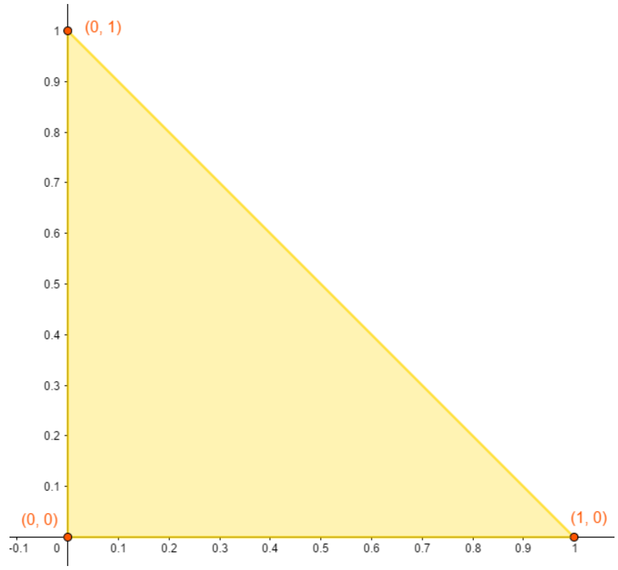
Métodos numéricos Diego Emilio Dominguez Tableros
Hw12

5.24 Given the triangular domain T with vertices $(0,0)$, $(1,0)$, and $(0,1)$, we would like to approximate $\iint f(x,y)\,dx\,dy$ over T.



a) **Derive a quadrature approximation $A_1 f(0,0) + A_2 f(0,1) + A_3 f(0,1)$ where the coefficients are chosen to make the approximation exact for $f = 1, x$ y y.**

b) **Derive a corresponding composite quadrature formula for the subdivision obtained by cutting T into four subtriangles by connecting the midpoints of the edges of T.**

a) Si queremos una fórmula de cuadratura exacta para los casos presentados se efectúan las siguientes integrales.

$$\iint_T 1\,dx\,dy = \int_0^1 \int_0^{1-y} dx\,dy = \int_0^1 (1-y)\,dy = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\iint_T x\,dx\,dy = \int_0^1 \int_0^{1-y} x\,dx\,dy = \frac{1}{2}\int_0^1 (1-y)^2\,dy = \left[-\frac{1}{6}(1-y)^3\right]_0^1 = \frac{1}{6}$$

$$\iint_T x\,dx\,dy = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx = \left[-\frac{1}{6}(1-x)^3\right]_0^1 = \frac{1}{6}$$

Igualando a la fórmula de cuadratura

$$\frac{1}{2} = A_1 f(0{,}0) + A_2 f(1{,}0) + A_3 f(0{,}1) = A_1 + A_2 + A_3$$

$$\frac{1}{6} = A_1 f(0{,}0) + A_2 f(1{,}0) + A_3 f(0{,}1) = A_2$$

$$\frac{1}{6} = A_1 f(0{,}0) + A_2 f(1{,}0) + A_3 f(0{,}1) = A_3$$

Resolvemos el sistema lineal obtenido para A_1, A_2 y A_3

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/6 \\ 1/6 \end{pmatrix}$$

Obteniendo $A_3 = \frac{1}{6}$, $A_2 = \frac{1}{6}$ y $A_1 = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6}$. Por lo tanto, la fórmula de cuadratura es

$$\iint_T f(x,y)\,dx\,dy = \frac{1}{6}\big(f(0{,}0) + f(1{,}0) + f(0{,}1)\big)$$

b) Ahora la región T se ha dividido en cuatro triángulos. Desarrollemos formulas de cuadratura para cada subtriángulo.

a. Triángulo $\left\{(0,1),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$

$$\iint_T 1\,dx\,dy=\int_{\frac{1}{2}}^{1}\int_0^{1-y}dx\,dy=\int_{\frac{1}{2}}^{1}(1-y)\,dy=\left[-\frac{(1-y)^2}{2}\right]_{\frac{1}{2}}^{1}=\frac{1}{8}$$

$$\iint_T x\,dx\,dy=\int_{\frac{1}{2}}^{1}\int_0^{1-y}x\,dx\,dy=\frac{1}{2}\int_{\frac{1}{2}}^{1}(1-y)^2\,dy=\left[-\frac{1}{6}(1-y)^3\right]_{\frac{1}{2}}^{1}=\frac{1}{48}$$

$$\iint_T y\,dx\,dy=\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1-x}y\,dy\,dx=\frac{1}{2}\int_{\frac{1}{2}}^{1}(1-x)^2\,dx=\frac{1}{12}$$

Igualando a la fórmula de cuadratura

$$\frac{1}{8}=A_1f(0,1)+A_2f\left(0,\frac{1}{2}\right)+A_3f\left(\frac{1}{2},\frac{1}{2}\right)=A_1+A_2+A_3$$

$$\frac{1}{48}=A_1f(0,1)+A_2f\left(0,\frac{1}{2}\right)+A_3f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{A_3}{2}$$

$$\frac{1}{48}=A_1f(0,1)+A_2f\left(0,\frac{1}{2}\right)+A_3f\left(\frac{1}{2},\frac{1}{2}\right)=A_1+\frac{A_2}{2}+\frac{A_3}{2}$$

Resolvemos el sistema lineal obtenido para A_1, A_2 y A_3, los resultados son $A_1=A_2=A_3=\frac{1}{24}$. Por lo tanto, la fórmula de cuadratura es

$$\boldsymbol{\iint_T f(x,y)\,dx\,dy=\frac{1}{24}\left(f(0,1)+f\left(0,\frac{1}{2}\right)+f\left(\frac{1}{2},\frac{1}{2}\right)\right)}$$

b. Triángulo $\left\{(0,0),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right)\right\}$

$$\iint_T 1\,dx\,dy=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-y}dx\,dy=\frac{1}{8}$$

$$\iint_T x\,dx\,dy=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-y}x\,dx\,dy=\frac{1}{48}$$

$$\iint_T y\,dx\,dy=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-x}y\,dy\,dx=\frac{1}{48}$$

Igualando a la fórmula de cuadratura

$$\frac{1}{8}=A_1f(0,0)+A_2f\left(0,\frac{1}{2}\right)+A_3f\left(\frac{1}{2},0\right)=A_1+A_2+A_3$$

$$\frac{1}{48}=A_1f(0,0)+A_2f\left(0,\frac{1}{2}\right)+A_3f\left(\frac{1}{2},0\right)=\frac{A_3}{2}$$

$$\frac{1}{48}=A_1f(0,0)+A_2f\left(0,\frac{1}{2}\right)+A_3f\left(\frac{1}{2},0\right)=\frac{A_2}{2}$$

Resolvemos el sistema lineal obtenido para A_1, A_2 y A_3, obteniendo $A_1=A_2=A_3=\frac{1}{24}$. Por lo tanto, la fórmula de cuadratura es

$$\boldsymbol{\iint_T f(x,y)\,dx\,dy=\frac{1}{24}\left(f(0,0)+f\left(0,\frac{1}{2}\right)+f\left(\frac{1}{2},1\right)\right)}$$

c. Triángulo $\left\{\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$

$$\iint_T 1\,dx\,dy = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}-y}^{\frac{1}{2}} dx\,dy = \frac{1}{8}$$

$$\iint_T x\,dx\,dy = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}-y}^{\frac{1}{2}} x\,dx\,dy = \frac{1}{24}$$

$$\iint_T y\,dx\,dy = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}-x}^{\frac{1}{2}} y\,dy\,dx = \frac{1}{24}$$

Igualando a la fórmula de cuadratura

$$\frac{1}{8} = A_1 f\left(0,\frac{1}{2}\right) + A_2 f\left(\frac{1}{2},0\right) + A_3 f\left(\frac{1}{2},\frac{1}{2}\right) = A_1 + A_2 + A_3$$

$$\frac{1}{24} = A_1 f\left(0,\frac{1}{2}\right) + A_2 f\left(\frac{1}{2},0\right) + A_3 f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{A_2}{2} + \frac{A_3}{2}$$

$$\frac{1}{24} = A_1 f\left(0,\frac{1}{2}\right) + A_2 f\left(\frac{1}{2},0\right) + A_3 f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{A_1}{2} + \frac{A_3}{2}$$

Resolvemos el sistema lineal obtenido para A_1, A_2 y A_3, obteniendo $A_1 = A_2 = A_3 = \frac{1}{24}$. Por lo tanto, la fórmula de cuadratura es

$$\boldsymbol{\iint_T f(x,y)\,dx\,dy = \frac{1}{24}\left(f\left(0,\frac{1}{2}\right) + f\left(\frac{1}{2},0\right) + f\left(\frac{1}{2},\frac{1}{2}\right)\right)}$$

d. Triángulo $\left\{(1,0),\left(\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$

$$\iint_T 1\,dx\,dy = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1-y} dx\,dy = \frac{1}{8}$$

$$\iint_T x\,dx\,dy = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1-y} x\,dx\,dy = \frac{1}{12}$$

$$\iint_T y\,dx\,dy = \int_0^{1}\int_{0}^{1-x} y\,dy\,dx = \frac{1}{48}$$

Igualando a la fórmula de cuadratura

$$\frac{1}{8} = A_1 f(1,0) + A_2 f\left(\frac{1}{2},0\right) + A_3 f\left(\frac{1}{2},\frac{1}{2}\right) = A_1 + A_2 + A_3$$

$$\frac{1}{12} = A_1 f(1,0) + A_2 f\left(\frac{1}{2},0\right) + A_3 f\left(\frac{1}{2},\frac{1}{2}\right) = A_1 + \frac{A_2}{2} + \frac{A_3}{2}$$

$$\frac{1}{48} = A_1 f(1,0) + A_2 f\left(\frac{1}{2},0\right) + A_3 f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{A_3}{2}$$

Resolvemos el sistema lineal obtenido para A_1, A_2 y A_3, obteniendo $A_1 = A_2 = A_3 = \frac{1}{24}$. Por lo tanto, la fórmula de cuadratura es

$$\boldsymbol{\iint_T f(x,y)\,dx\,dy = \frac{1}{24}\left(f(1,0) + f\left(\frac{1}{2},0\right) + f\left(\frac{1}{2},\frac{1}{2}\right)\right)}$$

Entonces la regla de cuadratura compuesta se obtiene sumando las cuatro reglas de cuadratura obtenidas para cada uno de los cuatro triángulos:

$$\iint_T f(x,y)\,dx\,dy$$
$$= \frac{1}{24}\left(f(0,1) + f\left(0,\frac{1}{2}\right) + f\left(\frac{1}{2},\frac{1}{2}\right)\right) + \frac{1}{24}\left(f(0,0) + f\left(0,\frac{1}{2}\right) + f\left(\frac{1}{2},0\right)\right)$$
$$+ \frac{1}{24}\left(f\left(\frac{1}{2},0\right) + f\left(0,\frac{1}{2}\right) + f\left(\frac{1}{2},\frac{1}{2}\right)\right) + \frac{1}{24}\left(f\left(\frac{1}{2},0\right) + f(1,0) + f\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

$$\boldsymbol{\iint_T f(x,y)\,dx\,dy = \frac{1}{24}(f(0,1) + f(0,0) + f(1,0)) + \frac{3}{24}\left(f\left(0,\frac{1}{2}\right) + f\left(\frac{1}{2},\frac{1}{2}\right) + f\left(\frac{1}{2},0\right)\right)}$$