Flow notes

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Let $G(V, E)$ be a Graph

Flows

An $(s-t)$ flow is a function $f: E \rightarrow \mathbb{R}$ that satisfies the following conditions:

- Skew Symmetry: $\forall\ (u, v) \in E,\ \ f(u, v) = -f(v, u)$
- Conservation constraint: $\forall\ u \in V - \{s, t\},\ \sum_{v \in V} f(u, v) = 0$
- Capacity constraints: $\forall\ (u, v) \in E, f(u, v) \leq c(u, v)$

Properties:

- **Flow Decomposition Theorem:** Any $(s-t)$ flow can be written as a linear combination of directed $(s-t)$ simple paths and directed cycles.
- **Uniqueness condition:** A maximum $(s-t)$ flow is unique iff the residual graph is acyclic
- Ford-Fulkerson algorithm may not terminate with irrational capacities. Edmonds-Karp and Dinic's work fine with irrational capacities.

Cuts

A cut (S, T) is a partition of V into S and $T = V - S$ such that s (source) belongs to S and t (sink) belongs to T.

The capacity of the cut is

$$\boxed{c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)}$$

Algorithm to find minimum cut

Find maximum flow and define S = {all vertices such that there exists a path from them to s in the final residual network} and $T = V - S$. Then (S, T) will be a minimum cut.

Properties:

- If (u, v) is part of any minimum edge cut, then (v, u) is part of no minimum edge cut.
- Let's (S, T), (S', T') be two minimum cuts. Then $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum cuts
- Let f be a maximum flow. Let C_s be the vertices reachable from s in the residual network of f. Let C_t be the vertices that can reach t in the residual network of f. Then $(C_s, V - C_s)$ and $(V - C_t, C_t)$ are minimum cuts, and for any minimum cut $(C, V - C)$ it holds that $C_s \subseteq C \subseteq V - C_t$
- **Uniqueness condition:** A maximum $(s - t)$ flow is unique iff $C_s = V - C_t$

Coverings, Matching, Independent Set

Source: https://www.epfl.ch/labs/dcg/wp-content/uploads/2018/10/GT-4-Covers.pdf

Preliminaries

Bipartiteness:

A graph is bipartite if its vertices can be divided into two disjoints sets such that there is no edge between vertices of the same set.

Necessary and sufficient condition:

A graph is bipartite iff it doesn't have and odd cycle.

Definitions

- **Matching :** Is a set $M \subset E$ such that the edges in M are pairwise disjoint
- **Vertex Cover:** Is a set $C \subset V$ such that every edge of G is incident to a vertex of C.
- **Edge Cover:** Is a set $C \subset E$ such that every vertex of G is incident to an edge in C (this concept is only defined in graph without isolated vertex)
- **Independent set:** Is a set $I \subset V$ such that no two vertices in I are adjacent.

Inequalities

For any arbitrary Graph:

$$\boxed{|maximum\ matching| \leq |minimum\ vertex\ cover|}$$

For any arbitrary Graph without isolated vertices:

$$\boxed{|maximum\ independent\ set| \leq |minimum\ edge\ cover|}$$

Gallai Theorem:

For any arbitrary Graph:

$$\boxed{|maximum\ independent\ set| + |minimum\ vertex\ cover| = |V|}$$

For any arbitrary Graph without isolated vertices:

$$\boxed{|maximum\ matching| + |minimum\ edge\ cover| = |V|}$$

Konig Theorem:

Source: https://www.epfl.ch/labs/dcg/wp-content/uploads/2018/10/GT-3-Matchings.pdf

If the graph is bipartite,

$$\boxed{|maximum\ matching| = |minimum\ vertex\ cover|}$$

If, additionally, doesn't have isolated vertices,

$$\boxed{|maximum\ independent\ set| = |minimum\ edge\ cover|}$$

Hall's Theorem:

- **Definition:** A matching M "covers" $A \subset V$ if every vertex in A is an endpoint of an edge of the matching.
- **Definition:** $N(S)$ is the set of neighbours of each node of S

Theorem: Let G be a bipartite graph with bipartition $V = A \cup B$. Then G has a matching that covers A if and only if for all $S \subset A$ we have $|N(S)| \geq |S|$.

Algorithm for finding each of them in Bipartite Graph:

Let say that our bipartite graph G has the partition $V = L \cup R$

- **Maximum matching**: Run the max flow algorithm on G. All the edges between $L\ and\ R$ that have flow are edges of a maximum matching

- **Minimum edge cover**: Let denote the maximum matching size by |M|. Take the |M| edges of the maximum matching. For the other |V| - 2 |M| unmatched vertices, take one of its edges (the other endpoint must be matched). This set of edges is a minimum edge covering.

- **Minimum vertex cover**: Find a minimum cut (S, T) . Take all the edges of the cut (those that goes from S to T). All the vertices that belong to those edges (except from the source and the sink) form a minimum vertex cover.

(Source: http://theory.stanford.edu/~trevisan/cs261/lecture14.pdf)

- **Maximum Independent set:** Take all the vertices that are not in the minimum vertex cover. These vertices form a maximum independent set.

Vertex/Edge Connectivity

Menger's Theorem:

- Maximum number of edge-disjoint paths from s to t equal the minimum $s - t$ edge cut (minimum number of edges whose removal disconnects s and t)
- Maximum number of vertex-disjoint paths from s to t equal the minimum $s - t$ vertex cut (minimum number of vertices whose removal disconnects s and t)

Both of these statements are also a consequence of the max flow min cut theorem.

Partially Ordered Sets

Definitions:

- **Partial Order:** A (strict) partial order over a set V is a binary relation, $\prec$, over V that is:
    1. irreflexive: for all $x, y \in V$ and $x \neq y$, $x \prec y$ implies $y \nprec x$
    2. transitive: for all $x, y, z \in V$, $x \prec y$ and $y \prec z$ implies $x \prec z$.

  Also, if $x \prec y$ or $y \prec x$, then we say that these elements are comparable; otherwise they are incomparable.

  We can represent a poset (partially ordered set) as a DAG.

- **Chain:** Is a subset of V such that every pair of elements is comparable
- **Antichain:** Is a set of V such that every pair of elements is incomparable.

  Note: A one element is both a chain and an antichain

- **Chain partition:** Is a partition of V (group of pairwise disjoint non-empty subsets of V) such that each subset is a chain.
- **Antichain partition:** Is a partition of V such that each subset is an antichain.
- **Height:** The size of the maximum chain
- **Width:** The size of the maximum antichain

Inequations:

$$\boxed{|any\ chain| \leq |any\ antichain\ partition|}$$

$$\boxed{|any\ anti\ chain| \leq |any\ chain\ partition|}$$

Mirsky's Theorem:

Statement: In a poset, it holds that

$$\boxed{|maximum\ chain| = |minimum\ antichain\ partition|}$$

That means that a poset of **height** H can be partitioned in H chains

Construction of the minimum antichain partition: Recursively remove the minimal (maximal) elements of the poset. Note that all minimal (maximal) elements at each iteration, form an antichain.

Minimal (maximal) elements in a DAG are the ones with outdegree (indegree) equals 0.

Construction of maximum chain: We can start with the nodes with indegree 0 and trying to pick the best choice of the chain using dp (or topological sorting).

Dilworth Theorem:

Inductive proof : https://pwp.gatech.edu/math3012openresources/lecture-videos/lecture-14/

Constructive proof : https://web.stanford.edu/class/cs361b/files/cs261-Jan2014-notes.pdf

Statement: In a poset, it holds that

$$\boxed{|maximum\ anti\ chain| = |minimum\ chain\ partition|}$$

That means that a poset of **width** W can be partitioned in W chains.

Also $\boxed{|maximum\ matching| + |minimum\ chain\ partition| = |V|}$

$$\boxed{|maximum\ matching| + |maximum\ antichain| = |V|}$$

Construction:

Let's denote the DAG of the poset as $G(V, E)$

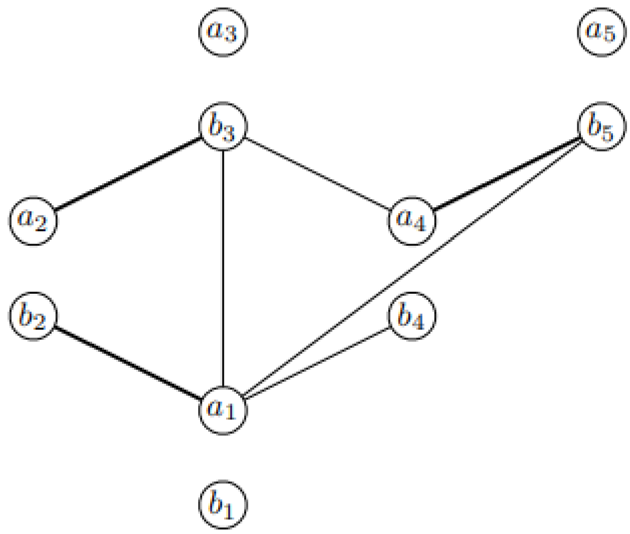
Let's construct the bipartite graph $G'(V', E')$ where

$V' = \{a_i, b_i \mid x_i \in V\}$, that means we create 2 nodes in G' for each node in G.

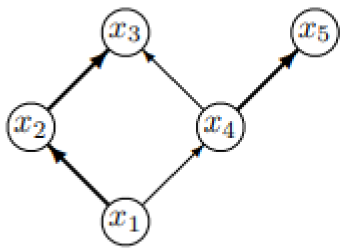
$E' = \{(a_i, b_j) \mid x_i \prec x_j\ in\ G\}$ that means that we create and edge in the G' for each pair of vertex in G such that x_i is an ancestor of x_j.

If we denote $n = |V|$. Then it holds that

- For any matching M' in G' , we can project each edge of the matching to an edge in G and it forms a chain partition ρ. Each chain of the partition if forms by the maximal union of edges that are adjacent in the projection of M'.
  Moreover: $n = |M'| + |\rho|$
  See the example below:



Bipartite Graph G' with a matching in bold



Original Graph G with the chain partition in bold

- For any vertex cover S' in G' , there exists an antichain U in G such that $|S'| + |U| \geq n$. The antichain is form in the following way: Project S' in G an denote this as S.
  Then $U = V \backslash S$
- If we denote M^* as the maximum matching, S^* as the minimum vertex cover , U^* as the maximum antichain , ρ^* as the minimum chain partition.

Then

$$\boxed{n = |M^*| + |\rho^*|}$$

$$\boxed{|\rho^*| = |U^*|}$$

Construction of minimum chain partition:

First build the maximum matching in G' with max flow algorithm. Then map each edge of this matching with an edge in G. If you consider only the mapped edges in G, each connected component form a chain, and the union of all of them is the minimum chain partition.

Construction of maximum antichain:

First build the minimum vertex cover in G' using the nodes of the min cut. Then map each node of this vertex cover with a node in G (some may be repeated) and call this set S. Then the antichain is form by the set of vertex that is not in S.