Homework 7

Math/CSCI 166, Fall 2022

Assigned: Friday, November 4, 2022
Due: Friday, November 11, 2022 by 1:30pm on Gradescope

(1) **(Norms and Condition Numbers)**

(a) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 2 & -1 \end{bmatrix}.$$

Compute the ∞-matrix norm of A. Find a vector x such that $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$. Then compute the 1-matrix norm of A. Find a vector x such that $\|A\|_1 = \|Ax\|_1 / \|x\|_1$.

(b) Show that for an induced matrix norm $\|AB\| \leq \|A\|\|B\|$. You can use as a fact the inequality discussed in class $\|A\vec{x}\| \leq \|A\|\|\vec{x}\|$.

(c) Prove that, if $(I - A)$ is invertible, then

$$\|(I - A)^{-1}\| \geq \frac{1}{1 + \|A\|}.$$

HINT: Write $I = (I - A)(I - A)^{-1}$ and then take matrix norms of both sides. The matrix norm of I is always 1. You may also find useful that

- $\|A + B\| \leq \|A\| + \|B\|$
- $\|kA\| = |k|\|A\|$ for any constant k.

(d) Show that $\kappa(AB) \leq \kappa(A)\kappa(B)$. ($\kappa$ means condition number)

(e) Show that $\kappa(\alpha A) = \kappa(A)$, where α is some nonzero scalar.

(2) **(Error Estimates)** Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

(a) Compute κ_∞.

(b) The equation $Ax = b$ has exact solution $x = [2 \;\; 2 \;\; 0]^T$ Take the approximate solution $\tilde{x} = [1.9 \;\; 2.1 \;\; -0.1]^T$. Compute the error and residual using this approximate solution.

(c) Using the ∞-norm, compare the relative error to the condition number times the relative residual. In other words, verify that this satisfies the theorem about condition numbers.

(3) **(The Hilbert Matrix)** The description of this problem uses MATLAB commands. If you are using another language, you should find a package that has commands to solve a linear system by LU decomposition and compute the condition number of a matrix (ask me if you need help).

(a) The infamous $n \times n$ Hilbert matrix H_n has entries $H_n(i,j) = 1/(i + j - 1)$. For a specific value of n, we define the right hand side vector b_n by having entries equal to the row sums of the Hilbert matrix. For example:

$$H_2 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \qquad\qquad b_2 = \begin{bmatrix} 1.5 \\ 0.83333 \end{bmatrix}$$

Write code that, given an input size n, generates the Hilbert matrix H_n and the right hand side vector b_n. Turn in your code and the output for $n = 4$.

(b) We will now solve the equation $H_n x = b_n$. It should be clear that the true solution to the system $H_n\, x = b_n$ is the n-dimensional vector $x_n = [1, 1, \ldots, 1]^T$.
Use the MATLAB commands (or equivalent for LU decomp in your programming language):

```
[Ln,Un] = lu(Hn)    %% This finds the LU decomp of Hn
xn = Un\(Ln\bn)     %% This solves for the solution xn using the decomp
```

to solve this linear system for $n = 5, 10, 15, 20$. For each n, provide the calculated solution vector $\widetilde{x}_n$. Then, compute the relative error and the relative residual using the ∞-norm. Report these errors and residuals along with the condition number of H_n in table form. The MATLAB command `cond` will be helpful. Write a sentence or two that discusses the results of these experiments.

(4) **(Preliminary Final Project Topic)** This is an easy part of the HW7 assignment grade, but I would like you to start thinking of your final project topic, if you haven't already. The final project is assigned on Camino, and was discussed in class. There are many sample project topics listed on the final project assignment sheet.

For credit on this problem, please write any ideas you have for your final project topic. If you are unsure about a project topic, this is your opportunity to ask for advice. If you have any interests in any topics (or something you've seen in another class), it would be helpful to write that down so I can help you find an appropriate, interesting topic.