{\alpha\beta\alpha'\beta'} \boldsymbol{\nabla}_{\mathbf{r}'}^{\beta'} \mathbf{v}^{\alpha'}(\mathbf{r}')$$

- The irreversible surface force density on the fluid $\mathcal{S}(\mathbf{r})$

$$\begin{aligned} \mathcal{S}^{\alpha}(\mathbf{r}) = &- \int d\mathbf{r}' \mathbf{G}_{\mathbf{r}\mathbf{r}'}^{\alpha\alpha'\beta'} \boldsymbol{\nabla}_{\mathbf{r}'}^{\beta'} \mathbf{v}^{\alpha'}(\mathbf{r}') + \boldsymbol{\nabla}_{\mathbf{r}}^{\beta} \int d\mathbf{r}' \mathbf{H}_{\mathbf{r}\mathbf{r}'}^{\alpha\beta\alpha'} (\mathbf{v}^{\alpha'}(\mathbf{r}') - \mathbf{V}^{\alpha'}) \\ &- \int d\mathbf{r}' \gamma_{\mathbf{r}\mathbf{r}'}^{\alpha\alpha'} (\mathbf{v}^{\alpha'}(\mathbf{r}') - \mathbf{V}^{\alpha'}) \end{aligned}$$

# The transport kernels

$$\eta_{\mathbf{r}\mathbf{r}'} \equiv \frac{1}{k_B T}\int_0^{\Delta t} dt' \langle \mathcal{Q}_t \hat{\boldsymbol{\sigma}}_{\mathbf{r}}(t') \mathcal{Q}_t \hat{\boldsymbol{\sigma}}_{\mathbf{r}'} \rangle^{\lambda(t)}$$

$$\mathbf{H}_{\mathbf{r}\mathbf{r}'} \equiv \frac{1}{k_B T}\int_0^{\Delta t} dt' \langle \mathcal{Q}_t \hat{\boldsymbol{\sigma}}_{\mathbf{r}}(t') \mathcal{Q}_t \hat{\mathbf{F}}^{\mathrm{s}\to\mathrm{l}}_{\mathbf{r}'} \rangle^{\lambda(t)}$$

$$\mathbf{G}_{\mathbf{r}\mathbf{r}'} \equiv \frac{1}{k_B T}\int_0^{\Delta t} dt' \langle \mathcal{Q}_t \hat{\mathbf{F}}^{\mathrm{s}\to\mathrm{l}}_{\mathbf{r}}(t') \mathcal{Q}_t \hat{\boldsymbol{\sigma}}_{\mathbf{r}'} \rangle^{\lambda(t)}$$

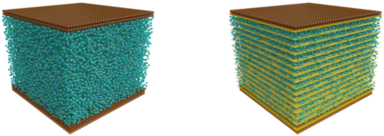
$$\gamma_{\mathbf{r}\mathbf{r}'} \equiv \frac{1}{k_B T}\int_0^{\Delta t} dt' \langle \mathcal{Q}_t \hat{\mathbf{F}}^{\mathrm{s}\to\mathrm{l}}_{\mathbf{r}}(t') \mathcal{Q}_t \hat{\mathbf{F}}^{\mathrm{s}\to\mathrm{l}}_{\mathbf{r}'} \rangle^{\lambda(t)}$$

# Simpler theory

- The amount of information to compute the hydrodynamic equations is exceedingly large:
    - $\boldsymbol{\eta}$ has 36 independent components.
    - **G** and **H** have 21 independent components.
    - $\gamma$ has 9 independent components.
- The interactions felt by the fluid due to the walls are statistically planar and isotropic. We restric ourselves to planar flows.
- In order to compare the hydrodynamic equations with the MD simulations we need a discrete version of the theory.

# Discrete basis function

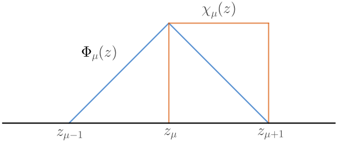
- $N_{\mathrm{bin}}$ bins with dimensions $L_x$, $L_y$, $\Delta z$ $(L_z/N_{\mathrm{bin}})$.



- Characteristic function $\chi_\mu(\mathbf{r})$ and finite element linear basis function $\Phi_\mu(\mathbf{r})$

$$\chi_\mu(\mathbf{r}) = \theta(z_{\mu+1} - z)\theta(z - z_\mu) = \chi_\mu(z)$$

$$\Phi_\mu(\mathbf{r}) = \chi_\mu(z)\frac{z_{\mu+1} - z}{\Delta z} + \chi_{\mu-1}(z)\frac{z - z_{\mu-1}}{\Delta z}$$



# Mass matrix and dual basis functions

- The usual mass matrix of the finite element method is

$$M^{\Phi}_{\mu\nu} = \Big(\Phi_\mu \Phi_\nu\Big)$$

where we have introduce the notation $\Big(\cdots\Big) = \int d\mathbf{r}...$

- We introduce the discrete velocity field in terms of $M^{\Phi}_{\mu\nu}$

$$\tilde{\mathbf{v}}_\mu = \sum_\nu \mathcal{V}_\mu [M^{\Phi}]^{-1}_{\mu\nu} \mathbf{v}_\nu$$

- We can contruct continuum and discrete fields from dual basis functions $\delta_\mu(\mathbf{r})$ and $\psi_\mu(\mathbf{r})$

$$v_\mu = \int d\mathbf{r} v(\mathbf{r}) \delta_\mu(\mathbf{r}), \qquad \overline{v}(\mathbf{r}) = \sum_\mu v_\mu \psi_\mu(\mathbf{r})$$

# Discrete equations of nanohydrodynamics

$$
\begin{aligned}
\frac{d}{dt}\rho_\mu &= \left(\overline{\rho}\ \overline{\mathbf{v}}\boldsymbol{\nabla}\delta_\mu\right) \\
\frac{d}{dt}\mathbf{g}_\mu &= \left(\overline{\rho\mathbf{v}}\ \overline{\mathbf{v}}\cdot\boldsymbol{\nabla}\delta_\mu\right) - \sum_\nu \left(\overline{\rho}\delta_\mu\boldsymbol{\nabla}\delta_\nu\right)\frac{\partial F}{\partial\rho_\nu}(\rho) \\
&\quad - \sum_\nu \mathcal{V}_\nu \frac{\mathbf{n}\cdot\left[\boldsymbol{\eta}_{\mu\nu} - \boldsymbol{\eta}_{\mu-1\nu} - \boldsymbol{\eta}_{\mu\nu-1} + \boldsymbol{\eta}_{\mu-1\nu-1}\right]}{\Delta z^2} : \mathbf{n}\tilde{\mathbf{v}}_\nu \\
&\quad + \sum_\nu \mathcal{V}_\nu \frac{\left[\mathbf{G}_{\mu\nu} - \mathbf{G}_{\mu\nu-1}\right]}{\Delta z}\cdot\mathbf{n}\tilde{\mathbf{v}}_\nu \\
&\quad + \sum_\nu \mathcal{V}_\nu \frac{\mathbf{n}\cdot\left[\mathbf{H}_{\mu\nu} - \mathbf{H}_{\mu-1\nu}\right]}{\Delta z}\cdot\tilde{\mathbf{v}}_\nu \\
&\quad - \sum_\nu \mathcal{V}_\nu \gamma_{\mu\nu}\cdot\tilde{\mathbf{v}}_\nu
\end{aligned}
$$

# Symmetry assumptions

- The system is isotropic when we rotate it with respect to an axis perpendicular to the walls...
- ... and reflect it with respect to a plane containing the axis.
- Large simplification of the structure of the tensors $\boldsymbol{\eta}_{\mu\nu}$, $\mathbf{G}_{\mu\nu}$, $\mathbf{H}_{\mu\nu}$ and $\gamma_{\mu\nu}$.
- Under the simplification we may separate the evolution of the selected variables in two contribution: normal and tangent.

## Normal and tangent evolution

- The normal evolution

$$\frac{d}{dt}\rho_\mu = \left(\bar{\rho}\ \bar{v}^z \nabla^z \delta_\mu\right)$$

$$\frac{d}{dt}\mathbf{g}_\mu^z = \left(\bar{\rho}\ \bar{v}^z\ \bar{v}^z \nabla^z \delta_\mu\right) - \left(\bar{\rho}\delta_\mu \nabla^z \delta_\nu\right) \frac{\partial F}{\partial \rho_\nu}(\rho) + M_{\mu\nu}^{\perp} \mathcal{V}_\nu \tilde{v}_\nu^z$$

- The parallel evolution for $\alpha = x, y$

$$\frac{d}{dt}\mathbf{g}_\mu^\alpha = -M_{\mu\nu}^{||} \mathcal{V}_\nu \tilde{v}_\nu^\alpha$$

- The dissipative matrix for $\odot = ||, \perp$

$$M_{\mu\nu}^{\odot} = -\frac{\eta_{\mu\nu}^{\odot} - \eta_{\mu-1\nu}^{\odot} - \eta_{\mu\nu-1}^{\odot} + \eta_{\mu-1\nu-1}^{\odot}}{\Delta z^2} + \frac{G_{\mu\nu}^{\odot} - G_{\mu\nu-1}^{\odot}}{\Delta z} + \frac{H_{\mu\nu}^{\odot} - H_{\mu-1\nu}^{\odot}}{\Delta z} - \gamma_{\mu\nu}^{\odot}$$

# The discrete transport kernels

$$\eta^{||}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\boldsymbol{\sigma}}^{xz}_\mu(t)\mathcal{Q}\hat{\boldsymbol{\sigma}}^{xz}_\nu \right\rangle \quad \eta^{\perp}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \langle \mathcal{Q}\hat{\boldsymbol{\sigma}}^{zz}_\mu(t)\mathcal{Q}\hat{\boldsymbol{\sigma}}^{zz}_\nu \rangle$$

$$G^{||}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\mathbf{F}}^{x}_\mu(t)\mathcal{Q}\hat{\boldsymbol{\sigma}}^{xz}_\nu \right\rangle \quad G^{\perp}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\mathbf{F}}^{z}_\mu(t)\mathcal{Q}\hat{\boldsymbol{\sigma}}^{zz}_\nu \right\rangle$$

$$H^{||}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\boldsymbol{\sigma}}^{xz}_\mu(t)\mathcal{Q}\hat{\mathbf{F}}^{x}_\nu \right\rangle \quad H^{\perp}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\boldsymbol{\sigma}}^{zz}_\mu(t)\mathcal{Q}\hat{\mathbf{F}}^{z}_\nu \right\rangle$$

$$\gamma^{||}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\mathbf{F}}^{x}_\mu(t)\mathcal{Q}\hat{\mathbf{F}}^{x}_\nu \right\rangle \quad \gamma^{\perp}_{\mu\nu} = \frac{1}{k_B T}\int_0^\tau dt \left\langle \mathcal{Q}\hat{\mathbf{F}}^{z}_\mu(t)\mathcal{Q}\hat{\mathbf{F}}^{z}_\nu \right\rangle$$

## Summary

- 
- We have obtained a set of nonlocal transport coefficients that are include in the nanohydrodynamic equations.

# Space and time locality for unconfined fluids

## Mori's theory

- Linear dynamic equations not only for the averages of the relevant variables but also for their correlations

$$\frac{d}{dt}C(t) = -L\cdot C^{-1}(0)\cdot C(t) - \int_0^t dt'\Gamma(t-t')\cdot C^{-1}(0)\cdot C(t')$$

where the following matrices have been introduced

$$L = \langle \hat{A} i\mathcal{L}\hat{A}^T\rangle$$
$$C(0) = \langle \hat{A}\hat{A}^T\rangle$$
$$\Gamma(t) = \langle F^+(t)F^{+T}(0)\rangle$$

- The projected forces are given by

$$F^+(t) = \exp\{\mathcal{Q}i\mathcal{L}t\}\mathcal{Q}i\mathcal{L}\hat{A}$$

- $\mathcal{Q} = 1 - \mathcal{P}$ where $\mathcal{P}$ is Mori's projector

$$\mathcal{P}\hat{F}(z) = \langle\hat{F}\rangle + \langle\hat{F}\hat{A}^T\rangle\cdot C^{-1}(0)\cdot\hat{A}(z)$$

# Markovian approximation

- Memory-less term

$$\int_0^t dt'\Gamma(t-t')\cdot C^{-1}(0)\cdot C(t') \simeq M^* C^{-1}(0)C(t)$$

- Expression for the correlations

$$\begin{aligned}\frac{d}{dt}C(t) &= -(L+M^*)\cdot C^{-1}(0)\cdot C(t)\\ &\equiv \Lambda^*\cdot C(t)\end{aligned}$$

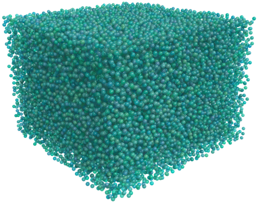
- For a linear Markovian theory the only possibility for a correlation is to decay in an exponential matrix way

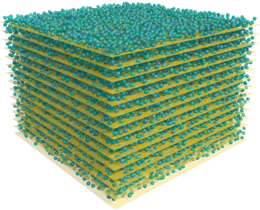
$$C(t) = \exp\{-\Lambda^*(t-\tau)\}\cdot C(\tau)$$

- **We need to find a constant matrix $\boldsymbol{\Lambda}^*$.**

# Simpler case: unconfined fluid

- The system





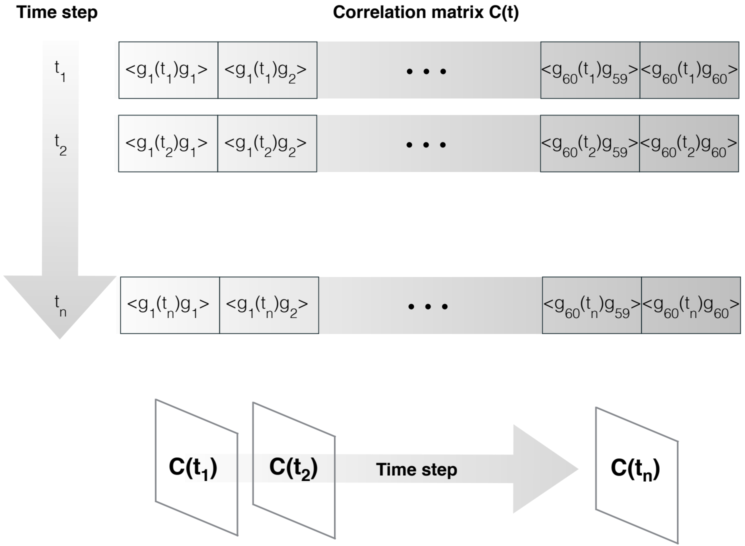
- The relevant variable

$$\hat{\mathbf{g}}_\mu(z) = \sum_i^N \mathbf{p}_i \delta_\mu(\mathbf{r}_i)$$

# Simulation set up

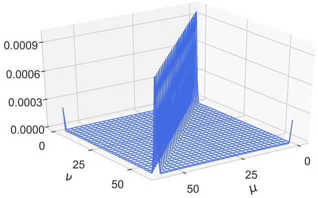
- Simulation of 28749 particles interacting with a LJ potential truncated at $\sigma = 2.5$.
- Box size $40x40x30$.
- $dt = 0.002$ in reduced units.
- Equilibration stage
  - Langevin thermostat for $10^5$ timesteps: $T = 2.0$, $\rho = 0.6$.
  - NVE microcanonical conditions for a further $10^5$ timesteps.
- Production stage
  - $1.5 \times 10^6$ timesteps.
  - $z$ axis binned in 60 bins $\mu$. $\boldsymbol{\Delta z = 0.5\sigma}$.
  - $g^x_\mu(t)$ recorded every 10 timesteps.

# Building the correlation matrix $C(t)$

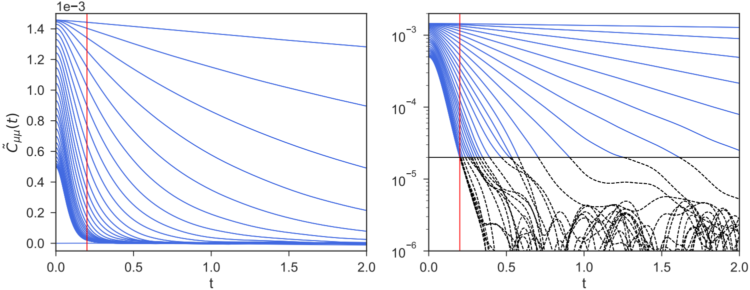


# The correlation matrix $C(t)$ and its eigenvalues $\tilde{C}_{\mu\mu}$

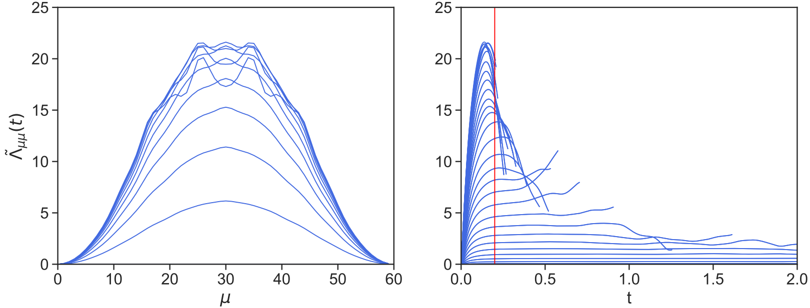
- The correlation matrix $C(t)$ at $t = 0$ (left) and $t = 0.6$ (right)




- The evolution of the different eigenvalues $\tilde{C}_{\mu\mu}(t)$.



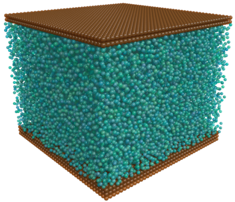
# Validation of the Markovian approximation

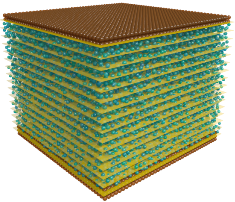



In the left panel, in ascending order the plotted times go from $t = 0$ to $t = 0.20$ in intervals of 0.02. In the right pannel the time evolution of $\tilde{\Lambda}_{\mu\mu}(t)$.

# Markovian behaviour near solids

- The system





- The CG variable and its correlation

$$\hat{\mathbf{g}}_\mu^x = \sum_i^N \mathbf{p}_i \delta_\mu(\mathbf{q}_i), \quad \hat{g}^T = (\hat{\mathbf{g}}_1^x, \cdots, \hat{\mathbf{g}}_{N_{\mathrm{bin}}}^x), \quad C(t) = \left\langle \hat{g}(t) \hat{g}^T \right\rangle$$

# Simulation set up

- Simulation of 28175 fluid particles interacting with a LJ potential truncated at $\sigma = 2.5$.
- Two solid walls in the $xy$ plane confine the fluid.
- Box size 40$x$40$x$33.
- $dt = 0.002$ in reduced units.
- Equilibration stage
    - Langevin thermostat for $10^5$ timesteps: $T = 2.0$, $\rho = 0.6$.
    - NVE microcanonical conditions for a further $10^5$ timesteps.
- Production stage
    - $12 \times 10^6$ timesteps.
    - $z$ axis binned in 66 bins $\mu$ $\mathbf{\Delta z = 0.5\boldsymbol{\sigma}}$ or 33 bins $\mu$ $\mathbf{\Delta z = 2\boldsymbol{\sigma}}$.
    - $g^x_\mu(t)$ recorded every 2 timesteps.

## Reciprocal space

- Eigenvalues $\tilde{C}_\mu$ and eigenvectors $u_\mu$

$$C(t) = \sum_{\mu}^{N_{\rm bin}} \tilde{C}_\mu(t) u_\mu(t) \otimes u_\mu^T(t)$$

- Unitary matrix $E(t)$

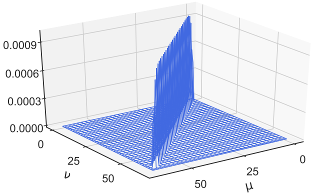
$$E^{-1}(t) \cdot C(t) \cdot E(t) = \tilde{C}(t),$$

- We observed that $\dot{E} \simeq 0$.
- The predictions of $C(t)$ in the reciprocal space

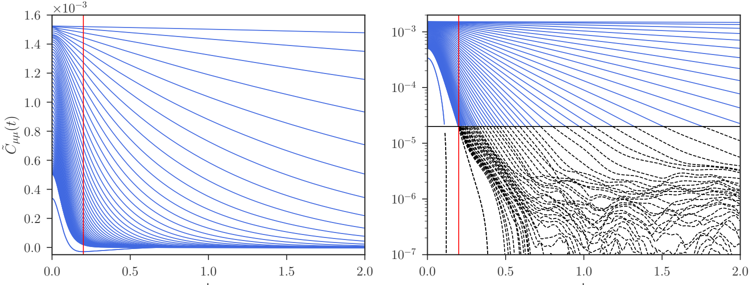
$$\tilde{C}_\mu(t) = \exp\{-\tilde{\Lambda}_{\mu\mu}(t-\tau)\} \tilde{C}_\mu(\tau)$$

# Thin bins ($\Delta z = 0.5\sigma$)

- $C_{\mu\nu}(t)$ for $t = 0$ (left) and $t = 0.6$ (right).

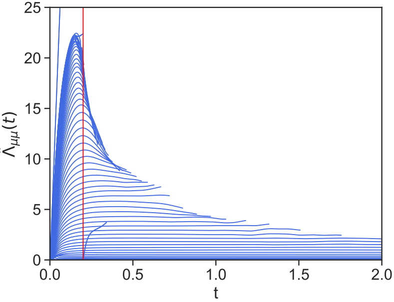



- Evolution of different eigenvalues $\tilde{C}_{\mu\nu}(t)$



# $\tilde{\Lambda}(t)$ ($\Delta z = 0.5\sigma$)

Diagonal elements $\tilde{\Lambda}_{\mu\mu}(t)$ of $\Lambda(t)$ in the reciprocal space. After a time $\tau = 0.2$ we observe a nice plateau for the lower modes.



# Eigenvalues and eigenvectors near the walls ($\Delta z = 0.5\sigma$)

The eigenvalues $\tilde{C}_\mu(t)$ of the correlation matrix $C(t)$ for $\mu = 59, 60$ which are identical and superimpose (left) and the corresponding eigenvectors $u_\mu$ in blue and orange, respectively (right).