CONSIDER THE HAMILTONIAN

$$H = -\frac{J}{2}\sum_{\langle i_A, j_B\rangle} S_{i_A} S_{j_B} + \frac{K}{2}\sum_{\langle i_A, j_B\rangle} S_{i_A}^2 S_{j_B}^2 + \mu \sum_i S_i^2$$
$$+ h\sum_{i_B} S_{i_B} - h\sum_{i_A} S_{i_A}$$

DEFINED ON A BIPARTITE LATTICE CONSIDER THE VARIATIONAL HAMILTONIAN

$$\hat{H} = -\gamma_A \sum_{i_A}^{N/2} S_{i_A} + \alpha_A \sum_{i_A}^{N/2} S_{i_A}^2 - \gamma_B \sum_{i_B}^{N/2} S_{i_B} + \alpha_B \sum_{i_B}^{N/2} S_{i_B}^2$$

THE PARTITION FUNCTION FOR THIS VARIATIONAL HAMILTONIAN IS

$$\hat{Z} = Z_A^{N/2} Z_B^{N/2}$$

WHERE THE SUBLATTICE ONE PARTICLE PARTITION FUNCTIONS

$$Z_A = 1 + 2e^{-\beta\alpha_A}\cosh(\beta\gamma_A)$$
$$Z_B = 1 + 2e^{-\beta\alpha_B}\cosh(\beta\gamma_B)$$

IT IS STRAIGHFORWARD TO COMPUTE SUBLATTICE MAGNETIZATION AND QUADRUPOLAR MOMENT

$$\langle S_{iA}^2 \rangle = -\frac{1}{\beta}\frac{1}{\hat{Z}}\frac{\partial Z}{\partial \alpha_A} = -\frac{1}{\beta}\frac{1}{Z_A}\frac{\partial Z_A}{\partial \alpha_A} = -\frac{1}{\beta}\frac{\partial}{\partial \alpha_A}\ln Z_A$$

$$\langle S_{iB}^2 \rangle = -\frac{1}{\beta}\frac{1}{\hat{Z}}\frac{\partial Z}{\partial \alpha_B} = -\frac{1}{\beta}\frac{1}{Z_B}\frac{\partial Z_B}{\partial \alpha_B} = -\frac{1}{\beta}\frac{\partial}{\partial \alpha_B}\ln Z_B$$

$$\langle S_{iA} \rangle = \frac{1}{\beta}\frac{1}{\hat{Z}}\frac{\partial \hat{Z}}{\partial \gamma_A} = \frac{1}{\beta}\frac{\partial}{\partial \gamma_A}\ln Z_A$$

$$\langle S_{iB} \rangle = \frac{1}{\beta}\frac{1}{\hat{Z}}\frac{\partial \hat{Z}}{\partial \gamma_B} = \frac{1}{\beta}\frac{\partial}{\partial \gamma_B}\ln Z_B$$

I THUS PROCEED TO COMPUTING FREE ENERGY ESTIMATE

$$\tilde{F} = \langle H - \hat{H} \rangle + \hat{F} = \langle H - \hat{H} \rangle - \frac{N}{2\beta}\ln Z_A - \frac{N}{2\beta}\ln Z_B$$

MINIMIZATION OF THIS ESTIMATE WITH RESPECT TO VARIATIONAL PARAMETERS SHOULD LEAD TO OPTIMAL VALUES AND A SET OF CONSISTENCY EQUATIONS TO BE SOLVED

GIVEN THAT

$$\langle H \rangle = -\frac{J}{2}\left\langle \sum_{iA} S_{iA} \sum_{\langle jB \rangle} S_{jB} \right\rangle - \frac{J}{2}\left\langle \sum_{iB} S_{iB} \sum_{\langle jA \rangle} S_{jA} \right\rangle$$
$$+ \frac{K}{2}\left\langle \sum_{iA} S_{iA}^2 \sum_{\langle jB \rangle} S_{jB}^2 \right\rangle + \frac{K}{2}\left\langle \sum_{iB} S_{iB}^2 \sum_{\langle jA \rangle} S_{jA}^2 \right\rangle$$
$$- h\left\langle \sum_{iA} S_{iA} \right\rangle + h\left\langle \sum_{iB} S_{iB} \right\rangle + \mu\left\langle \sum_i S_i^2 \right\rangle$$

AND ALL LATTICE SITE VARIABLES ARE INDEPENDENT BY HYPOTHESIS

$$\langle H \rangle = -\frac{JNZ}{2}\langle S_{iA}\rangle\langle S_{iB}\rangle + \frac{KNZ}{2}\langle S_{iA}^2\rangle\langle S_{iB}^2\rangle$$
$$- h\frac{N}{2}\langle S_{iA}\rangle + h\frac{N}{2}\langle S_{iB}\rangle + \frac{\mu N}{2}\langle S_{iA}^2\rangle + \frac{\mu N}{2}\langle S_{iB}^2\rangle$$

CLEARLY

$$\langle \hat{H} \rangle = -\frac{\gamma_A N}{2}\langle S_{iA}\rangle - \frac{\gamma_B N}{2}\langle S_{iB}\rangle + \frac{N\alpha_A}{2}\langle S_{iA}^2\rangle + \frac{N\alpha_B}{2}\langle S_{iB}^2\rangle$$

AND THUS VARIATIC AL FREE ENERGY ESTIMATE IS

$$\frac{\tilde{F}}{N} = \langle S_{iA}\rangle\left[\frac{\gamma_A}{2} - \frac{Jz}{4}\langle S_{iB}\rangle - \frac{h}{2}\right] + \langle S_{iA}^2\rangle\left[\frac{Kz}{4}\langle S_{iB}^2\rangle - \frac{\alpha_A}{2} + \frac{\mu}{2}\right]$$

$$\langle S_{iB}\rangle\left[\frac{\gamma_B}{2} - \frac{Jz}{4}\langle S_{iA}\rangle + \frac{h}{2}\right] + \langle S_{iB}^2\rangle\left[\frac{Kz}{4}\langle S_{iA}^2\rangle - \frac{\alpha_B}{2} + \frac{\mu}{2}\right]$$

$$-\frac{1}{2\beta}\ln \mathcal{Z}_A - \frac{1}{2\beta}\ln \mathcal{Z}_B$$

KEEPING IN MIND THAT

$$\langle S_{iA}\rangle = \frac{1}{\beta}\frac{1}{\hat{\mathcal{Z}}}\frac{\partial \hat{\mathcal{Z}}}{\partial \gamma_A} = \frac{1}{\beta}\frac{\partial}{\partial \gamma_A}\ln \mathcal{Z}_A$$

$$\langle S_{iB}\rangle = \frac{1}{\beta}\frac{1}{\hat{\mathcal{Z}}}\frac{\partial \hat{\mathcal{Z}}}{\partial \gamma_B} = \frac{1}{\beta}\frac{\partial}{\partial \gamma_B}\ln \mathcal{Z}_B$$

$$\langle S_{iA}^2\rangle = -\frac{1}{\beta}\frac{1}{\hat{\mathcal{Z}}}\frac{\partial \mathcal{Z}}{\partial \alpha_A} = -\frac{1}{\beta}\frac{1}{\mathcal{Z}_A}\frac{\partial \mathcal{Z}_A}{\partial \alpha_A} = -\frac{1}{\beta}\frac{\partial}{\partial \alpha_A}\ln \mathcal{Z}_A$$

$$\langle S_{iB}^2\rangle = -\frac{1}{\beta}\frac{1}{\hat{\mathcal{Z}}}\frac{\partial \mathcal{Z}}{\partial \alpha_B} = -\frac{1}{\beta}\frac{1}{\mathcal{Z}_B}\frac{\partial \mathcal{Z}_B}{\partial \alpha_B} = -\frac{1}{\beta}\frac{\partial}{\partial \alpha_B}\ln \mathcal{Z}_B$$

$$\frac{\partial\langle S_{iA}\rangle}{\partial\gamma_B} = \frac{\partial\langle S_{iA}\rangle}{\partial\alpha_B} = 0$$

$$\frac{\partial\langle S_{iB}\rangle}{\partial\gamma_A} = \frac{\partial\langle S_{iB}\rangle}{\partial\alpha_A} = 0$$

$$\frac{\partial\langle S_{iA}^2\rangle}{\partial\gamma_B} = \frac{\partial\langle S_{iA}^2\rangle}{\partial\alpha_B} = 0$$

$$\frac{\partial\langle S_{iB}^2\rangle}{\partial\gamma_A} = \frac{\partial\langle S_{iB}^2\rangle}{\partial\alpha_A} = 0$$

I OBTAIN THE FOLLOWING EQUATIONS BY MINIMIZING $\hat{F}$

$$\frac{\partial\langle S_{iA}\rangle}{\partial\gamma_A}\left(\gamma_A - Jz\langle S_{iB}\rangle - h\right) + \frac{\partial\langle S_{iA}^2\rangle}{\partial\gamma_A}\left(\kappa z\langle S_{iB}^2\rangle - \alpha_A + \mu\right) = 0$$

$$\frac{\partial\langle S_{iB}\rangle}{\partial\gamma_B}\left(\gamma_B - Jz\langle S_{iA}\rangle + h\right) + \frac{\partial\langle S_{iB}^2\rangle}{\partial\gamma_B}\left(\kappa z\langle S_{iA}^2\rangle - \alpha_B + \mu\right) = 0$$

AND SIMILAR ONES IMPOSING FIRST DERIVATIVE CRITERION WITH RESPECT TO α_A AND α_B

THEREFORE, OPTIMAL VARIATIONAL PARAMETERS ARE SUCH THAT

$$\gamma_A = Jz\langle S_{iB}\rangle + h$$

$$\gamma_B = Jz\langle S_{iA}\rangle - h$$

$$\alpha_A = Kz\langle S_{iB}^2\rangle + \mu$$

$$\alpha_B = Kz\langle S_{iA}^2\rangle + \mu$$

DEFINING VARIABLES $m_A = \langle S_{iA}\rangle$, $q_A = \langle S_{iA}^2\rangle$

$m_B = \langle S_{iB}\rangle$, $q_B = \langle S_{iB}^2\rangle$

I OBTAIN THE FOLLOWING CONSISTENCY EQUATIONS

$$m_A = \frac{2\sinh\left[\beta(Jzm_B+h)\right]}{e^{\beta(Kzq_B+\mu)} + 2\cosh\left[\beta(Jzm_B+h)\right]}$$

$$q_A = \frac{2\cosh\left[\beta(Jzm_B+h)\right]}{e^{\beta(Kzq_B+\mu)} + 2\cosh\left[\beta(Jzm_B+h)\right]}$$

$$m_B = \frac{2\sinh[\beta(Jzm_A - h)]}{e^{\beta(Kzq_A+\mu)} + 2\cosh[\beta(Jzm_A - h)]}$$

$$q_B = \frac{2\cosh[\beta(Jzm_A - h)]}{e^{\beta(Kzq_A+\mu)} + 2\cosh[\beta(Jzm_A - h)]}$$

NOTICE THAT IN THE LIMIT OF LOW VACANCY CONCENTRATION, $\mu = -\infty$, AND THE CONSISTENCY EQUATIONS REDUCE TO THOSE OF A SPIN 1/2 ISING MODEL, LIKE THE ONES USED BY PORTA AND TASTÁN

IF $h=0$, THE RESULTS ARE THOSE OBTAINED BY BLUME, EMERY AND GRIFFITHS CONSISTENCY EQUATIONS ARE ALMOT IDENTICAL TO THE ONES DERIVED ABOVE FOR $m_A = m_B$, $q_A = q_B$ IN THIS LIMIT, IT CAN BE SHOWN THAT FREE ENERGY EXPANSION LEADS TO CRITICAL CURVES FOR ORDERING TRANSITIONS ARE

$$2Jz\beta_c = e^{K/J} e^{\beta_c \mu_c} + 2 \qquad \text{ON } \mu\text{-}T \text{ PLANE}$$

$$1 - x_c = \frac{1}{1 + e^{Kz\beta_c(1-x_c)} e^{-K/J} (Jz\beta_c - 1)}$$

WHERE x IS VACANCY CONCENTRATION, WICH IS RELATED TO CHEMICAL POTENTIAL BY

$$1 - x = \frac{1}{1 + \frac{1}{2} e^{\beta(Kzq + \mu)}}$$

TRICRITICAL POINT IN THIS CASE WAS DERIVED BY BLUME, EMERY AND GRIFFITHS

$$T_{TP} = \frac{2K - 1}{2K - 3}$$

IN THEIR WORK CRITICAL CURVES FOR FIRST ORDER TRANSITIONS WHERE ALSO DERIVED, BUT THEIR STUDY IS NOT A MATTER OF THE PRESENT PROJECT

ALTHOUGH PORTA ET AL CONSIDERED A PHASE DIAGRAM OF A BEG MODEL LIKE THE PRESENT PROJECT, IN THE ABSENCE OF STAGGERED MAGNETIC FIELD, THEY REGARD THE CASE $K=0$ RATHER UNREALISTIC FOR MODELING VACANCY INTERACTION N DOMAIN GROWTH PROCESSES IN BINARY ALLOYS AS A FIRST APPROXIMATION, AND FOR SIMPLICITY, I SHALL ASSUME THAT $K=0$ HOWEVER UNREALISTIC, THIS COULD ALLOW FOR ANALYSIS OF ORDERING PHASE TRANSITIONS IN SUBSTITUTIONAL BINARY ALLOYS AT LEAST AS A FIRST APPROXIMATION

UNDER THE PREVIOUS ASSUMPTION, CONSISTENCY EQUATIONS SIMPLIFY DRAMATICALLY

$$m_A = \frac{2\sinh[\beta(Jzm_B + h)]}{e^{\beta\mu} + 2\cosh[\beta(Jzm_B + h)]}$$

$$q_A = \frac{2\cosh[\beta(Jzm_B + h)]}{e^{\beta\mu} + 2\cosh[\beta(Jzm_B + h)]}$$

$$m_B = \frac{2\sinh[\beta(Jzm_A - h)]}{e^{\beta\mu} + 2\cosh[\beta(Jzm_A - h)]}$$

$$q_B = \frac{2\cosh[\beta(Jzm_A - h)]}{e^{\beta\mu} + 2\cosh[\beta(Jzm_A - h)]}$$

CONSIDER THE SET OF EQUATIONS

$$m_A = \frac{2\sinh[\beta(Jzm_B+h)]}{e^{\beta\mu}+2\cosh[\beta(Jzm_B+h)]}$$

$$m_B = \frac{2\sinh[\beta(Jzm_A-h)]}{e^{\beta\mu}+2\cosh[\beta(Jzm_A-h)]}$$

I WILL NORMALIZE THEM DEFINING

$$\beta \longrightarrow Jz\beta \qquad h \longrightarrow h/Jz \qquad \mu \longrightarrow \mu/Jz$$

I THUS PROCEED TO REWRITE THE PAIR OF CONSISTENCY EQNS IN NORMALIZED FASHION ALSO, I REWRITE THEM IN TERMS OF THE ORDER PARAMETER OF THE TRANSITION, AND AN AUXILIAR VARIABLE SUCH THAT

$$m_A = m+n \quad , \quad m_B = m-n$$

NOTICE THAT $m = \frac{1}{2}(m_A + m_B)$ IS THE ORDER PARAMETER OF THE ORDERING PHASE TRANSITIONS

$$m + n = \frac{2\sinh[\beta(m-n+h)]}{e^{\beta\mu} + 2\cosh[\beta(m-n+h)]}$$

$$m - n = \frac{2\sinh[\beta(m+n-h)]}{e^{\beta\mu} + 2\cosh[\beta(m+n-h)]}$$

A TRIVIAL SOLUTION FOR THIS HIGHLY COMPLICATED SYSTEM OF EQUATIONS IS

$$m_0 = 0 \qquad\qquad n_0 = \frac{2\sinh[\beta(h-n_0)]}{e^{\beta\mu} + 2\cosh[\beta(h-n_0)]}$$

FOR COMPACTNESS I DEFINE

$$f(x) = \frac{2\sinh(\beta x)}{e^{\beta\mu} + 2\cosh(\beta x)}$$

AND THUS THE SYSTEM OF EQNS AMOUNTS TO

$$m + n = f(m + [h - n])$$
$$m - n = f(m - [h - n])$$

WITH A TRIVIAL SOLUTION VALID ABOVE A CRITICAL TEMPERATURE

$$m_0 = 0 \qquad\qquad n_0 = f(h - n_0)$$

TWO REMARKS ARE WORTHY OF MENTION FIRST, $-f(x) = f(-x)$, WHICH IS A DIRECT CONSEQUENCE OF THE SYMMETRY OF THE HAMILTONIAN UPON COLLECTIVE SPIN FLIPPING SECOND, ORDER PARAMETER IS THE IMPORTANT VARIABLE FOR ORDERING TRANSITIONS, SO, BY VIRTUE OF CONSISTENCY EQUATIONS, $n = n(m)$

ADDING THE TWO CONSISTENCY EQUATIONS

$$m = \frac{1}{2}\left[f(m+[h-n]) + f(m-[h-n])\right]$$

CONSIDER THIS EQUATION NEAR $m_0 = 0$

$$m \approx \frac{df}{dx}(h-n)\Big|_{n=n_0} m + O(m^2)$$

FOR THIS EQUATION TO HAVE NON TRIVIAL SOLUTION, IT MUST BE THAT

$$\frac{df}{dx}(h-n)\Big|_{n=n_0} > 1$$

BOUNDARY FOR ORDERING TRANSITIONS IS GIVEN BY

$$\frac{df}{dx}(h-n)\Big|_{n=n_0} = 1$$

A MORE RIGOROUS STATEMENT OF THIS CRITERION IS FOUND BY CONSIDERING DIRECTLY FREE ENERGY

$$\frac{\hat{F}}{N} = \frac{m_A m_B}{2} - \frac{1}{2\beta}\ln Z_A - \frac{1}{2\beta}\ln Z_B$$

WHERE IT IS THAT

$$Z_A = 1 + 2e^{-\beta\mu}\cosh(\beta[m_B + h])$$
$$Z_B = 1 + 2e^{-\beta\mu}\cosh(\beta[m_A - h])$$

IN TERMS OF m AND n

$$\frac{\hat{F}}{N} = \frac{m^2}{2} - \frac{n^2}{2} - \frac{1}{2\beta}\ln Z_A - \frac{1}{2\beta}\ln Z_B$$

WHERE IT IS THAT

$$Z_A = 1 + 2e^{-\beta\mu}\cosh(\beta[m - n + h])$$
$$Z_B = 1 + 2e^{-\beta\mu}\cosh(\beta[m + n - h])$$

CONSIDER THE FOLLOWING DERIVATIVES

$$\frac{1}{2}\frac{\partial(n^2)}{\partial m} = n\frac{\partial n}{\partial m}$$

$$\frac{1}{\beta}\frac{\partial}{\partial m}\ln Z_A = \left(1-\frac{\partial n}{\partial m}\right)(m+n)$$

$$\frac{1}{\beta}\frac{\partial}{\partial m}\ln Z_B = \left(1+\frac{\partial n}{\partial m}\right)(m-n)$$

$$\frac{1}{2\beta}\frac{\partial}{\partial m}\left[\ln Z_A+\ln Z_B\right] = m - n\frac{\partial n}{\partial m} = m - \frac{1}{2}\frac{\partial(n^2)}{\partial m}$$

IT IS CLEAR THAT A SET (m,n) THAT SATISFIES BOTH CONSISTENCY EQUATIONS PRODUCES A MINIMUM OF FREE ENERGY - I HEREBY STATE THAT THE FUNCTION THAT RELATES n TO m FOR ARBITRARY m IS

$$n = \frac{1}{2}\left[f(m+[h-n]) - f(m-[h-n])\right]$$

I NOW PROCEED TO COMPUTE SECOND DERIVATIVE OF FREE ENERGY AT THE POINT $m=0$

$$\frac{\partial \hat{g}}{\partial m} = m + n\frac{\partial n}{\partial m} - \frac{1}{2}\left[f(m-n+h)\left(1-\frac{\partial n}{\partial m}\right) + f(m+n-h)\left(1+\frac{\partial n}{\partial m}\right)\right]$$

$$\frac{\partial \hat{g}}{\partial m} = m - \frac{1}{2}\left[f(m-n+h) + f(m+n-h)\right]$$

IMPORTANT NOTE THAT THE CONDITION $\partial \hat{g} / \partial m = 0$ LEADS TO THE SET OF CONSISTENCY EQUATIONS

$$\frac{\partial^2 \hat{g}}{\partial m^2} = 1 - \frac{1}{2}\left[\frac{df}{dx}(m-n+h)\left(1-\frac{\partial n}{\partial m}\right) + \frac{df}{dx}(m+n-h)\left(1+\frac{\partial n}{\partial m}\right)\right]$$

AT $m=0$, AND GIVEN THE PARITY OF $f(x)$, THE CONDITION THAT THERE EXISTS A NON TRIVIAL MINIMUM IMPLIES

$$\left.\frac{df}{dx}(h-n)\right|_{n=n_0} > 1$$

AS A RESULT, BOUNDARY FOR ORDERING TRANSITIONS IS

$$\boxed{\left.\frac{df}{dx}(h-n)\right|_{n=n_0} = 1 \qquad \text{WITH} \qquad n_0 = f(h-n_0)}$$

$$\frac{d^3\hat{g}}{\partial m} = -\frac{1}{2}\left[\frac{d^2f}{dx^2}(m-n+h)\left(1-\frac{\partial n}{\partial m}\right)^2 - \cancel{\frac{df}{dx}(m-n+h)}\frac{\partial^2 n}{\partial m^2} + \frac{d^2f}{dx^2}(m+n-h)\left(1+\frac{\partial n}{\partial m}\right)^2 + \cancel{\frac{df}{dx}(m+n-h)}\frac{\partial^2 n}{\partial m^2}\right]$$

KEEPING IN MIND THE PARITY OF $f(x)$ IT IS THAT

$$\frac{d^3\hat{g}}{\partial m} = 2\,\frac{d^2f}{dx^2}(h-n)\Big|_{n=n_0}\,\frac{\partial n}{\partial m}\Big|_{n=n_0}$$

FROM CONSISTENCY EQUATIONS

$$Z_A = e^{-\beta\mu}\left[\frac{2\sinh[\beta(m-n+h)]}{m+n}\right]$$

$$Z_B = e^{-\beta\mu}\left[\frac{2\sinh[\beta(m+n-h)]}{m-n}\right]$$

AND THUS IT IS THAT

$$\frac{\hat{F}}{N} = \frac{m^2}{2} - \frac{n^2}{2} + \mu - \frac{1}{2\beta}\ln\left[\frac{2\sinh[\beta(m+n-h)]}{m+n}\right]$$

$$- \frac{1}{2\beta}\ln\left[\frac{2\sinh[\beta(m-n+h)]}{m-n}\right]$$

THE NUMBER OF LATTICE SITES, N, IS CONSTANT THERFORE, PERFORMING THE TRANSFORM

$$\frac{\hat{F}}{N} \longrightarrow \hat{g} = \frac{\hat{F}}{N} - \mu$$

AND THUS CONSISTENCY IMPLIES

$$C_A(\mu_c, \beta_c, h_c)\, C_B(\mu_c, \beta_c, h_c) = 1$$

USING EQUATION

$$e^{\beta\mu} = 2\left(\frac{x}{1-x}\right)\cosh(\beta h)$$

IT IS POSSIBLE TO DERIVE AN EQUATION OF THE FORM

$$C_A(x_c, \beta_c, h_c)\, C_B(x_c, \beta_c, h_c) = 1$$

BY PERFORMING TAYLOR SERIES EXPANSION, IT IS FOUND THAT THE DESIRED SURFACE EQUATION IS

$$\frac{4\beta_c^2(1-x_c)^2\left(x_c e^{\beta_c h_c}\cosh(\beta_c h_c) - 2x_c e^{\beta_c h_c} + x_c\cosh(\beta_c h_c) + 2e^{\beta_c h_c}\right)^2 e^{2\beta_c h_c}}{\left(-x_c e^{2\beta_c h_c} + 2x_c e^{\beta_c h_c}\cosh(\beta_c h_c) - x_c + e^{2\beta_c h_c} + 1\right)^4} = 1$$

IN THE LIMIT OF A STOICHIOMETRIC ALLOY $(h=0)$, THE EXPRESSION ABOVE REDUCES TO THAT OBTAINED BY BLUME, EMERY AND GRIFFITHS

$$1 - x_c = T_c$$

IN GENERAL, THE CRITICAL CONCENTRATION FOR GIVEN h AND T IS GIVEN BY

$$x_c = \frac{f_1(h/T) + \sqrt{f_2(h/T) + f_3(h/T)\,T}}{f_4(h/T)}$$

WITH THE DEFINITIONS

$$f_1(u) = e^{4u} - 6e^{2u} + 1$$

$$f_2(u) = e^{8u} + 4e^{6u} + 6e^{4u} + 4e^{2u} + 1$$

$$f_3(u) = 4e^{8u} - 8e^{4u} + 4$$

ON THE OTHER HAND, THE FREE ENERGY SIMPLIFIES TO

$$\frac{\hat{F}}{N} = \frac{m_A m_B}{2} - \frac{1}{2\beta}\ln Z_A - \frac{1}{2\beta}\ln Z_B$$