Math formulas for Competitive Programming

1 Number theory

Theorem 1.1 (Chicken McNugget Theorem). *Let a, b be two coprime positive integers. Then, the largest non-negative integer that cannot be expressed as $ax + by$ $(x \geq 0, y \geq 0)$ is equal to $ab - a - b$.*

Corollary 1.1.1. *Let a, b be two coprime positive integers. Then, there are exactly $\frac{(a-1)(b-1)}{2}$ non-negative integers that cannot be expressed as $ax + by$ $(x \geq 0, y \geq 0)$.*

Definition 1.2 (p-adic valuation of n). *For a given natural number n and a prime p, the p-adic valuation of n is the highest exponent of a power of p that divides n and is denoted as $v_p(n)$.*

Remark.

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \tag{1}$$

Theorem 1.3 (Legendre's Formula). *Let n be a natural number and p be a prime. Let $s_p(n)$ be the sum of digits of n in base p. Then,*

$$v_p(n!) = \frac{n - s_p(n)}{p - 1}$$

Corollary 1.3.1.

$$v_p(\binom{n}{m}) = \frac{s_p(m) + s_p(n-m) - s_p(n)}{p - 1}$$

Corollary 1.3.2.

$$v_p(\binom{n}{m}) = O(log_p(n))$$

Theorem 1.4 (Kummer's Theorem). *Given a prime p and integers $0 \leq m \leq n$, the p-adic valuation of the binomial coefficient, denoted as $v_p(\binom{n}{m})$, is equal to the number of carries when adding $n - m$ and m in base p.*

Lemma 1.5. *For any natural number n and any prime p, we have:*

$$(1 + x)^{p^n} \equiv 1 + x^{p^n} \pmod{p}$$

Theorem 1.6 (Lucas' Theorem)**.** *Given a prime p and non-negative integers n, m such that expressed in base p they are:*

$$n = \sum_{i=0}^{k} n_i p^i$$

$$m = \sum_{i=0}^{k} m_i p^i$$

where k is the maximum of the lengths of n and m in base p (we can complete with leading zeroes for the minimum length). Then, we have:

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}$$

(*using the convention that if $n < m$ then $\binom{n}{m} = 0$*)

Corollary 1.6.1. *Given two integers n, p such that $1 \leq p \leq n$ and p is prime, then*

$$\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod{p}$$

Corollary 1.6.2. *Given a prime p and a non-negative integers n with a p-expansion*

$$n = \sum_{i=0}^{k} n_i p^i$$

Then there are $\prod_{i=0}^{k} (n_i + 1)$ integers x such that $\binom{n}{x}$ is not divisible by p. In other words, the n-th row of Pascal's triangle has $\prod_{i=0}^{k} (n_i + 1)$ numbers that are not divisible by p.

Lemma 1.7 (Harmonic bounds)**.** *Let H_n be the harmonic sum ($H_n = \sum_{i=1}^{n} \frac{1}{i}$). Then, $\ln(n+1) \leq H_n \leq 1 + \ln(n)$*

Theorem 1.8 (Erdős–Szekeres Theorem)**.** *Any sequence of* ***distinct*** *real numbers with length at least $L = rs + 1$ (with r, s non-negative integers numbers) has*

- *A monotonically increasing subsequence with length at least $r + 1$, or*
- *A monotonically decreasing subsequence with length at least $s + 1$*

Corollary 1.8.1. *Any permutation of n numbers has*

- *A monotonically increasing subsequence with length at least $\lfloor \sqrt{n} \rfloor$, or*
- *A monotonically decreasing subsequence with length at least $\lfloor \sqrt{n} \rfloor$*

2 Graph

Lemma 2.1 (Euler's Formula - Planar Graph)**.** *For any planar graph with V vertices, E edges, F faces and C connected components, then $V - E + F = C + 1$*

Lemma 2.2 (Cayley's Formula)**.** *The number of spanning trees of a complete graph of n vertices is n^{n-2}*

Lemma 2.3 (Kirchoff's matrix tree theorem)**.** *Let A be the adjacency matrix where A_{uv} is the number of edges between u and v. Let D be the degree matrix where D_{uu} is the degree of vertex u. Let $L = D - A$ be the Laplacian Matrix. Then, all cofactors of L are equal to each other and they are equal to the number of spanning trees of the graph.*

3 Geometry

Lemma 3.1 (Euler's Formula)**.** *For any convex polyhedron with V vertices, E edges and F faces, then $V - E + F = 2$*