Homework 4

Math 166, Fall 2022

Assigned: Friday, October 7, 2022
Due: Friday, October 14, 2022 by 1:30pm on Gradescope

- In your submission, please label all problems (with answers boxed when appropriate) on Gradescope.
- Include printouts of all code (ideally with some comments).

Problem 1 requires programming Newton's method. Please turn in your code! Note you can hardcode in the function and its derivative if that's easier.
Also, remember that when trying to interpret algorithm output, it will not be as clean as the theory. Try to glean as much information as you can when answering the following questions.

(1) **(Finding Roots – Newton's Method)** *(Programming)*
  (a) The function $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero at x=1/3. Perform 10 iterations of Newton's method on this functions from $p_0 = 0$.
    (i) What is the apparent order of convergence? (It could be helpful to make a table like in HW 3)
    (ii) What is the asymptotic error constant?
    (iii) Without running the bisection method, would you expect it to converge faster than Newton's method here?
  (b) Now try to find the same root using the modified form of Newton's method $g(x) = x - 3\frac{f(x)}{f'(x)}$. Did it work? Can you determine the order of convergence?

(2) **(Fixed Point Iterations)** Consider the function $g(x) = \cos(x)$. This function has a unique fixed point at $x \approx 0.7390851332$.
  (a) Prove that the fixed point is unique using the theorem (find an interval for which g satisfies hypotheses A1 and A3).
  (b) (*Programming*) With a starting approximation of $p_0 = 1/2$, use the iteration scheme $p_n = \cos(p_n)$ to approximate the fixed point to within an absolute error of 1×10^{-7}. How many iterations did it take?
  (c) Use the theoretical error bound $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$ to obtain a theoretical bound on the number of iterations needed to approximate the fixed point to within 1×10^{-7}. How does this number compare with you answer in part (b)?

(3) **(Fixed Point Iterations 2)** Consider the function

$$g(x) = 2x(1 - x) ,$$

which has fixed points at $x = 0$ and $x = 1/2$. Use this function to define a fixed point iteration $p_{n+1} = g(p_n)$.
  (a) Why should we expect this fixed point iteration, starting even with a value very close to zero, will fail to converge toward $x = 0$?
  (b) Why should we expect this fixed point iteration, starting with p_0 in $(0, 1)$, to converge to $x = 1/2$. What order of convergence should we expect?

(c) (*Programming*) Perform seven iteration starting from an arbitrary p_0 (not $p_0 = 1/2$) in $(0, 1)$ and numerically confirm the order of convergence. Turn in your output and your code.

(4) **(Order of Convergence)** Apply the theorems on the order of convergence for fixed point iteration schemes to the following problems:

(a) Verify that $x = \sqrt{a}$ is a fixed point of the function

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) .$$

Determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ toward $x = \sqrt{a}$.

(b) Verify that $x = \sqrt{a}$ is a fixed point of the function

$$g(x) = \frac{x^3 + 3xa}{3x^2 + a} .$$

Determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ toward $x = \sqrt{a}$.