Homework 3

Math/CSCI 166, Fall 2022

Assigned: Friday, September 30
Due: Friday, October 7 by 1:30pm on Gradescope

- In your submission, please label all problems (with answers boxed when appropriate), and submit in the order assigned.
- Include printouts of all code (ideally with some comments).

(1) **(Order of convergence of a sequence)** *(Programming)* The following sequence converges to $\sqrt{5}$. Let $p_0 = 1$. For $n \geq 0$, define

$$p_{n+1} = \frac{1}{2}\left(p_n + \frac{5}{p_n}\right)$$

Define the absolute error $e_n = |p_n - \sqrt{5}|$. Make a table with the following columns, letting $n = 0, 1, \ldots, 6$. (Use eight or more decimal places.)

n	p_n	e_n	e_n/e_{n-1}	e_n/e_{n-1}^2	e_n/e_{n-1}^3

Use this table to decide the order of convergence of the sequence. Explain your reasoning. This is a real example, so the data table might be a little messy compared to theory, but try to justify your answer. Repeat this process for the sequence defined by

$$p_0 = 1 \qquad \text{and} \qquad p_{n+1} = \frac{p_n^3 + 15p_n}{3p_n^2 + 5} \qquad \text{for } n \geq 0,$$

which also converges to $\sqrt{5}$.

(2) **(Order of convergence of a sequence)** (*Not prgoramming*) Let a be a nonzero real number. For any x_0 satisfying $0 < x_0 < 2/a$, the recursive sequence defined by

$$x_{n+1} = x_n(2 - ax_n)$$

converges to $1/a$. Find the order of convergence and the asymptotic error constant by hand (not using programming).

HINT: Consider the limit $\lim_{n\to\infty} \frac{|x_{n+1}-1/a|}{|x_n-1/a|^\alpha}$. Replace x_{n+1} with the recursive definition given. Then try to factor the numerator in a way that makes it clear what α should be for the limit to converge.

(3) **(Rootfinding and Optimization)** *(Programming)*

(a) Suppose $f(x)$ is differentiable on $[a, b]$. Discuss how you might use a rootfinding method to identify a local extremum of $f(x)$ inside $[a, b]$.
(b) Let $f(x) = \ln x - \sin x$. Using calculus, prove that $f(x)$ has a unique maximum in the interval $[4, 6]$.
(c) Approximate this local maximum using 12 iterations of the enclosure methods (Bisection and False Position) with starting interval $[4, 6]$. You should code these two methods yourself.
(d) What is your best estimate for p, the location of the maximum?

(e) Compare the two algorithms using the following two tables. You can use the following value of p when computing the errors: $p = 4.9171859252871323$.

Table 1: Approximation p_n versus iteration number n

Iteration n	Bisection	False Position

Table 2: Absolute error $|p_n - p|$ versus iteration number n

Iteration n	Bisection	False Position

(f) Plot the log of the absolute error for the two methods on the same graph.
(g) Don't forget to turn in your code for both methods!

(4) **(Bisection Method Stopping Criteria)** Determine a formula which relates the number of iterations, n, required by the bisection method to converge to within an absolute error tolerance

$$\frac{b_n - a_n}{2} < \epsilon \ ,$$

starting from the initial interval $[a, b]$.

(5) **(Bisection in FPNS)**

Suppose that an equation is known to have a root in $(0, 1)$. How many iterations of the bisection method are needed to achieve full machine precision in the appoximation the location of th e root if the calculations are performed in the floating point number system $\beta = 2$, $k = 12$, $m = -7$, $M = 7$. Does the answer change if the root were known to be contained in the interval $(8, 9)$?

HINT: Consider the number of base 2 digits already known based on the initial enclosure and how many base 2 digits are available to choose in the indicated floating point number system.