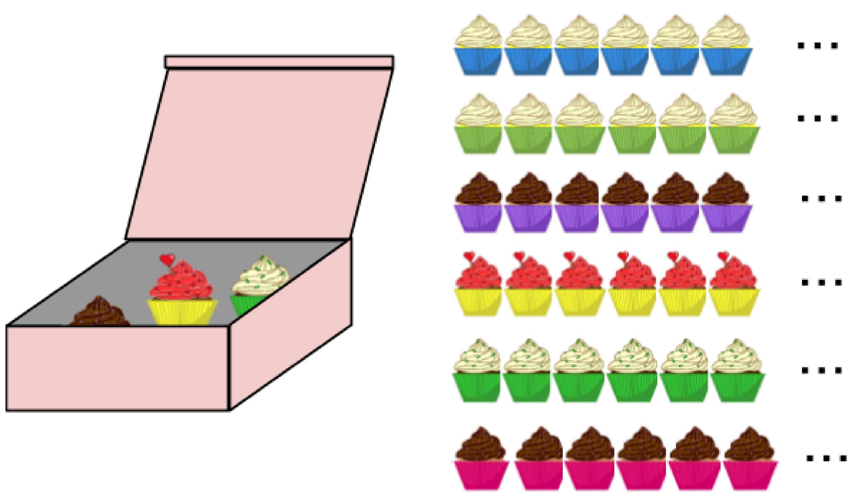
5.1 Sum and product rules

Introduction to counting

Counting, as simple as it may seem initially, is a central topic in discrete mathematics. Most children begin their education in mathematics by learning to count — 1, then 2, and so forth. In discrete mathematics the goal is to count the number of elements in (or the *cardinality* of) a finite set given a description of the set. Determining a set's cardinality often requires exploiting some mathematical structure of the set.

Figure 5.1.1: Example: Counting the number of cupcake selections.

A bakery sells 6 different varieties of cupcakes (chocolate, vanilla, red velvet, etc.). How many ways are there to fill a box with 24 cupcakes from the 6 varieties? The order in which the cupcakes are selected is unimportant; all that matters is the number of each variety in the box after they are chosen. The answer to the cupcake counting problem is 118755 — too many to count by hand.



The answer to the cupcake counting question is 118755. This material will cover a systematic technique to answer questions like the cupcake counting question. While counting cupcake selections may not seem like a compelling application, the same techniques can be used to count the number of ways 24 identical tasks can be assigned to a network of 6 processors which in turn can be used to calculate the probability that a random assignment distributes the workload evenly among the 6 processors.

Counting is an important mathematical tool to analyze many problems that arise in computer science. Counting is useful, for example, to understand the amount of a particular resource used in a computer system or computation. Counting is also used to determine the number of valid passwords for a security system or addresses in a network to ensure that there are enough unique choices to meet the demand. Counting is also at the heart of discrete probability, which is central to many areas of science.

The product rule

The two most basic rules of counting are the sum rule and the product rule. These two rules applied in different combinations can be used to handle a wide range of counting problems. The ***product rule*** provides a way to count sequences. While sequences may not seem like a particularly common type of object to count, many sets can be expressed as sets of sequences.

Theorem 5.1.1: The product rule.

Let $A_1, A_2, \ldots, A_n$ be finite sets.
Then,

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Consider a restaurant that has a breakfast special that includes a drink, a main course, and a side. The set of choices for each category are:

$$D = \{\text{coffee, orange juice}\}$$
$$M = \{\text{pancakes, eggs}\}$$
$$S = \{\text{bacon, sausage, hash browns}\}$$

Any particular breakfast selection can be described by a triplet indicating the choice of drink, main course, and side. For example, (coffee, pancakes, bacon) is one particular breakfast combination. The set of all possible choices is the same as the set of a triples where the first entry is a drink, the second entry is a main course, and the third entry is a side. The number of possible breakfast combinations is therefore:

$$|D \times M \times S| = |D| \cdot |M| \cdot |S| = 2 \cdot 2 \cdot 3 = 12$$

PARTICIPATION ACTIVITY | 5.1.1: An example of the product rule: counting breakfast selections.

Animation captions:

1. A breakfast special includes a choice of drink, main course, and side.
2. There are two choices for the drink: coffee or orange juice. There are two options so far for the breakfast special.
3. For each choice of drink, there are two choices for the main course: pancakes or eggs. There are $2 \cdot 2 = 4$ choices so far.
4. For each choice of drink and main course, there are three choices for the side: bacon, sausage, or hash browns. There are $4 \cdot 3 = 12$ choices total.

PARTICIPATION ACTIVITY | 5.1.2: Applying the product rule: Counting burrito selections.

A burrito stand sells burritos with different choices of stuffing. The set of choices for each category are:

- Filling choices = {chicken, beef, pork}
- Bean choices = {black, pinto}
- Salsa choices = {mild, medium, hot}

1) If every burrito has a filling, beans, and salsa, then how many possible burrito combinations are there?

Check Show answer

2) Suppose that the customer can also now select grilled veggies as a filling. Now how many selections are there?

Check Show answer

3) Now suppose that the burrito stand introduces a choice between plain flour or whole wheat tortillas. The additional option to select veggies as a filling is still available as well. Now how many selections are there?

Check Show answer

Counting strings

If Σ is a set of characters (called an ***alphabet***) then Σ^n is the set of all strings of length n whose characters come from the set Σ. For example, if $\Sigma = \{0, 1\}$, then Σ^6 is the set of all binary strings with 6 bits. The string 011101 is an example of an element in Σ^6. The strings xxyzx and zyyzy are examples of strings in the set $\{x, y, z\}^5$. The product rule can be applied directly to determine the number of strings of a given length over a finite alphabet:

$$|\Sigma^n| = |\underbrace{\Sigma \times \Sigma \times \cdots \times \Sigma}_{\text{n times}}| = \underbrace{|\Sigma| \cdot |\Sigma| \cdots |\Sigma|}_{\text{n times}} = |\Sigma|^n$$

For example, the number of binary strings of length n is 2^n since the size of the alphabet is 2 (e.g., $|\{0, 1\}| = 2$).

The product rule can also be used to determine the number of strings in a set when one or more of the characters are restricted. Define S to be the set of binary strings of length 5 that start and end with 0. A string is in the set S if it has the form 0***0, where each * could be a 0 or a 1.

$$|S| = |\{0\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0\}| = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 8$$

PARTICIPATION ACTIVITY

5.1.3: Using the product rule to count sets of strings.

1) How many six bit binary strings are there?

Check Show answer

2) How many six bit binary strings are there that begin with "01"?

Check Show answer

3) How many strings of length 4 are there over the alphabet {a, b, c}?

Check Show answer

4) How many strings of length 4 are there over the alphabet {a, b, c} that end with the character c?

Check Show answer

The sum rule

In the breakfast example, the product rule is applied because the customer selects a drink and a main course and a side. In contrast, the ***sum rule*** is applied when there are multiple choices but only one selection is made. For example, suppose a customer just orders a drink. The customer selects a hot drink or a cold drink. The hot drink selections are {coffee, hot cocoa, tea}. The cold drink selections are {milk, orange juice}. The total number of choices is 5, namely 3 hot drink choices plus 2 cold drink choices. Here is a formal statement of the sum rule, expressed in terms of sets:

Theorem 5.1.2: The sum rule.

Consider n sets, $A_1, A_2, \ldots, A_n$. If the sets are pairwise disjoint (which means that $A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

In the example with the customer selecting a drink, n = 2 since there are two categories of drinks: hot drinks and cold drinks. Let C be the set of cold drinks and H the set of hot drinks. The fact that H and C are disjoint ($C \cap H = \emptyset$) means that none of the drinks is categorized as both a hot drink and a cold drink. Applying the sum rule yields that the number of possible drinks is:

$$|C \cup H| = |C| + |H| = 3 + 2 = 5$$

Example 5.1.1: Product and sum rule in combination: counting passwords.

Consider a system in which a password must be a string of length between 6 and 8. The characters can be any lower case letter or digit.

Let L be the set of all lower case letters and D be the set of digits. $|L| = 26$ and $|D| = 10$. The set of all allowed characters is $C = L \cup D$. Since $D \cap L = \emptyset$, the sum rule can be applied to find the cardinality of C: $|C| = 26 + 10 = 36$.

Let A_j denote the strings of length j over the alphabet C. By the product rule, $|A_j| = 36^j$. Notice that for $j \neq k$, A_j and A_k are disjoint because a string can not have length j and length k at the same time. If the user must select a password of length 6 or 7 or 8, then the sum rule applies:

$$|A_6 \cup A_7 \cup A_8| = |A_6| + |A_7| + |A_8| = 36^6 + 36^7 + 36^8$$

In the next example, a customer purchasing a laptop can select three different sizes of screens and has a choice between two different processor speeds. For storage, the customer can select a hard disk drive or a solid state drive. The hard disk drive option comes in two sizes and the solid state option has three different sizes. The manufacturer would like to know how many different configurations are possible. The number of choices is worked out in the animation below:

PARTICIPATION ACTIVITY | 5.1.4: An example of the sum and product rule: counting laptop selections.

Animation captions:

1. Customizing a laptop includes the choice of screen size, processor speed, and storage. There are 3 screen sizes and 2 possible processor speeds.
2. The number of different laptops is $3 \cdot 2 \cdot \text{number of choices for storage}$. Storage can be SSD (3 different sizes) or HDD (2 different sizes).
3. The number of choices for SSD and HDD are combined by the sum rule. The total number of choices for the laptop is $3 \cdot 2 \cdot (3 + 2)$.

PARTICIPATION ACTIVITY | 5.1.5: Applying the sum and product rule in combination: Counting sets of bit strings.

A bit string consists of 0s and 1s. For example, 0101 is a bit string with four bits.

1) How many six bit strings are there that begin and end with a 1, or start with 00?

Check Show answer

2) How many bit strings of length five or six start with a 1?

Check Show answer

CHALLENGE ACTIVITY | 5.1.1: Counting password possibilities.

455912.3056722.qx3zqy7

Start

Each character in a password is either a digit [0-9] or lowercase letter [a-z]. How many valid passwords are there with the given restriction(s)?

Length is 20.

Ex: 26 * 36^21 Write a^b as: a^b

1 2 3 4 5

Check Next

Additional exercises

EXERCISE | 5.1.1: License plate combinations.

In a particular state, the license plates have 7 characters. Each character can be a capital letter or a digit except for 0. (The set of possible digits is {1; 2; 3; 4; 5; 6; 7; 8; 9}.) A person witnesses a crime and remembers some information about the license plate of the getaway car. The authorities would like to figure out how many license plates need to be checked in each case. For each constraint given below, indicate the number of license plates that satisfy that constraint.

Note: you do not need to calculate the number. You may keep the multiplications and powers in your answers

(a) No constraints

(b) The license plate starts with a digit

(c) First three are letters

(d) First three are letters and last four are numbers

EXERCISE 5.1.2: Counting passwords made up of letters, digits, and special characters.

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

(a) Strings of length 6. Characters can be special characters, digits, or letters.

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

EXERCISE 5.1.3: Selecting lunch specials for the week.

(a) A Chinese restaurant offers 10 different lunch specials. Each weekday for one week, Fiona goes to the restaurant and selects a lunch special. How many different ways are there for her to select her lunches for the week? Note that which lunch she orders on which day matters, so the following two selections are considered different.
One possible selection:
   - Mon: Kung pao chicken
   - Tues: Beef with broccoli
   - Wed: Kung pao chicken
   - Thurs: Moo shu pork
   - Fri: Beef with broccoli

A different selection:
   - Mon: Beef with broccoli
   - Tues: Kung pao chicken
   - Wed: Kung pao chicken
   - Thurs: Moo shu pork
   - Fri: Beef with broccoli

(b) Now suppose that out of the 10 dishes that the restaurant offers, only 3 of them are vegetarian. If Fiona must select a vegetarian option on Friday, how many ways are there for her to select her lunches?

(c) Again, suppose that out of the 10 dishes that the restaurant offers, only 3 of them are vegetarian. If Fiona must go with a vegetarian option on both Monday and Friday, how many ways are there for her to select her lunches?

(d) Now suppose that Fiona can select a vegetarian or a non-vegetarian lunch on any day of the week. However, in addition to selecting her main course, she must also selects between water or tea for her drink. How many ways are there for her to select her lunches?

EXERCISE 5.1.4: Dividing up a print job.

A 100-page document is being printed by four printers. Each page will be printed exactly once.

(a) Suppose that there are no restrictions on how many pages a printer can print. How many ways are there for the 100 pages to be assigned to the four printers?
One possible combination: printer A prints out pages 2-50, printer B prints out pages 1 and 51-60; printer C prints out 61-80 and 86-90; printer D prints out pages 81-85 and 91-100.

(b) Suppose the first and the last page of the document must be printed in color, and only two printers are able to print in color. The two color printers can also print black-and-white. How many ways are there for the 100 pages to be assigned to the four printers?

(c) Suppose that all the pages are black-and-white, but each group of 25 consecutive pages (1-25, 26-50, 51-75, 76-100) must be assigned to the same printer. Each printer can be assigned 0, 25, 50, 75, or 100 pages to print.
How many ways are there for the 100 pages to be assigned to the four printers?

5.2 The bijection rule

Some sets are easier to count than others. One way to approach a difficult counting problem is to show that the cardinality of the set to be counted is equal to the cardinality of a set that is easy to count. The ***bijection rule*** says that if there is a bijection from one set to another then the two sets have the same cardinality.

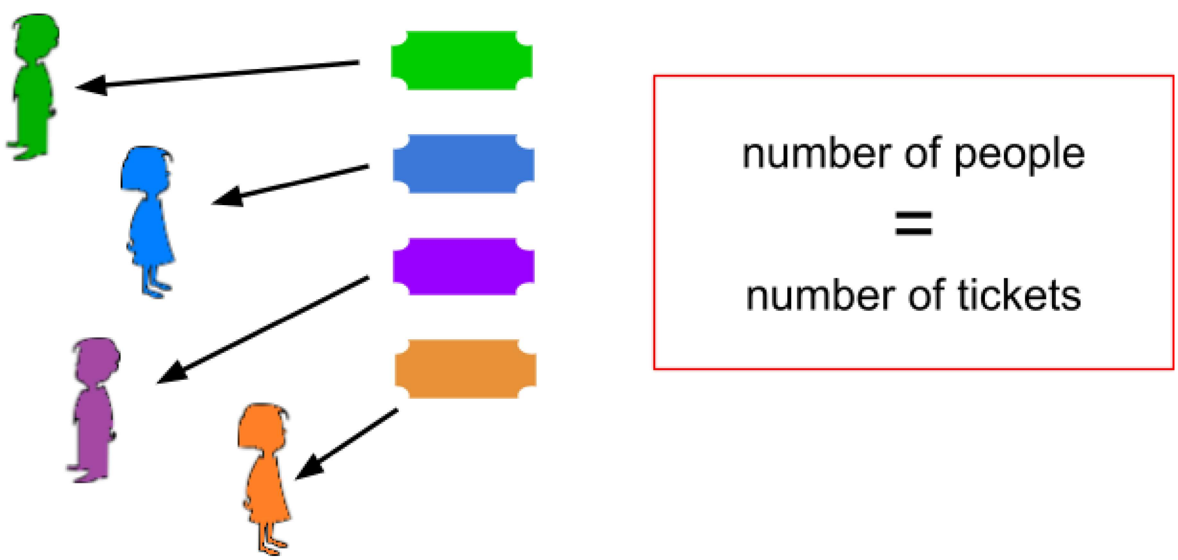
A function f from a set S to a set T is called a ***bijection*** if and only if f has a well defined inverse. The ***inverse*** of a function f that maps set S to set T is a function g that maps T to S such that for every s ∈ S and every t ∈ T, f(s) = t, if and only if g(t) = s. If a function f has an inverse, it is denoted by f^{-1}.

Definition 5.2.1: The bijection rule.

Let S and T be two finite sets. If there is a bijection from S to T, then |S| = |T|.

Suppose that every person in a theater must submit a ticket to an usher in order to enter. One way to count the number of people in the theater is to count the number of tickets submitted. In this case, the bijection is from the set of submitted tickets to the set of people in the theater. Each ticket is mapped to the person who submits the ticket to the usher. The inverse function maps each person to his or her ticket.

Figure 5.2.1: Bijection mapping tickets to people.



The next example uses the bijection rule to determine the cardinality of the power set of a finite set X. Recall that the power set of X (denoted P(X)) is the set of all subsets of X. Let |X| = n. The animation below illustrates a bijection f from P(X) to the set of binary strings of length n:

PARTICIPATION ACTIVITY | 5.2.1: An example of the bijection rule.

Animation captions:

1. A function f is a bijection from the power set of {a, b, c} to the set of 3-bit strings. The set Ø does not include a,b, or c. Therefore, f(Ø) = 000.
2. The set {a} includes a, so the first bit of f({a}) is 1. The set {a} does not include b or c, so the second and third bits of f({a}) are 0. f({a}) = 100.
3. The value of f can be determined the same way for all the subsets of {a,b,c}, ending with f({a,b,c}) = 111.
4. f defines a bijection between the set $\{0,1\}^3$ and the power set of {a, b, c}. Therefore $|P(X)| = |\{0, 1\}^3|$.

The bijection illustrated in the animation can be defined formally. Order the elements of X in some way: $x_1, x_2, ...x_n$. The order is arbitrary but it is important to pick one ordering and stick with it. For each $Y \subseteq X$, f(Y) is an n-bit string y whose bits are $y_1y_2...y_n$. The string y is defined by the rule: $y_i = 1$ if $x_i \in Y$, otherwise $y_i = 0$. The inverse of f maps binary strings of length n back to subsets of X. $f^{-1}(y)$ is a subset Y of X such that $x_i \in Y$ if $y_i = 1$, otherwise $x_i \notin Y$. For every $Y \subseteq X$ and every $y \in \{0, 1\}^n$,

$$f(Y) = y \;\leftrightarrow\; f^{-1}(y) = Y.$$

PARTICIPATION ACTIVITY | 5.2.2: The bijection mapping power sets to strings.

Let X = {1, 2, 3, 4}. Define the function f from P(X) to $\{0, 1\}^4$ as defined above.

1) What is f({1,4})?

Check Show answer

2) Which element is not in $f^{-1}(1101)$?

Check Show answer

3) How many elements are in the set $f^{-1}(0000)$?

Check Show answer

The k-to-1 rule

A group of kids at a slumber party all leave their shoes in a big pile at the door. One way to count the number of kids at the party is to count the number of shoes and divide by 2. Of course, it is important to establish that each kid has exactly one pair of shoes in the pile. Counting kids by counting shoes and dividing by 2 is an example of the k-to-1 rule with k = 2. Applying the k-to-1 rule requires a well defined function from objects we can count to objects we would like to count. In the example with the shoes, the function maps each shoe to the kid who owns it. Here is a definition of the kind of function that is required:

Definition 5.2.2: k-to-1 correspondence.

Let X and Y be finite sets. The function f:X→Y is a ***k-to-1 correspondence*** if for every $y \in Y$, there are exactly k different $x \in X$ such that f(x) = y.

A 1-to-1 correspondence is another term for a bijection, so a bijection is a k-to-1 correspondence with k = 1. The ***k-to-1 rule*** uses a k-to-1 correspondence to count the number of elements in the range by counting the number of elements in the domain and dividing by k.

Definition 5.2.3: k-to-1 rule.

Suppose there is a k-to-1 correspondence from a finite set A to a finite set B. Then |B| = |A|/k.

PARTICIPATION ACTIVITY | 5.2.3: An example of the k-to-1 rule.

Animation captions:

1. 6 cans of juice per pack. f(c) = p if can c belongs to pack p. f is a 6-to-1 function.
2. (# cans of juice)/6 = (# packs of juice).

PARTICIPATION ACTIVITY | 5.2.4: K-to-1 rule.

1) A farm orders x horse shoes for its horses. The farm does not order extras and all the horses will get new horse shoes. Apply the k-to-1 rule to determine the number of horses on the farm. Give your answer as a mathematical expression in terms of x.

Check Show answer

Additional exercises

EXERCISE | 5.2.1: Devising a 3-to-1 correspondence.

(a) Find a function from the set {1, 2, ..., 30} to {1, 2, ..., 10} that is a 3-to-1 correspondence. (You may find that the division, ceiling or floor operations are useful.)

EXERCISE | 5.2.2: Using the bijection rule to count palindromes.

If x is a string, then x^R is the reverse of the string. For example, if x = 1011, then x^R = 1101. A string is a palindrome if the string is the same backwards and forwards (i.e., if $x = x^R$). Let B = {0, 1}. The set B^n is the set of all n-bit strings. Let P_n be the set of all strings in B^n that are palindromes.

(a) Show a bijection between P_6 and B^3.

(b) What is $|P_6|$?

(c) Determine the cardinality of P_7 by showing a bijection between P_7 and B^n for some n.

EXERCISE 5.2.3: Using the bijection rule to count binary strings with even parity.

Let B = {0, 1}. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

(b) What is $|E_{10}|$?

EXERCISE 5.2.4: Using the bijection rule to count ternary strings whose digits sum to a multiple of 3.

Let T = {0, 1, 2}. A string $x \in T^n$ is said to be balanced if the sum of the digits is an integer multiple of 3.

(a) Show a bijection between the set of strings in T^6 that are balanced and T^5. Explain why your function is a bijection.

(b) How many strings in T^6 are balanced?

EXERCISE 5.2.5: Using the k-to-1 rule for counting ways to line up a group.

Ten kids line up for recess. The names of the kids are:

{Abe, Ben, Cam, Don, Eli, Fran, Gene, Hal, Ike, Jan}.

Let S be the set of all possible ways to line up the kids. For example, one ordering might be:

(Fran, Gene, Hal, Jan, Abe, Don, Cam, Eli, Ike, Ben)

The names are listed in order from left to right, so Fran is at the front of the line and Ben is at the end of the line.

Let T be the set of all possible ways to line up the kids in which Gene is ahead of Don in the line. Note that Gene does not have to be immediately ahead of Don. For example, the ordering shown above is an element in T.

Now define a function f whose domain is S and whose target is T. Let x be an element of S, so x is one possible way to order the kids. If Gene is ahead of Don in the ordering x, then f(x) = x. If Don is ahead of Gene in x, then f(x) is the ordering that is the same as x, except that Don and Gene have swapped places.

(a) What is the output of f on the following input?
(Fran, Gene, Hal, Jan, Abe, Don, Cam, Eli, Ike, Ben)

(b) What is the output of f on the following input?
(Eli, Ike, Don, Hal, Jan, Abe, Ben, Fran, Gene, Cam)

(c) Is the function f a k-to-1 correspondence for some positive integer k? If so, for what value of k? Justify your answer.

(d) There are 3628800 ways to line up the 10 kids with no restrictions on who comes before whom. That is, |S| = 3628800. Use this fact and the answer to the previous question to determine |T|.

(e) Let Q be the set of orderings in which Gene comes before Don and Jan comes before Abe (again, not necessarily immediately before). Define a k-to-1 correspondence from S to Q. Use the value of k to determine |Q|.

5.3 The generalized product rule

Consider a race with 20 runners. There is a first place, a second place and a third place trophy. An outcome of the race is defined to be who wins each of the three trophies. How many outcomes are possible?

All 20 of the runners are eligible to win the first place trophy. Once the first place runner is determined, there are 19 possibilities left for the second place trophy (since no one can place both first and second). Once the top two runners are determined, there are 18 possibilities for the third place trophy. The number of possibilities for the outcome of the race is $20 \cdot 19 \cdot 18 = 6840$.

The race example illustrates that a useful way to think about counting is to imagine selecting an element from the set to be counted. The selection process is carried out in a sequence of steps. In each step, one more decision is made about the item that will be selected. At the end of the process the item to be selected is fully specified. The ***generalized product rule*** says that in selecting an item from a set, if the number of choices at each step does not depend on previous choices made, then the number of items in the set is the product of the number of choices in each step.

Definition 5.3.1: Generalized product rule.

Consider a set S of sequences of k items. Suppose there are:

- n_1 choices for the first item.
- For every possible choice for the first item, there are n_2 choices for the second item.
- For every possible choice for the first and second items, there are n_3 choices for the third item.

⋮

- For every possible choice for the first k-1 items, there are n_k choices for the k^{th} item.

Then $|S| = n_1 \cdot n_2 \cdots n_k$.

The next animation illustrates the following example: A family of four (2 parents and 2 kids) goes on a hiking trip. The trail is narrow and they must walk single file. How many ways can they walk with a parent in the front and a parent in the rear?

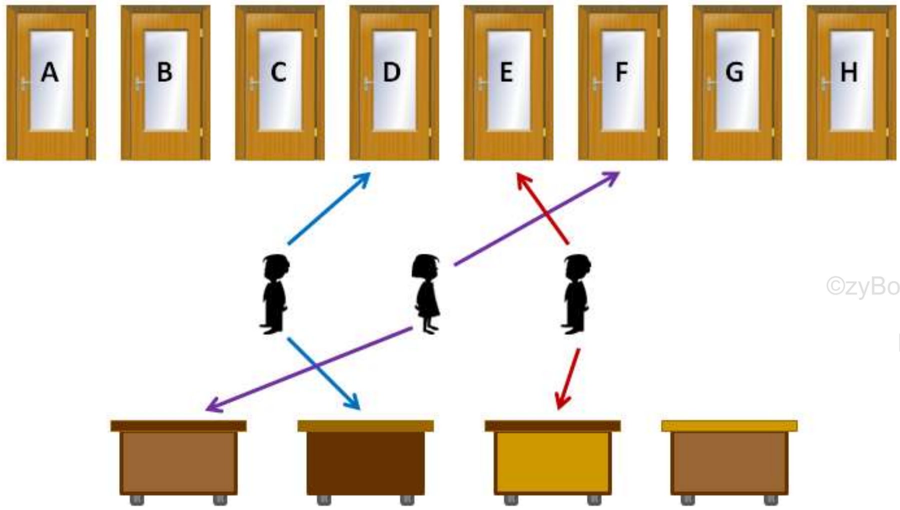
PARTICIPATION ACTIVITY 5.3.1: An example of the generalized product rule.

Animation captions:

1. The desired sequence is (Parent, Child, Child, Parent). The first person in the sequence can be Mom or Dad. (2 choices.)
2. For each choice for the first person, there are two choices for the second person, Sister or Brother. There are $2 \cdot 2$ choices so far.
3. If the second person is Sister, then the third person must be Brother. If the second person is Brother, the third person must be Sister. Only 1 choice exists for the third person.
4. The last person must be the parent who was not chosen to be the first person. Only 1 choice exists for the fourth person. The total number of choices is $2 \cdot 2 \cdot 1 \cdot 1 = 4$.

Example 5.3.1: Counting office and desk selections by the generalized product rule.

Suppose that there are three employees in a start-up. They rent an office space with 8 offices, anticipating growth. The office space comes with four desks. Each person can select an office and a desk. The selection is done in the order that the participants joined the company with the founder going first. How many ways are there for the selection to be done?



First the founder makes his selections. He has a choice of 8 different offices and four different desks. Using the product rule, this gives 8·4 different choices. Next the second employee selects. She has a choice of 7 different offices and 3 different desks (because she can not select either the office or the desk chosen by the founder). She has 7·3 choices for her office/desk combination. Finally the number three employee picks. He has a choice of 6 offices and two desks, for a total of 6·2 choices. Overall the number of possible selections is:

$$(8 \cdot 4) \cdot (7 \cdot 3) \cdot (6 \cdot 2) = 8064$$

PARTICIPATION ACTIVITY 5.3.2: Counting selections of officers by the generalized product rule.

In the following question, a club with 10 students elects a president, vice president, secretary and treasurer. No student can hold more than one position. Express your answer to each question as a number.

1) How many ways are there to select the class officers?

Check Show answer

2) Now suppose that there are five ninth graders and five tenth graders in the club. How many ways are there to elect the officers if the president is a tenth grader?

Check Show answer

3) Again suppose that there are five ninth graders and five tenth graders

in the club. How many ways are there to elect the officers if the president is a tenth grader and the VP is a ninth grader?

Check Show answer

Additional exercises

EXERCISE 5.3.1: Counting passwords without repeating characters.

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

(a) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters.

(b) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters. The first character can not be a special character.

EXERCISE 5.3.2: Strings with no repetitions.

(a) How many strings are there over the set {a, b, c} that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

EXERCISE 5.3.3: Counting license plate numbers.

License plate numbers in a certain state consists of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

(a) How many different license plate numbers are possible?

(b) How many license plate numbers are possible if no digit appears more than once?

(c) How many license plate numbers are possible if no digit or letter appears more than once?

EXERCISE 5.3.4: Selecting coders for 3 different projects.

(a) A manager must select three coders from her group to write three different software projects. There are 7 junior and 3 senior coders in her group. The first project can be written by any of the coders. The second project must be written by a senior person and the third project must be written by a junior person. How many ways are there for her to assign the three coders to the projects if no person can be assigned to more than one project?

5.4 Counting permutations

One of the most common applications of the generalized product rule is in counting permutations. An ***r-permutation*** is a sequence of r items with no repetitions, all taken from the same set. Consider the set X = {John, Paul, George, Ringo}. The sequences (Paul, Ringo, John) and (John, George, Paul) are both examples of 3-permutations over X. In a sequence, order matters, so the sequence (Paul, Ringo, John) is different from the sequence (Ringo, Paul, John).

PARTICIPATION ACTIVITY 5.4.1: Using the generalized product rule to count the number of 5-permutations from a set of size 8.

Animation captions:

1. Select a 5-permutation from a set with 8 elements {A, B, C, D, E, F, G, H}. 8 choices exist for the first item. If F is selected, the remaining choices are {A, B, C, D, E, G, H}.
2. There are 7 choices for the second item, 6 for the third, 5 for the fourth, and 4 for the fifth. There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$, 5 permutations from a set of 8.

Fact 5.4.1: The number of r-permutations from a set with n elements.

Let r and n be positive integers with r ≤ n. The number of r-permutations from a set with n elements is denoted by P(n, r):

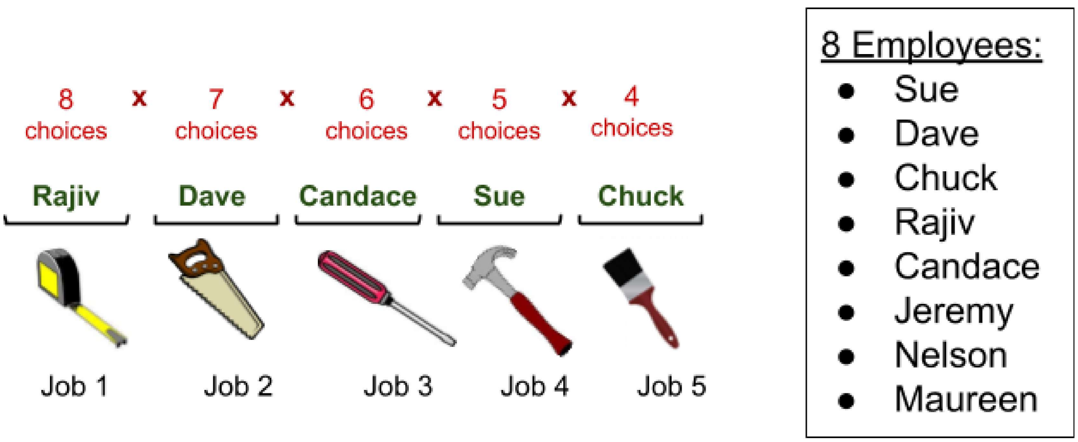
$$\begin{aligned} P(n,r) &= \frac{n!}{(n-r)!} \\ &= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 1}{(n-r)(n-r-1)\cdots 1} \\ &= n(n-1)\cdots(n-r+1) \end{aligned}$$

The closed form for P(n, r) is a consequence of the generalized product rule. There are n choices for the first item in the sequence because the set from which the items are drawn has n elements. Once the choice of the first item in the sequence is made, there are n - 1 choices for the next item because the first item in the sequence can not be repeated. In general, once the first i items in the sequence have been chosen, there are n - i remaining elements from which the next one can be chosen. The selection process continues until r items have been chosen for the sequence. Just before the last (r^{th}) item is chosen, r - 1 items have already been chosen and there are n - (r - 1) = n - r + 1 items from which to select the last item.

Example 5.4.1: Counting job assignments by counting r-permutations.

A manager has five different jobs that need to get done on a given day. She has eight employees whom she can assign to the jobs. A job only requires one person and no person can be assigned more than one job. How many possible ways can she do the assignment?

Order the jobs arbitrarily so that one job is first, one is second, etc. An assignment is a 5-permutation from the set of 8 employees. The first person gets the first job, the second person gets the second job,..., the fifth person gets the fifth job. The 5-permutation will look like:



Source: Tape measure (G.E.Sattler Creative Commons); other pictures are Public Domain.

The number of 5-permutations from a set of 8 people is P(8, 5) = 8 · 7 · 6 · 5 · 4 = 6720.

The number of assignments can also be derived using the generalized product rule directly. There are 8 choices for job 1. Once the person for job 1 has been selected, there are 7 remaining choices for job 2, then 6 choices for job 3, 5 choices for job 4, and 4 choices for job 5. Applying the product rule, the total number of assignments is 8 · 7 · 6 · 5 · 4 = 6720.

PARTICIPATION ACTIVITY 5.4.2: Counting r-permutations.

1) A red, blue, and green die are thrown. Each die has six possible outcomes. How many outcomes are possible in which the three dice all show different numbers?

Check Show answer

2) There are 5 computers and 3 students. How many ways are there for the students to sit at the computers if no computer has more than one student and each student is seated at a computer?

Check Show answer

3) A class has ten students. A teacher will give out three prizes: One student gets a gift card, one gets a book, and one gets a movie ticket. No student can receive more than one prize. How many ways can the teacher distribute the prizes?

Check Show answer

A ***permutation*** (without the parameter r) is a sequence that contains each element of a finite set exactly once. For example, the set {a, b, c} has six permutations:

Table 5.4.1: Permutations of the set {a, b, c}.

(a, b, c)	(b, a, c)	(c, a, b)
(a, c, b)	(b, c, a)	(c, b, a)

Fact 5.4.2: The number of permutations of a finite set.

The number of permutations of a finite set with n elements is

$$P(n, n) = n \times (n-1) \times \ldots \times 2 \times 1 = n!$$

PARTICIPATION ACTIVITY | 5.4.3: Counting line-ups: Permutations.

1) A wedding party consisting of a bride, a groom, two bridesmaids, and two groomsmen line up for a photo. How many ways are there for the wedding party to line up?

Check Show answer

The next example combines the product rule with counting permutations. Consider again the set {John, Paul, George, Ringo}. These four would like to sit on a bench together, but Paul and John would like to sit next to each other. How many possible seatings are there? In order to apply the generalized product rule, view the set of possibilities as a process in which a seating is specified. The first step is to determine whether Paul sits to the left or right of John. There are two possible choices: Paul is to the left of John or Paul is to the right of John. Then glue Paul and John together in the chosen order to satisfy the constraint that they sit together. Now there are three items to order: two of them are people (George and Ringo), the other is a pair that is bound together [John+Paul]. The next step is to select a permutation of the three items. The animation illustrates the final count:

PARTICIPATION ACTIVITY | 5.4.4: Using the product rule with counting permutations.

Animation captions:

1. First decide if John is to the left of Paul: John, Paul or Paul, John. There are 2 choices so far.
2. There are 3! permutations of George, Ringo, and (John+Paul).
3. Put the choices together. For each permutation of George, Ringo, and (John+Paul), there are 2 ways to order John and Paul. $(3!) \cdot 2 = 12$ choices.

PARTICIPATION ACTIVITY | 5.4.5: Counting line-ups: Permutations and the product rule.

1) A wedding party consisting of a bride, a groom, two bridesmaids, and two groomsmen line up for a photo. How many ways are there for the wedding party to line up so that the bride is next to the groom?

Check Show answer

CHALLENGE ACTIVITY | 5.4.1: Counting permutations of passwords.

455912.3056722.qx3zqy7

Start

Each character in a password is either a digit [0-9] or lowercase letter [a-z]. How many valid passwords are there with the given restriction(s)?

Length is 14.
No character repeats.

Ex: P(11, 4) * P(4, 3) Write permutations as: P(n, k)

1 2 3 4 5 6

Check Next

Additional exercises

EXERCISE | 5.4.1: Counting functions from a set to itself.

Count the number of different functions with the given domain, target and additional properties.

(a) f: $\{0,1\}^7 \to \{0,1\}^7$.

(b) f: $\{0,1\}^7 \to \{0,1\}^7$. The function f is one-to-one.

(c) f: $\{0,1\}^5 \to \{0,1\}^7$.

(d) f: $\{0,1\}^5 \to \{0,1\}^7$. The function f is one-to-one.

EXERCISE | 5.4.2: Counting telephone numbers.

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

(a) How many different phone numbers are possible?

(b) How many different phone numbers are there in which the last four digits are all different?

EXERCISE | 5.4.3: Lining up club members for a photo.

Ten members of a club are lining up in a row for a photograph. The club has one president, one VP, one secretary, and one treasurer.

(a) How many ways are there to line up the ten people?

(b) How many ways are there to line up the ten people if the VP must be beside the president in the photo?

(c) How many ways are there to line up the ten people if the president must be next to the secretary and the VP must be next to the treasurer?

EXERCISE 5.4.4: Lining up a girl scout troop.

(a) A girl scout troop with 10 girl scouts and 2 leaders goes on a hike. When the path narrows, they must walk in single file with a leader at the front and a leader at the back. How many ways are there for the entire troop (including the scouts and the leaders) to line up?

5.5 Counting subsets

Consider a class with 20 students who must elect three representatives to the student council. The teacher conducts a vote and reveals the names of the three students who received the most votes. He does not reveal how many votes each student received or which one received more votes than the other two. How many ways are there to select the three representatives?

The outcome of the election is a set of three students, not a sequence because there is no particular order imposed on the three representatives. The outcome {Joshua, Karen, Ingrid} is the same outcome as {Karen, Ingrid, Joshua}. A subset of size r is called an ***r-subset***. In counting the number of ways to elect the three representatives, we are counting the number of different 3-subsets of students from a class of size 20.

An r-subset is sometimes referred to as an ***r-combination***. The counting rules for sequences and subsets are commonly referred to as "permutations and combinations". The term "combination" in the context of counting is another word for "subset".

PARTICIPATION ACTIVITY 5.5.1: Distinguishing between r-subsets and r-permutations.

Let S = {a, b, c}.

1) Is (b, a) a 2-permutation or a 2-subset from S?

- 2-permutation
- 2-subset

2) Is {b, a} a 2-permutation or a 2-subset from S?

- 2-permutation
- 2-subset

3) How many different 2-permutations from S are there?
   - 6
   - 3
   - 1

4) How many different 2-subsets from S are there?
   - 6
   - 3
   - 1

Using the k-to-1 rule to count subsets

Consider a small example in which a subset of three colors is selected from the set

Colors = {blue, green, orange, pink, red}

The number of 3-permutations from the set of five colors is P(5, 3) = 5!/2! = 60. Now define a function mapping 3-permutations to 3-subsets. The function is defined by just removing the ordering, so (orange, pink, blue) and (blue, orange, pink) both map to the set {orange, blue, pink}. The animation below shows that the function is (3!)-to-1, so by the k-to-1 rule (with k = 3!):

$$\text{Number of 3-subsets of colors} = \frac{P(5,3)}{3!} = \frac{5!}{3!2!} = 10$$

PARTICIPATION ACTIVITY 5.5.2: Mapping r-permutations to r-subsets: A small example.

Animation captions:

1. How many permutations map to the selection {orange, blue, pink}? 3! permutations because there are 3! ways to permute the 3 colors.
2. 3! 3-permutations map to one 3-subset.

To derive the general rule for counting r-subsets, define a mapping between r-permutations from a set of size n and r-subsets. The k-to-1 rule will be applied with k = r! as illustrated in the animation below:

PARTICIPATION ACTIVITY 5.5.3: Mapping r-permutations to r-subsets: A general rule.

Animation captions:

1. There are P(n,r) r-permutations from set S = {1, 2, ..., n}.
2. r! r-permutations map to each r-subset from S.

3. Each r- subset from S ({1, 2, ..., r} through {n-r+1, ..., n}) has r! r-permutations that map onto it.
4. $(\text{\# r-subsets from S}) = \frac{(\text{\# r-permutations from S})}{\text{r!}} = \frac{\text{P(n, r)}}{\text{r!}} = \frac{\text{n!}}{(\text{r!(n-r)!})}$.

PARTICIPATION ACTIVITY | 5.5.4: Mapping r-permutations to r-subsets.

1) Consider a function that maps 5-permutations from a set S = {1, 2, ..., 20} to 5-subsets from S. The function takes a 5-permutation and removes the ordering on the elements. How many 5-permutations map on to the subset {2, 5, 13, 14, 19}?

Check Show answer

Counting subsets comes up so frequently that the formula for counting subsets has its own notation and terminology:

Definition 5.5.1: Counting subsets: 'n choose r' notation.

The number of ways of selecting an r-subset from a set of size n is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$ is read "n choose r". The notation C(n, r) is sometimes used for $\binom{n}{r}$.

PARTICIPATION ACTIVITY | 5.5.5: Calculating 'n choose r'.

1) Calculate a numerical value for $\binom{7}{3}$. (Hint: write out the factorials as products and cancel numbers before multiplying).

Check Show answer

2) Calculate a numerical value for $\binom{7}{4}$.

Check Show answer

3) Calculate a numerical value for $\binom{100}{1}$, the number of ways to select a subset of size 1 from a set of size 100.

Check Show answer

4) Calculate a numerical value for $\binom{100}{100}$, the number of ways to select a subset of size 100 from a set of size 100. Note that 0! = 1.

Check Show answer

We can calculate an expression for $\binom{n}{n-r}$ by replacing r with n - r in the expression for $\binom{n}{r}$:

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

An equation is called an ***identity*** if the equation holds for all values for which the expressions in the equation are well defined. The equation $\binom{n}{r} = \binom{n}{n-r}$ is an identity because the equality holds for any non-negative integer n and any integer r in the range from 0 through n. The identity means that for any set S with n elements, the number of r-subsets from S is equal to the number of (n - r)-subsets from S. In fact, there is a bijection between r-subsets of S and (n - r)-subsets of S: each r-subset X of S corresponds uniquely to a subset of (n - r) elements consisting of the elements that are not in X.

PARTICIPATION ACTIVITY 5.5.6: Examples of counting subsets.

For the following questions, express your answer as "n choose r". Ex: 8 choose 3.

1) A teacher must select four members of the math club to participate in an upcoming competition. How many ways are

there for her to make her selection if the club has 12 members?

Check Show answer

2) A file will be replicated on 3 different computers in a distributed network of 15 computers. How many ways are there to select the locations for the file?

Check Show answer

Counting binary strings with a fixed number of 1's

The ideas used to count subsets of a particular size can also be used to count binary strings with a particular number of 1's. The animation below shows how to count the number of 5-bit strings that have exactly two 1's by showing a bijection between the strings to be counted and the number of 2-subsets from a set of size 5.

PARTICIPATION ACTIVITY 5.5.7: Counting 5-bit strings with exactly two 1's.

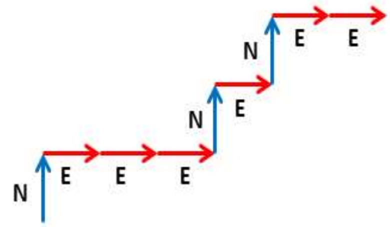
Animation captions:

1. Bijection from 5-bit strings with exactly 2 1's to 2-subsets of {1, 2, 3, 4, 5}. The bits are numbered 1 through 5 from left to right.
2. {1, 2} maps to 11000 because the two 1's are in places 1 and 2. Each 2-subset of {1, 2, 3, 4, 5} maps to a 5-bit string with 2 1's in the same way.
3. Since the mapping is a bijection, (# of 5-bit strings with exactly 2 1's) = (# of 2-subsets of {1, 2, 3, 4, 5} = $\binom{5}{2}$)

Example 5.5.1: Counting paths on grid.

Consider a city whose streets are laid out as a grid. Streets run north-south or east-west. A visitor is dropped off by a cab at a certain intersection and would like to visit a museum that is 6 blocks to the east and 3 blocks to the north of his current location. The visitor always takes a direct path and never travels west or south in the course of his walk. How many distinct paths are there for the visitor to walk to the museum?

The idea in counting paths is to show a bijection between distinct paths and certain kinds of sequences. The sequences are like the directions for the visitor, telling him in which direction he should walk at each block. The visitor needs to walk a total of nine blocks. For six of the blocks, he walks east and 3 blocks he walks north, so the sequences have length 9 and there are 6 E's and 3 N's. The path below corresponds to the sequence NEEENENEE:



Counting the number of sequences with 6 E's and 3 N's is similar to the problem of counting the number of binary strings with a particular number of 1's. There are a total of 6 + 3 = 9 characters in the sequence where each character is an N or an E. Once the location of the three N's is chosen from the 9 possible places in the sequence, E's are placed in the remaining six locations, and the sequence is determined.

There are $\binom{9}{3}$ sequences with 6 E's and 3 N's. In general, the number of paths that take the visitor n blocks north and m blocks east is $\binom{m+n}{m}$.

PARTICIPATION ACTIVITY | 5.5.8: Counting strings by counting subsets.

1) Define a bijection between 5-subsets of the set S = {1, 2, 3, 4, 5, 6, 7, 8} and 8-bit strings with exactly five 1's. A subset X of S with five elements maps on to a string x so that $j \in X$ if and only if the j^{th} bit of x is 1. What string corresponds to the set {1, 3, 4, 5, 8}?

Check Show answer

2) How many 8-bits strings have exactly five 1's?

Express your answer as 'n choose r'. Ex: 6 choose 2.

Check Show answer

3) How many strings over the alphabet {a, b} have length 20 and exactly 8

a's?

Express your answer as 'n choose r'.
Ex: 6 choose 2.

Check Show answer

CHALLENGE ACTIVITY | 5.5.1: Counting subsets of bit strings.

455912.3056722.qx3zqy7

Start

A bit string contains 1's and 0's. How many different bit strings can be constructed given the restriction(s)?

Length is 20.

Ex: 2^40 or 26 * C(36, 12)

Write a^b as: a^b
Write combination as: C(n, k)

1 2 3 4 5

Check Next

Additional exercises

EXERCISE | 5.5.1: Mapping permutations to subsets.

Consider a function f that maps 5-permutations from the set S = {1, ..., 20} to 5-subsets from S. The function takes a 5-permutation and creates an unordered set whose elements are the five numbers included in the permutation.

(a) What is the value of f on input (12, 1, 3, 15, 9)?

(b) Is (12, 3, 12, 4, 19) a 5-permutation? Why or why not?

(c) How many permutations are mapped onto the subset {12, 3, 13, 4, 19}?

EXERCISE | 5.5.2: Permutations and combinations from a set of letters.

Define the set S = {a, b, c, d, e, f, g}.

(a) Give an example of a 4-permutation from the set S.

(b) Give an example of a 4-subset from the set S.

(c) How many subsets of S have exactly four elements?

(d) How many subsets of S have either three or four elements?

EXERCISE 5.5.3: Counting bit strings.

How many 10-bit strings are there subject to each of the following restrictions?

(a) No restrictions.

(b) The string starts with 001.

(c) The string starts with 001 or 10.

(d) The first two bits are the same as the last two bits.

(e) The string has exactly six 0's.

(f) The string has exactly six 0's and the first bit is 1.

(g) There is exactly one 1 in the first half and exactly three 1's in the second half.

EXERCISE 5.5.4: Counting strings of letters.

How many different strings of length 12 containing exactly five a's can be chosen over the following alphabets?

(a) The alphabet {a, b}

(b) The alphabet {a, b, c}

EXERCISE 5.5.5: Choosing a chorus.

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

EXERCISE 5.5.6: Counting possible computer failures.

Suppose a network has 40 computers of which 5 fail.

(a) How many possibilities are there for the five that fail?

(b) Suppose that 3 of the computers in the network have a copy of a particular file. How many sets of failures wipe out all the copies of the file? That is, how many 5-subsets contain the three computers that have the file?

EXERCISE 5.5.7: Choosing a student committee.

14 students have volunteered for a committee. Eight of them are seniors and six of them are juniors.

(a) How many ways are there to select a committee of 5 students?

(b) How many ways are there to select a committee with 3 seniors and 2 juniors?

(c) Suppose the committee must have five students (either juniors or seniors) and that one of the five must be selected as chair. How many ways are there to make the selection?

EXERCISE 5.5.8: Counting five-card poker hands.

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(a) How many different five-card hands are there from a standard deck of 52 playing cards?

(b) How many five-card hands have exactly two hearts?

(c) How many five-card hands are made entirely of hearts and diamonds?

(d) How many five-card hands have four cards of the same rank?

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?

(f) How many five-card hands do not have any two cards of the same rank?

5.6 Subset and permutation examples

Questions about counting subsets and permutations often do not explicitly use the words "subset" or "permutation". Selecting a sample object from the set is helpful to determine whether the formulas P(n, k) or $\binom{n}{k}$ apply. If the order in which the elements of the permutation/subset are selected is important, then the question is asking about permutations. If order is not important, then the question is asking about subsets.

Consider as an example a distribution process in which a teacher is distributing a set of four prizes to the ten students in his class. Each student can get at most one prize. How many ways are there to distribute the prizes if:

- The prizes are all identical.
- The prizes are all different from each other.

PARTICIPATION ACTIVITY 5.6.1: Distributing identical vs. distinct prizes.

Animation captions:

1. Distribute 4 identical prizes. A sample outcome shows the subset of 4 students who get prizes. $\binom{10}{4}$ different choices.
2. Distribute 4 different prizes. A sample outcome shows which students get 1st, 2nd, 3rd, and 4th prizes.
3. The outcome is different if two students swap prizes. The outcome is a sequence of students who get prizes. P(10,4) choices.

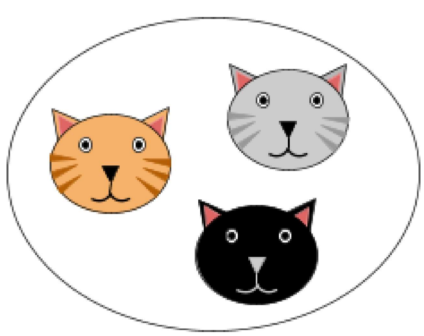
If the prizes are all different from each other, then the problem is asking about the number of 4-permutations from a set of 10 elements because the order in which prizes are distributed is important. If the prizes are all identical, then the problem is asking about the number of 4-subsets because the order in which prizes are distributed is not important.

Example 5.6.1: Two different cat selection problems: Subsets vs. permutations.

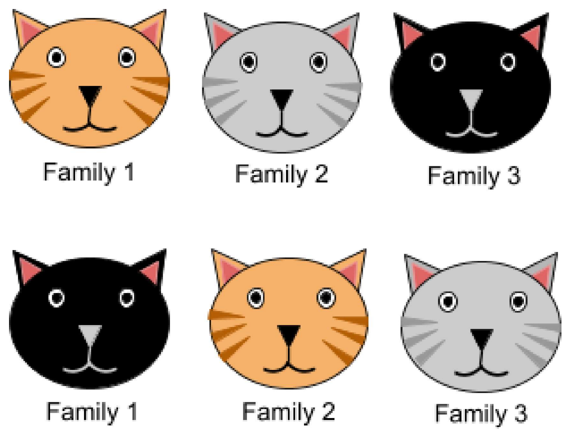
Consider two closely related counting problems:

1. A family goes to the animal shelter to adopt 3 cats. The shelter has 20 different cats from which to select. How many ways are there for the family to make their selection?
2. Three different families go to the animal shelter to adopt a cat. Each family will select one cat. How many ways are there for the families to make their selections? Note that which family gets which cat matters.

In the first problem in which one family selects three cats, the number of ways to make the selection is $\binom{20}{3}$ because the order in which the cats are selected is not important. The outcome of a selection is just the set of cats that the family selects.



In the second problem in which three families are each selecting one cat, the specific cat selected by each family is important. The order in which families make their selection is fixed but arbitrary. The first family selects their cat from the set of 20 cats at the shelter. Then, since no cat can belong to two families, the second family selects their cat from the 19 remaining cats, and finally the third family selects their cat from the 18 cats left after the first two families have chosen. Thus, there are $20 \cdot 19 \cdot 18 = P(20, 3)$ ways for the three families to make their selections. The fact that the order in which the cats are selected matters corresponds to the fact that which family gets which cat is important. The two sample selections below are considered different even though the same three cats leave the shelter that day.



PARTICIPATION ACTIVITY 5.6.2: Selecting subsets or permutations.

Provide solutions in either the form "P(n, r)" or "n choose r" (e.g. P(8, 3) or 8 choose 3).

1) Dave swims three times in the week. How many ways are there to plan his workout schedule (i.e. which days he will swim) for a given week?

Check Show answer

2) Dave will swim one day, run one day, and bike another day in a week. He does at most one activity on any particular day. How many ways are there for him to select his workout schedule (i.e. which activities he does which days)?

Check Show answer

3) The students in a class elect a president, vice president, secretary, and treasurer. There are 30 students in the class and no student can have more than one job. Note that it matters who is elected into which position. How many different outcomes are there from the election process?

Check Show answer

4) A class of 30 students elects four students to serve on a student leadership council. The teacher tallies the votes and only reveals the names of the four students who received the most votes. How many different outcomes are there from the election process?

Check Show answer

Additional exercises

EXERCISE 5.6.1: Selecting students for jobs.

(a) A teacher selects 4 students from her class of 37 to work together on a project. How many ways are there for her to select the students?

(b) A teacher selects students from her class of 37 students to do 4 different jobs in the classroom: pick up homework, hand out permission slips, staple worksheets, and organize the classroom library. Each job is performed by exactly one student in the class and no student can get more than one job. How many ways are there for her to select students and assign them to the jobs?

EXERCISE 5.6.2: Results of a piano competition.

120 pianists compete in a piano competition.

(a) In the first round, 30 of the 120 are selected to go on to the next round. How many different outcomes are there for the first round?

(b) In the second round, the judges select the first, second, third, fourth and fifth place winners of the competition from among the 30 pianists who advanced to the second round. How many outcomes are there for the second round of the competition?

EXERCISE 5.6.3: Choosing a lineup for a traveling basketball team.

There are 20 members of a basketball team.

(a) The coach must select 12 players to travel to an away game. How many ways are there to select the players who will travel?

(b) From the 12 players who will travel, the coach must select her starting line-up. She will select a player for each of the five positions: center, power forward, small forward, shooting guard and point guard. How many ways are there for her to select the starting line-up?

(c) From the 12 players who will travel, the coach must select her starting line-up. She will select a player for each of the five positions: center, power forward, small forward, shooting guard and point guard. However, there are only three of the 12 players who can play center. Otherwise, there are no restrictions. How many ways are there for her to select the starting line-up?

EXERCISE 5.6.4: Hiring a software engineer.

A search committee is formed to find a new software engineer.

(a) If 100 applicants apply for the job, how many ways are there to select a subset of 9 for a short list?

(b) If 6 of the 9 are selected for an interview, how many ways are there to pick the set of people who are interviewed? (You can assume that the short list is already decided).

(c) Based on the interview, the committee will rank the top three candidates and submit the list to their boss who will make the final decision. (You can assume that the interviewees are already decided.) How many ways are there to select the list from the 6 interviewees?

EXERCISE 5.6.5: Counting meals at a restaurant.

A group of five friends goes to a restaurant for dinner. The restaurant offers 20 different main dishes.

(a) Suppose that the group collectively selects five different dishes to share. The waiter just needs to place all five dishes in the center of the table. How many different possible meals are there for the group?

(b) Suppose that each individual selects a main course. The waiter must remember who selected which dish. It's possible for more than one person to select the same dish. How many different possible meals are there for the group?

(c) Suppose that each individual selects a main course. The waiter must remember who selected which dish. However, the friends agree that no two people will select the same dish. How many different possible meals are there for the group?

EXERCISE 5.6.6: Selecting a committee of senators.

A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

5.7 Counting by complement

Suppose we want to count the number of people in a room with red hair. We know that there are 20 people in the room and exactly 12 of them do not have red hair. Then we can deduce that the number of people in the room with red hair is 20 - 12 = 8. ***Counting by complement*** is a technique for counting the number of elements in a set S that have a property by counting the total number of elements in S and subtracting the number of elements in S that do not have the property. The principle of counting by complement can be written using set notation where P is the subset of elements in S that have the property.

$$|P| = |S| - |\overline{P}|$$

PARTICIPATION ACTIVITY 5.7.1: Counting by complement: The number of 8-bit strings with at least one 0.

Animation captions:

1. How many 8-bit strings have at least one 0? S = the set of 8-bit strings. $|S| = 2^8 = 256$.
2. P = the set of 8-bit strings with at least one 0. $\overline{P}$ = the set of 8-bit strings with no 0's.
3. Only one 8-bit string has no 0's: 11111111. $|\overline{P}| = 1$. $|P| = |S| - |\overline{P}| = 256 - 1 = 255$. 255 8-bit strings have at least one 0.

PARTICIPATION ACTIVITY 5.7.2: Counting by complement.

Give numerical answers for the questions below.

1) There are 10 kids on the math team. Two kids will be selected from the team to compete in the state competition. How many ways are there to select the 2 competitors?

Check Show answer

2) The math team has 6 girls and 4 boys. How many ways are there to select the two competitors if they are both girls?

Check Show answer

3) The math team has 6 girls and 4 boys. How many ways are there to select the two competitors so that at least one boy is chosen?

Check Show answer

4) Four people (John, Paul, George, and Ringo) are seated in a row on a bench. The number of ways to order the four people so that John is next to Paul is 12. How many ways are there to order the four people on the bench so that John is not next to Paul?

Check Show answer

CHALLENGE ACTIVITY 5.7.1: Counting possibilities by complement.

455912.3056722.qx3zqy7

Start

An auto dealer has 8 different cars and 7 different trucks.

How many ways are there to select two vehicles?

Ex: 26 * C(36, 12)

Write combination as: C(n, k)

1 2 3 4

Check Next

Additional exercises

EXERCISE 5.7.1: Counting passwords.

How many different passwords are there that contain only digits and lower-case letters and satisfy the given restrictions?

(a) Length is 6 and the password must contain at least one digit.

(b) Length is 6 and the password must contain at least one digit and at least one letter.

EXERCISE 5.7.2: Counting 5-card hands from a deck of standard playing cards.

A 5-card hand is drawn from a deck of standard playing cards.

(a) How many 5-card hands have at least one club?

(b) How many 5-card hands have at least two cards with the same rank?

EXERCISE 5.7.3: Counting bit strings.

(a) How many 8-bit strings have at least two consecutive 0's or two consecutive 1's?

(b) How many 8-bit strings do not begin with 000?

EXERCISE 5.7.4: Lining up club members for a photo.

Ten members of a club are lining up in a row for a photograph. The club has one president and one VP.

(a) How many ways are there for the club members to line up in which the president is not next to the VP?

(b) How many ways are there for the club members to line up if the VP is not in the leftmost position?

(c) How many ways are there for the club members to line up if the VP is not at one end (i.e. in the leftmost or rightmost positions)?

5.8 Permutations with repetitions

How many ways are there to scramble the letters in the word MISSISSIPPI? The question is concerned about counting different orderings which suggests counting permutations. However, a permutation is defined as an ordering of distinct objects. The letters in MISSISSIPPI have multiple repetitions: there are four S's, four I's, two P's, and one M. A ***permutation with repetition*** is an ordering of a set of items in which some of the items may be identical to each other. To illustrate with a smaller example, there are 3! = 6 permutations of the letters CAT because the letters in CAT are all different. However, there are only 3 different ways to scramble the letters in DAD: ADD, DAD, DDA. The animation below shows how to count the number of distinct permutations of the letters in MISSISSIPPI by repeatedly applying the formula for counting r-subsets and putting together the choices using the product rule.

PARTICIPATION ACTIVITY 5.8.1: Counting the number of permutations of the letters in MISSISSIPPI.

Animation captions:

1. How many ways exist to scramble MISSISSIPPI? 11 possible locations for the 2 P's. $\binom{11}{2}$ choices to place the P's.
2. 9 locations left for the 4 I's. $\binom{9}{4}$ choices to place the I's. 5 locations left for the 4 S's. $\binom{5}{4}$ choices to place the S's. 1 location left for the M. $\binom{1}{1}$ choices to place the M.
3. $\binom{11}{2}\binom{9}{4}\binom{5}{4}\binom{1}{1} = \frac{11!}{2!9!} \cdot \frac{9!}{4!5!} \cdot \frac{5!}{4!1!} \cdot \frac{1!}{1!0!} = \frac{11!}{2!4!4!1!}$ ways exist to scramble MISSISSIPPI.

Fact 5.8.1: Formula for counting permutations with repetition.

The number of distinct sequences with n_1 1's, n_2 2's, ..., n_k k's, where $n = n_1 + n_2 + ... + n_k$ is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

The formula for permutations with repetition is derived from repeated use of the formula for counting r-subsets:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-n_2\cdots-n_{k-1})!}{n_k!0!}$$

$$= \frac{n!}{n_1!n_2!\cdots n_k!}$$

The last 0! in the denominator on the second line comes from the fact that $n - n_1 - n_2 - \ldots - n_k = 0$. Recall that 0! = 1.

PARTICIPATION ACTIVITY 5.8.2: Permutations of letters with repetition.

1) How many ways are there to permute the letters in PEPPER? (Give a numerical answer)

Check Show answer

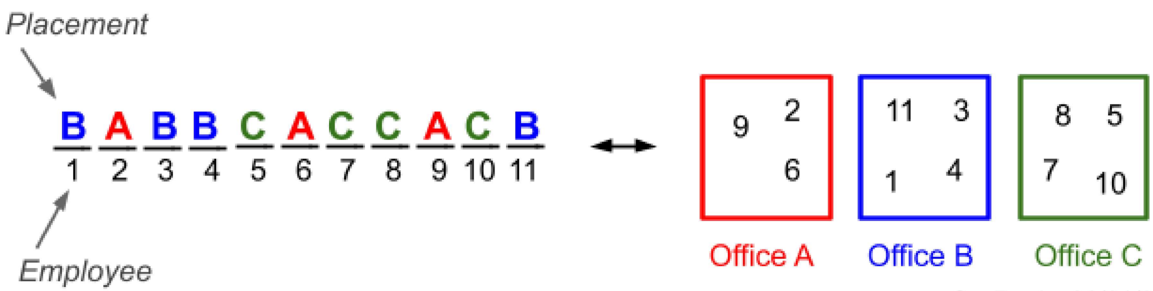
2) How many ways are there to permute the letters in HAPPY? (Give a numerical answer)

Check Show answer

Example 5.8.1: Counting assignments using permutations with repetition.

Consider a company with 11 employees that rents an office space with three offices. The first office (office A) can hold 3 people, the second office (office B) can hold 4 people and the third office (office C) can hold 4 people. How many different assignments are there for employees to offices?

Define S to be the set of permutations of the 11 characters: AAABBBBCCCC (3 A's, 4 B's, and 4 C's). There is a bijection between the set of office assignments and permutations in S. Order the employees from 1 to 11. The i^{th} character in the permutation is the office assignment for employee i:



Each permutation in S corresponds uniquely to an office assignment. Each office assignment corresponds uniquely to a permutation in S. The number of distinct assignments is therefore:

$$\frac{11!}{3!4!4!}$$

An alternative way to solve the office assignment problem is to apply the generalized product rule directly. There are $\binom{11}{3}$ ways to select the 3 people who will go into office A. After office A is filled, there are 8 people left from which to pick the 4 people for office B which results in $\binom{8}{4}$ choices. After offices A and B are filled, there are four people left from which to pick the 4 people for office C which results in $\binom{4}{4}$ choices. The choices made in each step are put together using the product rule.

$$\binom{11}{3}\binom{8}{4}\binom{4}{4} = \frac{11!}{3!4!4!}$$

The equation above can be verified by expanding the "n choose r" expressions using factorials and then cancelling.

PARTICIPATION ACTIVITY 5.8.3: Counting assignments using permutations with repetitions.

1) Suppose that 9 desserts are handed out to 9 kids. Each kid gets one dessert. There are three ice cream sandwiches, four cupcakes and two bowls of pudding. How many ways are there to hand out the desserts to the kids?

- $\binom{9}{3}$
- $\frac{9!}{3!4!2!}$
- $9!$

2) Fifteen kids arrive at camp and are assigned a place to sleep. There are three different cabins each of which can hold five kids. How many ways are there to assign kids to cabins?

- $\frac{15!}{5!5!5!}$
- $15!$
- $\binom{15}{5}$

Additional exercises

EXERCISE 5.8.1: Permuting letters in words.

How many ways are there to permute the letters in each of the following words?

(a) NUMBER

(b) DISCRETE

(c) SUBSETS

EXERCISE 5.8.2: Counting ternary strings.

(a) How many ternary strings (digits 0,1, or 2) are there with exactly seven 0's, five 1's and four 2's?

EXERCISE 5.8.3: Dealing cards to four players.

How many ways are there to deal hands from a standard playing deck to four players if:

(a) Each player gets exactly 13 cards.

(b) Each player gets seven cards and the rest of the cards remain in the deck?

EXERCISE 5.8.4: Distributing comic books.

20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

(b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

EXERCISE

5.8.5: Assigning bedrooms to daughters.

(a) A family has four daughters. Their home has three bedrooms for the girls. Two of the bedrooms are only big enough for one girl. The other bedroom will have two girls. How many ways are there to assign the girls to bedrooms?

EXERCISE 5.8.6: Assigning summer camp activities.

(a) A camp offers 4 different activities for an elective: archery, hiking, crafts and swimming. The capacity in each activity is limited so that at most 35 kids can do archery, 20 can do hiking, 25 can do crafts and 20 can do swimming. There are 100 kids in the camp. How many ways are there to assign the kids to the activities?

EXERCISE 5.8.7: Scheduling meals at a school.

A school cook plans her calendar for the month of February in which there are 20 school days. She plans exactly one meal per school day. Unfortunately, she only knows how to cook ten different meals.

(a) How many ways are there for her to plan her schedule of menus for the 20 school days if there are no restrictions on the number of times she cooks a particular type of meal?

(b) How many ways are there for her to plan her schedule of menus if she wants to cook each meal the same number of times?

5.9 Counting multisets

This section has been set as optional by your instructor.

A set is a collection of distinct items. A ***multiset*** is a collection that can have multiple instances of the same kind of item. When the expression {1, 2, 2, 3} is viewed as a set, the repetitions don't matter and {1, 2, 2, 3} = {1, 2, 3}. However, when the expression {1, 2, 2, 3} is viewed as a multiset, then the fact that there are two occurrences of 2 is important, and {1, 2, 2, 3} ≠ {1, 2, 3}. Two multisets are equal if they have the same number of each type of element. The curly braces denote the fact that the order in which the elements are listed does not matter, so {1, 2, 2, 3} is equal to {2, 1, 2, 3}.

Multisets are useful in modeling situations in which there are several varieties of objects and one can have multiple instances of the same variety. Suppose that a customer at a bakery is selecting a dozen cookies to buy. There are four varieties of cookies: chocolate chip, sugar, ginger, and oatmeal raisin. Cookies of the same variety are indistinguishable, so one sugar cookie is the same as any other sugar cookie. A selection of cookies is a multiset of size 12 in which the elements are cookies chosen from the four different varieties. An example of a selection of 12 cookies would be 3 chocolate chip, 2 sugar, 2 ginger, and 5 oatmeal, which can be denoted by the multiset {C, C, C, S, S, G, G, O, O, O, O, O}.

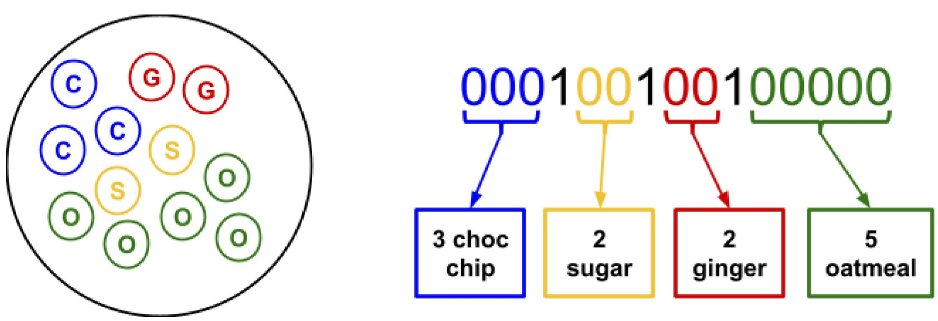
PARTICIPATION ACTIVITY 5.9.1: Multisets and sets.

1) Which multiset is equivalent to the multiset {a, b, b, b}?
   - {a, b}
   - {b, a, b, b}
   - {a, b, b, b, c}

2) Which multiset is equivalent to the multiset {1, 2, 2, 3, 3, 3}?
   - {1, 2, 2, 3, 3}
   - {1, 1, 2, 2, 3, 3}
   - {2, 2, 1, 3, 3, 3}

Suppose we want to count the number of ways to select 12 cookies from 4 varieties. We define a bijection between the set of possible cookie selections and a certain set of binary strings called code words. Each string in the set uniquely encodes a cookie selection. Every cookie selection is encoded by a unique code word. Therefore, the number of code words is equal to the number of distinct cookie selections. The encoding requires that the varieties be ordered in an arbitrary but fixed order. For the cookie selection example, the varieties are ordered: chocolate chip, sugar, ginger, oatmeal. The diagram below shows an example of a cookie selection and its corresponding code word:

Figure 5.9.1: A sample cookie selection and corresponding code word.



The table below summarizes how to encode a selection using code words. The different varieties are numbered from 1 to m.

Table 5.9.1: Rules for encoding a selection of n objects from m varieties.

Selections	Code words

n = number of items to select	n = number of 0's in code word
m = number of varieties	m - 1 = number of 1's in code word
Number selected from the first variety	Number of 0's before the first 1
Number selected from the i^{th} variety, for 1 < i < m	Number of 0's between the i-1^{st} and i^{th} 1
Number selected from the last variety	Number of 0's after the last 1

If the mapping of selections to code words is a bijection, then by the bijection rule, the number of distinct code words is equal to the number of distinct selections. If the number of objects to select is n, and the number of varieties of object is m, each code word has n 0's and m-1 1's, for a total of n+m-1 bits. The number of binary strings of length n+m-1 with exactly m-1 1's is

$$\binom{n+m-1}{m-1}$$

For the cookie selections problem, n = 12 and m = 4, so the length of the code words is 15. The number of binary strings of length 15 with exactly three 1's is

$$\binom{15}{3}$$

The two animations below illustrate why the mapping of selections to code words is a bijection. The first animation shows that each code word unambiguously specifies a selection of cookies. The second animation shows that a selection of cookies unambiguously corresponds to a code word.

PARTICIPATION ACTIVITY | 5.9.2: Counting multisets: mapping code words to cookie selections.

Animation captions:

1. Code word 001000010100000. The two 0's before the first 1 represent two chocolate chip cookies.
2. The four 0's between the first and second 1 represent four sugar cookies.
3. The one 0 between the second and third 1 represent one ginger cookie. The five 0's after the last 1 represent five oatmeal cookies.
4. The cookie selection consists of 2 chocolate chip, 4 sugar, 1 ginger, and 5 oatmeal cookies.
5. Code word 110000000100000. There are no 0's until after the second 1, so there are no chocolate chip or sugar cookies. There are 7 ginger and 5 oatmeal.
6. The cookie selection consists of 7 ginger and 5 oatmeal cookies.

PARTICIPATION ACTIVITY | 5.9.3: Counting subsets with repetitions: mapping cookie selections to code words.

Animation content:

undefined

Animation captions:

1. The cookie selection with 3 chocolate chip, 4 sugar, 3 ginger, and 2 oatmeal cookies corresponds to the code word 000100001000100.
2. Each cookie selection corresponds to a unique code word. Each code word corresponds to a unique cookie selection. Thus, the mapping is a bijection.

PARTICIPATION ACTIVITY 5.9.4: Encoding multiset selections with binary strings.

10 balls are selected from a large set of balls that come in three colors: blue, red and green. Balls of the same color are identical. Number the colors so that blue is variety number 1, red is variety number 2 and green is variety number 3.

1) How many bits will there be in an encoding of a selection?

- 12
- 13

2) Which string corresponds to the selection consisting of 5 blue balls, 2 red balls and 3 green balls?

- 00000100000
- 000001001000
- 0000010010001
- 001000001000

3) Which selection corresponds to the string 000100000001?

- 2 blue balls, 7 red balls, 1 green balls.
- 3 red balls and 7 green balls.
- 3 blue balls, 7 red balls, 0 green balls.

Fact 5.9.1: Counting multisets.

The number of ways to select n objects from a set of m varieties is

$$\binom{n+m-1}{m-1},$$

if there is no limitation on the number of each variety available and objects of the same variety are indistinguishable.

Here is another application for counting multisets that may appear to be unrelated to the initial application.

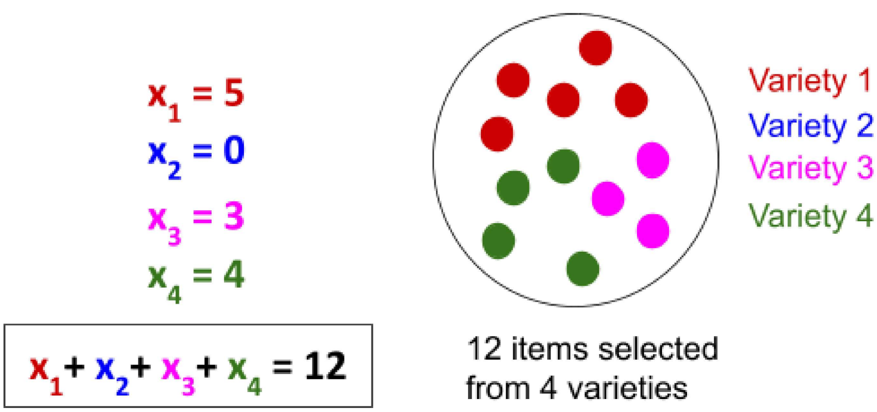
Example 5.9.1: Counting solutions to variables with a fixed sum.

Consider the equation:

$$x_1 + x_2 + x_3 + x_4 = 12$$

How many solutions are there to the above equation if all the variables are non-negative integers?

There is a bijection between solutions to the equation and the number of ways to select 12 objects from four varieties. The varieties of objects are numbered from 1 to 4. For i = 1, 2, 3, 4, x_i is the number of objects selected from the i^{th} variety.



Since each x_i is a non-negative integer, an assignment of values to variables corresponds to a valid selection of objects. The variables sum to 12, so there are a total of 12 objects selected. Similarly, a selection of 12 objects from 4 varieties, corresponds to a valid assignment of values to variables, so that the sum of the variables is 12. Therefore, the number of solutions to the equation is equal to the number of ways to select 12 objects from 4 varieties:

$$\binom{12+4-1}{4-1} = \binom{15}{3}$$

Explanation of solution (2:02)

PARTICIPATION ACTIVITY 5.9.5: Counting solutions to equations with sums of non-negative integer variables.

Give your answers in the form of "n choose k".

1) How many solutions are there to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30,$$

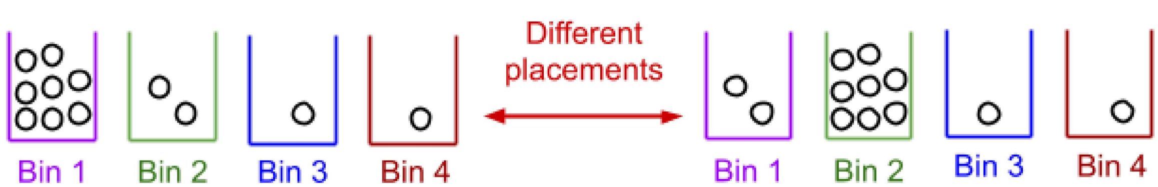
where each of the variables x_1 through x_5 is a non-negative integer?

Check Show answer

A set of identical items are called ***indistinguishable*** because it is impossible to distinguish one of the items from another. A set of different or distinct items are called ***distinguishable*** because it is possible to distinguish one of the items from the others.

Example 5.9.2: Counting the number of ways to place indistinguishable balls in distinguishable bins.

Suppose that 12 indistinguishable balls are to be placed in one of four bins. The bins are numbered, making them distinguishable, so putting 8 balls in bin 1 and 2 balls in bin 2 is different than putting 2 balls in bin 1 and 8 balls in bin 2 as illustrated in the diagram below:



How many ways are there to place the balls in the bins?

There is a bijection between placements of balls into bins and the number of ways to select 12 objects from four varieties. The varieties of objects are numbered from 1 to 4. For i = 1, 2, 3, 4, the number of balls placed in the i^{th} bin is equal the number of objects selected from the i^{th} variety. A selection of 12 objects from 4 varieties, corresponds to a valid placement of 12 identical balls into 4 distinct bins. Similarly, a placement of balls into bins corresponds to a unique way to select objects. Therefore, the number of placements of 12 identical balls into 4 distinct bins is equal to the number of ways to select 12 objects from 4 varieties:

$$\binom{12+4-1}{4-1} = \binom{15}{3}$$

Explanation of solution (1:43)

Fact 5.9.2: Indistinguishable balls and distinguishable bins.

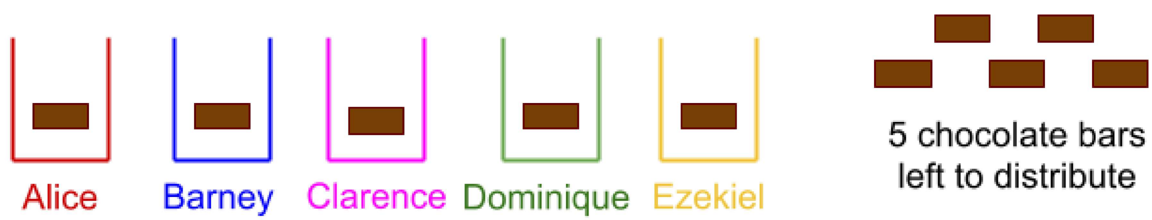
The number of ways to place n indistinguishable balls into m distinguishable bins is

$$\binom{n+m-1}{m-1}$$

Consider a situation where indistinguishable balls are placed into distinguishable bins, with the constraint that there must be at least a certain number in some of the bins. For example, suppose that a teacher distributes ten identical chocolate bars to five different kids but wants to make sure that each kid gets at least one chocolate bar. How many ways are there for the teacher distribute the chocolate bars with the additional constraint?

The constraint that each kid gets at least one chocolate bar can be satisfied by first giving a chocolate bar to each kid. Since the chocolate bars are identical, it doesn't matter which chocolate bar goes to which kid. The figure below illustrates the situation after one chocolate bar has been given to each kid. Now there are five remaining chocolate bars to distribute. The kids are represented by bins. Giving a chocolate bar to a kid is accomplished by placing the chocolate bar in his or her bin. For the purposes of counting, distributing indistinguishable chocolate bars is the same as distributing indistinguishable balls.

Figure 5.9.2: Distributing indistinguishable chocolate bars to kids with at least one to each kid.



Example 5.9.3: Counting the number of ways to distribute identical items to different people.

A teacher distributes ten identical chocolate bars to five different kids.

- How many ways are there for the teacher to distribute the chocolate bars among the five kids if each kid gets at least one chocolate bar?

  To satisfy the constraint that each kid gets at least one chocolate bar, start by giving each kid a chocolate bar. Since the chocolate bars are all the same, it does not matter who gets which one. Now there are 5 chocolate bars remaining (n = 5) to be distributed among the five kids (m = 5) and they can be distributed in any way. There are $\binom{5+5-1}{5-1} = \binom{9}{4}$ ways to distribute the remaining chocolate bars.

  Explanation of solution (1:21)

- How many ways are there for the teacher to distribute the chocolate bars among the five kids if one particular kid (say Alice) gets at least two chocolate bars? (Besides the fact that Alice gets at least two chocolate bars, there are no other restrictions on how the chocolate bars are distributed).

  To satisfy the constraint that Alice gets at least two chocolate bars, start by giving Alice two chocolate bars. Since the chocolate bars are all the same, it does not matter which chocolate bars are given to Alice. Now there are 8 chocolate bars remaining (n = 8) to be distributed among the five kids (m = 5) which can be distributed in any way. In particular, Alice can be given more chocolate bars than the two she received initially. There are $\binom{8+5-1}{5-1} = \binom{12}{4}$ ways to distribute the remaining chocolate bars.

  Explanation of solution (1:14)

PARTICIPATION ACTIVITY | 5.9.6: Multisets and balls into bins.

Give your answers in the form of "n choose k".

1) There are seven varieties of donuts sold at a bakery. How many ways are there to select a dozen donuts? The order in which the donuts are selected does not matter and donuts of the same variety are all the same.

   Check Show answer

2) There are seven varieties of donuts sold at a bakery. How many ways are there to select a dozen donuts if the selection must have at least one of each variety? The order in which they are selected does not matter and donuts of the same variety are all the same.

   Check Show answer

3) Susan must do exactly 100 push ups in the course of a seven day week. How many different schedules are there for her to do her

push ups? A schedule consists of the number of push ups she does on each of the seven days of the week, for example, M:10, T: 10, W: 20, Th:25, F: 5, Sa: 0, Su:30.

Check Show answer

4) Susan must do exactly 100 push ups in the course of a seven day week. How many different schedules are there for her to do her push ups if she must do at least 20 on each weekend day (Saturday and Sunday)?

Check Show answer

PARTICIPATION ACTIVITY | 5.9.7: Combining counting by complement and counting multisets.

Give numerical answers for the questions below.

1) How many ways are there to select a set of 8 donuts from 3 varieties? Donuts of the same variety are indistinguishable.

Check Show answer

2) How many ways are there to select a set of 8 donuts from 3 varieties in which at least 3 chocolate donuts are chosen? Give a numerical answer.

Check Show answer

3) How many ways are there to select a set of 8 donuts from 3 varieties in

which at most 2 chocolate donuts are selected?

Check Show answer

Additional exercises

EXERCISE 5.9.1: Showings at a movie theater.

(a) A movie theater offers 6 showings of a movie each day. A total of 1000 people come to see the movie on a particular day. The theater is interested in the number of people who attended each of the six showings. How many possibilities are there for the tallies for each showing for that day?

EXERCISE 5.9.2: Selecting cookies.

A cookie store sells 6 varieties of cookies. It has a large supply of each kind.

(a) How many ways are there to select 15 cookies?

(b) How many ways are there to select 15 cookies if at least three must be chocolate chip?

(c) How many ways are there to select 15 cookies if at most 2 can be sugar cookies?

(d) How many ways are there to select 15 cookies if at most 2 can be sugar cookies and at least 3 must be chocolate chip?

EXERCISE 5.9.3: Selecting pieces of taffy.

(a) Sally goes into a candy store and selects 12 pieces of taffy. The candy store offers 75 varieties of taffy. How many ways are there for Sally to select her 12 pieces of taffy?

EXERCISE 5.9.4: Selecting coins from piles.

A large number of coins are divided into four piles according to whether they are pennies, nickels, dimes or quarters.

(a) How many ways are there to select 25 coins from the piles?

(b) How many ways are there to select 25 coins if at least 5 of the chosen coins must be quarters?

(c) Suppose the pile of quarters only has 10 quarters, so at most 10 quarters can be selected. How many ways are there to select 25 coins?

EXERCISE | 5.9.5: Ordering soda at a grocery store.

An employee of a grocery store is placing an order for soda. There are 8 varieties of soda and they are sold in cases. Each case contains all the same variety of soda. The store will order 50 cases total.

(a) How many ways are there to place the order?

(b) How many ways are there to place the order if she orders at least 3 of each variety?

(c) How many ways are there to place the order if she does not order more than 20 cases of Coke?

EXERCISE | 5.9.6: Counting solutions to integer equations.

In the following question, we will count distinct integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 35$.

(a) How many solutions are there if all the variables must be non-negative?

(b) How many solutions are there if all the variables must be positive?

(c) How many solutions are there if $x_1 \geq 2$, $x_2 \geq 4$, $x_3 \geq 0$, and $x_4 \geq 0$?

(d) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 \leq 35$ in which all the x_i are non-negative? (Hint: add an extra variable y such that y is non-negative and $x_1 + x_2 + x_3 + x_4 + y = 35$.)

EXERCISE | 5.9.7: Sending jobs to printers.

20 identical printing jobs are sent to 5 different printers.

(a) One of the printers is much faster than the other four. How many ways are there to distribute the jobs if the faster printer gets at least 7 jobs?

(b) One of the printers is almost out of paper. How many ways are there to distribute the jobs so that the printer that is almost out of paper does not get more than three jobs? (In addition to the fact that the faster one gets at least 7 jobs.)

EXERCISE | 5.9.8: Coefficients and terms in a multinomial expansion.

Consider the product $(x + y + z)^2$. If the expression is multiplied out and like terms collected, the result is:

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

Suppose we do the same to the product $(v + w + x + y + z)^{25}$

(a) What is the coefficient of the term $v^9w^2x^5y^7z^2$?

(b) How many different terms are there? (Two terms are the same if the degree of each variable is the same.)

5.10 Assignment problems: Balls in bins

This section has been set as optional by your instructor.

Many counting problems that ask about the number of ways to assign or distribute a set of items can be expressed abstractly by asking about the number of ways to place n balls into m different bins. In all the problems presented in this material, the bins are numbered, so placing a ball in bin 1 is considered different than placing a ball in bin 2. Some problems place different constraints on the number of balls that can be placed into the bins. Problems also vary according to whether the balls are all the same (indistinguishable) or all different (distinguishable). If the balls are different, they are numbered 1 through n, and which ball gets placed in which bin matters.

Figure 5.10.1: Six different balls-into-bins problems.

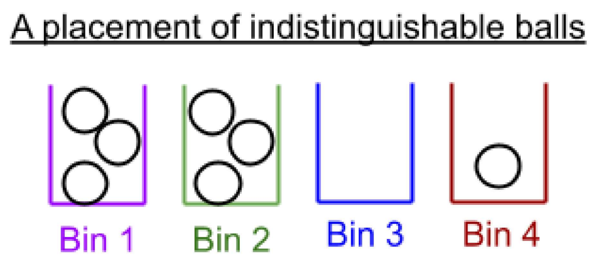
The table below shows the number of ways to place n balls into m distinguishable bins. Some of the restrictions on the number of balls per bins imply a relationship between m and n. For example, if there can be at most one ball per bin, then m, the number of bins, must be at least n, the number of balls. If the same number of balls must be placed in each bin, then m, the number of bins, must evenly divide n, the number of balls.

Each of the six formulas in the table is explained below based on counting techniques.

	No restrictions	At most one ball per bin	Same number of balls in each bin
	(any positive m and n)	(m must be at least n)	(m must evenly divide n)
Indistinguishable balls	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	1
Distinguishable balls	m^n	$P(m, n)$	$\frac{n!}{((n/m)!)^m}$

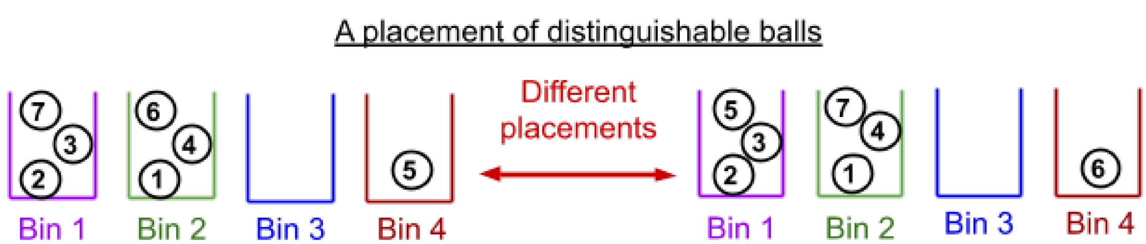
Figure 5.10.2: Balls in bins: No restrictions on the number of balls per bin.

The picture below shows a placement of 7 indistinguishable balls into bins. The only thing that makes one placement different from another is the number of balls in each bin. A placement can therefore be represented as a multi-set. For example, the placement shown below can be expressed as {1, 1, 1, 2, 2, 2, 4}. The order of the numbers in the multi-set is not important because the only factor that makes one multi-set different from another is the number of 1's, 2's, 3's, and 4's.



For n balls and m bins, the number of different placements is $\binom{n+m-1}{m-1}$.

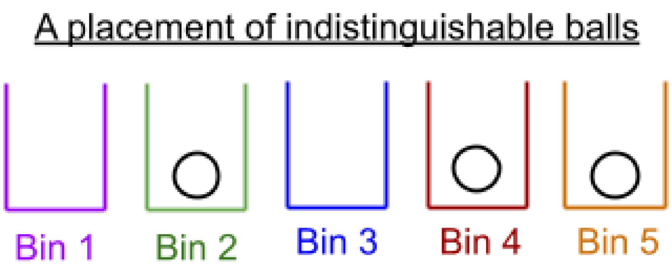
The next picture shows two different placements of 7 distinguishable balls into 4 bins. The number of balls in each bin is the same in the two placements but the specific balls placed in each bin differs.



To count the number of ways to place n distinguishable balls into m distinguishable bins, number the balls 1 through n and imagine placing each ball in turn. There are m choices of bins in which to place the first ball. Since there are no restrictions on the number of balls in each bin, there are also m choices of bins in which to place the second ball, the third ball, etc. The number of choices for each ball are put together by the product rule, so the number of ways to place all n balls is: m·m·m...m·m (n times), which is m^n.

Figure 5.10.3: Balls in bins: At most one ball per bin.

The first picture shows a placement of 3 indistinguishable balls (n=3) into m distinguishable bins (m=5) with at most one ball per bin. Each bin has 1 or 0 balls, so a placement can be specified by indicating which subset of the bins gets a ball. For example, the placement below can be expressed as {2, 4, 5}. Since the balls are all the same, the order of the bin numbers is not important. The number of different placements of n balls into m bins with at most one ball per bin is equal to the number of n-subsets of the m bins which is $\binom{m}{n}$.



The picture below shows a placement of 3 distinguishable balls (n=3) into m distinguishable bins (m=5) with at most one ball per bin. The placement can be specified by the sequence (4, 5, 2) which is a 3-permutation from the set {1, 2, 3, 4, 5}. The first number is the bin in which ball 1 is placed, the second number is the bin in which ball 2 is placed and the third number is the bin in which ball 3 is placed. Bin numbers can not be repeated because at most one ball can be placed in each bin.

To count the number of ways to place the n balls, imagine placing the balls in order. There are m choices of bins in which to place the first ball. Once the first ball has been placed, there are m-1 choices of bins in which to place the second ball, m-2 for the third, etc. The number of different placements is m·(m-1)·(m-2)...(m-n+1) = P(m, n).

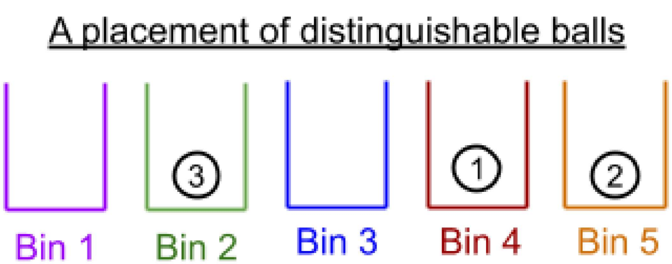
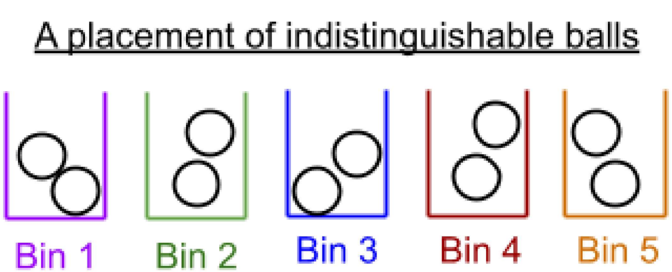


Figure 5.10.4: Balls in bins: Same number of balls per bin.

If the number of balls in each bin is the same, there must be n/m balls in each bin. The picture below shows a placement of 10 indistinguishable balls into 5 bins with exactly 2 balls in each bin. Since the number of balls in each bin is determined and the balls are all identical, there is nothing to decide about the placement. Therefore, there is only one way to place the balls.

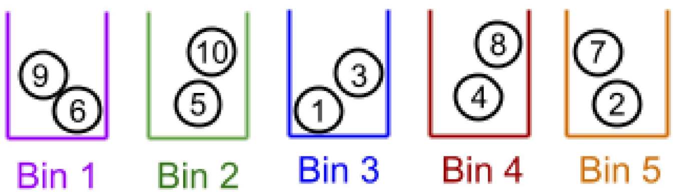


The picture below shows a placement of 10 distinguishable balls into 5 bins with exactly 2 balls in each bin. An assignment can be expressed as a permutation with repetitions: (3, 5, 3, 4, 2, 1, 5, 4, 1, 2). The j^{th} number is the bin in which ball j is placed. Since there are exactly 2 balls in each bin, there are two of each number 1 through 5.

To count the number of placements, first select the two balls that will go in bin 1. Then from the remaining 8 balls, select the 2 that will go in bin 2. Keep selecting two balls for each bin until there are only two remaining balls for the last bin. The total number of ways to select the placement is

$$\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{10!}{2!2!2!2!2!}$$

A placement of distinguishable balls



For n balls and m bins, let x = n/m:

$$\binom{n}{x}\binom{n-x}{x}\binom{n-2x}{x}\cdots\binom{2x}{x}\binom{x}{x} = \frac{n!}{\underbrace{x!x!\cdots x!}_{\text{m times}}} = \frac{n!}{(x!)^m} = \frac{n!}{\left(\left(\frac{n}{m}\right)!\right)^m}$$

PARTICIPATION ACTIVITY 5.10.1: Applying balls in bins formulas in word problems.

1) 10 identical coupons will be given out to 20 different shoppers in a store. There is a limit of at most one coupon per shopper. How many ways are there to distribute the coupons?

- 20^{10}
- P(20, 10)
- $\binom{20}{10}$
- $\binom{29}{19}$

2) 10 identical coupons will be given out to 20 different shoppers in a store. There is no limit on the number of coupons

that can be given to a customer. How many ways are there to distribute the coupons?

- P(20, 10)
- $\binom{29}{9}$
- $\binom{29}{19}$
- $\binom{20}{10}$

3) 8 different tasks are assigned to 20 employees at a company. Each of the 8 tasks goes to exactly one employee. There is no limit to the number of different tasks that can be given to any particular employee. How many ways are there to make the assignments?

- $P(20, 8)$
- 20^8
- $\binom{27}{19}$
- $\binom{20}{8}$

4) 8 different tasks are assigned to 20 employees at a company. Each of the 8 tasks goes to exactly one employee. Each person can be assigned at most one task. How many ways are there to make the assignments?

- $P(20, 8)$
- 20^8

- $\binom{27}{19}$
- $\binom{20}{8}$

5) 15 identical chocolate bars are given out to 5 kids so that each kid gets the same number of chocolate bars. How many ways are there to hand out the chocolate bars to the kids?

- $\frac{15!}{3!3!3!3!3!}$
- $\binom{19}{4}$
- 1

6) A grandfather is giving away all 45 coins in his coin collection to his nine grandchildren. The coins are all different, and he wants to give the same number of coins to each grandchild. How many different ways are there for him to distribute his collection?

- $\frac{45!}{(5!)^9}$
- 1
- 9^{45}
- $\binom{53}{8}$

Additional exercises

EXERCISE 5.10.1: Distributing exams to TAs.

Three TAs are grading a final exam. There are a total of 60 exams to grade.

(a) How many ways are there to distribute the exams among the TAs if all that matters is how many exams go to each TA?

(b) Now suppose it matters which students' exams go to which TAs. How many ways are there to distribute the exams?

(c) Suppose again that we are counting the ways to distribute exams to TAs and it matters which students' exams go to which TAs. The TAs grade at different rates, so the first TA will grade 25 exams, the second TA will grade 20 exams and the third TA will grade 15 exams. How many ways are there to distribute the exams?

EXERCISE 5.10.2: Distributing a coin collection.

A man is distributing his coin collection with 35 coins to his five grandchildren. How many ways are there to distribute the coins if:

(a) The coins are all the same.

(b) The coins are all distinct.

(c) The coins are the same and each grandchild gets the same number of coins

(d) The coins are all distinct and each grandchild gets the same number of coins

EXERCISE 5.10.3: Distributing books to children.

A kindergarten teacher has five books to distribute to 20 children in her class.

(a) How many ways are there for her to distribute the books if they are all the same and no child gets more than one?

(b) How many ways are there for her to distribute the books if they are different and no child gets more than one? If Charlie gets *Green Eggs and Ham* and Amanda gets *The Cat in the Hat*, that is a different distribution from one in which Amanda gets *Green Eggs and Ham* and Charlie gets *The Cat in the Hat*.

(c) How many ways are there for her to distribute the books if they are all the same and there is no restriction on the number of books that can be given to any child?

EXERCISE 5.10.4: Distributing prizes.

Ten prizes are given to a class with 100 students. Each student can receive at most one prize. Alice and Bob are two students in the class.

(a) If the prizes are identical, how many ways are there to distribute the prizes so that either Alice or Bob (or both) receive a prize?

(b) If the prizes are different, how many ways are there to distribute the prizes so that either Alice or Bob (or both) receive a prize?

EXERCISE 5.10.5: Selecting boxed lunches.

A company is placing an order for boxed lunches for a meeting of its executive board. There are 10 members of the board and each will have one lunch. The company is getting the lunches from a restaurant that has 25 varieties of boxed lunches.

(a) How many different ways are there for the company to place the lunch order for the 10 lunches? Note that it is not important who gets what lunch. All that matters is how many of each of the 25 possible varieties are purchased.

(b) How many ways are there for the company to place the lunch order if the 10 lunches purchased are all different? (Again, who gets what lunch is not important, just how many of each lunch are purchased).

EXERCISE 5.10.6: Planning a summer study schedule.

Susan plans on studying four different subjects (Math, Science, French, and Social Studies) over the course of the summer. There are 100 days in her summer break and each day she will study one of the four subjects. A schedule consists of a plan for which subject she will study on each day.

(a) How many ways are there for her to plan her schedule if there are no restrictions on the number of days she studies each of the four subjects?

(b) How many ways are there for her to plan her schedule if she decides that the number of days she studies each subject will be the same?

5.11 Inclusion-exclusion principle

This section has been set as optional by your instructor.

A high school gives free admission to football games to any student who is either a senior or on the honor roll. The school is trying to determine the number of students who will be admitted for free. They know the number of seniors (denoted by the variable s) as well as the number of students on the honor roll (denoted by the variable h). Is this enough information to determine the number of students eligible to attend for free? The sum (s + h) results in over counting because seniors on the honor roll are counted twice, once for being a senior and once for being on the honor roll. The school must also know the number of students who are both seniors and on the honor roll in order to be able to determine the number of people in either group.

Define the set S to be the set of all seniors. Define H to be the set of students on the honor roll. The school would like to count the number of students in the set S ∪ H. The ***principle of inclusion-exclusion*** is a technique for determining the cardinality of the union of sets that uses the cardinality of each individual set as well as the cardinality of their intersections. The animation below illustrates the idea with two sets:

PARTICIPATION ACTIVITY 5.11.1: Inclusion-exclusion principle illustrated with two sets.

Animation captions:

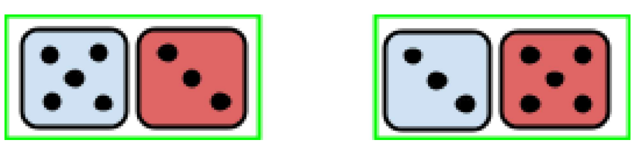
1. The number in each area of the Venn diagram is the number of times the people in that area have been counted. |S|: everyone in S is counted once.
2. Add |H|. Now everyone in (S - H) and (H - S) is counted once and everyone in (S ∩ H) is counted twice.
3. Subtract |S ∩ H|. Everyone in (S ∩ H) is counted (2 - 1 = 1) time.
4. Now, everyone in (S ∪ H) has been counted once. $|S \cup H| = |S| + |H| - |S \cap H|$.

Principle 5.11.1: The inclusion-exclusion principle with two sets.

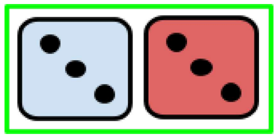
Let A and B be two finite sets, then |A ∪ B| = |A| + |B| - |A ∩ B|

Example 5.11.1: Example of the inclusion-exclusion principle: a roll of two dice.

Suppose that two six-sided dice are thrown. One is colored red and the other is colored blue. An outcome of a roll of the two dice is determined by the number that shows up on the blue die and the number that shows up on the red die. The picture below shows two different outcomes of a roll of the dice:



How many outcomes are there in which at least one of the dice comes up 3? Define B to be the set of outcomes in which the blue die comes up 3. |B| = 6 because after the blue die is determined to be 3 there are still six possibilities for the outcome of the red die. Define R to be the set of outcomes in which the red die comes up 3. Similarly, |R| = 6. There is only one element in B ∩ R, the outcomes in which both the blue and red dice are 3:



Applying the principle of inclusion-exclusion gives that the number outcomes in which at least one of the die comes up 3 is:

$$|B \cup R| = |B| + |R| - |B \cap R| = 6 + 6 - 1 = 11$$

PARTICIPATION ACTIVITY | 5.11.2: Applying the inclusion-exclusion with two sets.

1) In a group, there are 10 women, 8 blondes and 3 blonde women. How many people are either blonde or a woman?

Check Show answer

2) Erica goes swimming three out of the seven days of the week. How many possibilities are there for her swim schedule if she goes swimming on Monday or Tuesday or both? (Define M to be the set of schedules in which Erica goes swimming on Monday. Let T be the set of schedules in which Erica goes swimming on Tuesday. Determine the numerical value of $|M \cup T|$.)

Check Show answer

3) How many positive integers less than 100 have at least one digit that is a 9? (Let T be the set of positive integers less than 100 with a 9 in the ten's place. Let O be the set of positive integers less than 100 with a 9 in the one's place. Now determine the numerical value of $|T \cup O|$).

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Diego Rosenberg
NYUCSBRWinter2023

Check Show answer

The inclusion-exclusion principle with three sets

The inclusion-exclusion principle can be applied to count the number of elements in a union of more than two sets. As the number of sets grows, the expression becomes more complex. The animation below illustrates how to compute the cardinality of the union of three sets:

PARTICIPATION ACTIVITY | 5.11.3: Inclusion-exclusion principle illustrated with three sets.

Animation captions:

1. The number in each area of the Venn diagram is the number of times the people in that area have been counted. $|A|$: everyone in A is counted once.
2. Add $|B|$. Now everyone in $(A - B)$, and $(B - A)$ is counted once and everyone in $(A \cap B)$ is counted twice.
3. Add $|C|$. Now everyone in $(A \cap B - C)$, $(A \cap C - B)$, and $(B \cap C - A)$ is counted twice and everyone in $(A \cap B \cap C)$ is counted three times.
4. Subtract $|A \cap B|$. Everyone in $(A \cap B - C)$ is counted once. Everyone in $(A \cap B \cap C)$ is counted twice.
5. Subtract $|B \cap C|$. Everyone in $(B \cap C - A)$ is counted once. Everyone in $(A \cap B \cap C)$ is counted once.
6. Subtract $|A \cap C|$. Everyone in $(A \cap C - B)$ is counted once. Everyone in $(A \cap B \cap C)$ is counted zero times.
7. Add $|A \cap B \cap C|$. Everyone in $(A \cup B \cup C)$ is counted once. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Principle 5.11.2: Inclusion-exclusion with three sets.

Let A, B and C be three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Example 5.11.2: The inclusion-exclusion with three sets: course enrollments.

A computer science department offers three lower division classes in a given quarter: discrete math, digital logic, and introductory programming. The department would like to know the number of students enrolled in any of the three courses. Let M be the set of students enrolled in discrete math, L be the set of students enrolled in digital logic, and P the set of students enrolled in introductory programming. Here are numbers for enrollments in the courses:

- |M| = 112 students enrolled in discrete math.
- |L| = 138 students enrolled in digital logic.
- |P| = 142 students enrolled in introductory programming.
- |M ∩ L| = 25 students enrolled in both discrete math and digital logic.
- |L ∩ P| = 17 students enrolled in both digital logic and introductory programming.
- |M ∩ P| = 32 students enrolled in both introductory programming and discrete math.
- |M ∩ L ∩ P| =7 students are enrolled in all three classes.

The total number of students enrolled in any of the three courses is:

|M ∪ L ∪ P| = |M|+ |L| + |P| - |M ∩ L| - |L ∩ P| - |M ∩ P| + |M ∩ L ∩ P|

= 112 + 138 + 142 - 25 - 17 - 32 + 7 = 325

PARTICIPATION ACTIVITY | 5.11.4: Inclusion-exclusion principle example.

Animation captions:

1. How many integers from 1 to 30 are divisible by 2, 3, or 5? $A_i =$ the set of integers between 1 and 30 divisible by i. Find $|A_2 \cup A_3 \cup A_5|$.
2. A_2 is the set {2, 4, ... , 30}. $|A_2| = 15$.
3. Add $|A_3|$. There are 30 DIV 3 = 10 numbers between 1 and 30 divisible by 3, so $|A_3| = 10$.
4. Add $|A_5|$. There are 30 DIV 5 = 6 numbers between 1 and 30 divisible by 5, so $|A_5| = 6$.
5. Subtract $|A_2 \cap A_3|$. There are 30 DIV 6 = 5 numbers between 1 and 30 divisible by 2 and 3, so $|A_2 \cap A_3| = 5$.
6. Subtract $|A_5 \cap A_3|$. There are 30 DIV 15 = 2 numbers between 1 and 30 divisible by 5 and 3, so $|A_5 \cap A_3| = 2$.
7. Subtract $|A_5 \cap A_2|$. There are 30 DIV 10 = 3 numbers between 1 and 30 divisible by 5 and 2, so $|A_5 \cap A_2| = 3$.
8. Add $|A_2 \cap A_3 \cap A_5|$. There are 30 DIV 30 = 1 number between 1 and 30 divisible by 2, 3 and 5, so $|A_2 \cap A_3 \cap A_5| = 1$.
9. $|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_5 \cap A_3| - |A_5 \cap A_2| + |A_2 \cap A_3 \cap A_5| = 15 + 10 + 6 - 5 - 2 - 3 + 1 = 22$.

PARTICIPATION ACTIVITY | 5.11.5: Applying the inclusion-exclusion principle with three sets.

1) Three employees of a company, Anna, Fred and Jose, have each worked on 12 projects. Each pair of people have worked on 4 together, including one project that all three have worked on as a team. What is the total number of projects?

Check Show answer

2) How many numbers in the range from 1 through 42 are divisible by 2, 3, or 7?

Check Show answer

The general inclusion-exclusion principle

The inclusion-exclusion principle can be generalized to work for any number of sets. The pattern is the same as with two or three subsets. First, add in the size of each subset individually. Then consider each pair of sets and subtract the size of the intersection of each pair. Then consider every set of three sets and add in the three-way intersection of the triplets of sets. Continue with the pattern until the final term which is the intersection of all the sets. If the number of sets is even, the last term is subtracted. If the number of sets is odd, the last term is added. The general inclusion-principle is stated mathematically below.

Principle 5.11.3: Inclusion-exclusion with an arbitrary number of sets.

Let $A_1, A_2, \ldots, A_n$ be a set of n finite sets.

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{j=1}^{n} |A_j| \\
&- \sum_{1 \le j < k \le n} |A_j \cap A_k| \\
&+ \sum_{1 \le j < k < l \le n} |A_j \cap A_k \cap A_l| \\
\cdots \\
&+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}
$$

Figure 5.11.1: The general inclusion-exclusion principle applied to four sets.

$$
\begin{aligned}
&|A \cup B \cup C \cup D| \\
&= |A| + |B| + |C| + |D| && \text{(add the sizes of the sets)} \\
&- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| && \text{(minus pairwise intersections)} \\
&+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| && \text{(plus 3-way intersections)} \\
&- |A \cap B \cap C \cap D| && \text{(minus 4-way intersection)}
\end{aligned}
$$

PARTICIPATION ACTIVITY 5.11.6: Applying the general inclusion-exclusion principle.

1) Suppose you are using the inclusion-exclusion principle to compute the number of elements in the union of four sets. Each set has 15 elements. The pair-wise intersections have 5 elements each. The three-way intersections have 2 elements each. There is only one element in the intersection of all four sets. What is the size of the union?

Check Show answer

CHALLENGE ACTIVITY 5.11.1: The general inclusion-exclusion principle.

455912.3056722.qx3zqy7

Start

Given two sets: A and B.
A has 6 elements.
B has 7.
A and B share 3 elements.
How many elements are there in total?

Ex: 60

1 2 3

Check Next

The inclusion-exclusion principle and the sum rule

A collection of sets is ***mutually disjoint*** if the intersection of every pair of sets in the collection is empty. If we apply the principle of inclusion-exclusion to determine the union of a collection of mutually disjoint sets, then all the terms with the intersections are zero. Thus, for a collection of mutually disjoint sets, the cardinality of the union of the sets is just equal to the sum of the cardinality of each of the individual sets:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

The equation above is a restatement of the sum rule which only applies when the sets are mutually disjoint.

PARTICIPATION ACTIVITY | 5.11.7: The Inclusion-exclusion principle with mutually disjoint sets.

How many 5-bit strings contain the string "100" as a consecutive substring?

Define the following sets:

- A = set of 5-bit strings of the form 100**
- B = set of 5-bit strings of the form *100*
- C = set of 5-bit strings of the form **100

The *'s can be either 0 or 1.

1) Sets A, B, and C all have the same cardinality. What is the cardinality of each set?

Check Show answer

2) What is $|A \cap B|$?

Check Show answer

3) What is $|A \cap C|$?

Check Show answer

4) What is $|B \cap C|$?

Check Show answer

5) How many 5-bit strings have "100" consecutively as a substring?

Check Show answer

Determining the cardinality of a union by complement

While the inclusion-exclusion principle is a useful tool in many situations, there are also cases in which it is not the most efficient way to determine the cardinality of a union of sets. Consider a situation in which a person has forgotten his 4-digit PIN to access his bank account. A PIN can be any string of 4 digits, e.g. 0032 or 3801. If U is the set of all possible PINs, then $|U| = 10^4$ because each digit can be any one of 10 choices from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. He does remember, however, that at least one of the digits is an 8. How much does the information that at least one of the digits is an 8 help him narrow down his search?

Let P_i be the set of PINs with an 8 in the i^{th} digit. For example, 0872 would be an element of P_2 because the second digit is an 8. The PIN 0882 would be in P_2 and P_3. The goal is to find $|P_1 \cup P_2 \cup P_3 \cup P_4|$. Using inclusion-exclusion would result in a mathematical expression with many terms.

Instead, counting by complement can be used to express the size of the union as:

$$|U| - |P_1 \cup P_2 \cup \cdots \cup P_n| = |\overline{P_1 \cup P_2 \cup \cdots \cup P_n}|$$

The animation below illustrates:

PARTICIPATION ACTIVITY 5.11.8: Finding the cardinality of a union by complement.

Animation content:

undefined

Animation captions:

1. The Universe set U is the set of all 4-digit PINs. $P_i =$ 4-digit PINs with an 8 in place i. For example 0982 $\in P_3$.
2. $P_1 \cup P_2 \cup P_3 \cup P_4 =$ PINs with an 8 in any location.
3. $\overline{P_1 \cup P_2 \cup P_3 \cup P_4} =$ PINs with no 8 in any location.
4. $|P_1 \cup P_2 \cup P_3 \cup P_4| = |U| - |\overline{P_1 \cup P_2 \cup P_3 \cup P_4}| = 10^4 - 9^4$. Thus, 9^4 is the number of length 4 strings from {0, 1, 2, 3, 4, 5, 6, 7, 9} (no 8).

PARTICIPATION ACTIVITY 5.11.9: Calculating the cardinality of a union by complement.

1) How many 4-bit strings have a 1 in at least one of the first two places?

Check Show answer

2) Erica goes swimming three out of the seven days of the week. How many possibilities are there for her swim schedule if she goes swimming on Monday or Tuesday or both?

Check Show answer

CHALLENGE ACTIVITY | 5.11.2: The inclusion-exclusion principle.

455912.3056722.qx3zqy7

Start

Lisa works 3 out of the 7 days of the week. How many possible schedules are there Monday or Wednesday or both?

Ex: 5

1 2 3 4 5

Check Next

Additional exercises

EXERCISE | 5.11.1: Counting strings over {a, b, c}.

Count the number of strings of length 9 over the alphabet {a, b, c} subject to each of the following restrictions.

(a) The first or the last character is a.

(b) The string contains at least 8 consecutive a's.

(c) The string contains at least 8 consecutive identical characters.

(d) The first character is the same as the last character, or the last character is a, or the first character is a.

(e) The string contains at least seven consecutive a's.

(f) The characters in the string "abababa" appear consecutively somewhere in the 9-character string. (So "ccabababa" would be such a 9-character string, but "cababcaba" would not.)

(g) The string has exactly 2 a's or exactly 3 b's.

(h) The string has exactly 2 a's or exactly 2 b's or exactly 2 c's

EXERCISE 5.11.2: Counting binary strings.

Count the number of binary strings of length 10 subject to each of the following restrictions.

(a) The string has at least one 1.

(b) The string has at least one 1 and at least one 0.

(c) The string contains exactly five 1's or it begins with a 0.

EXERCISE 5.11.3: Counting calculus students.

A university offers 3 calculus classes: Math 2A, 2B and 2C. In both parts, you are given data about a group of students who have all taken at least one of the three classes.

(a) Group A contains 157 students. Of these, 51 students in Group A have taken Math 2A, 80 have taken Math 2B, and 70 have taken Math 2C. 15 have taken both Math 2A and 2B, 20 have taken both Math 2A and 2C, and 13 have taken both Math 2B and 2C. How many students in Group A have taken all three classes?

(b) You are given the following data about Group B. 28 students in Group B have taken Math 2A, 28 have taken Math 2B, and 25 have taken Math 2C. 11 have taken both Math 2A and 2B, 9 have taken both Math 2A and 2C, and 10 have taken both Math 2B and 2C. 3 have taken all three classes. How many students are in Group B?

EXERCISE 5.11.4: Counting integer multiples.

(a) How many integers in the range 1 through 120 are integer multiples of 2, 3, or 5?

(b) How many integers in the range 1 through 140 are integer multiples of 2, 5, or 7?

EXERCISE | 5.11.5: Counting ways to line up for a family photo.

A family lines up for a photograph. In each of the following situations, how many ways are there for the family to line up so that the mother is next to at least one of her daughters?

(a) The family consists of two parents, two daughters and two sons.

(b) The family consists of two parents, three daughters and four sons.

EXERCISE | 5.11.6: Counting permutations of 100 numbers.

(a) S = {1, 2, ..., 100}. How many permutations are there of S in which the number 1 is next to at least one even number?

5.12 Counting problem examples

This section has been set as optional by your instructor.

Counting problems may require multiple counting techniques in combination. For example, Cheryl goes to the bakery to buy a dozen donuts. The bakery sells 5 varieties of donuts: plain, chocolate, jelly, maple, and custard. The bakery has a large selection of plain, chocolate, and custard donuts. However, the bakery only has 3 jelly and 5 maple donuts left. How many different selections are possible for the dozen donuts?

PARTICIPATION ACTIVITY | 5.12.1: Counting donut selections with limits on two varieties.

Animation content:

undefined

Animation captions:

1. The number of ways to select 12 donuts from 5 varieties with at most 3 jelly AND at most 5 maple = # selections with no restrictions - # selections ¬(≤ 3 jelly AND ≤ 5 maple).
2. # selections with no restrictions - # selections (¬(≤ 3 jelly) OR ¬(≤ 5 maple)). By De Morgan's Law.
3. # selections with no restrictions - # selections (≥ 4 jelly OR ≥ 6 maple). Not selecting 3 or fewer jelly = selecting 4 or more jelly.
4. # selections with no restrictions - [# selections (≥ 4 jelly) + # selections (≥ 6 maple) - # selections (≥ 4 jelly AND ≥ 6 maple)]. Inclusion/exclusion.

5. $\binom{12+5-1}{5-1} - \left[\binom{8+5-1}{5-1} + \binom{6+5-1}{5-1} - \binom{2+5-1}{5-1}\right]$. In all terms $m = 5$. In the first term $n = 12$. In the second term $n = 12 - 4 = 8$. In the third term $n = 12 - 6 = 6$. In the last term $n = 12 - 4 - 6 = 2$.

Example 5.12.1: Solution to the license plate counting problem.

A license plate number consists of 7 characters that are either digits or capital letters. A witness of a crime briefly sees the license plate of the getaway car. She remembers that the license plate number starts with a digit and has at least two Q's. How many license plate numbers satisfy her description?

Let D be the set of license plate numbers that start with a digit, with no other restrictions on the remaining characters. Apply the principle of counting by complement within the set D by subtracting the number of license plates that have zero or one Q. Define:

- D_0 is the set of license plate numbers that start with a digit and have zero Q's.
- D_1 is the set of license plate numbers that start with a digit and have one Q.

Since the sets D_0 and D_1 are disjoint, the sum rule applies to count the cardinality of their union: $|D_0 \cup D_1| = |D_0| + |D_1|$. The solution to the problem is then:

$$|D| - |D_0 \cup D_1| = |D| - (|D_0| + |D_1|) = |D| - |D_0| - |D_1|$$

The cardinality of D, D_0, and D_1 is determined by:

- $|D| = 10 \cdot 36^6$. There are 10 choices for the first character, which is a digit from 0 to 9. There are 36 choices for each of the remaining six characters, which are digits or capital letters (i.e. 0 to 9 or A to Z).
- $|D_0| = 10 \cdot 35^6$. There are 10 choices for the first character, which is a digit from 0 to 9. There are 35 choices for each of the remaining six characters, which are digits or capital letters other than Q (i.e. 0 to 9, A to P, or R to Z).
- $|D_1| = 10 \cdot 6 \cdot 35^5$. There are 10 choices for the first character, which is a digit from 0 to 9. There are $\binom{6}{1} = 6$ choices for the location of the one Q among the remaining six characters, and 35 choices for each of the remaining five characters, which are digits or capital letters other than Q (i.e. 0 to 9, A to P, or R to Z).

The final solution is:

$$|D| - |D_0| - |D_1| = 10 \cdot 36^6 - 10 \cdot 35^6 - 10 \cdot 6 \cdot 35^5 = 10 \cdot (36^6 - 35^6 - 6 \cdot 35^5) = 233854610$$

PARTICIPATION ACTIVITY 5.12.2: A counting problem using multiple techniques.

A manager of a grocery store purchases 50 crates of soda to stock the store. There are 5 varieties of soda. How many ways are there for her to purchase the 50 crates of soda if she does not order more than 25 of any particular variety?

The following series of questions will lead to the final answer.

1) How many ways are there for the manager to purchase the soda if there are no restrictions on the number of each variety she chooses?

- $\binom{50}{5}$
- $\binom{54}{4}$
- $\binom{54}{49}$

2) The number of ways that the manager can purchase 50 crates of soda with at most 25 chosen from any variety is $\binom{54}{4} - X$. What is the correct description of X?

- The number of ways to purchase soda with at least 25 of some variety chosen.
- The number of ways to purchase soda with at least 26 of some variety chosen.
- The number of ways to purchase soda with at least 26 of every variety chosen.

3) Let V_i be the set of ways to choose 50 crates of soda from 5 varieties with at least 26 chosen of the i^{th} variety. What is the correct expression for X from the previous problem?

- $|V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5|$
- $|V_1 \cap V_2 \cap V_3 \cap V_4 \cap V_5|$

4) $|V_1|$ is the number of ways to choose 50 crates of soda from 5 varieties with at least 26 chosen of the first variety. What is $|V_1|$?

- $\binom{29}{4}$
- $\binom{28}{5}$
- $\binom{28}{4}$

5) What is $|V_1 \cap V_2|$?

- 0
- $\binom{28}{4}$

6) What is $|V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5|$?

- $\binom{28}{4}$
- $5 \cdot \binom{28}{4}$
- $5 \cdot \binom{28}{4} - 10$

7) How many ways are there for the manager to purchase 50 crates of soda from 5 varieties if she does not purchase more than 25 of any particular variety?

- $\binom{54}{4} - 5 \cdot \binom{28}{4}$
- $\binom{54}{5} - 5 \cdot \binom{28}{4}$
- $\binom{54}{5} - \binom{28}{4}$

Additional exercises

EXERCISE 5.12.1: Ordering soda at a grocery store.

An employee of a grocery store is placing an order for soda. There are 8 varieties of soda and they are sold in cases. Each case contains all the same variety. The store will order 50 cases total.

(a) How many ways are there to place the order?

(b) How many ways are there to place the order if she does not order more than 20 of any single variety?

EXERCISE 5.12.2: Counting solutions to integer equations.

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 25$ in which each x_i is a non-negative integer and ...

(a) There are no other restrictions.

(b) $x_i \geq 3$ for $i = 1, 2, 3, 4, 5, 6$

(c) $3 \leq x_1 \leq 10$

(d) $3 \leq x_1 \leq 10$ and $2 \leq x_2 \leq 7$

EXERCISE 5.12.3: More on distributing coupons.

10 coupons are given to 20 shoppers in a store. Each shopper can receive at most one coupon. 5 of the shoppers are women and 15 of the shoppers are men.

(a) If the coupons are identical, how many ways are there to distribute the coupons so that at least one woman receives a coupon?

(b) If the coupons are different, how many ways are there to distribute the coupons so that at least one woman receives a coupon?

(c) If the coupons are identical, how many ways are there to distribute the coupons so that at least one woman does not receive a coupon?

(d) If the coupons are different, how many ways are there to distribute the coupons so that at least one woman does not receive a coupon?

EXERCISE 5.12.4: Selecting a championship soccer team.

Two soccer teams in a youth league tie for first place. The "Breakaways" have 20 players and the "Cyclones" have 18 players. The league must select 12 players to form a new team that will go on to the regional championship.

(a) How many ways are there to select the 12 players from the Cyclone and Breakaway players?

(b) How many ways are there to select the 12 players so that the new team has the same number of players from the Breakaways as from the Cyclones?

(c) How many ways are there to select the 12 players so that none of the players from the Breakaways are chosen?

EXERCISE 5.12.5: Ordering jobs in a printer queue.

There are eight different jobs in a printer queue. Each job has a distinct tag which is a string of three upper case letters. The tags for the eight jobs are:

{ LPW, QKJ, CDP, USU, BBD, PST, LSA, RHR }

(a) How many different ways are there to order the eight jobs in the queue?

(b) How many different ways are there to order the eight jobs in the queue so that job USU comes immediately before CDP?

(c) How many different ways are there to order the eight jobs in the queue so that either QKJ or LPW come last?

(d) How many different ways are there to order the eight jobs in the queue so that QKJ is either last or second-to-last?

(e) How many different ways are there to order the eight jobs in the queue so that job USU comes somewhere before CDP in the queue, although not necessarily immediately before?

(f) How many different ways are there to order the eight jobs in the queue so that job USU comes somewhere before CDP in the queue (although not necessarily immediately before) and CDP comes somewhere before BBD (again, not necessarily immediately before)?

EXERCISE | 5.12.6: Counting strings.

This question concerns strings over the alphabet { a, b, c, d, e }.

(a) How many strings have length 9 and exactly four a's?

(b) How many strings have length 9 and exactly four a's and exactly two b's?

(c) How many strings have length 9 and begin with "ab" or "cab"?

(d) How many strings have length 9 and begin with "ab" or "cab" and have exactly three d's?

(e) How many strings have length 9 and begin with "ab" or "cab" and have exactly three or four d's?

EXERCISE | 5.12.7: Selecting long-haired and short-haired cats.

A family goes to the animal shelter to select three pet cats. The shelter currently has 27 cats. At the shelter there are 10 long-haired cats and 17 short-haired cats.

(a) How many ways are there for the family to select their cats?

(b) How many ways are there for the family to make their selection if they want two long-haired cats and one short-haired cat?

(c) How many ways can they make their selection if they want at least one long-haired cat?

EXERCISE | 5.12.8: Distributing identical homework passes.

A teacher distributes 10 identical homework passes to her class of 20 students. How many ways are there for her to distribute the passes if:

(a) Each student gets at most one homework pass.

(b) There are no restrictions on the number of homework passes a student can get.

(c) There are no restrictions on the number of homework passes a student can get, except that one particular student, Sam, gets at least two homework passes.

(d) There are no restrictions on the number of homework passes a student can get, except that one particular student, Sam, gets at most two homework passes.

EXERCISE | 5.12.9: Distributing donuts to kids.

Two dozen donuts are given out to 24 kids so that each kid gets exactly one donut.

(a) How many ways are there to distribute the donuts if the donuts are all different?

(b) One of the kids is name Antonio. One of the donuts is lemon-filled. How many ways are there to distribute the donuts if the donuts are all different and Antonio must get the lemon-filled donut?

(c) One of the kids is name Antonio and one of the kids is named Rachel. One of the donuts is lemon-filled. How many ways are there to distribute the donuts if the donuts are all different and either Antonio or Rachel must get the lemon-filled donut?

(d) How many ways are there to distribute the donuts if there are four varieties of donuts and exactly six of each variety (and there are no restrictions on who can get which variety)?

EXERCISE | 5.12.10: Counting ternary strings.

A ternary string has characters from the set {0, 1, 2}. For example 122010 and 0011210 are examples of ternary strings.

(a) How many ternary strings are there whose length is in the range 6 through 8?

(b) How many ternary strings of length seven start with a 1 or a 2?

(c) How many ternary strings are there of length 8 with exactly three 1's?

EXERCISE | 5.12.11: Photo line up.

Eight kids line up for a photo. One of the kids is named Felicia.

(a) How many ways are there to line up the eight kids?

(b) Felicia has one best friend named Bob. How many ways are there to line up the eight kids so that Felicia is next to Bob?

(c) Felicia has two best friends named Bob and Hubert. How many ways are there to line up the eight kids so that Felicia is next to Bob and Hubert?

(d) Felicia has three best friends named Bob, Cassandra, and Hubert. How many ways are there to line up the eight kids so that Felicia is next to exactly one of her three best friends?

EXERCISE | 5.12.12: Distributing gifts to a set of shoppers.

A grocery store offers a promotion in which five customers visiting the store on a particular day are each given a gift. No customer can receive more than one gift. 250 customers enter the store on the day of the promotion.

(a) If the gifts are all identical, how many different ways are there to distribute the gifts?

(b) If the gifts are all different from each other, how many different ways are there to distribute the gifts?

EXERCISE | 5.12.13: Selecting an honors choir.

There are 30 singers in a school choir. 20 of the singers are women, and the other 10 are men. The choir director must select 12 singers from the choir to be in the honors choir.

(a) How many ways are there to select the honors choir if there is no restriction on the number of men and women selected?

(b) How many ways are there to select the honors choir if there must be at least one man in the honors choir?

(c) How many ways are there to select the honors choir if there must be the same number of men and women?

EXERCISE | 5.12.14: Counting strings with a fixed number of c's.

(a) How many strings over the alphabet {a, b, c, d, e} of length 13 have exactly four c's?

EXERCISE | 5.12.15: Selecting actors for roles in a play.

(a) 30 girls audition for a play. There are four different female roles in the play. How many possible outcomes are there for the auditions? Note that it is important which girl gets assigned which part.

EXERCISE | 5.12.16: Lining up kids - boys ahead of girls.

(a) Ten kids line up for recess. There are five boys and five girls. How many ways are there for the 10 kids to line up so that all the boys are ahead of all the girls? That is, none of the girls are ahead of any of the boys.

EXERCISE 5.12.17: Counting binary strings with a fixed number of 1's.

(a) How many binary strings of length 12 have six 1's or seven 1's?

EXERCISE 5.12.18: Counting schedules for school lunches.

(a) A chef is deciding on the schedule for the week of lunch specials for her restaurant. She makes 22 different dishes that she can put on the schedule. For each of the seven days of the week, she must select one dish for the lunch special. How many ways are there for her to select the schedule for the week if she does not select the same dish more than once? Note that it matters which dish is served on which day for the schedule of daily specials.

EXERCISE 5.12.19: Counting PINs.

A PIN is a string of four digits. Each of the four digits can be any digit from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} except that the last digit must be even and the second-to-last digit must be odd. (Note that 0 is even.)

(a) How many different choices are there for a PIN if there are no restrictions on the number of times a digit can appear in the PIN?

(b) How many different choices are there for a PIN if the four digits in the PIN must all be different?

EXERCISE 5.12.20: Counting binary strings, cont.

(a) How many binary strings of length 12 do not have exactly four 1's?

(b) How many binary strings of length 12 start with 101 or 1110?

(c) How many binary strings of length 12 start with 101 or 1110 and have exactly four 1's?

(d) How many binary strings of length 12 start with 101 or 1110 and do not have exactly four 1's?

(e) How many binary strings of length 12 start with 00 or end with 00 or both?

EXERCISE 5.12.21: Assigning tasks to processors.

A set of 30 tasks are assigned to a set of 10 processors. The processors are all distinct. Each task is assigned to exactly one processor. A processor can be assigned more than one task.

(a) How many ways are there to assign the tasks if the tasks are all different and there are no restrictions on the number of tasks that can go to any particular processor?

(b) How many ways are there to assign the tasks if the tasks are all identical and there are no restrictions on the number of tasks that can go to any particular processor?

(c) How many ways are there to assign the tasks if the tasks are all different and each processor must receive the same number of tasks?

(d) How many ways are there to assign the tasks if the tasks are all identical and each processor must receive the same number of tasks?

EXERCISE 5.12.22: Cardinality of a power set.

(a) S = {a, b, c, d, e, f, g}. What is |P(S)|?

EXERCISE 5.12.23: Purchasing candy bars.

Felicity goes to a grocery store to purchase 13 candy bars. The store sells 8 varieties of candy bars. One of the varieties is Snickers and one of the varieties is Twix.

(a) How many ways are there for Felicity to make her selection?

(b) Suppose that Felicity would like at least two Snickers bars. Then how many ways are there for Felicity to make her selection?

(c) Suppose that Felicity would like exactly two Snickers bars. Then how many ways are there for Felicity to make her selection?

(d) Suppose that Felicity would like at most three Twix bars. Then how many ways are there for Felicity to make her selection?

(e) Suppose that Felicity would like at most three Twix bars and at least two Snickers bars. Then how many ways are there for Felicity to make her selection?

EXERCISE 5.12.24: Solutions to equations involving a sum of integer valued variables.

(a) How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 37$
where each x_i is a non-negative integer?

(b) How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 37$
where each x_i is an integer that satisfies $x_i \geq 2$?

(c) How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 37$
where each x_i is an integer that satisfies $x_i \geq 2$?

EXERCISE | 5.12.25: Storing widgets in warehouses.

A company has different 10 warehouses for storing their inventory. They will be storing 100 identical crates of widgets in the 10 warehouses.

(a) How many ways are there for the company to distribute the crates of widgets among the warehouses?

(b) How many ways are there for the company to distribute the crates of widgets among the warehouses if at least 5 crates must be stored at each warehouse?

(c) How many ways are there for the company to distribute the crates of widgets among the warehouses if at most 50 crates can be stored at any one warehouse?

(d) How many ways are there for the company to distribute the crates of widgets among the warehouses if at most 40 crates can be stored at any one warehouse?