Chaotic time series prediction with Reservoir computing

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***Abstract*— Time series are ubiquitous in physical problems. Some of them are classified as chaotic, a term referring to the property of such systems in which very small changes in the initial conditions result in a deep change in the dynamics. In this paper a novel approach for time series predictions is presented, its capabilities are then analysed through several physical examples.**

I. Introduction

The aim of this report is to show the capabilities of reservoir computing in tackling chaotic time series. Initially a simple non-chaotic problem, namely the damped harmonic oscillator, is presented to show the potential of the current approach. Afterward, systems with increasing complexity and chaoticity are studied to show the capabilities and the limitations of this novel approach.

II. Datasets

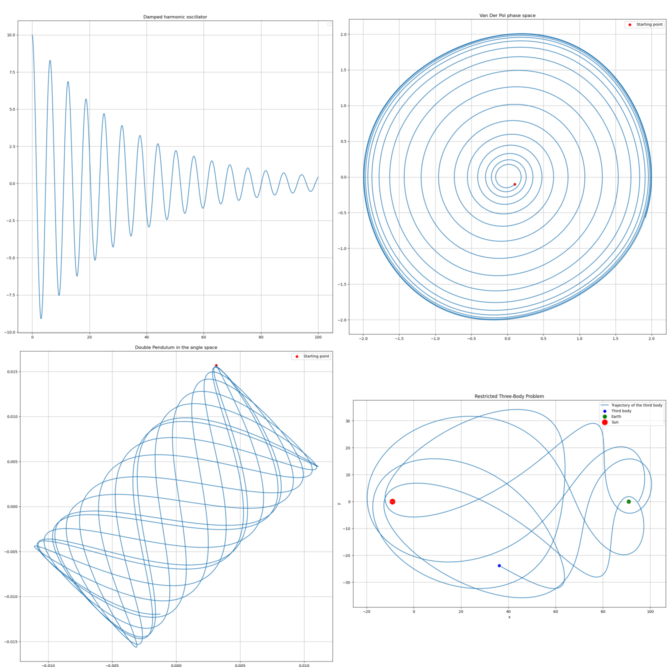

Figure 1: Four synthetic datasets.
(first): Damped harmonic oscillator.
(second): Van Der Pol oscillator in the phase space.
(third): Double pendulum in the angle space.
(fourth): Restricted Planar Three Body Problem.

Besides the Damped Harmonic Oscillator case every other dataset has been generated using the standard *Runge-Kutta 4/5* algorithm, implemented via the library *scikit-learn*.

A. Damped Harmonic oscillator

This system, shown in the first entry of Figure 1 is one of the simplest and oldest in physics, it has been used to model a multitude of physical situations. This problem is used as first test on the Reservoir Computing performance. The data are generated following the reported particular analytical solution for the ODE

$$x(t) = 10e^{-0.03t}\cos(t) \qquad \text{for } t \in [0, 100] \tag{1}$$

and no numerical integration is needed.

B. Van Der Pol Oscillator

The ODE in consideration models a non-linearly damped oscillator

$$x''(t) = \mu(1 - x^2(t))x'(t) - x(t) \tag{2}$$

It can be proved that for fixed $\mu > 0$ and for every initial condition the system converges to a stable orbit in the phase space (x, x') as the second image shown in Figure 1. From this consideration it is evident that, despite the equation is chaotic, the problem present a strong regularity. A value of $\mu = 0.1$ was set.

C. Double Pendulum

One of the most famous chaotic systems in physics, it presents strong chaoticity and no stable solutions exists (besides vertical stationary pendula). The data has been generated in the small angle regime, thus obtaining a much more regular trajectory, described by

$$\begin{aligned} (m_1 + m_2)l_1^2\theta_1'' + m_2 l_1 l_2 \theta_2'' &= -(m_1 + m_2)gl_1\theta_1 \\ m_2 l_1 l_2 \theta_1'' + m_2 l_2^2 \theta_2'' &= -m_2 g l_2 \theta_2 \end{aligned} \tag{3}$$

To generate the data a standard choice of the parameters is $l_1 = l_2 = 1$ and $m_1 = m_2 = 1$ obtaining the third orbit in Figure 1.

D. Restricted Planar Three Body Problem

In the restricted three-body problem, a body of negligible mass (the "planetoid") moves under the influence of two massive bodies. Having negligible mass, the planetoid exerts force on the two massive bodies that may be neglected; placing the reference frame in the rotating center of mass of the two bigger bodies, the system can be described as follows:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -m_1\frac{x - x_1}{r_1^3} - m_2\frac{x - x_2}{r_2^3} \\ \frac{d^2y}{dt^2} &= -m_1\frac{y - y_1}{r_1^3} - m_2\frac{y - y_2}{r_2^3} \end{aligned} \tag{4}$$

where $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$.
The planetoid's mass is set equal to the unit, whilst the "Earth" and

"Sun" masses are set respectively to 100 and 1000, not realistically our Sun and Earth. The numerical integration applied as before results in the last trajectory shown in Figure 1.

III. The model

A. Structure

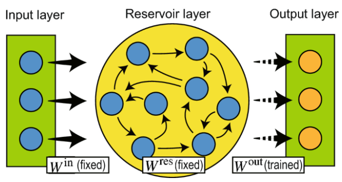

Figure 2: Reservoir Computing generic structure.

The core of the model implemented consists in applying a standard scaler to the hidden states generated recursively by the reservoir and finally squeezing the information applying a PCA on the scaled states. This allows to extract the useful information from the reservoir without discarding (as for LASSO regression) or suppressing (as for Ridge regression) the components of the reservoir deemed as not meaningful, but instead by combining them in the most informative way. After the PCA has been applied a simple Linear regression concludes the training process, the obtained fitted matrix and bias will compose the final step of the reservoir scheme in Figure 2.
The procedure works as follows

$$\begin{aligned} h_t &= (1 - lr) \cdot h_{t-1} + lr \cdot (\tanh(W^{\text{in}} x_t + W^{\text{res}} h_{t-1})) \\ \tilde{h}_t &= PCA(h_t) \ \rightarrow \ y_t = W^{\text{out}} \tilde{h}_t + b \end{aligned} \tag{5}$$

where lr is the *leaking rate*, which describe how much of the previous step information is passed onto the current one. Tuning the leaking rate allows to change the memory of the reservoir, thus it is a fundamental parameter of such model. The number of components and the reservoir size (the dimensionality of h_t) are also fundamental parameters in order to extract the most possible information from the reservoir. The spectral radius of the W^{res} matrix is associated to the memory capacity of the model but also to the *Echo State Property* and it has to be finetuned too.

B. Echo State Property & Fading memory

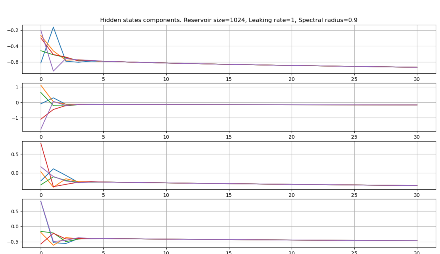

Figure 3: Successful Echo State Property.

To ensure the correctness of the model implemented, the Echo State Property and the Fading Memory of the reservoir were verified in Figure 3 through the visualization of some components of the reservoir's hidden states throughout the timesteps.
The image reported below shows how the hyperparameters choice affects the ESP.

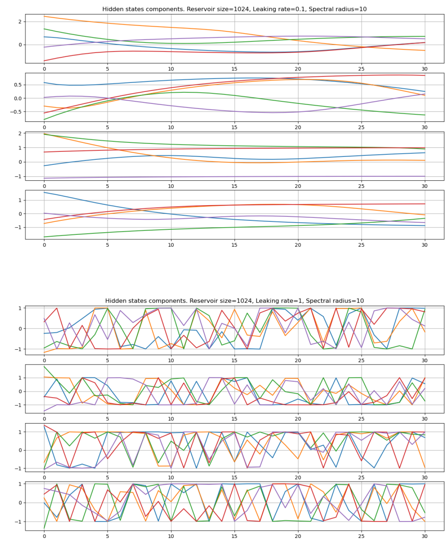

Figure 4: Failed Echo State Property.

This property is fundamental since it ensures that the unnatural choice of the initial hidden state (usually $h_0 = \vec{0}$) is forgotten after a thermalization time.
Since it is not possible to plot every component of the reservoir a warmup size of 100 was set to be confident that each of them reached the thermalization.

IV. Results

Here are presented the results obtained for the various problems.

A. Damped Harmonic Oscillator

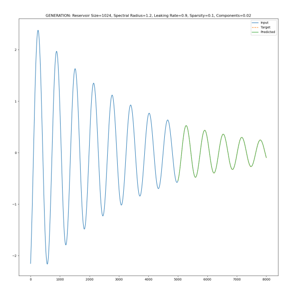

Figure 5: Generated data for the Damped Harmonic Oscillator.

The pattern of this sequence of data is explicit and pretty simple, indeed the model has no difficulties in correctly predict unseen points following the given input orbit even with a relatively small reservoir.

B. Van Der Pol Oscillator

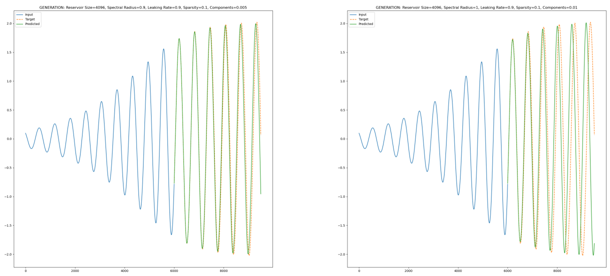

Figure 6: Generated data for the Van Der Pol Oscillator.

As the Figure 6 shows, the model correctly understand the dynamics underlying the input trajectory and it is capable of following it in the generation process.
On the *left* side a very precise predicted orbit is shown.
On the *right* side, instead, one may think that the model failed in understanding the dynamics but this is not the case. The chaotic behaviour of the system causes a divergence of the predicted orbit from the target one, due to very small deviation from the orbit in the first steps of the generation, even though the model learned noticeably well the undelying physics. Indeed this second prediction cannot be considered wrong.

C. Double Pendulum

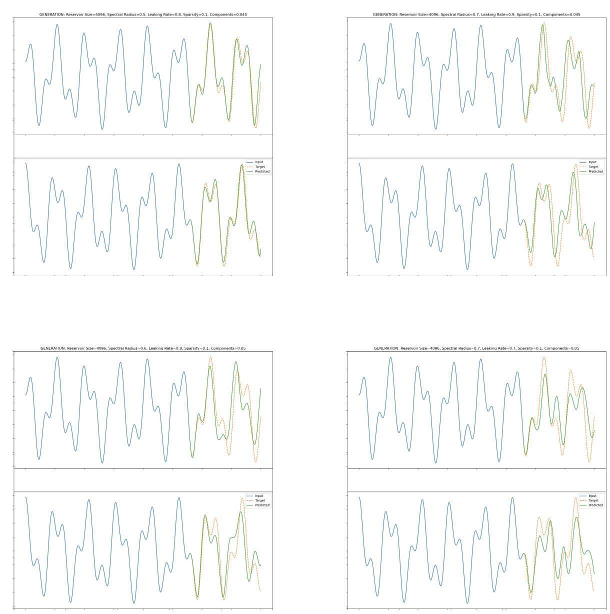

Figure 7: Generated data for the Double Pendulum, component-wise view.

Exactly as for the previous system the model has been fitted on the data, shown in Figure 1 at the third place. After hyperoptimizing the parameters of the model, via a grid search, the results are reported in Figure 7. Four results have been shown, to enhance again the importance of the chaotic behaviour of the problem in the prediction task. No one of these four situations can be deemed as completely wrong even though the true trajectories are not closely followed.

D. Restricted Planar Three Body Problem

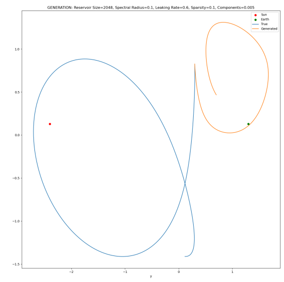

Figure 8: Data generation in orange of the RPTBP.

Despite the simplifications made to the original Three Body Problem this chaotic system is still too complex to be tackled with the implemented model which did not learn the physics governing the planetoid's motion. The reservoir cannot interpret the irregular orbit of the planetoid and grasp the gravitational field generated by the two masses.
A future improvement in this direction could be to aggregate the information of different trajectories on the same physical situation. Another possible step forward could be achieved by embedding some physical knowledge in the model, to improve its capabilities in understanding the physics behind the scenes, similarly to the *Physics Informed Neural Networks*.

V. Conclusions

In this report the Reservoir Computing capabilities has been exploited to predict chaotic time series. Due to the nature of these time series, the generation process is highly unstable and rapidly deviates from the desired trajectory, this is the reason why the standard validation techniques based on the error on a validation set cannot be applied.
Almost every problem presented in this paper has been successfully simulated and the model was able to understand the underlying dynamics. Regarding the RPTBP the model did not learn the physics governing the motion of the planetoid, resulting in a completely wrong trajectory. This is due to a strong irregular trajectory, without any form of periodicity or pattern.
In the end we can say that Reservoir Computing is a valid alternative to recurrent neural network such as LSTM and GRU for time trajectories that are chaotic but still with some kind of regularity (such as for Van Der Pol, Double Pendulum in the small angle regime but also, for instance, Lorenz Attractor).
An important feature of the reservoir computing is that the training process can be accomplished in just few minutes. This is fundamental from an energy consumption point of view but also for an easier and more accurated fine-tuning of the parameters.