Journal of Macroeconomics 33 (2011) 477–489

Contents lists available at ScienceDirect

Journal of Macroeconomics

journal homepage: www.elsevier.com/locate/jmacro

Business cycle synchronization and the Euro: A wavelet analysis

Luís Aguiar-Conraria [a,*], Maria Joana Soares [b]

[a] NIPE and Economics Department, University of Minho, Portugal
[b] NIPE and Department of Mathematics and Applications, University of Minho, Portugal

ARTICLE INFO

Article history:
Received 20 May 2010
Accepted 14 February 2011
Available online 21 March 2011

JEL classification:
E32
C22
F02
F15
F41

Keywords:
Business cycle synchronization
European Union integration
Continuous wavelet transform
Wavelet distance matrix
Multidimensional scaling

ABSTRACT

We use wavelet analysis to study business cycle synchronization across the EU-15 and the Euro-12 countries. Based on the wavelet transform, we propose a metric to measure and test for business cycles synchronization. Several conclusions emerge. France and Germany form the core of the Euro land, being the most synchronized countries with the rest of Europe. Portugal, Greece, Ireland and Finland do not show statistically relevant degrees of synchronization with Europe. We also show that some countries (like Spain) have a French accent, while others have a German accent (e.g., Austria). Perhaps surprisingly, we find that the French business cycle has been leading the German business cycle as well as the rest of Europe. Among the countries that may, in the future, join the Euro, the Czech Republic seems the most promising candidate.

1. Introduction

The literature on business cycle synchronization is related to the literature on optimal currency areas and, more broadly, on economic unions. With the recent enlargement of the European Union and the challenges that EMU currently faces, the interest on this topic is guaranteed for a while. If several countries delegate on some supranational institution the power to perform a common monetary (or fiscal) policy, then they lose this policy stabilization instrument. If countries have asymmetric business cycles then it will not be optimal to have the same decision applied to every country. Business cycle synchronization is not sufficient to guarantee that a monetary union is desirable. However, it is a necessary condition: a country with an asynchronous business cycle will face several difficulties in a monetary union, because of the 'wrong' stabilization policies.

The literature is large, and growing, and may be subdivided into several branches, which are not isolated between themselves. One branch is concerned with the best way to estimate a common business cycle. For example, the EuroCOIN, a coincident indicator that measures European Economic Activity, is based on the work of Forni et al. (2000), who rely on a dynamic factor model to extract the common European Activity Index. Another example of this type of approach is given by Artis et al. (2004), who use Markov switching vector autoregressions to identify a common unobserved component that determines the dynamics of an European business cycle. Another branch does not take as given the existence of a common

* Corresponding author. Address: Departamento de Economia, Universidade do Minho, Campus de Gualtar, Braga 4710-008, Portugal
E-mail addresses: lfaguiar@eeg.uminho.pt (L. Aguiar-Conraria), jsoares@math.uminho.pt (M. Joana Soares).

0164-0704/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmacro.2011.02.005

business cycle. Instead, it tries to answer the question of whether or not there is a common business cycle. Camacho et al. (2006) and Harding and Pagan (2006) discuss how the degree of synchronization among business cycles of different countries can be measured and tested. So far, the empirical evidence on the existence of an European business cycle is mixed. For example, while Clark and Wincoop (2001) document that business cycles of U.S. Census regions are substantially more synchronized than those of European countries and Camacho et al. (2006) conclude that there is no common business cycle across Europe, Artis et al. (2004) find a common component governing European business cycle dynamics. Finally, some authors are concerned with the determinants of business cycle comovements. Selover and Jensen (1999) take a purely mathematical modeling approach to conclude that the world business cycle may result from a mode-locking phenomenon (a nonlinear process by which weak coupling between oscillating systems tends to synchronize oscillations in the systems). Other authors look for economic reasons. Frankel and Rose (1998) focus on the effects of international trade and Rose and Engel (2002) argue that, because currency union members have more trade, business cycles are more synchronized across currency union countries. According to Imbs (2004), economic regions with strong financial links are significantly more synchronized. Relevant to our analysis are the results of Inklaar et al. (2008), who conclude that convergence in monetary and fiscal policies has a significant impact on business cycle synchronization. However, again, evidence on this topic is mixed. Baxter and Kouparitsas (2005), for example, argue that currency unions are not important determinants of business cycle synchronization (or at least this effect is not robust) and Camacho et al. (2008) present evidence that differences between business cycles in Europe have not been disappearing.

We use wavelet analysis to study business cycle synchronization. As we will show, wavelet analysis is particularly well suited to study this issue because it performs the estimation of the spectral characteristics of a time-series as a function of time, revealing how the different periodic components of a particular time-series evolve over time. By comparing the wavelet spectra of two countries, we are able to check if the contribution of cycles at each frequency to the total variance is similar between both countries, if this contribution happens at the same time or not, and, finally, if the ups and downs of each cycle occur simultaneously. As a coherent mathematical body, wavelets were born in the mid-1980s (Grossmann and Morlet, 1984; Goupillaud et al., 1984). The literature rapidly expanded and wavelet analysis is now extensively used in physics, epidemiology, signal processing, etc. Still, this technique is infrequently used in Economics.[1] One peculiarity of the applications of wavelet to economics is the almost exclusive use of the discrete wavelet transform, e.g., Gençay et al. (2005), instead of the continuous transform that we use in this paper. To our knowledge, Raihan et al. (2005), Crowley et al. (2006), Crowley and Mayes (2008), Aguiar-Conraria et al. (2008) and Rua (2010) are among the few exceptions to this rule.

We use data on the Industrial Production for the countries in the EU-15, i.e., the first twelve countries joining the Euro and the three countries that could have joined the Euro in 1999 (the United Kingdom, Sweden and Denmark). We propose a metric to compare the wavelet spectra and measure the degree of synchronization among countries. We test if that synchronization is statistically significant by Monte Carlo simulations. With that metric, we fill a dissimilarity or distance matrix which is used to map the countries into a two dimensional axis in terms of business cycle synchronization. *En passant*, we show that business cycle dissimilarities are highly correlated with geographical physical distances. Then, we use cross-wavelets, wavelet-phase and phase difference analysis, to study in more detail when and at what frequencies each country is synchronized or not. We also extend the analysis to a set of countries that have recently joined the European Union and, in some cases, have adopted the Euro.

The paper proceeds as follows. In Section 2, we present the continuous wavelet transform, discuss some of its properties and the optimal characteristics of the Morlet wavelet. We also describe the wavelet power spectrum, the cross-wavelet power spectrum, the wavelet coherency and the phase-difference. Finally, we propose a metric for measuring business cycle synchronization. In Section 3, we apply these tools to study and test for business cycle synchronization across the EU-15. Section 4 extends part of the analysis to the new European Union members for which we could gather at least 15 years of data: Hungary, Poland, Cyprus, Romania, Slovakia, and the Czech Republic. Section 5 concludes.

2. Wavelets: frequency analysis across time

2.1. The continuous wavelet transform

Wavelet analysis[2] performs the estimation of the spectral characteristics of a time-series as a function of time, revealing how the different periodic components of a particular time-series evolve over time. While the Fourier transform breaks down a time-series into constituent sinusoids of different frequencies and infinite duration in time,[3] a wavelet function drops towards zero. For most of the applications, it is enough to require that the wavelet function, ψ, called the mother wavelet, has zero mean, i.e., $\int_{-\infty}^{\infty}\psi(t)dt = 0$, and satisfies a decaying property. This means that the function has to wiggle up and down the t-axis while it approaches zero; i.e., it must behave like a small wave that loses its strength as it moves away from the center. It is this property that allows, contrary to the Fourier transform, for an effective localization in both time and frequency.

[1] For a detailed review of wavelet applications to economic and financial data, the reader is referred to Crowley (2007).

[2] The technical details related to wavelet analysis are thoroughly explained in Aguiar-Conraria and Soares (2010). Associated with that paper, there is Matlab wavelet toolbox that we wrote. It is freely available at http://sites.google.com/site/aguiarconraria/joanasoares-wavelets.

[3] The Fourier basis functions are sines and cosines.

Given a time-series x_t, its continuous wavelet transform (CWT) with respect to the wavelet ψ is a function of two variables, $W_x(\tau,s)$:

$$W_x(\tau,s) = \int x_t \left[\frac{1}{\sqrt{|s|}}\overline{\psi}\left(\frac{t-\tau}{s}\right)\right]dt, \tag{1}$$

where the bar denotes complex conjugation, s is a scaling factor that controls the width of the wavelet and τ is a translation parameter controlling its location. Scaling a wavelet simply means stretching it (if $|s| > 1$) or compressing it (if $|s| < 1$), while translating it simply means shifting its position in time. [4]

2.1.1. The choice of the mother wavelet

There are several types of wavelet functions available with different characteristics, such as, Morlet, Mexican Hat, Haar, and Daubechies. Since the wavelet coefficients $W_x(s,\tau)$ contain combined information on both x_t and $\psi(t)$, the choice of the wavelet is an important aspect to be taken into account, which depends on the particular application one has in mind. To study synchronism between different time-series, one has to select a complex-valued wavelet, because its corresponding transform contains information on both amplitude and phase. Among these, analytic wavelets are ideal.[5] Analytic wavelets are ideal for the analysis of oscillatory signals, since the continuous analytic wavelet transform provides an estimate of the instantaneous amplitude and instantaneous phase of the signal in the vicinity of each time/scale location (τ,s). In such case, reconstruction formulas involving only positive values of the scale parameter s are available. In particular, if $0 < K_\psi := \int_0^\infty \frac{\bar{\hat{\psi}}(f)}{f} df < \infty$, one can use the reconstruction formula, given by

$$x(t) = 2\Re\left[\frac{1}{K_\psi}\int_0^\infty W_x(t,s)\frac{ds}{s^{3/2}}\right].^6 \tag{2}$$

The Morlet wavelet became the most popular of the complex-valued wavelets mainly because of four properties. First, as we explain in Aguiar-Conraria and Soares (2010) there are three sensible ways to convert wavelet scales into frequencies. One uses the peak frequency, the other uses the energy frequency and, finally, the central instantaneous frequency. In the case of the Morlet wavelet, these are all equal, facilitating the conversion from scales to frequencies. Second, the Morlet wavelet has optimal joint time-frequency concentration.[7] Third, the time radius and the frequency radius are equal, and, therefore, this wavelet represents the best compromise between time and frequency concentration. Finally, as long as it is conveniently parametrized, for numerical purposes, it is an analytic wavelet. The Morlet wavelet is given by

$$\psi_{\omega_0}(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-\frac{t^2}{2}}. \tag{3}$$

All our results are obtained with the particular choice $\omega_0 = 6$. For this parametrization of the Morlet wavelet, there is an inverse relation between wavelet scales and frequencies, $f \approx \frac{1}{s}$, greatly simplifying the interpretation of the empirical results. Thanks to this very simple one-to-one relation between scale and frequency we can use both terms interchangeably.[8]

2.1.2. Wavelet tools

In analogy with the terminology used in the Fourier case, the (local) **wavelet power spectrum** (sometimes called scalogram or wavelet periodogram) is defined as

$$(WPS)_x(\tau,s) = |W_x(\tau,s)|^2. \tag{4}$$

This gives us a measure of the variance distribution of the time-series in the time-scale/frequency plane.[9]

The concepts of cross wavelet power, wavelet coherency and phase-difference are natural generalizations of the basic wavelet analysis tools that enable us to deal with the time-frequency dependencies between two time-series. The cross-wavelet transform of two time-series, $x(t)$ and $y(t)$, is defined as

$$W_{xy}(\tau,s) = W_x(\tau,s)\overline{W_y}(\tau,s), \tag{5}$$

---

[4] In practice, we deal with a discrete time-series $x = \{x_t, t = 0, \ldots, T-1\}$ of T observations with a uniform time step. The integral in (1) has to be discretized and is replaced by a summation over the T time steps.

[5] $\psi(t)$ is analytic if its Fourier transform, $\hat{\psi}(f)$, is such that $\hat{\psi}(f) = 0$, for $f < 0$.

[6] Note that by adjusting the limits of the interval this reconstruction formula can be used as a band-pass filter.

[7] By this we mean that the Heisenberg box area reaches its lower bound with this wavelet, i.e., the uncertainty attains the minimum possible theoretical value.

[8] To our knowledge, every application of the continuous wavelet transform in Economics have used this choice. Another important family of analytic wavelets, the so-called Generalized Morse Wavelets (GMWs), is also becoming popular in physical sciences. GMWs are a two-parameter family of wavelets. As a robustness check we also tried this wavelet for a range of reasonable parameter values, namely values that imply that the Heisenberg box area was close to its lower bound (see, Aguiar-Conraria and Soares, 2010). Depending on the parameters, the results were either qualitatively similar or almost indistiguishable from the ones presented in the paper.

[9] Sometimes the wavelet power spectrum is averaged over time for comparison with classical spectral methods. When the average is taken over all times, we obtain the global wavelet power spectrum, $(GWPS)_x(s,\tau) = \int_{-\infty}^{\infty} |W_x(\tau,s)|^2 d\tau$.

where W_x and W_y are the wavelet transforms of x and y, respectively. We define the **cross wavelet power**, as $|W_{xy}(\tau,s)|$. The cross-wavelet power of two time-series depicts the local covariance between two time-series at each time and frequency. When compared with the cross wavelet power, the **wavelet coherency** has the advantage of being normalized by the power spectrum of the two time-series. In analogy with the concept of coherency used in Fourier analysis, given two time-series $x(t)$ and $y(t)$ one defines their wavelet coherency:

$$R_{xy}(\tau,s) = \frac{|S(W_{xy}(\tau,s))\|}{\sqrt{S(|W_{xx}(\tau,s)|)S(|W_{yy}(\tau,s)|)}}, \tag{6}$$

where S denotes a smoothing operator in both time and scale.[10]

Although there is some work done on the theoretical distribution of the wavelet power and on the distribution of cross-wavelets, the available tests imply null hypotheses that are too restrictive to deal with economic data. Therefore, we will rely on Monte Carlo simulations for statistical inference.

As we have discussed, one of the major advantages of using a complex-valued wavelet is that we can compute the phase of the wavelet transform of each series and thus obtain information about the possible delays of the oscillations of the two series as a function of time and scale/frequency, by computing the phase-difference. The **phase difference** can be computed from the cross wavelet transform, by using the formula

$$\phi_{x,y}(s,\tau) = \tan^{-1}\left(\frac{\Im(W_{xy}(s,\tau))}{\Re(W_{xy}(s,\tau))}\right), \tag{7}$$

and information on the signs of each part to completely determine the value of $\phi_{xy} \in [-\pi,\pi]$. A phase-difference of zero indicates that the time-series move together at the specified frequency; if $\phi_{xy} \in \left(0,\frac{\pi}{2}\right)$, then the series move in phase, but the time-series y leads x; if $\phi_{xy} \in \left(-\frac{\pi}{2},0\right)$, then it is x that is leading; a phase-difference of π(or $-\pi$) indicates an anti-phase relation; if $\phi_{xy} \in \left(\frac{\pi}{2},\pi\right)$, then x is leading; time-series y is leading if $\phi_{xy} \in \left(-\pi,-\frac{\pi}{2}\right)$.

2.2. Wavelet spectra distance matrix

In this section, we propose a metric for measuring the distance between a pair of given wavelet spectra. The measure can then be applied to each pair of spectra of all the countries in our dataset, thus allowing us to fill in a distance/dissimilarity matrix, suitable for cluster analysis.

Comparing time-series based on their wavelet spectra is, in a sense, like comparing two images. Direct comparison is not suitable because there is no guarantee that regions of low power will not overshadow the comparison. It would be like comparing two pencil-drawing sketches based mainly on the color of the paper, disregarding the sketches themselves. We build on the work of Rouyer et al. (2008) and use the singular value decomposition (SVD) of a matrix to focus on the common high power time-frequency regions. This method is analogous to Principal Component Analysis, but while with the latter one finds linear combinations that maximize the variance, subject to some orthogonality conditions, the method we use extracts the components that maximize covariances instead. Therefore, the first extracted components correspond to the most important common patterns between the wavelet spectra. With that information, we just need to define a metric to measure the pairwise distance between the several extracted components

2.2.1. Leading vectors and leading patterns

Given two $F \times T$ wavelet spectral matrices W_x and W_y, let $C_{xy} := W_x W_y^H$, where W_y^H is the conjugate transpose of W_y, be their covariance matrix. Performing an SVD of this matrix yields

$$C_{xy} = U\Sigma V^H, \tag{8}$$

where the matrices U and V are unitary matrices (i.e., $U^HU = V^HV = I$), and $\Sigma = \text{diag}(\sigma_i)$ is a diagonal matrix with non-negative diagonal elements ordered from highest to lowest, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_F \geqslant 0$.

The columns, $\mathbf{u}_k$ of the matrix U and the columns $\mathbf{v}_k$ of V are known, respectively, as the singular vectors for W_x and W_y, and the σ_i are the singular values. The number of nonzero singular values is equal to the rank of the matrix C_{xy}.

The singular vectors $\mathbf{u}_k$ and $\mathbf{v}_k$ satisfy an important variational property. For each k, they are such that

$$\mathbf{u}_k^H C_{xy} \mathbf{v}_k = \max_{\mathbf{p}_k,\mathbf{q}_k \in S} \{\mathbf{p}_k^H C_{xy} \mathbf{q}_k\} \tag{9}$$

where S is the set of all vectors satisfying the following orthogonality conditions:

$$\mathbf{p}_k^H \mathbf{p}_j = \mathbf{q}_k^H \mathbf{q}_j = \delta_{k,j}, \quad \text{for } j = 1,\ldots,k, \tag{10}$$

with $\delta_{k,j}$ denoting the Kronecker delta symbol.

[10] In our codes we allow smoothing to be done with several types of windows, including Hanning, Hamming, Blackman and Bartlett windows. The particular choice has no relevant impact in our results.

Let $\mathbf{l}_x^k$ and $\mathbf{l}_y^k$ be the leading patterns, i.e., the $1 \times T$ vectors obtained by projecting each spectrum W_x and W_y onto the respective kth singular vector (axis):

$$\mathbf{l}_x^k := \mathbf{u}_k^{\mathrm{H}} W_x \quad \text{and} \quad \mathbf{l}_y^k := \mathbf{v}_k^H W_y. \tag{11}$$

Note that $\mathbf{l}_x^k$ is a linear combination of the rows of W_x whose weights are the conjugates of the components of the kth singular vector $\mathbf{u_k}$ (and similarly for $\mathbf{l}_y^k$). Then, since

$$\mathbf{u}_k^{\mathrm{H}} C_{xy} \mathbf{v}_k = \mathbf{u}_k{}^{\mathrm{H}} W_x W_y^{\mathrm{H}} \mathbf{v}_k = \mathbf{u}_k^{\mathrm{H}} W_x \left(\mathbf{v}_k^{\mathrm{H}} W_y\right)^{\mathrm{H}} = \mathbf{l}_x^k \left(\mathbf{l}_y^k\right)^{\mathrm{H}}, \tag{12}$$

we can conclude that the leading patterns are the linear combinations of the rows of W_x and W_y, respectively, that maximize their mutual covariance (subject to the referred orthogonality constraints).

Eq. (8) can be written equivalently as

$$U^H C_{xy} V = \Sigma. \tag{13}$$

By equating the diagonal elements of the matrices on each side of this equation, one gets that the covariance of the kth leading patterns is given by

$$\left|\mathbf{l}_x^k \left(\mathbf{l}_y^k\right)^H\right|^2 = \left|\mathbf{u}_k^{\mathrm{H}} C_{xy} \mathbf{v}_k\right|^2 = \sigma_k^2. \tag{14}$$

On the other hand, the (squared) covariance of W_x and W_y is given by $\|C_{xy}\|_{Fro}^2$, where $\|\cdot\|_{Fro}$ is the Frobenius matrix norm, defined by $\|A\|_{Fro} := \sqrt{\sum_{ij} |a_{ij}|^2}$. Since this norm is invariant under a unitary transformation, we have

$$\|C_{xy}\|_{Fro}^2 = \|U^H C_{xy} V\|_{Fro}^2 = \|\Sigma\|_{Fro}^2 = \sum_{i=1}^{F} \sigma_i^2.$$

The (squared) singular values, σ_k^2, are the weights to be attributed to each leading pattern and are equal to the (squared) covariance explained by each pair of singular vectors.

If we denote by L_x and L_y the matrices whose rows are the leading patterns $\mathbf{l}_x^k$ and $\mathbf{l}_y^k$, Eq. (11) shows that $L_x = U^H W_x$ and $L_y = V^H W_y$, from where we immediately obtain

$$W_x = U L_x = \sum_{k=1}^{F} \mathbf{u}_k \mathbf{l}_x^k, \qquad W_y = V L_y = \sum_{k=1}^{F} \mathbf{v}_k \mathbf{l}_y^k.$$

In practice, we select a certain number $K < F$ (K usually much smaller than F) of leading patterns, guaranteeing, for example, that the fraction of covariance $\left(\sum_{k=1}^{K} \sigma_k^2\right) / \left(\sum_{k=1}^{F} \sigma_k^2\right)$ is above a certain threshold,[11] and use

$$W_x \approx \sum_{k=1}^{K} \mathbf{u}_k \mathbf{l}_x^k, \quad W_y \approx \sum_{k=1}^{K} \mathbf{v}_k \mathbf{l}_y^k.$$

2.2.2. Distance between two spectra

We have reduced the information contained in the two wavelet spectra to a few components: the K most relevant leading patterns and leading vectors. Now, the idea is to define a distance between the two spectra, by appropriately measuring the distances from these components. To do so, we compute the distance between two-vectors (leading patterns or leading vectors) by measuring the angle between each pair of corresponding segments, defined by the consecutive points of the two-vectors, and take the mean of these values. This would be easy to perform if all the values were real. In our case, because we use a complex wavelet, we need to define an angle in a complex vector space. Unfortunately, not much guidance is available in the mathematical literature on angles in complex vector spaces. Scharnhorst (2001) summarizes several possible definitions. We will consider two possibilities.

Recall that, given two-vectors $\mathbf{a}$ and $\mathbf{b}$ in the Euclidian vector space $\mathbb{R}^n$, with the usual inner product $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{R}} = \mathbf{a}^T \mathbf{b}$ and norm $\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_{\mathbb{R}}}$, the angle between the two-vectors, $\Theta = \Theta(\mathbf{a},\mathbf{b})$, can be found using the formula:

$$\cos(\Theta) = \frac{\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{R}}}{\|\mathbf{a}\| \|\mathbf{b}\|}, \quad \Theta \in [0, \pi]. \tag{15}$$

Now, assume that $\mathbf{a}$ and $\mathbf{b}$ are vectors in the vector space $\mathbb{C}^n$. There are two reasonable approaches to define a (real)-valued angle between $\mathbf{a}$ and $\mathbf{b}$. The first one is to consider the natural isomorphism $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ given by $\phi(\mathbf{a}) = \phi((a_1, \ldots, a_n)) = (\Re(a_1), \Im(a_1), \cdots, \Re(a_n), \Im(a_n))$ and simply define the Euclidean angle between the complex vectors $\mathbf{a}$ and $\mathbf{b}$ as the angle (defined by using formula (15)) between the real vectors $\phi(\mathbf{a})$ and $\phi(\mathbf{b})$.

[11] In our paper, we used $K = 3$. Three leading patterns were enough to guarantee a fraction above 90%. Using larger values for K yields indistinguishable results.

The other approach is based on the use of the Hermitian inner product $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{C}} = \mathbf{a}^H \mathbf{b}$ and corresponding norm $\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_{\mathbb{C}}}$. We can then define the so-called Hermitian angle between the complex vectors **a** and **b**, $\Theta_H(\mathbf{a}, \mathbf{b})$, by the formula

$$\cos(\Theta_H) = \frac{|\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{C}}|}{\|\mathbf{a}\| \|\mathbf{b}\|}, \quad \Theta_H \in \left[0, \frac{\pi}{2}\right]. \tag{16}$$

The measures are not equal, but they are related; see Scharnhorst (2001) for details. In all our numerical computations, we use the Hermitian angle.[12]

The distance between two-vectors $\mathbf{p} = (p_1, \ldots, p_M)$ and $\mathbf{q} = (q_1, \ldots, q_M)$ with M components in $\mathbb{C}$ (applicable to the leading patterns and leading vectors) is simply defined by

$$d(\mathbf{p}, \mathbf{q}) = \frac{1}{M-1} \sum_{i=1}^{M-1} \Theta_H(\mathbf{s}_i^{\mathbf{p}}, \mathbf{s}_i^{\mathbf{q}}) \tag{17}$$

where the ith segment $\mathbf{s}_i^{\mathbf{p}}$ is the two-vector $\mathbf{s}^{\mathbf{p}}{}_i := (i + 1, p_{i+1}) - (i, p_i) = (1, p_{i+1} - p_i)$.

To compare the wavelet spectra of country x and country y, we then compute the following distance:

$$\mathrm{dist}(W_x, W_y) = \frac{\sum_{k=1}^{K} \sigma_k^2 \left[d\left(\mathbf{l}_x^k, \mathbf{l}_y^k\right) + d(\mathbf{u}_k, \mathbf{v}_k) \right]}{\sum_{k=1}^{K} \sigma_k^2}, \tag{18}$$

where σ_k^2 are the weights equal to the squared covariance explained by each axis.

The above distance is computed for each pair of countries and, with this information, we can then fill a matrix of distances.

3. Business cycle synchronization in the Euro land

We analyze the cycles of the core of the Euro area looking both at the frequency content and phasing of cycles. In our analysis, we first consider the EU-15. The 12 countries that first joined the Euro – Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal and Spain – and the three countries that were part of the European Union in 1999, but chose not to join the monetary union – United Kingdom, Sweden, Denmark.

For this type of purpose, to measure real economic activity, most studies use either real GDP or an Industrial Production Index. We will use the Industrial Production Index because wavelet analysis is quite data demanding, and to have monthly data is a bonus. Using the International Financial Statistics database of the IMF, we gather non seasonally adjusted data from July 1975 until May 2010.[13] To derive an Euro-12 Industrial Production Index, we calculate a weighted average of the industrial production of the 12 countries. As weights, we use each country's GDP in year 2000. We remove seasonal effects using the reconstruction formula (2).[14] Because we want to focus our analysis on business cycle frequencies, we estimate the wavelet power spectra between 1.5 and 8 years frequencies. [15]

In Fig. 1, we see the continuous wavelet power spectrum of the Euro-12 Industrial Production. To save space, we omit the wavelet power spectra for the member countries. Not much information is lost with this omission because the most notorious common patterns are also apparent when we look at the European aggregate in Fig. 1: first, with the exception of Greece, every country shows a spike around the 6-years frequency. This spike is stronger in the 1980s for several countries (like Ireland, Luxembourg, Germany, Belgium, Netherlands and Austria), while for others, like Portugal and Finland, the high power region is situated between 1990 and 1995. Second, when we look at 3-year frequencies, we observe a spike in the 1990s that is common to several countries (although not all of them). Finally, it is also apparent that after 2005, volatility increased at all frequencies and across all countries.

3.1. Business cycles geography

Although suggestive, the wavelet power spectrum is not the best tool to analyze business cycle synchronization, because information about the phase, which depends on the imaginary part, is lost when we take the absolute value. Therefore, even if two countries share a similar high power region, one cannot be positive that their business cycles look alike. In the upper triangle of Table 1, we have information on the physical distance between countries.[16] In the lower triangle of Table 1, we can find countries dissimilarities, based on formula (18) and computed with the whole dataset. Given that we use the Hermitian angle, the highest value the distance can take is $\pi/2$. As explained in the previous section, this measure takes into account both the real and the imaginary part of the wavelet transform. A value very close to zero means that two countries have a very similar

[12] The Euclidian angle approach delivers similar results.

[13] In a previous version of this paper, we have avoided using data affected by the current financial and economic crisis. The results were qualitatively similar.

[14] Using different filters yields similar results. For example, had we used the Baxter and King band-pass filter, the correlation between the distances shown in Table 1 and the alternative Table 1 would be above 88%.

[15] It is also common to consider that business cycle frequencies are between 3 and 8 years. Doing so yields similar results.

[16] To estimate the physical distance (in kilometers) between the capitals of these countries, we used the software developed by Byers (2003).

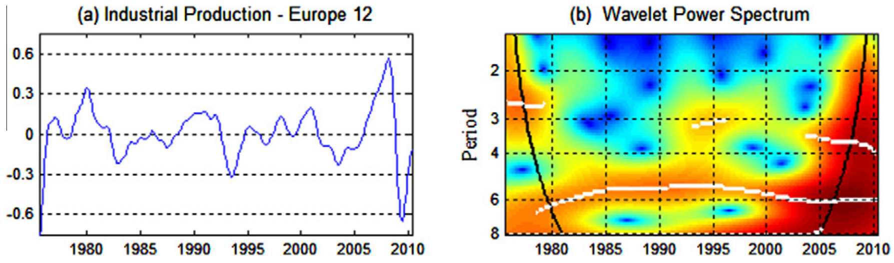

Fig. 1. (a) Euro-12 Industrial Production Index. (b) Wavelet power spectrum – the cone of influence, which indicates the region affected by edge effects, is shown with a black line. The color code for power ranges from blue (low power) to red (high power). The white lines show the maxima of the undulations of the wavelet power spectrum. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Lower triangle – business cycle dissimilarities. Upper triangle – distance in Kms.

	Sp	Pt	Gr	Fi	Ne	Lx	It	Ge	Fr	Be	Au	Ir	UK	DK	Sw
Spain		503	2366	2948	1480	1279	1370	1870	1053	1314	1809	1450	1263	2072	2592
Portugal	0.467		2849	3359	1861	1710	1869	2312	1452	1709	2298	1639	1582	2476	2986
Greece	0.438	0.733		2468	2162	1904	1038	1804	2097	2089	1287	2852	2393	2135	2408
Finland	0.448	0.516	0.457		1503	1670	2208	1104	1908	1650	1434	2024	1824	883	398
Netherlands	0.416	0.393	0.452	0.495		318	1307	577	427	173	931	756	360	622	1126
Luxembourg	0.354	0.452	0.435	0.435	0.199		1000	603	289	188	760	952	493	801	1323
Italy	0.367	0.519	0.415	0.230	0.427	0.365		1194	1121	1186	775	1897	1448	1541	1984
Germany	0.349	0.506	0.475	0.412	0.288	0.222	0.334		879	653	520	1316	935	352	808
France	0.218	0.473	0.410	0.441	0.372	0.289	0.308	0.273		261	1033	777	342	1026	1542
Belgium	0.317	0.410	0.474	0.470	0.358	0.247	0.354	0.272	0.281		913	773	322	767	1281
Austria	0.335	0.530	0.388	0.396	0.217	0.198	0.355	0.172	0.284	0.329		1678	1235	864	1236
Ireland	0.406	0.375	0.492	0.435	0.286	0.261	0.402	0.417	0.407	0.327	0.374		460	1239	1627
UK	0.355	0.397	0.411	0.331	0.285	0.275	0.312	0.411	0.350	0.323	0.343	0.190		958	1434
DK	0.328	0.446	0.403	0.428	0.285	0.256	0.333	0.244	0.246	0.306	0.221	0.355	0.344		522
Sw	0.434	0.714	0.427	0.351	0.442	0.412	0.379	0.337	0.418	0.442	0.299	0.468	0.405	0.413	
Eur12	0.276	0.510	0.423	0.392	0.334	0.253	0.309	0.246	0.183	0.268	0.214	0.390	0.355	0.201	0.335
p < 0.10	p < 0.05	p < 0.01													

wavelet transform. This, in turn, implies that the two countries share the same high power regions and also that their phases are aligned. This means that (1) the contribution of cycles at each frequency to the total variance is similar between both countries, (2) this contribution happens at the same time in both countries and, finally, (3) the ups and downs of each cycle occur simultaneously in both countries. In this sense, we say that a value close to zero between countries means that their business cycles are highly synchronized.

One can argue that what really matters for synchronization in the context of the euro area is synchronization between each member state and the euro area aggregate minus its domestic output. We compute that. In Table 1, in the last row, when we measure the distance between each country and the Euro aggregate, that country is excluded from the aggregate, except, of course, for the UK, Sweden and Denmark, which are compared to the Euro-12. To assess if synchronization is statistically significant, we rely on Monte Carlo methods.[17]

From the lower triangle of Table 1, one can see that the tighter pair of countries is Germany and Austria, followed by the United Kingdom and Ireland. The most dissimilar are Portugal and Greece, and Portugal and Sweden.

Looking at pairwise distances and also at the distance between each country and the rest of Europe, we are able to identify what we can call the European core. There are four countries whose synchronism with the rest of Europe is significant at 1% level: France, Germany, Luxembourg and Austria. In this latter case, the reason is that Austria is very close to Germany, a

[17] We fit an ARMA(1,1) model and construct new samples by drawing errors from a Gaussian distribution, with the same variance as the estimated error terms. For each pair of countries, we do this 5000 times and compute the distance for each trial. From the computed distances, we extract the critical values at 1, 5 and 10%.

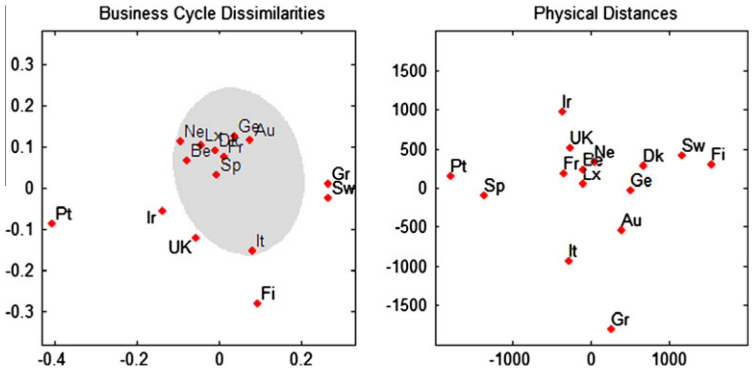

Fig. 2. Multidimensional scaling maps.

country with a GDP that accounts for 30% of the European GDP. If we exclude Germany from the aggregate index, then Austria is no longer synchronous at the 1% level. Therefore, it is fair to say that these results identify Germany and France as the core countries of the Euro zone (Luxembourg is too small to play that role).

Portugal, Greece, Finland and Ireland are four countries whose business cycle seem independent of the rest of Euro zone. More than that, the only country with which Portugal is synchronized with at the 10% level is Ireland and Greece is not statistically synchronized with any other country. Finally, it is interesting to note that Ireland is synchronized at 1% with the UK (suggesting that Ireland should be sharing the currency with it and not with Germany and France). The other countries that have adopted the Euro have a business cycle synchronized with the rest of Europe at 5% significance.

Three countries could but did not adopt the Euro. Denmark chose so by referendum. The UK decided not to join the Euro based on five tests, which included business cycle synchronization (HMT, 2003). The Swedish government, based on the Calmfors report, Calmfors et al. (1997), reached similar conclusions for Sweden. According to our results, Sweden and the UK are synchronized at 10%, suggesting that on this regard, they would be in better position to be part of the Euro than Portugal, Greece, Finland and Ireland; but not as good as the other member countries. Therefore, our results lend some support to the decision of the British and Swedish governments. Denmark, on its turn, could perfectly well have adopted the Euro. If it had, it would be in the Euro core, as it is the second most synchronized country with the Euro-12. In fact, Denmark gave up on having an independent monetary policy, by keeping a fixed exchange rate with the Euro.

To visualize Table 1, we follow Camacho et al. (2006). We reduce the dissimilarity matrix to a two-column matrix. This new matrix, the configuration matrix, contains the position of each country in two orthogonal axes. Therefore, we can position each country on a plane. We apply the same procedure to the physical distances, using the information in the upper triangle of Table 1. Naturally, this cannot be performed with perfect accuracy because distances are not Euclidean.[18]

Fig. 2 shows these maps. From the picture on the left, it is clear that there is an Euro land core, formed by Germany, Austria, the Benelux countries, France, Spain, and, to a lesser extent, Italy. Like Camacho et al. (2006), we also conclude that Portugal, Greece and Finland are the countries exhibiting the less "European" cycles. It is comforting to observe that quite different approaches lead to some overlapping results.

This European business cycle map is not independent from geography. Looking at the map based on physical distances (right picture) Finland, Portugal and Greece are the countries that are further away from the core. Spain, Italy and Ireland cannot be considered central countries either. Finally, the closest country to Ireland, on both measures, is the United Kingdom. Comparing the business cycle map with the geographical map, the biggest surprise is Spain, which was able to cross the Pyrenees. This eyeball procedure can be formally confirmed: the Spearman Rank Correlation between physical and business cycle distances (i.e., between the upper and the lower triangle of Table 1) is 0.67 (p-value of 4×10^{-15}).

3.2. Phase-difference and cross-wavelets

The phase-difference gives us information on the delay between oscillations of two time-series, the coherency cross-wavelet transform will tell us if the correlation is strong or not. To perform the cross-wavelet analysis we focus on the wavelet coherency, instead of the wavelet cross spectrum, because there is some redundancy between both measures and the wavelet coherency has the advantage of being normalized by the power spectrum of the two time-series. Regions of high coherency between two countries are synonym of strong local correlation.

To assess if coherency is statistically significant, we follow the same method we used to test for business cycle distances. The test for the phase-difference is not straightforward. In fact, Ge (2008) showed that under the null of no linear relation

[18] On the one hand, the business cycle dissimilarity matrix is obviously not Euclidean. On the other hand, even the physical distances are not Euclidean because of Earth's curvature.

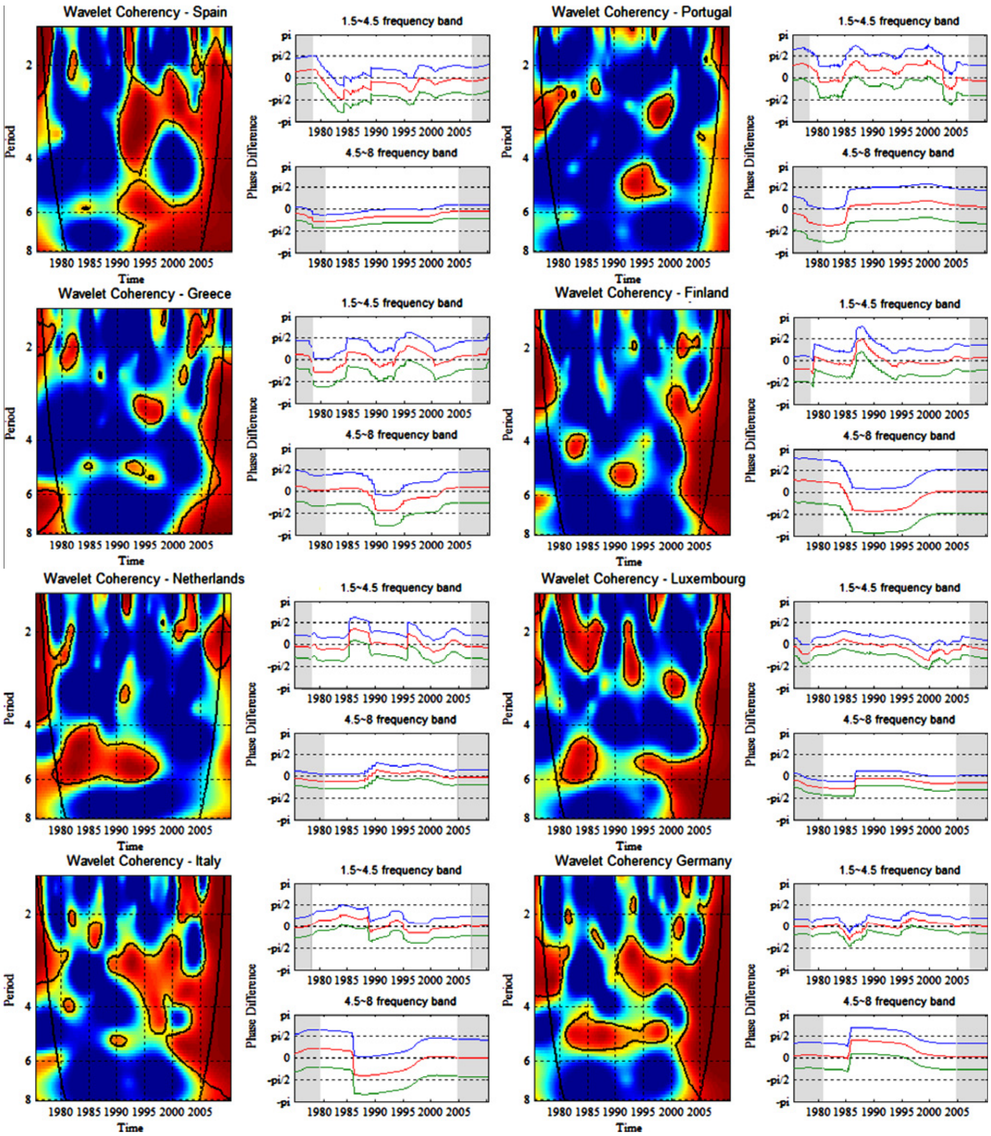

Fig. 3. On the Left: Wavelet coherency – the cone of influence is shown with a black line. The contour designates the 5% significance level. Coherency ranges from blue (low coherency) to red (high coherency). On the right: Phase-difference (red line), plus or minus two standard deviations. The shaded area may be affected edge effects. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

between two variables the phase angle will be uniformly distributed. Hence it will be dispersed between $-\pi$ and π. Because of that, Ge argues that one should not use significance tests for the wavelet-phase-difference. Instead, its analysis should be complemented by inspection of the coherence significance. For reference, and without claiming to be rigorous, we construct confidence intervals based on adding and subtracting two standard deviations.

In Fig. 3 we have, on the left, the coherency between each country's industrial production and the rest of Europe. On the right, for each country, we present two graphs. On top, we have the phase-difference between the two time-series calculated for the 1.5–4.5 years frequency-band. In the bottom, the analysis is performed in the longer run, 4.5–8 years frequency-band. A phase-difference between $-\pi/2$ and $\pi/2$ means that both series are in-phase: between zero and $\pi/2$, Europe is leading; between $-\pi/2$ and zero, it is the country that is leading.

One common feature to every country, with the exception of Portugal and the Netherlands, is that there is a high coherency region in the last years of the 2000's decade. This is not surprising. With the global crisis hitting simultaneously several countries, they behave as if they were highly synchronized.

Starting with the countries that we have identified as not belonging to the euro-cycle, Portugal, Greece, Finland and Ireland. These countries, specially the first three, do not exhibit many regions of high coherency, confirming the results we had already obtained. Ireland is a different case. Ireland exhibits some regions of high coherency. This is not surprising. In fact,

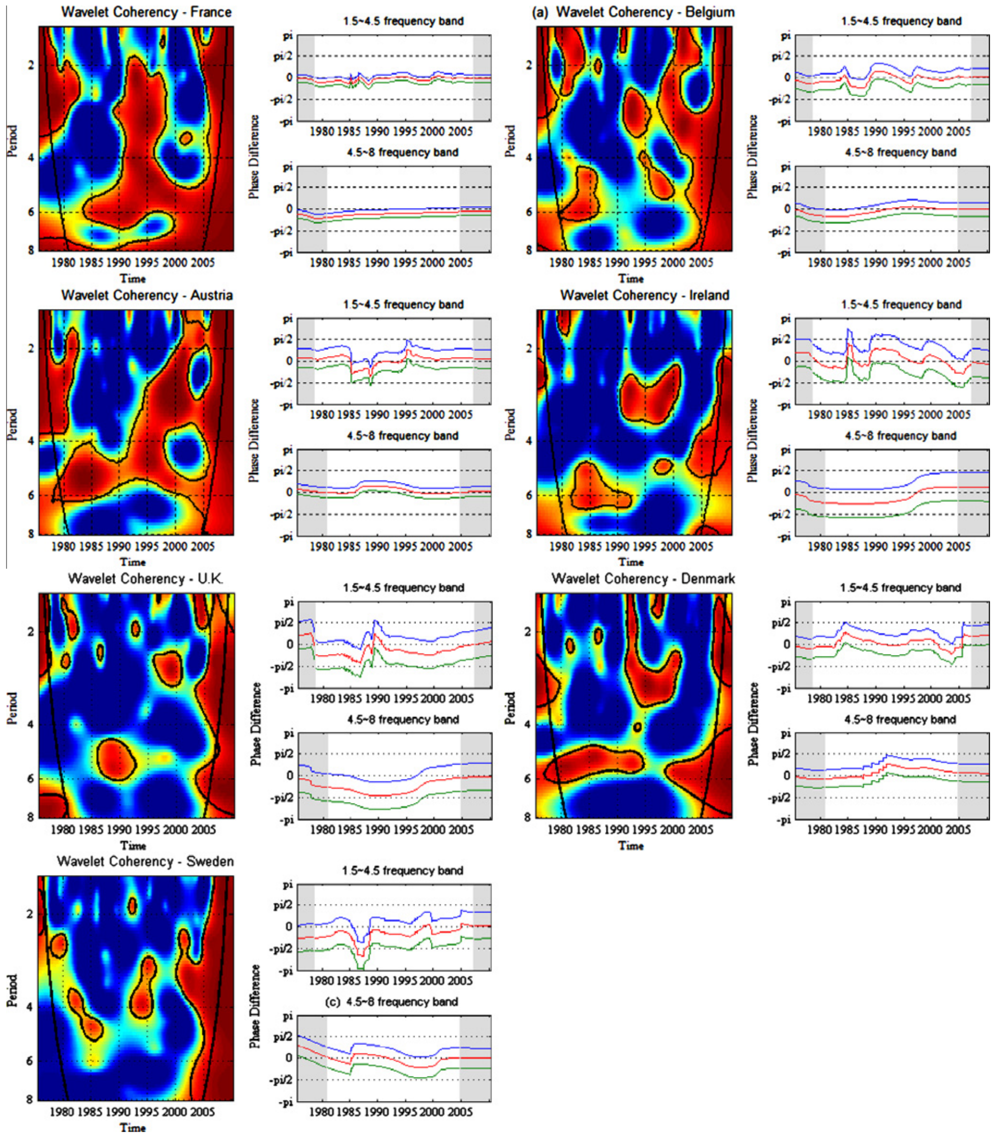

Fig. 3 (*continued*)

looking at Table 1, one can see that Ireland is synchronized with some individual European countries (like Luxembourg or the Netherlands).[19] Finland exhibits some interesting dynamics. Between 1990 and 1994 it shows a small region of high coherency at low frequencies but the phase-difference calculated for the 4.5–8 year frequency-band reveals that Finland was almost out of phase with the rest of Europe. After 1995, the phase-difference analysis shows that the cycles became more synchronized and, after 2000, the regions of high coherency extend across all frequencies. Therefore, there is evidence that the Finish business cycle started approaching the European cycle after mid-1990s.

Looking at the European core, which we have identified as being formed by Germany and France, one concludes, perhaps surprisingly, that France shows more regions of high coherency. On top of that, while in the shorter run (1.5–4.5 frequencies) both France and Germany are very much in-phase with the rest of Europe, when one looks at longer-run frequencies (4.5–8 year frequency-band) one concludes that it is France, not Germany, that has been leading the European cycle. This result is confirmed in Fig. 4, when we directly compare France with Germany: at lower frequencies, France has been leading the German cycle.

[19] In a previous version of the paper, we found the Irish business cycle to be synchronized at 10% with Europe. Even in that previous version, evidence for Ireland was mixed. So it is fair to say that our results for Ireland are not robust.

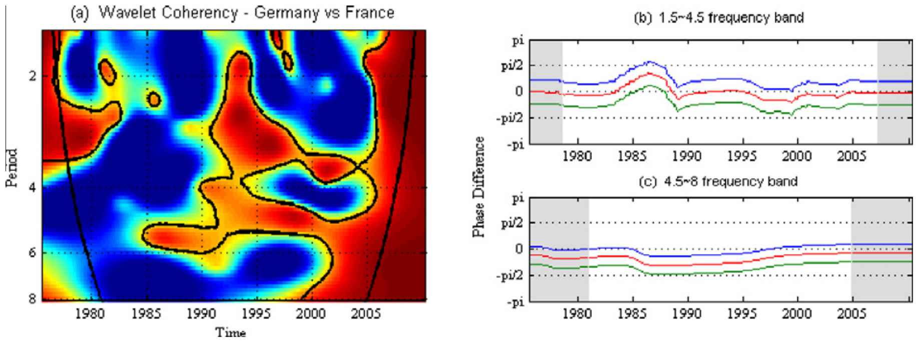

Fig. 4. On the Left: Wavelet coherency – the cone of influence is shown with a black line. The contour designates the 5% significance level. Coherency ranges from blue (low coherency) to red (high coherency). On the right: Phase-difference (red line), plus or minus two standard deviations. A phase-difference between $-\pi/2$ and $\pi/2$ means that both series are in-phase. Between $-\pi/2$ and zero, France is leading. The shaded area may be affected edge effects. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

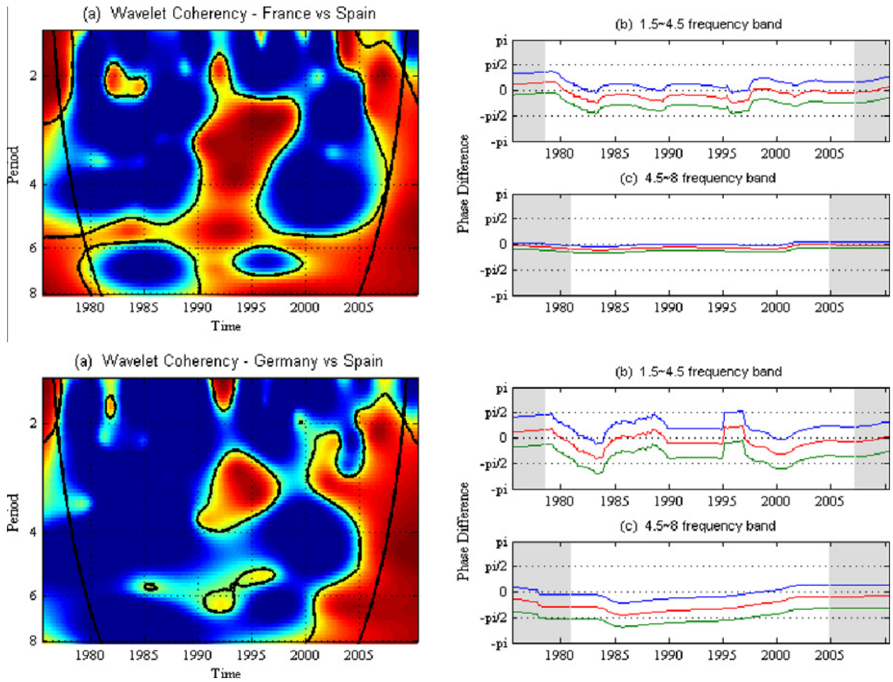

Fig. 5. On the Left: Wavelet coherency – the cone of influence is shown with a black line. The contour designates the 5% significance level. Coherency ranges from blue (low coherency) to red (high coherency). On the right: Phase-difference (red line), plus or minus two standard deviations. A phase-difference between $-\pi/2$ and zero means that Spain is leading. The shaded area may be affected edge effects. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Spain, Italy and Belgium show more regions of high coherency after 1990 suggesting that it was in that decade that they started approaching the core. In Fig. 5, we estimate the coherency and phase-difference between Spain and the two Euro-core countries. It is clear that Spain, after 1990, shows regions of strong coherency with Germany but specially with France, and that the longer-run cycle is particularly aligned with France. Therefore, one can infer that Spain has approached the Euro-core mainly because it has been getting closer to France. A similar analysis (not shown) would lead to similar conclusions about Italy (although not as evident). In the case of Belgium, coherency is stronger with Germany but, in spite of that, the Belgium phases seem more aligned with the French phases.

Finally, looking at the countries that are not in the Euro – the UK, Sweden and Denmark –, it is clear that Denmark is the one that is the most synchronized with the Euro area. In particular, note that the there are large coherency regions around the 6 year frequency, which we identified in Fig. 1 as being very relevant.

Table 2
Business cycle dissimilarities between several countries and the Euro 12.

	Hungary	Poland	Cyprus	Romania	Slovakia	Czech
Dissimilarity	0.474	0.571	0.453	0.464	0.329	0.376
Critical value (10%)	0.339	0.314	0.299	0.290	0.258	0.257

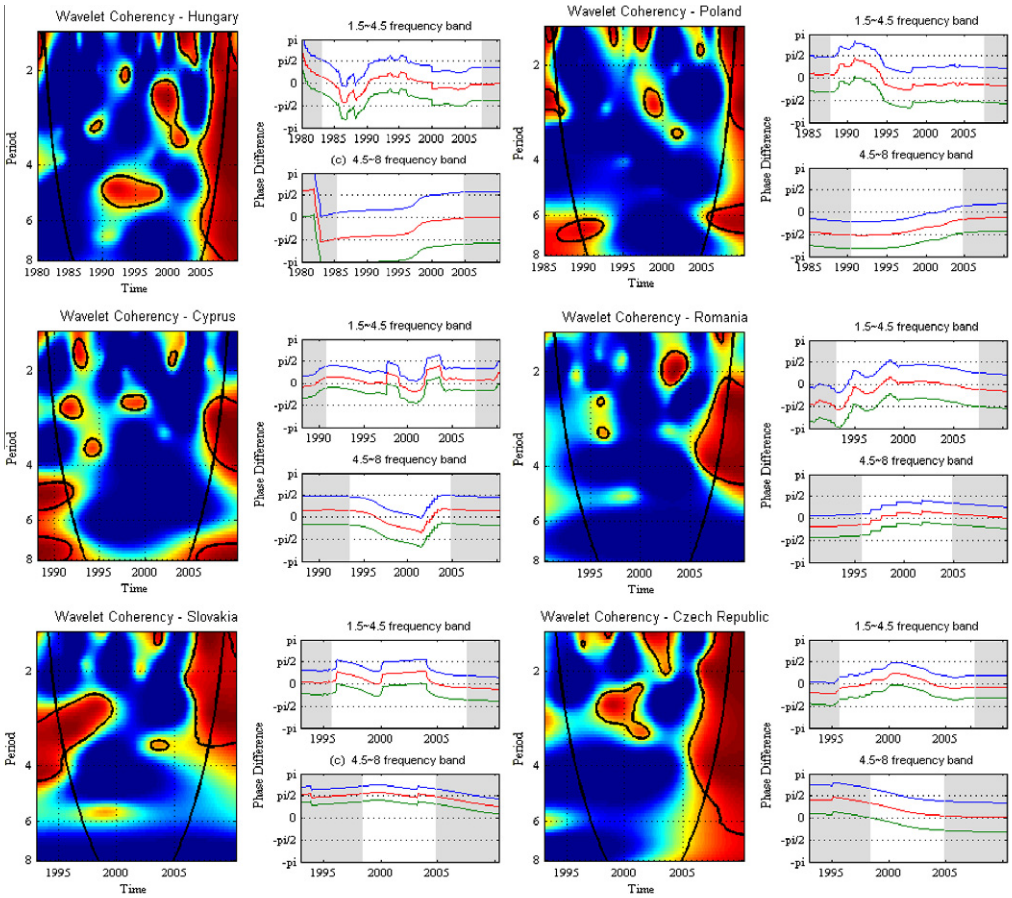

Fig. 6. On the Left: Wavelet coherency – the cone of influence is shown with a black line. The contour designates the 5% significance level. Coherency ranges from blue (low coherency) to red (high coherency). On the right: Phase-difference (red line), plus or minus two standard deviations. The shaded area may be affected edge effects. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4. Which other countries could join the Euro?

We have already assessed the degree of synchronization of three countries that are in the European Union but not in the Euro zone. In this section we extend the analysis for some other countries for which we could gather a decent sample size: Hungary, Poland, Cyprus, Romania, Slovakia, and the Czech Republic.[20]

As before, we computed business cycle dissimilarities based on Eq. (18). At 10% significance, no country is synchronized with the Euro-12 (Table 2).[21] This information is complemented by Fig. 6. Again, most regions of high coherency are located after 2005. In Hungary and Czech Republic, this high coherency region occurs at all frequencies.

Cyprus and Slovakia have already joined the Euro. In spite of that, these two countries do not show any signs of strong convergence. In the case of Cyprus there are not many relevant regions of high coherency and, in the case of Slovakia, phases are not much aligned with the European ones.

[20] For Lithuania, Latvia, Bulgaria, Estonia, Slovenia and Malta the data started somewhere between 1997 and 2005.

[21] These dissimilarities are not comparable between each other, because the sample sizes differ.

The country that shows the strongest convergence to the Euro-12 cycle is the Czech Republic. It is the country with the largest high coherency region after 2005. It is also clear that the phase-differences are approaching zero (after 2005, they are almost zero), showing that the phases are getting more aligned with the Euro area.

5. Conclusions

Wavelet analysis is particularly well suited to study business cycles, because it estimates the spectrum as a function of time, revealing how the different periodic components of the time-series change over time. We used the wavelet tools to investigate business cycles synchronization among the countries that have adopted, could have adopted and that, in the near future, may adopt the Euro.

We derived a Euro-core and a Euro-periphery in terms of business cycles synchronism. Our results indicate that business cycle proximity is highly correlated with physical proximity. Nearby countries have more synchronized economic cycle. Not surprisingly, Germany and France form the Euro-core around which the other countries gravitate. Perhaps surprisingly, it is France, not Germany, which has been leading the European cycle. This result is confirmed when we directly compare France with Germany.

Portugal, Greece, Ireland and Finland are in the Euro-periphery. Their cycles are not in sync with the rest of Euro-12. In particular, Ireland is highly synchronized with the UK. One should also note that Finland has been converging to the Euro-core.

All the other Euro-12 countries are synchronized at 5% significance. Finally, looking at the EU-15 countries that are not in the Euro, Denmark is highly synchronized with the Euro area, while Sweden and the UK are in the limbo, with a synchronization that is significant at 10%.

Among the new member countries, Cyprus and Slovakia have already joined the Euro. These countries business cycles are not very aligned with the Euro-12. Among the countries that have not joined the Euro, the Czech Republic seems the most promising candidate to join it.

Acknowledgements

Luís Aguiar-Conraria acknowledges financial support from Fundação para a Ciência e a Tecnologia, project ‘‘Oil shocks and the Macroeconomy: Econometric Estimation, Economic Modeling and Policy Implications’’, PTDC/ECO/64750/2006.

References

Aguiar-Conraria, L., Azevedo, N., Soares, M.J., 2008. Using wavelets to decompose the time-frequency effects of monetary policy. Physica A: Statistical Mechanics and its Applications 387, 2863–2878.
Aguiar-Conraria, L., Soares, M.J., 2010. The continuous wavelet transform: a primer. NIPE-WP 23/2010. <http://www3.eeg.uminho.pt/economia/nipe/docs/2010/NIPE_WP_23_2010.pdf>.
Artis, M., Krolzig, H.-M., Toro, J., 2004. The European business cycle. Oxford Economic Papers 56, 1–44.
Baxter, M., Kouparitsas, M., 2005. Determinants of business cycle comovement: a robust analysis. Journal of Monetary Economics 52, 113–157.
Byers, J., 2003. Great Circle Distances. <http://chemical-ecology.net/>.
Calmfors, L., Flam, H., Gottfries, N., Matlary, J., Jerneck, M., Lindahl, R., Berntsson, C., Rabinowitcz, E., Vredin, A., 1997. EMU: A Swedish Perspective. Kluwer Academic Publishers.
Camacho, M., Perez-Quirós, G., Saiz, L., 2006. Are European business cycles close enough to be just one? Journal of Economics Dynamics and Control 30, 1687–1706.
Camacho, M., Perez-Quirós, G., Saiz, L., 2008. Do European business cycles look like one? Journal of Economic Dynamics and Control 32, 2165–2190.
Clark, T.E., Wincoop, E., 2001. Borders and business cycles. Journal of International Economics 55, 59–85.
Crowley, P., 2007. A guide to wavelets for economists. Journal of Economic Surveys 21, 207–267.
Crowley, P., Maraun, D., Mayes, D., 2006 How hard is the euro area core? An evaluation of growth cycles using wavelet analysis. Bank of Finland Research Discussion Papers 18.
Crowley, P., Mayes, D., 2008. How fused is the Euro area core? An evaluation of growth cycle co-movement and synchronization using wavelet analysis. Journal of Business Cycle Measurement and Analysis 4, 63–95.
Forni, M., Hallin, M., Lippi, M., Reichlin, L., 2000. The generalized dynamic-factor model: identification and estimation. The Review of Economics and Statistics 82, 540–554.
Frankel, J.A., Rose, A.K., 1998. The endogeneity of the optimum currency area criteria. The Economic Journal 108, 1009–1025.
Ge, Z., 2008. Significance tests for the wavelet cross spectrum power and wavelet linear coherence. Annales Geophysicae 26, 3819–3829.
Gençay, R., Selçuk, F., Witcher, B., 2005. Multiscale systematic risk. Journal of International Money and Finance 24, 55–70.
Goupillaud, P., Grossman, A., Morlet, J., 1984. Cycle-octave and related transforms in seismic signal analysis. Geoexploration 23, 85–102.
Grossmann, A., Morlet, J., 1984. Decomposition of Hardy functions into square integrable wavelets of constant shape. SIAM Journal on Mathematical Analysis 15, 723–736.
Harding, D., Pagan, A., 2006. Synchronization of cycles. Journal of Econometrics 132, 59–79.
HMT, 2003. UK membership of the single currency: assessment of the five economic tests. H.M. Treasury, June.
Imbs, J., 2004. Trade, finance, specialization, and synchronization. The Review of Economics and Statistics 86, 723–734.
Inklaar, R., Jong-A-Pin, R., de Haan, J., 2008. Trade and business cycle synchronization in OECD countries – a re-examination. European Economic Review 52, 646–666.
Raihan, S., Wen, Y., Zeng, B., 2005. Wavelet: a new tool for business cycle analysis. Working Paper 2005-050A, Federal Reserve Bank of St. Louis.
Rose, A., Engel, C., 2002. Currency unions and international integration. Journal of Money, Credit and Banking 34, 1067–1089.
Rouyer, T., Fromentin, J.-M., Stenseth, N., Cazelles, B., 2008. Analysing multiple time series and extending significance testing in wavelet analysis. Marine Ecology Progress Series 359, 11–23.
Rua, A., 2010. Measuring comovement in the time-frequency space. Journal of Macroeconomics 32, 685–691.
Scharnhorst, K., 2001. Angles in complex vector spaces. Acta Applicandae Mathematicae 69, 95–103.
Selover, D., Jensen, R., 1999. ‘Mode-locking’ and international business cycle transmission. Journal of Economic Dynamics and Control 23, 591–618.