Homework 7

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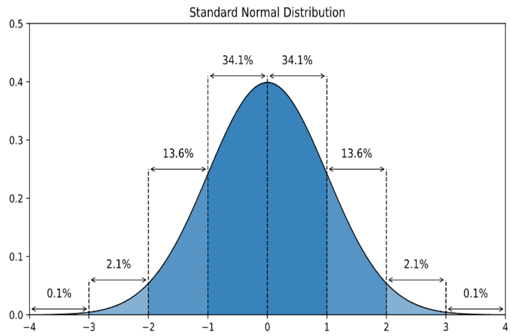
Contents

1 *Fair Price for Binary Options*

$$f_t = e^{-r(T-t)}\mathbb{E}[f_T]$$

For discrete:
$\mathbb{E}[X] = \sum_{x=1}^{n} x_i p(x_i)$
For continuous:
$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\ p(x)\ dx$
For a normal distribution $x \sim N(\mu_x, \sigma_x^2)$, therefore:

$$p(x) = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(\frac{x-\mu_x}{\sigma_x})^2}}{\sqrt{2\pi\sigma_x^2}}\ dx$$



If $\mu_x = 0$ and $\sigma_x^2 = 1$:

$$N(z_i) = \int_{-\infty}^{z_i} \frac{e^{-\frac{1}{2}(z_i)^2}}{\sqrt{2\pi}}\ dz$$

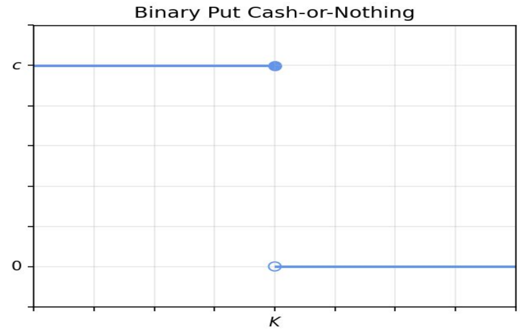
Calculating derivatives:

$$\frac{\partial N(z_i)}{\partial z} = \frac{e^{-\frac{1}{2}(z_i)^2}}{\sqrt{2\pi}} = N'(z_i)$$

$$\frac{\partial^2 N(z_i)}{\partial z^2} = -z_i N'(z_i) = N''(z_i)$$

2 Put Cash or Nothing

$$f_T^{\text{Put CN}} = f(S_T, T) = \begin{cases} 0 & \text{if } S_T > K \\ c & \text{if } S_T \leq K \end{cases}$$



2.1 1 Validate S_T to $\ln(S_T)$ in utility function

$$f_T^{\text{Put CN}} = f(ln(S_T), T) = \begin{cases} 0 & \text{if } ln(S_T) > ln(K) \\ c & \text{if } ln(S_T) \leq ln(K) \end{cases}$$

We define $f_T = f(ln(S_T), T) = f(x, T)$, therefore $x = ln(S_T) \to S_T = e^x$

2.2 2 Get μ and σ^2 of $\ln(S_T)$

Recalling model 2 log-form:

$$ln(S_T) = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T - t) + \sqrt{T - t}\sigma W_1$$

$\mathbb{E}[x]$ is applied to both sides to obtain μ

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T - t) + \sqrt{T - t}\sigma W_1\right]$$

With help of properties:

1. $\mathbb{E}[x + y] = \mathbb{E}[x] + \mathbb{E}[y]$
2. $\mathbb{E}[ax] = a\mathbb{E}[x]$
3. $\mathbb{E}[a] = a$

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t)\right] + \mathbb{E}\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T - t)\right] + \mathbb{E}\left[\sqrt{T - t}\sigma W_1\right]$$

$$\mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T - t) + \sqrt{T - t}\sigma\mathbb{E}\left[W_1\right]$$

$$W_1 \sim \mathcal{N}(0, 1)$$

$$\mu_x = \mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T - t)$$

Now, $Var[x]$ is applied to both sides to obtain σ^2

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

4. $Var[x+y] = Var[x] + Var[y] \quad x, y = \text{independent}$
5. $Var[ax] = a^2 Var[x]$
6. $Var[a] = 0$

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t)\right] + Var\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + Var\left[\sqrt{T-t}\sigma W_1\right]$$

$$Var\left[ln(S_T)\right] = (T-t)\sigma^2 Var\left[W_1\right]$$

$$\sigma_x^2 = Var\left[ln(S_T)\right] = (T-t)\sigma^2$$

With these values, we can write the x distribution as:

$$x \sim \mathcal{N}\left(ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t), (T-t)\sigma^2\right)$$

2.3 *3 Model 7, expand* $\mathbb{E}[X]$

Recalling model 7:

$$f_t = e^{-r(T-t)}\mathbb{E}[f_t]$$

$$f_t = e^{-r(T-t)} \int_{-\infty}^{\infty} f(x)p(x)dx$$

2.4 *4 Split integral based on utility function*

$$f_t = e^{-r(T-t)}\left[\int_{-\infty}^{ln(K)} cp(x)dx + \int_{ln(K)}^{\infty} 0p(x)dx\right]$$

$$f_t = e^{-r(T-t)} \int_{-\infty}^{ln(K)} cp(x)dx$$

2.5 *5 Expand* $p(x)$

$$f_t = e^{-r(T-t)} \int_{-\infty}^{ln(K)} c\frac{e^{-\frac{1}{2}\left(\frac{x_i - \left[ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]}{\sqrt{T-t}\sigma}\right)^2}}{\sqrt{2\pi\sigma^2(T-t)}}dx$$

2.6 *6 Change variable to get standard model*

We define a new variable z:

$$z = \frac{x - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}$$

Differential change for the new variable:

$$\frac{dz}{dx} = \frac{1}{\sqrt{T-t}\sigma} \Rightarrow dx = \sqrt{T-t}\sigma dz$$

Upper integral limit:

$$\lim_{x \to ln(K)} = \frac{ln(K) - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}$$

$$\lim_{x \to ln(K)} = -\left(\frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}\right) = -d_2$$

Lower integral limit:

$$\lim_{x \to -\infty} = \frac{-\infty - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma} = -\infty$$

2.6.1 *6.1 New variable in integral*

$$f_t = e^{-r(T-t)} \int_{-\infty}^{-d_2} c\frac{e^{-\frac{1}{2}(z)^2}\sigma\sqrt{T-t}}{\sqrt{2\pi}\sigma\sqrt{T-t}}dz$$

$$f_t = e^{-r(T-t)} \int_{-\infty}^{-d_2} c\frac{e^{-\frac{1}{2}(z)^2}}{\sqrt{2\pi}}dz$$

Finally, the fair price for a Put Cash of Nothing option is:

$$f_t = ce^{-r(T-t)}\mathcal{N}(-d_2)$$

2.7 *7 Example*

Data:

$S_t = 100$

$K = 120$

$\sigma = 30\%$

$r = 3\%$

$T = 1$

$t = 0$

$c = 500$

Formulas:

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Substitution:

$$d_1 = \frac{\ln(\frac{100}{120}) + (0.03 + \frac{0.3^2}{2})(1-0)}{0.3\sqrt{1-0}} = -0.3577$$

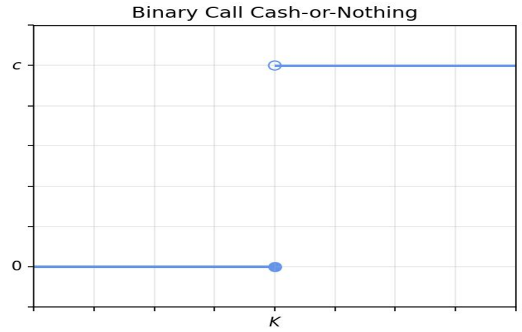
$$d_2 = -0.3577 - 0.3\sqrt{1-0} = -0.6577$$

$$f_t = 500e^{-0.03(1-0)}\mathcal{N}(-(-0.6577))$$

$$f_t = 500e^{-0.03(1-0)} \cdot 0.7446 = \$361.29$$

3 Call Cash or Nothing

$$f_T^{\text{Call CN}} = f(S_T, T) = \begin{cases} c & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases}$$



3.1 1 Validate S_T to $\ln(S_T)$ in utility function

$$f_T^{\text{Call CN}} = f(ln(S_T), T) = \begin{cases} c & \text{if } ln(S_T) > ln(K) \\ 0 & \text{if } ln(S_T) \leq ln(K) \end{cases}$$

We define $f_T = f(ln(S_T), T) = f(x, T)$, therefore $x = ln(S_T) \rightarrow S_T = e^x$

3.2 2 Get μ and σ^2 of $\ln(S_T)$

Recalling model 2 log-form:

$$ln(S_T) = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1$$

$\mathbb{E}[x]$ is applied to both sides to obtain μ

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

1. $\mathbb{E}[x+y] = \mathbb{E}[x] + \mathbb{E}[y]$
2. $\mathbb{E}[ax] = a\mathbb{E}[x]$
3. $\mathbb{E}[a] = a$

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t)\right] + \mathbb{E}\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + \mathbb{E}\left[\sqrt{T-t}\sigma W_1\right]$$

$$\mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma\mathbb{E}\left[W_1\right]$$

$$W_1 \sim \mathcal{N}(0,1)$$

$$\mu_x = \mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)$$

Now, $Var[x]$ is applied to both sides to obtain σ^2

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

4. $Var[x+y] = Var[x] + Var[y] \quad x, y = \text{independent}$
5. $Var[ax] = a^2 Var[x]$
6. $Var[a] = 0$

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t)\right] + Var\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + Var\left[\sqrt{T-t}\sigma W_1\right]$$

$$Var\left[ln(S_T)\right] = (T-t)\sigma^2 Var\left[W_1\right]$$

$$\sigma_x^2 = Var\left[ln(S_T)\right] = (T-t)\sigma^2$$

With these values, we can write the x distribution as:

$$x \sim \mathcal{N}\left(ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t), (T-t)\sigma^2\right)$$

3.3 *3 Model 7, expand $\mathbb{E}[X]$*

Recalling model 7:

$$f_t = e^{-r(T-t)}\mathbb{E}[f_t]$$

$$f_t = e^{-r(T-t)}\int_{-\infty}^{\infty} f(x)p(x)dx$$

3.4 *4 Split integral based on utility function*

$$f_t = e^{-r(T-t)}\left[\int_{-\infty}^{ln(K)} 0p(x)dx + \int_{ln(K)}^{\infty} cp(x)dx\right]$$

$$f_t = e^{-r(T-t)}\int_{ln(K)}^{\infty} cp(x)dx$$

3.5 *5 Expand $p(x)$*

$$f_t = e^{-r(T-t)}\int_{ln(K)}^{\infty} c\frac{e^{-\frac{1}{2}\left(\frac{x_i - \left[ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]}{\sqrt{T-t}\sigma}\right)^2}}{\sqrt{2\pi\sigma^2(T-t)}}dx$$

3.6 *6 Change variable to get standard model*

We define a new variable z:

$$z = \frac{x - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}$$

Differential change for the new variable:

$$\frac{dz}{dx} = \frac{1}{\sqrt{T-t}\sigma} \Rightarrow dx = \sqrt{T-t}\sigma dz$$

Lower integral limit:

$$\lim_{x \to ln(K)} = \frac{ln(K) - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}$$

$$\lim_{x \to ln(K)} = -\left(\frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}\right) = -d_2$$

Upper integral limit:

$$\lim_{x \to \infty} = \frac{\infty - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma} = \infty$$

3.6.1 *6.1 New variable in integral*

$$f_t = e^{-r(T-t)} \int_{-d_2}^{\infty} c\frac{e^{-\frac{1}{2}(z)^2}\sigma\sqrt{T-t}}{\sqrt{2\pi}\sigma\sqrt{T-t}}dz$$

$$f_t = e^{-r(T-t)} \int_{-d_2}^{\infty} c\frac{e^{-\frac{1}{2}(z)^2}}{\sqrt{2\pi}}dz$$

By symmetry:

$$\int_{-\infty}^{-x} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} = \int_{x}^{\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$$

$$f_t = e^{-r(T-t)} \int_{\infty}^{d_2} c\frac{e^{-\frac{1}{2}(z)^2}}{\sqrt{2\pi}}dz$$

Finally, the fair price for a Put Cash of Nothing option is:

$$f_t = ce^{-r(T-t)}\mathcal{N}(d_2)$$

3.7 *7 Example*

Data:

$S_t = 100$

$K = 120$

$\sigma = 30\%$

$r = 3\%$

$T = 1$

$t = 0$

$C = 500$

Formulas:

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Substitution:

$$d_1 = \frac{\ln(\frac{100}{120}) + (0.03 + \frac{0.3^2}{2})(1-0)}{0.3\sqrt{1-0}} = -0.3577$$

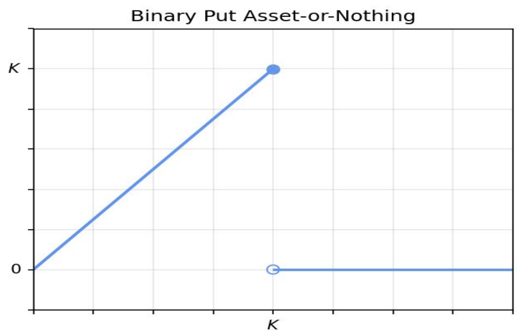
$$d_2 = -0.3577 - 0.3\sqrt{1-0} = -0.6577$$

$$f_t = 500e^{-0.03(1-0)}\mathcal{N}(-0.6577)$$

$$f_t = 500e^{-0.03(1-0)} \cdot 0.2553 = \$123.87$$

4 *Put Asset or Nothing*

$$f_T^{\text{Put AN}} = f(S_T, T) = \begin{cases} 0 & \text{if } S_T > K \\ S_T & \text{if } S_T \leq K \end{cases}$$



4.1 *1 Validate S_T to $\ln(S_T)$ in utility function*

$$f_T^{\text{Put AN}} = f(\ln(S_T), T) = \begin{cases} 0 & \text{if } \ln(S_T) > \ln(K) \\ S_T & \text{if } \ln(S_T) \leq \ln(K) \end{cases}$$

$$x = \ln(S_T) \Rightarrow S_T = e^x$$

4.2 *2 Get μ and σ^2 of $\ln(S_T)$*

Recalling model 2 log-form:

$$ln(S_T) = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1$$

$\mathbb{E}[x]$ is applied to both sides to obtain μ

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

1. $\mathbb{E}[x+y] = \mathbb{E}[x] + \mathbb{E}[y]$
2. $\mathbb{E}[ax] = a\mathbb{E}[x]$
3. $\mathbb{E}[a] = a$

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t)\right] + \mathbb{E}\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + \mathbb{E}\left[\sqrt{T-t}\sigma W_1\right]$$

$$\mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma\mathbb{E}\left[W_1\right]$$

$$W_1 \sim \mathcal{N}(0,1)$$

$$\mu_x = \mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)$$

Now, $Var[x]$ is applied to both sides to obtain σ^2

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

4. $Var[x+y] = Var[x] + Var[y] \quad x, y = \text{independent}$
5. $Var[ax] = a^2 Var[x]$
6. $Var[a] = 0$

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t)\right] + Var\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + Var\left[\sqrt{T-t}\sigma W_1\right]$$

$$Var\left[ln(S_T)\right] = (T-t)\sigma^2 Var\left[W_1\right]$$

$$\sigma_x^2 = Var\left[ln(S_T)\right] = (T-t)\sigma^2$$

With these values, we can write the x distribution as:

$$x \sim \mathcal{N}\left(ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t), (T-t)\sigma^2\right)$$

4.3 3 Model 7, expand $\mathbb{E}[X]$

Recalling model 7:

$$f_t = e^{-r(T-t)}\mathbb{E}[f_t]$$

$$f_t = e^{-r(T-t)} \int_{-\infty}^{\infty} f(x)p(x)dx$$

4.4 4 Split integral based on utility function

$$f_t = e^{-r(t-t)} \left[\int_{-\infty}^{\ln(K)} e^x \; p(x) \; dx + \int_{\ln(K)}^{\infty} 0 \; p(x) \; dx\right]$$

4.5 5 *Expand p(x)*

$$f_t = e^{-r(T-t)} \int_{-\infty}^{ln(K)} e^x \frac{e^{-\frac{1}{2}\left(\frac{x_i - \left[ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]}{\sqrt{T-t}\sigma}\right)^2}}{\sqrt{2\pi\sigma^2(T-t)}} dx$$

4.6 6 *Change variable to get standard model*

We define a new variable y:

$$y = \frac{x - \ln(S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Differential change for the new variable:

$$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{T-t}} \Longrightarrow dx = dy\ \sigma\sqrt{T-t}$$

$$x = y\sigma\sqrt{T-t} + \ln(S_t) + (r - \frac{1}{2}\sigma^2)(T-t)$$

Upper limit:

$$\lim_{x \to ln(K)} = \frac{ln(K) - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}$$

$$\lim_{x \to ln(K)} = -\left(\frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}\right) = -d_2$$

Lower limit:

$$\lim_{x \to -\infty} = \frac{-\infty - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma} = -\infty$$

4.7 7 *New variable in integral*

$$f_t = e^{-r(T-t)} \int_{-\infty}^{-d_2} e^x \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}\sigma\sqrt{T-t}} dy\ \sigma\sqrt{T-t}$$

Important properties:

$$e^a \cdot e^{b+c} = e^{a+b+c}$$
$$e^a \cdot e^{b+\ln(c)} = ce^{a+b} = ce^a \cdot e^b$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$f_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{[y\sigma\sqrt{T-t} + \ln(S_t) + (r - \frac{1}{2}\sigma^2)(T-t)]} \cdot e^{-\frac{1}{2}y^2}\ dy$$

$$f_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{[-\frac{1}{2}y^2 + y\sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t) + r(T-t) + \ln(S_t)]}\ dy$$

$$f_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-\frac{1}{2}(y^2 - 2y\sigma\sqrt{T-t} + \sigma^2(T-t))} \cdot e^{r(T-t)} \cdot S_t\ dy$$

Some more properties:

$$a = y$$
$$b = \sigma\sqrt{T-t}$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$f_t = e^{-r(T-t)} \cdot e^{r(t-t)} \cdot S_t \int_{-\infty}^{-d_2} \frac{e^{-\frac{1}{2}(y-\sigma\sqrt{T-t})^2}}{\sqrt{2\pi}} dy$$

4.8 8 *Another change of variable*

We define a new variable z:

$$z = y - \sigma\sqrt{T-t}$$

Differential change for the new variable:

$$\frac{dz}{dy} = 1 \Rightarrow dz = dy$$

Upper limit:

$$\lim_{y \to -d_2} = -d_2 - \sigma\sqrt{T-t} = -d_1$$

d_1 and d_2 properties:

$$d_2 = d1 - \sigma\sqrt{T-t}$$
$$-d_1 = -d_2 - \sigma\sqrt{T-t}$$

Lower limit:

$$\lim_{y \to -\infty} = -\infty - \sigma\sqrt{T-t} = -\infty$$

4.9 9 *New variable in integral*

$$f_t = S_t \int_{-\infty}^{-d_1} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

$$f_t = S_t \mathcal{N}(-d_1)$$

4.10 10 *Example*

Data:

$S_t = 100$

$K = 120$

$\sigma = 30\%$

$r = 3\%$

$T = 1$
$t = 0$
$C = 500$

Formulas:

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

Substitution:

$$d_1 = \frac{\ln(\frac{100}{120}) + (0.03 + \frac{0.3^2}{2})(1 - 0)}{0.3\sqrt{1 - 0}} = -0.3577$$

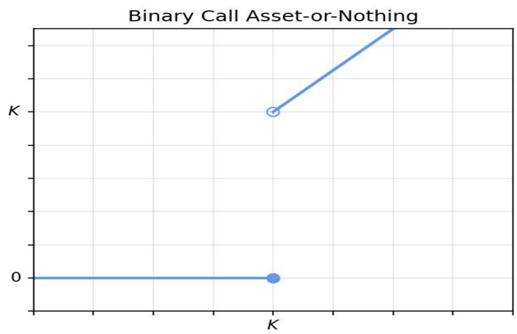
$$d_2 = -0.3577 - 0.3\sqrt{1 - 0} = -0.6577$$

$$f_t = 100\mathcal{N}(-(-0.3577))$$

$$f_t = 100 \cdot 0.6397 = \$63.97$$

5 *Call Asset or Nothing*

$$f_T^{\text{Call AN}} = f(S_T, T) = \begin{cases} S_T & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases}$$



5.1 *1 Validate S_T to $\ln(S_T)$ in utility function*

$$f_T^{\text{Call AN}} = f(\ln(S_T), T) = \begin{cases} S_T & \text{if } \ln(S_T) > \ln(K) \\ 0 & \text{if } \ln(S_T) \leq \ln(K) \end{cases}$$

$$x = \ln(S_T) \Rightarrow S_T = e^x$$

5.2 *2 Get μ and σ^2 of $\ln(S_T)$*

Recalling model 2 log-form:

$$ln(S_T) = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1$$

$\mathbb{E}[x]$ is applied to both sides to obtain μ

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

1. $\mathbb{E}[x+y] = \mathbb{E}[x] + \mathbb{E}[y]$
2. $\mathbb{E}[ax] = a\mathbb{E}[x]$
3. $\mathbb{E}[a] = a$

$$\mathbb{E}\left[ln(S_T)\right] = \mathbb{E}\left[ln(S_t)\right] + \mathbb{E}\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + \mathbb{E}\left[\sqrt{T-t}\sigma W_1\right]$$

$$\mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma\mathbb{E}\left[W_1\right]$$

$$W_1 \sim \mathcal{N}(0,1)$$

$$\mu_x = \mathbb{E}\left[ln(S_T)\right] = ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)$$

Now, $Var[x]$ is applied to both sides to obtain σ^2

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t) + \sqrt{T-t}\sigma W_1\right]$$

With help of properties:

4. $Var[x+y] = Var[x] + Var[y] \quad x, y = \text{independent}$
5. $Var[ax] = a^2 Var[x]$
6. $Var[a] = 0$

$$Var\left[ln(S_T)\right] = Var\left[ln(S_t)\right] + Var\left[\left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t)\right] + Var\left[\sqrt{T-t}\sigma W_1\right]$$

$$Var\left[ln(S_T)\right] = (T-t)\sigma^2 Var\left[W_1\right]$$

$$\sigma_x^2 = Var\left[ln(S_T)\right] = (T-t)\sigma^2$$

With these values, we can write the x distribution as:

$$x \sim \mathcal{N}\left(ln(S_t) + \left(\mu_R - \frac{1}{2}\sigma^2\right)(T-t), (T-t)\sigma^2\right)$$

5.3 3 Model 7, expand $\mathbb{E}[X]$

Recalling model 7:

$$f_t = e^{-r(T-t)}\mathbb{E}[f_t]$$

$$f_t = e^{-r(T-t)} \int_{-\infty}^{\infty} f(x)p(x)dx$$

5.4 4 Split integral based on utility function

$$f_t = e^{-r(T-t)} \left[\int_{-\infty}^{\ln k} 0p(x)dx + \int_{\ln k}^{\infty} e^x p(x)dx \right]$$

5.5 5 Expand $p(x)$

$$f_t = e^{-r(T-t)} \int_{\ln k}^{\infty} e^x \cdot \frac{e^{-\frac{1}{2}\left(\frac{x_i - \left[ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]}{\sqrt{T-t}\sigma}\right)^2}}{\sqrt{2\pi\sigma^2(T-t)}} dx$$

5.6 6 Change variable to get standard model

We define a new variable y:

$$y = \frac{x - \ln(S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Differential change for the new variable:

$$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{T-t}} \implies dx = dy\ \sigma\sqrt{T-t}$$

$$x = y\sigma\sqrt{T-t} + \ln(S_t) + (r - \frac{1}{2}\sigma^2)(T-t)$$

Lower limit:

$$\lim_{x \to ln(K)} = \frac{ln(K) - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma}$$

$$\lim_{x \to ln(K)} = -\left(\frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma} \right) = -d_2$$

Upper limit:

$$\lim_{x \to \infty} = \frac{\infty - ln(S_t) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{T-t}\sigma} = \infty$$

5.7 7 New variable in integral

$$f_t = e^{-r(T-t)} \int_{-d_2}^{\infty} e^x \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi\sigma\sqrt{T-t}}} dy\ \sigma\sqrt{T-t}$$

Important properties:

$$e^a \cdot e^{b+c} = e^{a+b+c}$$
$$e^a \cdot e^{b+\ln(c)} = ce^{a+b} = ce^a \cdot e^b$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$f_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{[y\sigma\sqrt{T-t}+\ln(S_t)+(r-\frac{1}{2}\sigma^2)(T-t)]} \cdot e^{-\frac{1}{2}y^2}\ dy$$

$$f_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{[-\frac{1}{2}y^2+y\sigma\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)+r(T-t)+\ln(S_t)]}\ dy$$

$$f_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y^2-2y\sigma\sqrt{T-t}+\sigma^2(T-t))} \cdot e^{r(T-t)} \cdot S_t\ dy$$

Some more properties:

$$a = y$$
$$b = \sigma\sqrt{T-t}$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$f_t = e^{-r(T-t)} \cdot e^{r(t-t)} \cdot S_t \int_{-d_2}^{\infty} \frac{e^{-\frac{1}{2}(y-\sigma\sqrt{T-t})^2}}{\sqrt{2\pi}} dy$$

5.8 8 *Another change of variable*

We define a new variable z:

$$z = y - \sigma\sqrt{T-t}$$

Differential change for the new variable:

$$\frac{dz}{dy} = 1 \Rightarrow dz = dy$$

Lower limit:

$$\lim_{y \to -d_2} = -d_2 - \sigma\sqrt{T-t} = -d_1$$

d_1 and d_2 properties:

$$d_2 = d_1 - \sigma\sqrt{T-t}$$
$$-d_1 = -d_2 - \sigma\sqrt{T-t}$$

Upper limit:

$$\lim_{y \to \infty} = \infty - \sigma\sqrt{T-t} = \infty$$

5.9 9 *New variable in integral*

$$f_t = S_t \int_{-d_1}^{\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

By symmetry:

$$\int_{-\infty}^{-x} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} = \int_{x}^{\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$$

$$f_t = S_t \int_{-\infty}^{d_1} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

$$f_t = S_t \mathcal{N}(d_1)$$

5.10 *10 Example*

Data:

$S_t = 100$
$K = 120$
$\sigma = 30\%$
$r = 3\%$
$T = 1$
$t = 0$
$C = 500$

Formulas:

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

Substitution:

$$d_1 = \frac{\ln(\frac{100}{120}) + (0.03 + \frac{0.3^2}{2})(1 - 0)}{0.3\sqrt{1 - 0}} = -0.3577$$

$$d_2 = -0.3577 - 0.3\sqrt{1 - 0} = -0.6577$$

$$f_t = 100\mathcal{N}(-0.3577)$$

$$f_t = 100 \cdot 0.3602 = \$36.02$$