Readings about DP and Recursion

Diego Linares - kiwiAipom

I. Everything about Dynamic Programming

II. Some general approach for solving recursive problems

Step One: Think about any input for which you know what your function should return.
Now suppose you have a task, related to a similar one. Keep calling that function to solve it: *I'll solve the problem if you give me this subproblem first*. Which is done by a call to the same function.
Example: With factorial, you only know that $0! = 1$ and $n! = n(n-1)!$. So the function that gives me factorial of n just needs the results of one that returns $(n-1)!$. This will keep going as long as we don't know whaat value to return.

```
factorial(n):
    if n = 0:
        return 1 // I know this, so I don't
            want my function to go any further
    else:
        return n*factorial(n-1) // just reuse
            the function
```

Step Two: They can do the same as loops, a simple `for` can be implemented as:

```
for(i, n):
    if i = n:
        return // Terminates
    // Do whatever needed
    for(i+1, n) // Next iteration
```

And for backwards:

```
rof(i, n):
    if i = n:
        return // Terminates
    rof(i+1, n) // Next iteration
    // Do whatever needed
```

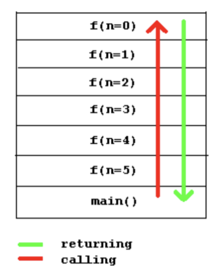
Since the function calls itself again until reaching a limit value and then starts returning.
Example: To print numbers backwards you may do this:

```
function(i, n):
    if i <= n:
        function(i+1, n)
        print(i)
```

Which for numbers from 1 to 5 would work like:

```
call function_1 with i=1
    call function_2 with i=2
        call function_3 with i=3
            call function_4 with i=4
                call function_5 with i=5
                    call function_6 with i=6
                        i breaks condition, no more calls
                    return to function_5
                    print 5
                return to function_4
                print 4
            return to function_3
            print 3
        return to function_2
        print 2
    return to function_1
    print 1
return to main, done!
```

Step Three: There's a stack call which looks like this:



The memory of $f(3)$ for example, won't be freed until $f(2)$ is done. Serves the purpose of using an array, since the functions store variables and values.
Step Four: Be careful, in CP they are generally avoided since most can be done iteratively, andd they may exceed time and memory. Since every function is alloted a space at the moment it is called, might run into RTE. Use only $O(\lg n)$ and small $O(n)$ recursions.
Step Five: When there are overlapping branches in the recursion tree we store computed values (DP).

III. Dynamic programming 2

IV. Matrix

A. Cutting to the Chase

It can be slow if not done properly.
Example: Obtain x^n if multiplying is $O(1)$. Then $x^n = x * x * \ldots = O(N)$. We can reduce it like: $x^n = x^2 * x^2 * \ldots$, and now the constant is now half. If we do $x^{\sqrt{n}} * x^{\sqrt{n}} * \ldots$ and now the constant is $O(\sqrt{N})$
So let's go faster. For even numbers $x^n = x^{\frac{n}{2}} * x^{\frac{n}{2}}$ and for odds $x^n = x^{\frac{n}{2}} * x^{\frac{n}{2}} * x$.
Since the terms keep repeating themselves, we only need to calculate $x^{\frac{n}{2}}$ once and so on. This creates $O(\lg n)$ complexity.
Since matrices are an **associative mathematical structure**, this applies to it too.

B. Don't get stuck with struct

Matrix multiplication goes as such:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} * \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 * b_1 + a_2 * b_3 & a_1 * b_2 + a_2 * b_4 \\ a_3 * b_1 + a_4 * b_3 & a_3 * b_2 + a_4 * b_4 \end{bmatrix}$$

Basically, each row times each column of the other matrix. Now we can define a struct like:

```
struct Matrix
{
    int m[N][N];
    matrix()
    {
        memset(m,0,sizeof(m));
    }
    matrix operator * (matrix b)
    {
        matrix c = matrix();
        for(int i = 0; i < n; i++)
            for(int j = 0; j < n; j++)
                for(int k = 0; k < n; k++)
                    c.m[i][j] = c.m[i][j] + m[
                        i][k] * b.m[k][j]
        return c;
    }
    matrix modPow(matrix m, int n)
    {
        if(n == 0)
            return unit;
        matrix half = modPow(m,n/2);
        matrix out = half * half;
        if(n % 2)
            out *= m;
        return out;
    }
}
```

The operator $*$ was defined so the matrix could be treated as a number. Unit refers to the unit matrix.

C. N-th Fibonacci Term

Knowing how Fibonacci works, we can put it in the form of a matrix. Knowing that each term **is dependent on the previous two** we need a 2 row matrix. From the pair (F_{n-2}, Fn_1) we compute (F_{n-1}, Fn), like so:

$$F_n = F_{n-1} * 1 + F_{n-2} * 1$$

$$F_{n-1} = F_{n-1} * 1 + F_{n-2} * 0$$

In the form of a matrix:

$$\begin{bmatrix} F_{n-2} & F_{n-1} \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_{n-1} & F_n \\ 0 & 0 \end{bmatrix}$$

We can go one step back to see the pattern:

$$\begin{bmatrix} F_{n-3} & F_{n-2} \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} F_{n-2} & F_{n-1} \\ 0 & 0 \end{bmatrix}$$

Finally

$$\begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-2} = \begin{bmatrix} F_{n-1} & F_n \\ 0 & 0 \end{bmatrix}$$

And we already know the power of a matrix is logarithmic.

D. Bits and Pieces

How many arrays of length n with maximum k consecutive 0 bits are.
Supposing we can use DP for the problem. $D_{n.k}$ is the number of arrays of length n which **end** in k 0s.
Considering a string can only be added a 0 or a 1. From (n, k) we can go to $(n+1, k+1)$ (if we add 0) and $(n+1, 0)$ (we add 1).
So $D_{n,0} = \sum_{i=0}^{k} D_{n-1,i}$ (all the strings of length n which do not end in 0). And $D_{n,k} = D_{n-1,k-1}$. So in recursive form:

$$D_{n,0} = \sum_{i=0}^{k} D_{n-1,i}$$

$$D_{n,1} = D_{n-1,0}$$

$$D_{n,2} = D_{n-1,2}$$

$$\ldots$$

$$D_{n,k} = D_{n-1,k-1}$$

And this can be explained through a matrix.

$$\begin{bmatrix} D_{n-1,0} & D_{n-1,1} & \cdots & D_{n-1,k} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 1 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Yielding the result:

$$\begin{bmatrix} D_{n,0} & D_{n,1} & \cdots & D_{n,k} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The complexity is reduced from $O(n * k)$ to $O(\lg n * k^3)$, where k^3 is the complexity of multiplying the matrices.

E. How big can it get?

This is a solution to the problem *Chimney* from TopCoder. **Worth checking out**.

```
Layer 1      Layer 2
+-----+--+   +--+-----+
|  1  |  |   |  |  B  |
+--+--|2 |   | A+--+--+
|  |  |  |   |  |  |  |
| 4+--+--+   +--+--+C |
|  |  3  |   |  D  |  |
+--+-----+   +-----+--+
```

We need to find out the different orders in which bricks can be placed for n layers, only restriction is that the 2 bricks that support the brick above are set before it.
We can show it as a matrix here:

```
+-----+--+   +-----+--+   +-----+--+   +-----+--+
|xxxxx|  |   |     |  |   |xxxxx|  |   |     |xx|
+--+--|  |   +--+--|  |   +--+--|  |   +--+--|xx|
|  |  |  |   |xx|  |  |   |  |  |  |   |xx|  |xx|
|  +--+--+   |xx+--+--+   |  +--+--+   |xx+--+--+
|  |     |   |xx|xxxxx|   |  |xxxxx|   |xx|xxxxx|
+--+-----+   +--+-----+   +--+-----+   +--+-----+
```

```
    1            2           3           4

+-----+--+  +-----+--+  +-----+--+  +-----+--+
|     |  |  |     |xx|  |     |xx|  |     |xx|
+--+--|  |  +--+--|xx|  +--+--|xx|  +--+--|oo|
+--+--|  |  +--+--|xx|  +--+--|xx|  +--+--|oo|
|xx|  |  |  |xx|  |oo|  |xx|  |oo|  |xx|  |oo|
|xx+--+--+  |xx+--+oo+  |xx+--+oo+  |xx+--+oo+
|=====xxx|  |xx|xxxoo|  |===== oo|  |===@@@@@|
+--+-----+  +--+-----+  +--+-----+  +--+-----+
    5            6           7           8
```

Where x are the bricks in n layer, o and $=$ in the $n+1$ and @ in the $n+2$. After we place 2 bricks next to the other we can put a brick in the layer above.
Can be solved with matrix multiplication, $9*9$ matrix and multiplying it logarithmically. The base matrix being:

```
int mat[9][9] =  // constructing matrix column
     by column
    {{0,0,0,0,1,0,0,0,0},
    {4,0,0,0,0,0,1,0,0},
    {0,2,0,0,0,0,0,1,0},
    {0,1,0,0,0,0,0,0,0},
    {0,0,2,2,0,0,0,0,0},
    {0,0,1,0,0,0,0,0,1},
    {0,0,0,0,2,2,0,0,0},
    {0,0,0,0,0,0,1,0,0},
    {0,0,0,0,0,0,0,1,0}};
```

F. *Summing Up*

Useful tool but not reccommended outside of contests.

References

[1] *Attacking Recursions*, I, Me and Myself. From: https://zobayer.blogspot.com/2009/12/cse-102-attacking-recursion.html
[2] *Matrix*, DanAlex's blog, From: https://codeforces.com/blog/entry/21189