Burnside's Lemma and Polya's Enumeration theorem

1 Burnside's Lemma (in simple terms)

Let D be a set of arrays of a fixed length of objects where each object has only k possible values.

Let's say that π is a permutation an let's define $\pi(S)$, for som $S \in D$, as the array of objects formed by permuting the elements of S using permutation π. That is, that the i-th object in the new set will be the π_i-th one in S.

Let's says that two arrays of objects S_1, S_2 are equivalent ($S1 \equiv S2$) under some group of permutations G if there exists some permutation $\pi \in G$ such that $\pi(S_1) = S_2$. The permutations in G are called **invariant permutations**.

Let's say that S is a fixed point for permutation π if $\pi(S) = S$. And let's define $F(\pi)$ as the number of fixed points of permutation π.

Let's denote as $|Classes|$ the number of equivalence classes of D under a set of invariant permutations G , then burnside's lemma state that :

$$|Classes| = \frac{1}{|G|} \sum_{\pi \in G} F(\pi)$$

2 Polya's Enumeration theorem (special case)

Let's denote $C(\pi)$ as the number of cycles in the permutation π and remember that each object in any set of D has only k possible values. Then:

$$|Classes| = \frac{1}{|G|} \sum_{\pi \in G} k^{C(\pi)}$$

3 Example

Count the number of different circled arrays that consist of n numbers and each number has m possible values. Two circles are considered equal if we can apply some (possible none) cycle shifts to the right to one of them so that it become equal to the other.

There are n possible permutation that represents x $(0 \leq x \leq n-1)$ cycle shifts to the right. A permutation that makes x cycle shifts has cycles of length $\frac{n}{gcd(n,x)}$ so it has $gcd(n, x)$ cycles (using Bezout's Identity). Then the answer is:

$$|Classes| = \frac{1}{n}\sum_{x=0}^{n-1} m^{gcd(n,x)}$$

4 Reference

https://cp-algorithms.com/combinatorics/burnside.html