QUANTUM FOURIER TRANSFORM

CONSIDER A M-QBIT SYSTEM AND ITS COMPUTATIONAL BASIS $\{|0\rangle, |1\rangle, |2\rangle, \quad , |N-1\rangle\}$ WITH $N = 2^M$ NOTE THAT EACH M-BIT BINARY STRING CORRESPONDS TO A NUMBER FROM 1 TO N−1 USING BINARY REPRESENTATION

DEFINE A TRANSLATION OPERATOR SUCH THAT

$$T_1 |n\rangle = |(n+1) \bmod N\rangle$$

IF ADDITION IS ASSUMED TO BE mod N, THEN

$$T_1 |n\rangle = |n+1\rangle$$

GIVEN THAT $T_1^N = 1$, IT IS CLEAR THAT THE EIGENVALUES OF T_1 ARE

$$\lambda_k = e^{-i\frac{2\pi}{N}k} \qquad k = 0, \quad , N-1$$

ALSO, IF ITS EIGENVECTORS ARE DENOTED BY $|\tilde{k}\rangle$, THEN

$$T_1 = \sum_k e^{-i\frac{2\pi}{N}k} |\tilde{k}\rangle\langle\tilde{k}|$$

AS A RESULT, IT IS POSSIBLE TO DEFINE A MOMENTUM OPERATOR SUCH THAT

$$T_1 = e^{-ip}$$

NOTE T_1 PRESERVES INNER PRODUCTS AND THUS $|\tilde{0}\rangle, \quad , |\widetilde{N-1}\rangle$ IS A ORTHONORMAL BASIS

$$P = \sum_k \frac{2\pi k}{N} |\tilde{k}\rangle\langle\tilde{k}|$$

THE OPERATION THAT TRANSFORM FROM $|n\rangle$ BASIS TO $|\tilde{k}\rangle$ BASIS IS THE QUANTUM FOURIER TRANSFORM SUPPOSE

$$|\tilde{k}\rangle = \sum_n C_{kn} |n\rangle$$

$$e^{-i\frac{2\pi k}{N}} |\tilde{k}\rangle = T_1 |\tilde{k}\rangle = \sum_n C_{kn} |n+1\rangle$$

FROM WHICH IT IS DEDUCED THAT

$$C_n = e^{i\frac{2\pi}{N}k} C_{n-1}$$

$$C_n = e^{i\frac{2\pi}{N}kn} C_0$$

SINCE $\langle\tilde{k}|\tilde{k}\rangle = 1$, THEN $C_0 = 1/\sqrt{N}$, AND

$$|\tilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_n e^{i\frac{2\pi}{N}kn} |n\rangle$$

THIS EQUATION FORMALY DEFINES THE QUANTUM FOURIER TRANSFORM THE IMPORTANCE OF THIS OPERATION RELIES ON THE OBSERVATION THAT THE EIGENVALUES OF A HERMITEAN OPERATOR CAN BE DIRECTLY MEASURED BY PUSHING THEM TO A QUANTUM REGISTER USING THE MOMENTUM OPERATOR TO SEE THIS CONSIDER THE OPERATOR

$$H' = H \otimes P$$

WITH H SOME HERMITEAN OPERATOR A CIRCUIT MAY BE DEVISED TO BUILD

$$u = e^{-iH'} = e^{-iH \otimes p}$$

SUPPOSE THE INPUT STATE ON THE CIRCUIT IS

$$|\psi_0\rangle = |\tilde{h}\rangle|0\rangle$$

NOTE H IS A m-QBIT OPERATOR, p IS A ℓ QBIT MOMENTUM OPERATOR

WHERE $H|\tilde{h}\rangle = h|\tilde{h}\rangle$ THEN

$$
\begin{aligned}
u|\psi_0\rangle &= \sum_j \frac{(-i)^j}{j!}(H \otimes p)^j |\tilde{h}\rangle|0\rangle \\
&= \sum_j \frac{(-i)^j}{j!} H^j \otimes p^j |\tilde{h}\rangle|0\rangle \\
&= \sum_j \frac{(-ihp)^j}{j!} |\tilde{h}\rangle|0\rangle \\
&= |\tilde{h}\rangle \left(e^{-ihp}|0\rangle\right) \\
&= |\tilde{h}\rangle T_1^h|0\rangle \\
&= |\tilde{h}\rangle|h\rangle
\end{aligned}
$$

IN THIS WAY, IT IS POSSIBLE TO PUSH THE EIGENVALUE TO AN AUXILIARY REGISTER

NOTE THE ABOVE DEDUCTION IS EXACT FOR CONTINOUS QUANTUM SYSTEMS, OR IN CASE h IS AN INTEGER BETWEEN 0 AND $N-1$

BEFORE DIVING INTO THIS TOPIC, LETS CONSIDER HOW TO CHANGE FROM COMPUTATIONAL BASIS TO MOMENTUM BASIS

IF $n = \sum_{l=0}^{m-1} n_l 2^l$, $k = \sum_{l=0}^{m-1} k_l 2^l$, $n_l, k_l = 0,1$

$$\frac{n}{2^m} = \sum_{l=0}^{m-1} n_l 2^{l-m}$$

$$e^{i 2\pi k \frac{n}{2^m}} = \prod_{l=0}^{m-1} e^{i 2\pi n_l (k 2^{l-m})}$$

SINCE $|n\rangle = \bigotimes_{l=0}^{n-1} |n_l\rangle$

$$e^{i \frac{2\pi}{N} jk} |k\rangle = \bigotimes_{l=0}^{m-1} e^{i 2\pi n_l (k 2^{l-m})} |n_l\rangle$$

ALSO, SINCE

$$\sum_{n=0}^{N-1} = \sum_{n_0, n_1, \ , n_{m-1} = 0}^{1}$$

IT IS SO THAT

$$|\hat{k}\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{i \frac{2\pi}{N} k n} |n\rangle$$

$$= \frac{1}{\sqrt{N}} \bigotimes_{l=0}^{m-1} \left(|0\rangle + e^{i \frac{2\pi}{N} k 2^{l-m}} |1\rangle \right)$$

NOTICE THAT

$$k 2^{\iota-m} = \sum_{\ell=0}^{m-1} k_\ell 2^{\ell+\iota-m}$$

$$e^{i2\pi k 2^{\iota-m}} = \prod_{\ell=0}^{m-1} e^{i2\pi k_\ell 2^{\ell+\iota-m}}$$

IF $\ell \geq m-\iota$, THE ADDITIONAL PHASE IS IRRELEVANT, THEREFORE

$$e^{i2\pi k 2^{\iota-m}} = \prod_{\ell=0}^{m-\iota-1} e^{i2\pi k_\ell 2^{\ell-(m-\iota)}}$$

$$= e^{i2\pi 0.\underbrace{k_{m-(\iota+1)} k_{m-(\iota+2)} \dots k_0}_{m-\iota \text{ BITS}}}$$

AND THEN

$$|\tilde{k}\rangle = \frac{1}{\sqrt{N}} \bigotimes_{\iota=0}^{m-1} \left(|0\rangle + \prod_{\ell=0}^{m-\iota} e^{i2\pi k_\ell 2^{\ell-(m-\iota)}} |1\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \left(|0\rangle + e^{i2\pi 0.k_{m-1}k_{m-2}\dots k_0} |1\rangle \right) \otimes \left(|0\rangle + e^{i2\pi 0.k_{m-2}k_{m-3}\dots k_0} |1\rangle \right) \otimes \dots \otimes \left(|0\rangle + e^{i2\pi 0.k_0} |1\rangle \right)$$

THIS RESULT LEADS TO AN EFFICIENT ALGORITHM FOR COMPUTING THE QUANTUM FOURIER TRANSFORM

FOR A SINGLE QBIT

$$|\tilde{k}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i 2\pi 0.k}|1\rangle\right)$$

BUT, FROM DISCUSSION BEFORE

$$e^{i 2\pi 0.k} = 1 \qquad \text{IF} \quad k = 0$$

$$e^{i 2\pi 0.k} = -1 \qquad \text{IF} \quad k = 1$$

AS A RESULT, A CIRCUIT THAT TAKES AS INPUT STATE $|0\rangle$ ($|1\rangle$) AND OUTPUTS $|\tilde{0}\rangle$ OR $|\tilde{1}\rangle$ IS COMPOSED ONLY BY A HADAMARD GATE

IN GENERAL, THE RESULTS SHOW THAT THE MOMENTUM BASE VECTORS ARE TENSOR PRODUCTS OF QBIT STATES OF THE FORM

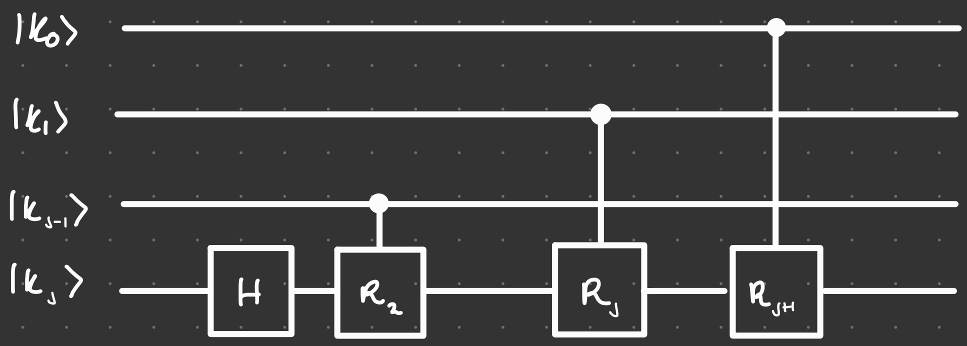
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i 2\pi 0.k_j k_{j-1} \quad k_0}|1\rangle\right)$$

THE STATE

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i 2\pi 0.k_j}|1\rangle\right)$$

CAN BE PRODUCED USING A HADAMARD GATE THE REMAINING PHASES MAY BE PUSHED USING CONTROLLED ROTATIONS ON Z AXIS

USING m-j OF THIS GATES, WITH CONTROL QBIT THE (m-j-k-1)TH, THE DESIRED STATE IS READILLY SEEN TO BE OBTAINED BY THE FOLLOWING CIRCUIT



BY PERFORMING THE ABOVE OPERATION FOR ALL INPUT QBITS $|k_j\rangle$, THE STATE BECOMES

$$|k\rangle = |k_0 k_1 \ldots k_{m-1}\rangle \rightarrow = \frac{1}{\sqrt{N}}\left(|0\rangle + e^{i2\pi 0.k_0}|1\rangle\right) \otimes \ldots \otimes \left(|0\rangle + e^{i2\pi 0.k_{m-2}k_{m-3}\ldots k_0}|1\rangle\right) \otimes \left(|0\rangle + e^{i2\pi 0.k_{m-1}k_{m-2}\ldots k_0}|1\rangle\right)$$

BY SWAPPING QBITS j AND $m-j-1$, FOR $0 \leq j \leq \lfloor m/2 \rfloor$, THE MOMENTUM EIGENSTATE $|\tilde{k}\rangle$ CAN BE OBTAINED

QUANTUM PHASE ESTIMATION

THE IMPORTANCE OF THE IDENTITY

$$e^{-iH\otimes p}|\psi\rangle|0\rangle = |\psi\rangle|h\rangle$$

M-QBIT TARGET REGISTER

IF $H|\psi\rangle = h|\psi\rangle$, WAS POINTED OUT THIS IDENTITY ALLOWS MEASUREMENT OF THE EIGENVALUES OF A HERMITEAN OPERATOR H USING A QUANTUM COMPUTATION TO SEE HOW SUCH AN ALGORITHM MAY BE DEVISED, LETS CONSIDER THE STATE OF THE TARGET REGISTER IN THE MOMENTUM BASIS

$$e^{-iH\otimes p}|\psi\rangle|0\rangle = \frac{1}{2^{m/2}}\sum_{n=0}^{2^m-1} e^{-iH\otimes p}|\psi\rangle|\tilde{n}\rangle$$

$$= \frac{1}{2^{m/2}}\sum_{n=0}^{2^m-1}\sum_{k=0}^{\infty}\frac{(-iH\otimes p)^k}{k!}|\psi\rangle|\tilde{n}\rangle$$

$$= \frac{1}{2^{m/2}}\sum_{n=0}^{2^m-1}\sum_{k=0}^{\infty}\left(\frac{-i2\pi n H}{2^m}\right)^k\frac{1}{k!}|\tilde{n}\rangle$$

$$= \frac{1}{2^{m/2}}\sum_{n=0}^{2^m-1} e^{-i\frac{2\pi n}{2^m}H}|\psi\rangle|\tilde{n}\rangle$$

SUPPOSE THERE EXISTS AN ALGORITHM FOR BUILDING THE UNITARY

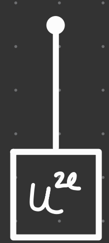
$$U = e^{-i\frac{2\pi}{2^m}H} = e^{-iH'}$$

SO THAT $H'|\psi\rangle = \frac{2^m h}{2\pi}|\psi\rangle = 2^m h'|\psi\rangle$

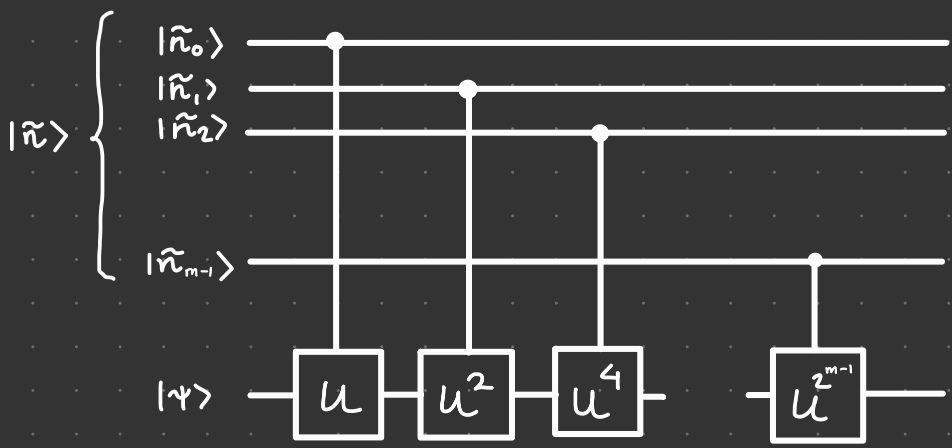
FROM THE ABOVE CALCULATIONS

$$e^{-iH'\otimes p}|\psi\rangle|0\rangle = \frac{1}{2^{m/2}}\sum_{m=0}^{2^m-1} U^n|\psi\rangle|\tilde{n}\rangle$$

SINCE $n = \sum_{\ell=0}^{m-1} n_\ell 2^\ell$, IT IS POSSIBLE TO IMPLEMENT U^n, USING CONTROLLED GATES



USING AS CONTROLS, THE BITS OF THE TARGET REGISTER AS FOLLOWS



BY RETURNING THE TARGET REGISTER ($|\tilde{n}\rangle$) TO THE COMPUTATIONAL BASIS, THE STATE

$$|\psi\rangle|2^m h/2\pi\rangle$$

IS OBTAINED DEFINING $\phi = h/2\pi$, THIS STATE CORRESPONDS TO

$$|\psi\rangle|2^m\phi\rangle$$

NOTICE THAT $2^m\phi$ CORRESPONDS TO THE BEST m-DIGIT BINARY REPRESENTATION OF ϕ AS A RESULT, THIS ALGORITHM FOR COMPUTING

$$e^{-iH'\otimes P}|\psi\rangle|0\rangle$$

ALLOWS FOR DETERMINING THE BEST m-BIT REPRESENTATION OF ϕ THE QUANTUM PHASE ESTIMATION ALGORITHM MAY BE SUMMARISED AS FOLLOWS

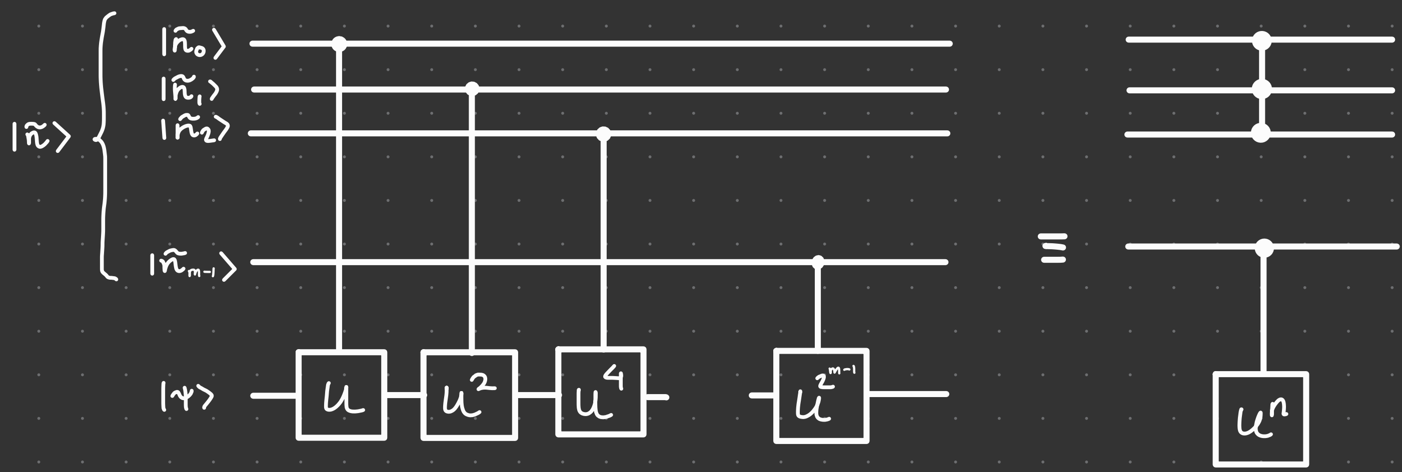
1 CREATE A TARGET REGISTER INITIALISED ON STATE $|0\rangle$, AND AN AUXILIARY REGISTER INITIALISED ON STATE $|\psi\rangle$

2 CHANGE TARGET REGISTER TO MOMENTUM BASIS USING QUANTUM FOURIER TRANSFORM

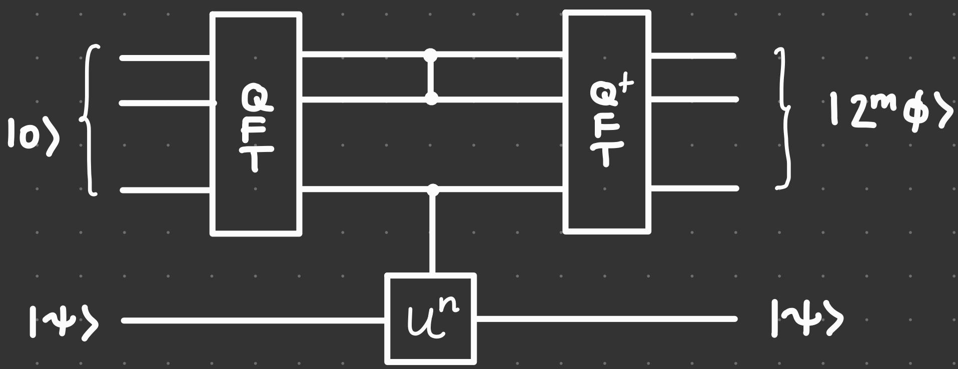
3 COMPUTE $e^{-iH'\otimes P}|\psi\rangle|0\rangle$ USING THE CIRCUIT ILLUSTRATED BEFORE

4 RETURN TARGET REGISTER TO COMPUTATIONAL BASIS USING QFT

DEFINE



A CIRCUIT THAT IMPLEMENTS QPE ALGORITHM MAY LOOK LIKE



BY NOTICING THAT $QFT\,|0\rangle = H^{\otimes m}|0\rangle$, THE ABOVE CIRCUIT CAN BE SIMPLIFIED TO