Readings about Strings and Tries

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I. Manacher's Algorithm

Statement: Find all pairs (i, j) which make a palyndrome substring.

A. Detailed Statement

Worst case we would have $O(n^2)$ palyndromes.

Compact way of keeping palyndromes: $i = 0 \ldots n-1$, $d_1[i], d_2[i]$ represent the number of palyndromes with odd and even length with their center in i.

Ex. *abababc*, has $d_1 = 3$ (odd length), and *cbaabd* has $d_2[3] = 2$ (even length).

Sub-palindrome with l size with center in i, we also have with $l-2, l-4, \ldots$.

Both palyndrom arrays can be calculated in linear time.

Solution: Can be done with string hashing and suffix trees, but this has a smaller constant and memory complexity.

B. Trivial Algorithm

Tries to increase the answer by one until it's possible for each center i. It is $O(n^2)$ in time. Implementation being:

```
vector<int> d1(n),  d2(n);
for (int i = 0; i < n; i++) {
    d1[i] = 1; // Pair with itself
    while (0 <= i - d1[i] && i + d1[i] < n &&
        s[i - d1[i]] == s[i + d1[i]]) {
        d1[i]++;
    }

    d2[i] = 0;
    while (0 <= i - d2[i] - 1 && i + d2[i] < n
         && s[i - d2[i] - 1] == s[i + d2[i]])
        {
        d2[i]++;
    }
}
```

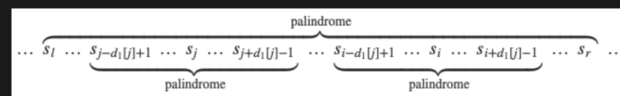
C. Manacher's Algorithm

Allows to find all the sub-palyndromes with odd length (even length is just a small mod).

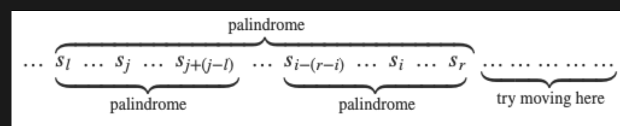
Borders: (l, r) of the palyndrome with maximal r. Initially we assume $l = 0, r = -1$

We want to calculate $d_1[i]$ with all the previous i already calculated.

- If it is ourside of the current rightmost sub-palyndrome $i > r$, we launch the trivial algorithm, allowing us to calculate $d_i[i]$ and updating (l, r).
- $i \leq r$. We can flip i into $j = l + (r - i)$ and since they are symmetrical, we can **almost always** do $d_1[i] = d_1[j]$. Like so:



- **Trick case:** When the inner palyndrome reaches the border of the outer one. $j - d_1[j] + 1 \leq l$. Since the symmetry of the outer palyndrome is not guaranteed, we assign $d_1[i] = r - i + 1$, and then run the trivial algorithm to increase the value if necessary, updating in the end. Like so:



Note we only use the part of the palyndrome with guaranteed symmetry before moving to the *try moving here part*.

A similar algorithm is then used for the even part.

D. Complexity

Not as intuitive, but if we look at *Z-function building algorithm* we can check it's similar and linear.

We can also notice that r can only increase by one ine very iteration of the trivial algorithm, and that it can't decrease. $O(n)$ iterations. The rest is linear.

E. Implementation

Here is the implementation, which is fairly similar for both d_1, d_2.

```
vector<int> d1(n), d_2(n);
for (int i = 0, l = 0, r = -1; i < n; i++) {
    int k = (i > r) ? 1 : min(d1[l + r - i], r
         - i + 1);
    while (0 <= i - k && i + k < n && s[i - k]
         == s[i + k]) {
        k++;
    }
    d1[i] = k--;
    if (i + k > r) {
        l = i - k;
        r = i + k;
    }
}
for (int i = 0, l = 0, r = -1; i < n; i++) {
    int k = (i > r) ? 0 : min(d2[l + r - i +
        1], r - i + 1);
    while (0 <= i - k - 1 && i + k < n && s[i
        - k - 1] == s[i + k]) {
        k++;
    }
    d2[i] = k--;
    if (i + k > r) {
```

```
        l = i - k - 1;
        r = i + k ;
    }
}
```

II. Z-FUNCTION

String s of length n. The function returns an array of length n where $Z[i]$ is equal to **greatest number of characters starting from position** i**, that coincide with the first characters of** s. Longest common prefix between s, and suffix of s starting at i.

We assume 0 based indexing and $z[0] = 0$. It is $O(n)$ time.

Ex. $aaaaa = [0, 4, 3, 2, 1], aaabaab = [0, 2, 1, 0, 2, 1, 0], abacaba = [0, 0, 1, 0, 3, 0, 1]$.

A. Trivial Algorithm

This one is $O(n^2)$:

```
vector<int> z_function_trivial(string s) {
    int n = (int) s.length();
    vector<int> z(n);
    for (int i = 1; i < n; ++i)
        while (i + z[i] < n && s[z[i]] == s[i
            + z[i]])
            ++z[i];
    return z;
}
```

Basically calculating $z[i]$ for each i independently.

B. Efficient

We make use of the previously computed values.

Segment Match: a substring which coincides with a prefix of s. So $z[i]$ is the length of the segment match that starts at i at ends at $i + z[i] - 1$.

(l, r) will be the indices of our rightmost segment match. r can be seen as the boundary to which our string has been scanned by the algorithm.

If we are in index i, there are two options to calculate:

- $i > r$, so we haven't processed the current position yet. So we run the trivial algorithm. If $z[i] > 0$ we have to update $r = i + z[i] - 1$.
- Otherwise, we are inside the current segment match. We can use the previou values to give us a head start, through the value $z[i - l]$, since $sl..r$ and $s[0..l - r]$ match. However, the value might be too large, and when applied to i we could get an overflow. For example $aaaabaa$, in $i = 6$ we are not able to initialize it with $z[1] = 3$ since we would overflow the array. So we instead do:

  $$z_0[i] = min(r - i + 1, z[i - l])$$

  Then we increment $z[i]$ with the trivial algorithm, to see if it continues to match or not.

The only difference between the cases is the initial value of $z[i]$, then both branches use the trivial algorithm.

C. Implementation

```
vector<int> z_function(string s) {
  int n = (int) s.length();
  vector<int> z(n);
  for (int i = 1, l = 0, r = 0; i < n; ++i) {
    if (i <= r)
      z[i] = min (r - i + 1, z[i - l]);
    while (i + z[i] < n && s[z[i]] == s[i + z[
        i]])
      ++z[i];
    if (i + z[i] - 1 > r)
      l = i, r = i + z[i] - 1;
  }
  return z;
}
```

Returns an array of size n.

The initial array is initially filled with 0 and the rightmost match is $[0, 0]$. We then first check what option of the algorithm we are gonna use, and in the end if necessary, we update the rightmost segment $(i + z[i] - 1 > r)$.

D. Asymptotic Behavior

It's $O(n)$, and we only need to focus on the `while` loop, showing that each iteration will increase the right border r. So we consider both branches of the algorithm.

- $i > r$: If $s[0] \neq s[i]$ it won't make any iterations, otherwise it will necessarily move to the right and update r.
- $i \leq r$. We initialize $z[i]$ to z_0, and we need to compare this to $r - i + 1$ so we have a trichotomy:
  - $z_0 < r - i + 1$: No iteration of the loop will take place. If it did, the initial z_0 approximation was inaccurate. But that is not true because we know $s[l..r]$ and $s[0...r - l]$ match.
  - $z_0 = r - i + 1$: In this case, it will make a few iterations, but each will lead to an increase of r since we will start comparing from $s[r + 1]$ increasing the (l, r) interval.
  - $z_0 > r - i + 1$: By definition, can't happen.

Each iteration of the inner loop makes r increase, and at most $n - 1$ times, since it won't be able to overflow the array.

The rest of the algorithm clearly runs in $O(n)$ so it is proven.

E. Applications

Very similar to the prefix function ones.

1) Search the Substring: Find all the ocurrences of a pattern p inside a text t.

We create a new string $p + \alpha + t$, α being a separator character we are sure won't appear.

Compute the function for this string, then for every i in the range $[0; length(t) - 1]$ consider the corresponding value.

$$k = z[i + length(p) + 1]$$

If k is equal to $length(p)$ we know there is one ocurrence in the i position of the text.

Running time and memory consumption of $O(length(t) + length(p))$

2) Number of distinct substrings: **Iterative approach:** Knowing the current number of different substrigns, recalculate after adding the end of s one character.

k being the current number, we add c to s. There can be new substrings which end in c (all the ones that end with this symbol and we haven't encountered yet).

Take a string $t = s + c$ and invert it. So, now we have to count how many prefixes in t are not found anywhere else in t.

Compute the Z-function of t and find its max value, z_{max}. The number of new substrings that appear when symbol c is appended to s is equal to:

$$length(t) - z_{max}$$

Running time being $O(n^2)$ We can recalculate in $O(n)$ time the amount of substrings when adding at the beginning and removing.

3) String Compression: In s find a string t of shortest length, such that s can be represented as concatenations of t.

Z-function solution: compute de function of s loop through all i which divide n, stop at the first i such that $i + z[i] = n$. The string then can be compressed to length i.

Can also be done with prefix function, check that for the proof.

References

[1] *Manacher's Algorithm - Finding all sub-palindromes in $O(N)$*, jakobkogler for E-maxx. From: https://cp-algorithms.com/string/manacher.html

[2] *Z-function and its calculation*, E-maxx. From: https://cp-algorithms.com/string/z-function.html