The computation aspects of the equivalent-layer technique: review and perspective

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1 FUNDAMENTALS

2 Let $\mathbf{d}$ be a $D \times 1$ vector, whose i-th element d_i is the observed potential field at the position (x_i, y_i, z_i),
3 $i \in \{1 : D\}$. Consider that d_i can be satisfactorily approximated by a harmonic function

$$f_i = \sum_{j=1}^{P} g_{ij}\, p_j\,, \quad i \in \{1 : D\}\,, \tag{1}$$

4 where, p_j represents the scalar physical property of a virtual source (i.e., monopole, dipole, prism) located
5 at (x_j, y_j, z_j), $j \in \{1 : P\}$ and

$$g_{ij} \equiv g(x_i - x_j, y_i - y_j, z_i - z_j)\,, \quad z_i < \min\{z_j\}\,, \quad \forall i \in \{1 : D\}\,, \tag{2}$$

6 is a harmonic function, where $\min\{z_j\}$ denotes the minimum z_j, or the vertical coordinate of the shallowest
7 virtual source. These virtual sources are called *equivalent sources* and they form an *equivalent layer*. In
8 matrix notation, the potential field produced by all equivalent sources at all points (x_i, y_i, z_i), $i \in \{1 : D\}$,
9 is given by:

$$\mathbf{f} = \mathbf{G}\mathbf{p}\,, \tag{3}$$

10 where $\mathbf{p}$ is a $P \times 1$ vector with j-th element p_j representing the scalar physical property of the j-th
11 equivalent source and $\mathbf{G}$ is a $D \times P$ matrix with element g_{ij} given by equation 2.

12 The equivalent-layer technique consists in solving a linear inverse problem to determine a parameter
13 vector $\mathbf{p}$ leading to a predicted data vector $\mathbf{f}$ (equation 3) *sufficiently close to* the observed data vector $\mathbf{d}$,
14 whose i-th element d_i is the observed potential field at (x_i, y_i, z_i). The notion of *closeness* is intrinsically
15 related to the concept of *vector norm* (e.g., Golub and Van Loan, 2013, p. 68) or *measure of length* (e.g.,
16 Menke, 2018, p. 41). Because of that, almost all methods for determining $\mathbf{p}$ actually estimate a parameter
17 vector $\tilde{\mathbf{p}}$ minimizing a length measure of the difference between $\mathbf{f}$ and $\mathbf{d}$ (see subsection 1.3). Given an
18 estimate $\tilde{\mathbf{p}}$, it is then possible to compute a potential field transformation

$$\mathbf{t} = \mathbf{A}\tilde{\mathbf{p}}\,, \tag{4}$$

19 where $\mathbf{t}$ is a $T \times 1$ vector with k-th element t_k representing the transformed potential field at the position
20 (x_k, y_y, z_k), $k \in \{1 : T\}$, and

$$a_{kj} \equiv a(x_k - x_j, y_k - y_j, z_k - z_j)\,, \quad z_k < \min\{z_j\}\,, \quad \forall k \in \{1 : T\}\,, \tag{5}$$

21 is a harmonic function representing the kj-th element of the $T \times P$ matrix $\mathbf{A}$.

1.1 Spatial distribution and total number of equivalent sources

23 There is no well-established criteria to define the optimum number P or the spatial distribution of the
24 equivalent sources. We know that setting an equivalent layer with more (less) sources than potential-field
25 data usually leads to an underdetermined (overdetermined) inverse problem (e.g., Menke, 2018, p. 52–53).
26 Concerning the spatial distribution of the equivalent sources, the only condition is that they must rely on a
27 surface that is located below and does not cross that containing the potential field data. Soler and Ueda
28 (2021) present a practical discussion about this topic.

29 From a theoretical point of view, the equivalent layer reproducing a given potential field data set cannot
30 cross the true gravity or magnetic sources. This condition is a consequence of recognizing that the equivalent
31 layer is essentially an indirect solution of a boundary value problem of potential theory (e.g., Roy, 1962;
32 Zidarov, 1965; Dampney, 1969; Li et al., 2014; Reis et al., 2020). In practical applications, however, there
33 is no guarantee that this condition is satisfied. Actually, its is widely known from practical experience (e.g.,
34 Gonzalez et al., 2022) that the equivalent-layer technique works even for the case in which the layer cross
35 the true sources.

36 CRITÉRIOS PARA DEFINIR A PROFUNDIDADE DA CAMADA: DAMPNEY (ESPAÇAMENTO
37 DO GRID) E REIS (ESPAÇAMENTO DAS LINHAS)

1.2 **Matrix** G

39 Generally, the harmonic function g_{ij} (equation 2) is defined in terms of the inverse distance between the
40 observation point (x_i, y_i, z_i) and the j-th equivalent source at (x_j, y_j, z_j),

$$\frac{1}{r_{ij}} \equiv \frac{1}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \, , \tag{6}$$

41 or by its partial derivatives of first and second orders, respectively given by

$$\partial_\alpha \frac{1}{r_{ij}} \equiv \frac{-(\alpha_i - \alpha_j)}{r_{ij}^3} \, , \quad \alpha \in \{x, y, z\} \, , \tag{7}$$

42 and

$$\partial_{\alpha\beta} \frac{1}{r_{ij}} \equiv \begin{cases} \frac{3(\alpha_i - \alpha_j)^2}{r_{ij}^5} \, , & \alpha = \beta \, , \\ \frac{3(\alpha_i - \alpha_j)(\beta_i - \beta_j)}{r_{ij}^5} - \frac{1}{r_{ij}^3} \, , & \alpha \neq \beta \, , \end{cases} \quad \alpha, \beta \in \{x, y, z\} \, . \tag{8}$$

43 In this case, the equivalent layer is formed by punctual sources representing monopoles or dipoles (e.g.,
44 Dampney, 1969; Emilia, 1973; Leão and Silva, 1989; Cordell, 1992; Oliveira Jr. et al., 2013; Siqueira et al.,
45 2017; Reis et al., 2020; Takahashi et al., 2020; Soler and Uieda, 2021; Takahashi et al., 2022). Another
46 common approach consists in not defining g_{ij} by using equations 6–8, but other harmonic functions
47 obtained by integrating them over the volume of regular prisms (e.g., Li and Oldenburg, 2010; Barnes and
48 Lumley, 2011; Li et al., 2014; Jirigalatu and Ebbing, 2019). There are also some less common approaches
49 defining the harmonic function g_{ij} (equation 2) as the potential field due to plane faces with constant
50 physical property (Hansen and Miyazaki, 1984), doublets (Silva, 1986) or by computing the double
51 integration of the inverse distance function with respect to z (Guspí and Novara, 2009).

52 A common assumption for most of the equivalent-layer methods is that the harmonic function g_{ij}
53 (equation 2) is independent on the actual physical relationship between the observed potential field and
54 their true sources (e.g., Cordell, 1992; Guspí and Novara, 2009; Li et al., 2014). Hence, g_{ij} can be
55 defined according to the problem. The only condition imposed to this function is that it decays to zero
56 as the observation point (x_i, y_i, z_i) goes away from the position (x_j, y_j, z_j) of the j-th equivalent source.
57 However, several methods use a function g_{ij} that preserves the physical relationship between the observed
58 potential field and their true sources. For the case in which the observed potential field is gravity data, g_{ij}
59 is commonly defined as a component of the gravitational field produced at (x_i, y_i, z_i) by a point mass or
60 prism located at (x_j, y_j, z_j), with unit density. On the other hand, g_{ij} is commonly defined as a component

61 of the magnetic induction field produced at (x_i, y_i, z_i) by a dipole or prism located at (x_j, y_j, z_j), with unit
62 magnetization intensity, when the observed potential field is magnetic data.

63   For all harmonic functions discussed above, the sensitivity matrix $\mathbf{G}$ (equation 3) is always dense. For
64 scattered potential-field data, $\mathbf{G}$ does not have a well-defined structure, regardless of whether the spatial
65 distribution of the equivalent sources is set. Nevertheless, for the particular case in which (i) there is a
66 single equivalent source right below each potential-field datum and (ii) both data and sources rely on
67 planar and regularly spaced grids, Takahashi et al. (2020, 2022) show that $\mathbf{G}$ assumes a block-Toeplitz
68 Toeplitz-block (BTTB) structure. In this case, the product of $\mathbf{G}$ and an arbitrary vector can be efficiently
69 computed via 2D fast Fourier transform as a discrete convolution.

1.3 General formulation

71   A general formulation for almost all equivalent-layer methods can be achieved by first considering that
72 the $P \times 1$ parameter vector $\mathbf{p}$ (equation 3) can be reparameterized into a $Q \times 1$ vector $\mathbf{q}$ according to:

$$\mathbf{p} = \mathbf{H}\,\mathbf{q}\,, \tag{9}$$

73 where $\mathbf{H}$ is a $P \times Q$ matrix. The predicted data vector $\mathbf{f}$ (equation 3) can then be rewritten as follows:

$$\mathbf{f} = \mathbf{G}\,\mathbf{H}\,\mathbf{q}\,. \tag{10}$$

74   Then, the problem of estimating a parameter vector $\tilde{\mathbf{p}}$ minimizing a length measure of the difference
75 between $\mathbf{f}$ (equation 3) and $\mathbf{d}$ is replaced by that of estimating an auxiliary vector $\tilde{\mathbf{q}}$ minimizing the goal
76 function

$$\Gamma(\mathbf{q}) = \Phi(\mathbf{q}) + \mu\,\Theta(\mathbf{q})\,, \tag{11}$$

77 which is a combination of particular measures of length given by

$$\Phi(\mathbf{q}) = (\mathbf{d} - \mathbf{f})^\top \mathbf{W}_d\,(\mathbf{d} - \mathbf{f})\,, \tag{12}$$

78 and

$$\Theta(\mathbf{q}) = (\mathbf{q} - \bar{\mathbf{q}})^\top \mathbf{W}_q\,(\mathbf{q} - \bar{\mathbf{q}})\,, \tag{13}$$

79 where μ is a positive scalar controlling the trade-off between $\Phi(\mathbf{q})$ and $\Theta(\mathbf{q})$; $\mathbf{W}_q$ is a $Q \times Q$ symmetric
80 matrix imposing prior information on $\mathbf{q}$ given by

$$\mathbf{W}_q = \mathbf{H}^\top \mathbf{W}_p\,\mathbf{H}\,, \tag{14}$$

81 with $\mathbf{W}_p$ being a $P \times P$ symmetric matrix imposing prior information on $\mathbf{p}$; $\bar{\mathbf{q}}$ is a $Q \times 1$ vector of reference
82 values for $\mathbf{q}$ satisfying

$$\bar{\mathbf{p}} = \mathbf{H}\,\bar{\mathbf{q}}\,, \tag{15}$$

83 with $\bar{\mathbf{p}}$ being a $P \times 1$ vector containing reference values for the original parameter vector $\mathbf{p}$; and $\mathbf{W}_d$ is a
84 $D \times D$ symmetric matrix defining the relative importance of each observed datum d_i. After obtaining an
85 estimate $\tilde{\mathbf{q}}$ for the reparameterized parameter vector $\mathbf{q}$ (equation 9) minimizing $\Gamma(\mathbf{q})$ (equation 11), the
86 estimate $\tilde{\mathbf{p}}$ for the original parameter vector (equation 3) is computed by

$$\tilde{\mathbf{p}} = \mathbf{H}\,\tilde{\mathbf{q}}\,. \tag{16}$$

87   The reparameterized vector $\tilde{\mathbf{q}}$ is obtained by first computing the gradient of $\Gamma(\mathbf{q})$,

$$\boldsymbol{\nabla}\Gamma(\mathbf{q}) = -2\,\mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{W}_d\,(\mathbf{d}-\mathbf{f}) + 2\,\mu\,\mathbf{W}_q\,(\mathbf{q}-\bar{\mathbf{q}})\;. \tag{17}$$

88   Then, by considering that $\boldsymbol{\nabla}\Gamma(\tilde{\mathbf{q}}) = \mathbf{0}$ (equation 17), where $\mathbf{0}$ is a vector of zeros, as well as adding and
89   subtracting the term $\left(\mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{W}_d\mathbf{G}\,\mathbf{H}\right)\bar{\mathbf{q}}$, we obtain

$$\tilde{\boldsymbol{\delta}}_q = \mathbf{B}\,\tilde{\boldsymbol{\delta}}_d\,, \tag{18}$$

90   where

$$\tilde{\boldsymbol{\delta}}_q = \tilde{\mathbf{q}} - \bar{\mathbf{q}}\,, \tag{19}$$

91

$$\tilde{\boldsymbol{\delta}}_d = \mathbf{d} - \mathbf{G}\,\mathbf{H}\,\bar{\mathbf{q}}\,, \tag{20}$$

92

$$\mathbf{B} = \left(\mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{W}_d\,\mathbf{G}\,\mathbf{H} + \mu\,\mathbf{W}_q\right)^{-1}\mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{W}_d\,, \tag{21}$$

93   or, equivalently (Menke, 2018, p. 62),

$$\mathbf{B} = \mathbf{W}_q^{-1}\,\mathbf{H}^{\top}\mathbf{G}^{\top}\left(\mathbf{G}\,\mathbf{H}\,\mathbf{W}_q^{-1}\,\mathbf{H}^{\top}\mathbf{G}^{\top} + \mu\mathbf{W}_d^{-1}\right)^{-1}\;. \tag{22}$$

94   Evidently, we have considered that all inverses exist in equations 21 and 22.

95   Matrix $\mathbf{B}$ defined by equation 21 is commonly used for the cases in which $D > P$, i.e., when there are
96   more data than parameters (overdetermined problems). In this case, we consider that the estimate $\tilde{\mathbf{q}}$ is
97   obtained by solving the following linear system for $\tilde{\boldsymbol{\delta}}_q$ (equation 19):

$$\left(\mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{W}_d\,\mathbf{G}\,\mathbf{H} + \mu\,\mathbf{W}_q\right)\tilde{\boldsymbol{\delta}}_q = \mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{W}_d\,\tilde{\boldsymbol{\delta}}_d\,. \tag{23}$$

98   On the other hand, for the cases in which $D < P$ (underdetermined problems), matrix $\mathbf{B}$ is usually defined
99   according to equation 22. In this case, we consider that the the estimate $\tilde{\mathbf{q}}$ is obtained in two steps, which
100   consists in first solving a linear system for a dummy vector $\mathbf{u}$ and then computing a matrix-vector product
101   as follows:

$$\begin{aligned}\left(\mathbf{G}\,\mathbf{H}\,\mathbf{W}_q^{-1}\,\mathbf{H}^{\top}\mathbf{G}^{\top} + \mu\,\mathbf{W}_d^{-1}\right)\mathbf{u} &= \tilde{\boldsymbol{\delta}}_d \\ \tilde{\boldsymbol{\delta}}_q &= \mathbf{W}_q^{-1}\mathbf{H}^{\top}\mathbf{G}^{\top}\mathbf{u}\end{aligned}\,. \tag{24}$$

102   After obtaining $\tilde{\boldsymbol{\delta}}_q$ (equations 23 and 24), the estimate $\tilde{\mathbf{q}}$ is computed with equation 19.

2   COMPUTATIONAL STRATEGIES

103   Two important factors affecting the efficiency of a given matrix algorithm are the storage and amount of
104   required arithmetic. Here, we quantify this last factor by counting flops. A flop is a floating point addition,
105   subtraction, multiplication or division (Golub and Van Loan, 2013, p. 12–14).

106   NÃO SEI SE TÁ BOM AQUI: To investigate the efficiency of equivalent-layer methods, we consider
107   how they (i) set up and (ii) solve the linear inverse problem to estimate the physical property distribution
108   on the equivalent layer, as well as (iii) perform potential field transformations (equation 4).

109   We focus on the overall strategies used by the selected methods

110 **2.1 Notation**

111   Here, we use a notation inspired on that presented by (Van Loan, 1992, p. 4).

112   Subvectors of $\mathbf{d}$, for example, are specified by $\mathbf{d}[\mathbf{i}]$, where $\mathbf{i}$ is a list of integer numbers that "pick
113 out" the elements of $\mathbf{d}$ forming the subvector $\mathbf{d}[\mathbf{i}]$. For example, $\mathbf{i} = (1, 6, 4, 6)$ gives the subvector
114 $\mathbf{d}[\mathbf{i}] = [d_1\ d_6\ d_4\ d_6]^\top$. Note that the list $\mathbf{i}$ of indices may be sorted or not and it may also have repeated
115 indices.

We may define regular lists of indices by using the colon notation. For example,

$$\mathbf{i} = (3 : 8) \Leftrightarrow \mathbf{d}[3 : 8] = [d_3\ d_4\ \ldots\ d_8]^\top$$
$$\mathbf{i} = (: 8) \Leftrightarrow \mathbf{d}[: 8] = [d_1\ d_2\ \ldots\ d_7]^\top \quad ,$$
$$\mathbf{i} = (3 :) \Leftrightarrow \mathbf{d}[3 :] = [d_3\ d_4\ \ldots\ d_D]^\top$$

116 where D is the number of elements forming $\mathbf{d}$.

The notation above can also be used to define submatrices. For example, $\mathbf{i} = (2, 7, 4, 6)$ and $\mathbf{j} = (1, 3, 8)$
lead to the submatrix

$$\mathbf{G}[\mathbf{i}, \mathbf{j}] = \begin{bmatrix} g_{21} & g_{23} & g_{28} \\ g_{71} & g_{73} & g_{78} \\ g_{41} & g_{43} & g_{48} \\ g_{61} & g_{63} & g_{68} \end{bmatrix} .$$

The i-th row and the j-th column of $\mathbf{G}$ can be defined respectively by $\mathbf{G}[i, :]$ and $\mathbf{G}[:, j]$. Finally, we may
use the colon notation to define the following submatrix of $\mathbf{G}$:

$$\mathbf{G}[2 : 5, 3 : 7] = \begin{bmatrix} g_{23} & g_{24} & g_{25} & g_{26} & g_{27} \\ g_{33} & g_{34} & g_{35} & g_{36} & g_{37} \\ g_{43} & g_{44} & g_{45} & g_{46} & g_{47} \\ g_{53} & g_{54} & g_{55} & g_{56} & g_{57} \end{bmatrix} .$$

117 **2.2 Moving windows**

118   Split the observed data d_i, $i \in \{1 : D\}$, into M overlapping subsets (or data windows) formed by D^m
119 data each, $m \in \{1 : M\}$.

120   The data inside the m-th window have indices defined by an integer list $\mathbf{i}^m$ having D^m elements.

121   The number of data D^m forming the data windows are not necessarily equal to each other.

122   The data forming a given window are usually adjacent to each other.

123   Each data window has a $D^m \times 1$ observed data vector $\mathbf{d}^m \equiv \mathbf{d}[\mathbf{i}^m]$.

124   Define a set of P equivalent sources with scalar physical property p_j, $j \in \{1 : P\}$.

125   Split the equivalent sources into M overlapping subsets (or source windows) formed by P^m data each,
126 $m \in \{1 : M\}$.

127   The sources inside the m-th window have indices defined by an integer list $\mathbf{j}^m$ having P^m elements.

128 Each source window has a $P^m \times 1$ parameter vector $\mathbf{p}^m \equiv \mathbf{p}[\mathbf{j}^m]$.

129 The m-th source window is located right below the corresponding m-th data window.

130 Let each $\mathbf{d}^m$ be approximated by

$$\mathbf{f}^m = \mathbf{G}^m \mathbf{p}^m \,, \tag{25}$$

131 where $\mathbf{G}^m \equiv \mathbf{G}[\mathbf{i}^m, \mathbf{j}^m]$ is a submatrix of $\mathbf{G}$ (equation 3) formed by the elements computed with equation
132 2 by considering only the data and equivalent sources located inside the window m-th.

133 The main advantage of this approach is that the estimated parameter vector $\tilde{\mathbf{p}}$ is not obtained by solving
134 the full linear system, but several smaller ones.

135 Leão and Silva (1989) presented a pioneer work using this approach.

136 Their method requires a regularly-spaced grid of observed data on a horizontal plane z_0.

137 The data windows are defined by square local grids of $\sqrt{D'} \times \sqrt{D'}$ adjacent points, all of them having
138 the same number of points D'.

139 The equivalent sources in the m-th data window are located below the observation plane, at a constant
140 vertical distance Δz_0. They are arranged on a regular grid of $\sqrt{P'} \times \sqrt{P'}$ adjacent points following the
141 same grid pattern of the observed data. The local grid of sources for all data windows have the same
142 number of elements P'. Besides, they are vertically aligned, but expands the limits of their corresponding
143 data windows, so that $D' < P'$.

144 Because of this spatial configuration of observed data and equivalent sources, we have that $\mathbf{G}^m = \mathbf{G}'$
145 (equation 25) for all data windows (i.e., $\forall\, m \in \{1 : M\}$), where $\mathbf{G}'$ is a $D' \times P'$ constant matrix.

146 By omitting the normalization strategy used by Leão and Silva (1989), their method consists in directly
147 computing the transformed potential field t_c^m at the central point $(x_c^m, y_c^m, z_0 + \Delta z_0)$ of each data window
148 as follows:

$$t_c^m = \left(\mathbf{G}'\mathbf{a}'\right)^\top \left[\mathbf{G}'\left(\mathbf{G}'\right)^\top + \mu\,\mathbf{I}_{D'}\right]^{-1} \mathbf{d}^m \,, \quad m \in \{1 : M\} \,, \tag{26}$$

149 where $\mathbf{I}_{D'}$ is the identity matrix of order D' and $\mathbf{a}'$ is a $P' \times 1$ vector with elements computed by equation
150 5 by using all equivalent sources in the m-th subset and only the coordinate of the central point in the
151 m-th data window. Due to the presumed spatial configuration of the observed data and equivalent sources,
152 $\mathbf{a}'$ is the same for all data windows. Note that equation 26 combines the potential-field transformation
153 (equation 4) with the solution of the undetermined problem (equation 24) for the particular case in which
154 $\mathbf{H} = \mathbf{W}_p = \mathbf{I}_{P'}$ (equations 9 and 14), $\mathbf{W}_d = \mathbf{I}_{D'}$ (equation 12), $\bar{p} = \mathbf{0}$ (equation 15), where $\mathbf{I}_{P'}$ and $\mathbf{I}_{D'}$
155 are identity matrices of order P' and D', respectively, and $\mathbf{0}$ is a vector of zeros.

156 The method proposed by Leão and Silva (1989) can be outlined by the Algorithm 1. Note that Leão
157 and Silva (1989) directly compute the transformed potential t_c^m at the central point of each data window
158 without explicitly computing and storing $\mathbf{p}^m$ (equation 25). It means that their method allows computing
159 a single potential-field transformation. A different transformation or the same one evaluated at different
160 points require running their moving-data window method again.

161 Soler and Uieda (2021) generalized the method proposed by Leão and Silva (1989) for irregularly spaced
162 data on an undulating surface.

163 A direct consequence of this generalization is that a different matrix $\mathbf{G}^m$ (equation 25) must be computed
164 for each window.

Algorithm 1: Generic pseudo-code for the method proposed by Leão and Silva (1989).

Initialization :
1 Set the indices $\mathbf{i}^m$ for each data window, $m \in \{1 : M\}$;
2 Set the indices $\mathbf{j}^m$ for each source window, $m \in \{1 : M\}$;
3 Set the constant depth $z_0 + \Delta z_0$ for all equivalent sources ;
4 Compute the vector $\mathbf{a}'$ associated with the desired potential-field transformation ;
5 Compute the matrix $\mathbf{G}'$;
6 Compute $(\mathbf{G}'\mathbf{a}')^\top \left[\mathbf{G}' (\mathbf{G}')^\top + \mu \, \mathbf{I}_{D'} \right]^{-1}$;
7 $m = 1$;
8 **while** $m < M$ **do**
9 $\quad$ Compute t_c^m (equation 26) ;
10 $\quad$ $m \leftarrow m + 1$;
11 **end**

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165 Differently from Leão and Silva (1989), Soler and Uieda (2021) store the computed $\tilde{\mathbf{p}}^m$ for all windows
166 and subsequently use them to compute a desired potential-field transformation (equation 4) as the
167 superposed effect of all windows.

168 The computed $\tilde{\mathbf{p}}^m$ for all windows are combined to form a single $P \times 1$ vector $\tilde{\mathbf{p}}$, which is an estimate
169 for original parameter vector $\mathbf{p}$ (equation 3).

170 For each data window, Soler and Uieda (2021) solve a overdetermined problem (equation 23) for $\tilde{\mathbf{p}}^m$ by
171 using $\mathbf{H} = \mathbf{W}_p = \mathbf{I}_{P^m}$ (equations 9 and 14), $\mathbf{W}_d^m$ (equation 12) equal to a diagonal matrix of weights for
172 the data inside the m-th window and $\bar{p} = \mathbf{0}$ (equation 15), so that

$$\left[(\mathbf{G}^m)^\top \mathbf{W}_d^m \mathbf{G}^m + \mu \, \mathbf{I}_{P'} \right] \tilde{\mathbf{p}}^m = (\mathbf{G}^m)^\top \mathbf{W}_d^m \mathbf{d}^m . \tag{27}$$

173 The overall steps of their method are defined by the Algorithm 2. Note that Algorithm 2 starts with a
174 residuals vector $\mathbf{r}$ that is iteratively updated. At each iteration, the potential field predicted a source window
175 is computed at all observation points and removed from the residuals vector $\mathbf{r}$.

Algorithm 2: Generic pseudo-code for the method proposed by Soler and Uieda (2021).

Initialization :
1 Set the indices $\mathbf{i}^m$ for each data window, $m \in \{1 : M\}$;
2 Set the indices $\mathbf{j}^m$ for each source window, $m \in \{1 : M\}$;
3 Set the depth of all equivalent sources ;
4 Set a $D \times 1$ residuals vector $\mathbf{r} = \mathbf{d}$;
5 Set a $P \times 1$ vector $\tilde{\mathbf{p}} = \mathbf{0}$;
6 $m = 1$;
7 **while** $m < M$ **do**
8 $\quad$ Set the matrix $\mathbf{W}_d^m$;
9 $\quad$ Compute the matrix $\mathbf{G}^m$;
10 $\quad$ Compute $\tilde{\mathbf{p}}^m$ (equation 27) ;
11 $\quad$ $\tilde{\mathbf{p}}[\mathbf{j}^m] \leftarrow \tilde{\mathbf{p}}[\mathbf{j}^m] + \tilde{\mathbf{p}}^m$;
12 $\quad$ $\mathbf{r} \leftarrow \mathbf{r} - \mathbf{G}[:, \mathbf{j}^m] \, \tilde{\mathbf{p}}^m$;
13 $\quad$ $m \leftarrow m + 1$;
14 **end**

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176   PAREI AQUI - APARENTEMENTE, A COLON NOTATION VAI SER UMA MÃO NA RODA

177   **2.3   Column update**

178   Cordell (1992)

179   Guspí and Novara (2009)

180   **2.4   Row update**

181   Algebraic reconstruction techniques (ART) van der Sluis and van der Vorst (2004)

182   Mendonça and Silva (1994)

183   **2.5   Reparameterization**

184   Barnes and Lumley (2011)

185   Oliveira Jr. et al. (2013)

186   Mendonça (2020)

187   **2.6   Wavelet compression**

188   Li and Oldenburg (2010)

189   **2.7   Iterative methods using the original $\mathbf{G}$**

190   Xia and Sprowl (1991)

191   Xia et al. (1993)

192   Siqueira et al. (2017)

193   Jirigalatu and Ebbing (2019)

194   **2.8   Discrete convolution**

195   Takahashi et al. (2020)

196   Takahashi et al. (2022)

3   TEXTO ANTIGO

197 Leão and Silva (1989) reduced the total processing time and memory usage of equivalent-layer technique
198 by means of a moving data-window scheme. A small moving data window with N_w observations and
199 a small equivalent layer with M_w equivalent sources ($M_w > N_w$) located below the observations are
200 established. For each position of a moving-data window, Leão and Silva (1989) estimate a stable solution
201 $\mathbf{p_w}^*$ by using a data-space approach with the zeroth-order Tikhonov regularization (**?**), i.e.,

$$\left(\mathbf{A_w} \mathbf{A_w}^\top + \mu \mathbf{I} \right) \mathbf{w} \;=\; \mathbf{d_w}^o , \tag{28a}$$

$$\mathbf{A_w}^\top \mathbf{w} \;=\; \mathbf{p_w}^* , \tag{28b}$$

202 where $\mathbf{w}$ is a dummy vector, μ is a regularizing parameter, $\mathbf{d_w}^o$ is an N_w-dimensional vector containing
203 the observed potential-field data, $\mathbf{A_w}$ is an $N_w \times M_w$ sensitivity matrix related to a moving-data window, $\mathbf{I}$

204 is an identity matrix of order N_w and the superscript $\top$ stands for a transpose. After estimating an $M_w \times 1$
205 parameter vector $\mathbf{p_w}^*$ (equation 28b) the desired transformation of the data is only calculated at the central
206 point of each moving-data window, i.e.:

$$\hat{t}_k = \mathbf{t}_k^\top \, \mathbf{p_w}^* , \tag{29}$$

207 where $\hat{t}_k$ is the transformed data calculated at the central point k of the data window and $\mathbf{t}_k$ is an $M_w \times 1$
208 vector whose elements form the kth row of the $N_w \times N_w$ matrix of Green's functions $\mathbf{T}$ (equation **??**) of
209 the desired linear transformation of the data.

210   By shifting the moving-data window with a shift size of one data spacing, a new position of a data
211 window is set up. Next, the aforementioned process (equations 28b and 29) is repeated for each position of
212 a moving-data window, until the entire data have been processed. Hence, instead of solving a large inverse
213 problem, Leão and Silva (1989) solve several much smaller ones.

214   To reduce the size of the linear system to be solved, Soler and Uieda (2021) adopted the same strategy
215 proposed, originally, by Leão and Silva (1989) of using a small moving-data window sweeping the whole
216 data. In Leão and Silva (1989), a moving-data window slides to the next adjacent data window following a
217 sequential movement, the predicted data is calculated inside the data window and the desired transformation
218 are only calculated at the center of the moving-data window. Unlike Leão and Silva (1989), Soler and
219 Uieda (2021) do not adopt a sequential order of the data windows; rather, they adopt a randomized order of
220 windows in the iterations of the gradient-boosting algorithm (**?**, **?** and **?**). The gradient-boosting algorithm
221 in Soler and Uieda (2021) estimates a stable solution using the data and the equivalent sources that fall
222 within a moving-data window; however, it calculates the predicted data and the residual data in the whole
223 survey data. Next, the residual data that fall within a new position of the data window is used as input
224 data to estimate a new stable solution within the data window which in turn is used to calculated a new
225 predicted data and a new residual data in the whole survey data. Finally, unlike Leão and Silva (1989),
226 in Soler and Uieda (2021) neither the data nor the equivalent sources need to be distributed in regular
227 grids. Indeed, Leão and Silva (1989) built their method using regular grids, but in fact regular grids are not
228 necessary. Regarding the equivalent-source layout, Soler and Uieda (2021) proposed the block-averaged
229 sources locations in which the survey area is divided into horizontal blocks and one single equivalent
230 source is assigned to each block. Each single source per block is placed over the layer with its horizontal
231 coordinates given by the average horizontal positions of observation points. According to Soler and Uieda
232 (2021), the block-averaged sources layout reduces the number of equivalent sources significantly and the
233 gradient-boosting algorithm provides even greater efficiency in terms of data fitting.

3.0.1 The equivalent-data concept

235   To reduced the total processing time and memory usage of equivalent-layer technique, Mendonça and
236 Silva (1994) proposed a strategy called 'equivalent data concept'. The equivalent data concept is grounded
237 on the principle that there is a subset of redundant data that does not contribute to the final solution and
238 thus can be dispensed. Conversely, there is a subset of observations, called equivalent data, that contributes
239 effectively to the final solution and fits the remaining observations (redundant data). Iteratively, Mendonça
240 and Silva (1994) selected the subset of equivalent data that is substantially smaller than the original dataset.
241 This selection is carried out by incorporating one data point at a time.

242   According to Mendonça and Silva (1994), the number of equivalent data is about one-tenth of the total
243 number of observations. These authors used the equivalent data concept to carry out an interpolation of
244 gravity data. They showed a reduction of the total processing time and memory usage by, at least, two

245 orders of magnitude as opposed to using all observations in the interpolation process via the classical
246 equivalent-layer technique.

3.0.2 The wavelet compression and lower-dimensional subspace

248 For large data sets, the sensitivity matrix $\mathbf{A}$ (equation 3) is a drawback in applying the equivalent-layer
249 technique because it is a large and dense matrix.

250 **?** transformed a large and full sensitivity matrix into a sparse one by using fast wavelet transforms. In
251 the wavelet domain, **?** applyied a 2D wavelet transform to each row and column of the original sensitivity
252 matrix $\mathbf{A}$ to expand it in the wavelet bases. This operation can be done by premultiplying the original
253 sensitivity matrix $\mathbf{A}$ by a matrix representing the 2D wavelet transform $\mathbf{W_2}$ and then the resulting is
254 postmultiplied by the transpose of $\mathbf{W_2}$ (i.e., $\mathbf{W_2}^\top$).

$$\tilde{\mathbf{A}} = \mathbf{W_2}\,\mathbf{A}\,\mathbf{W_2}^\top , \tag{30}$$

255 where $\tilde{\mathbf{A}}$ is the expanded original sensitivity matrix in the wavelet bases with many elements zero or close
256 to zero. Next, the matrix $\tilde{\mathbf{A}}$ is replaced by its sparse version $\tilde{\mathbf{A}}_\mathbf{s}$ in the wavelet domain which in turn
257 is obtained by retaining only the large elements of the $\tilde{\mathbf{A}}$. Thus, the elements of $\tilde{\mathbf{A}}$ whose amplitudes
258 fall below a relative threshold are discarded. In **?**, the original sensitivity matrix $\mathbf{A}$ is high compressed
259 resulting in a sparce matrix $\tilde{\mathbf{A}}_\mathbf{s}$ with a few percent of nonzero elements and the the inverse problem is
260 solved in the wavelet domain by using $\tilde{\mathbf{A}}_\mathbf{s}$ and a incomplete conjugate gradient least squares, without an
261 explicit regularization parameter and a limited number of iterations. The solution is obtained by solving the
262 following linear system

$$\tilde{\mathbf{A}}_\mathbf{L}^\top\,\tilde{\mathbf{A}}_\mathbf{L}\,\tilde{\mathbf{p}}_\mathbf{L}^* = \tilde{\mathbf{A}}_\mathbf{L}^\top\,\tilde{\mathbf{d}}^o , \tag{31}$$

263 where $\tilde{\mathbf{p}}_\mathbf{L}^*$ is obtained by solving the linear system given by equation 31,

$$\tilde{\mathbf{A}}_\mathbf{L} = \tilde{\mathbf{A}}_\mathbf{s}\,\tilde{\mathbf{L}}^{-1}, \tag{32a}$$

$$\tilde{\mathbf{p}}_\mathbf{L} = \tilde{\mathbf{L}}\tilde{\mathbf{p}}, \tag{32b}$$

$$\tilde{\mathbf{d}}^o = \mathbf{W_2}\,\mathbf{d}^o, \tag{32c}$$

264 where $\tilde{\mathbf{L}}$ is a diagonal and invertible weighting matrix representing the finite-difference approximation in
265 the wavelet domain. Finally, the distribution over the equivalent layer in the space domain $\mathbf{p}$ is obtained by
266 applying an inverse wavelet transform in two steps, i.e.:

$$\tilde{\mathbf{p}} = \tilde{\mathbf{L}}^{-1}\,\tilde{\mathbf{p}}_\mathbf{L}^* , \tag{33}$$

267 and

$$\mathbf{p} = \mathbf{W_2}\,\tilde{\mathbf{p}} . \tag{34}$$

268 Although the data misfit quantifying the difference between the observed and predicted data by the
269 equivalent source is calculated in the wavelet domain, we understand that the desired transformation is
270 calculated via equation **??** which uses a full matrix of Green's functions $\mathbf{T}$.

271 **?** used the equivalent-layer technique with a wavelet compression to perform an upward continuation
272 of total-field anomaly between uneven surfaces. For regularly spaced grid of data, **?** reported that high
273 compression ratios are achived with insignificant loss of accuracy. As compared to the upward-continued

274 total-field anomaly by equivalent layer using the dense matrix, **?**'s (**?**) approach, using the Daubechies
275 wavelet, decreased CPU (central processing unit) time by up to two orders of magnitude.

276 **?** overcame the solution of intractable large-scale equivalent-layer problem by using the subspace method
277 (e.g., **?**, **?**; **?**, **?**; **?**, **?**; **?**, **?**). The subspace method reduces the dimension of the linear system of equations to
278 be solved. Given a higher-dimensional space (e.g., M-dimensional model space, $\mathbb{R}^M$), there exists many
279 lower-dimensional subspaces (e.g., Q-dimensional subspace) of $\mathbb{R}^M$. The linear inverse problem related
280 to the equivalent-layer technique consists in finding an M-dimension parameter vector $\mathbf{p} \in \mathbb{R}^M$ which
281 adequately fits the potential-field data. The subspace method looks for a parameter vector who lies in a
282 Q-dimensional subspace of $\mathbb{R}^M$ which, in turn, is spanned by a set of Q vectors $\mathbf{v}_i = 1, ..., Q$, where
283 $\mathbf{v}_i \in \mathbb{R}^M$ In matrix notation, the parameter vector in the subspace method can be written as

$$\mathbf{p} = \mathbf{V}\,\boldsymbol{\alpha}\,, \tag{35}$$

284 where $\mathbf{V}$ is an $M \times Q$ matrix whose columns $\mathbf{v}_i = 1, ..., Q$ form a basis vectors for a subspace Q of $\mathbb{R}^M$.
285 In equation 35, the parameter vector $\mathbf{p}$ is defined as a linear combination in the space spanned by Q basis
286 vectors $\mathbf{v}_i = 1, ..., Q$ and $\boldsymbol{\alpha}$ is a Q-dimensional unknown vector to be determined. The main advantage of
287 the subspace method is that the linear system of M equations in M unknowns to be originally solved is
288 reduced to a new linear system of Q equations in Q unknowns which requires much less computational
289 effort since $Q << M$, i.e.:

$$\mathbf{V}^\top \mathbf{A}^\top \mathbf{A} \mathbf{V} \boldsymbol{\alpha}^* = \mathbf{V}^\top \mathbf{d}^o\,. \tag{36}$$

290 To avoid the storage of matrices $\mathbf{A}$ and $\mathbf{V}$, **?** evaluates an element of the matrix $\mathbf{AV}$ by calculating the dot
291 product between the row of matrix $\mathbf{A}$ and the column of the matrix $\mathbf{B}$. After estimating $\boldsymbol{\alpha}^*$ (equation 36)
292 belonging to a Q-dimensional subspace of $\mathbb{R}^M$, the distribution over the equivalent layer $\mathbf{p}$ in the $\mathbb{R}^M$ is
293 obtained by applying equation 35. The choice of the Q basis vectors $\mathbf{v}_i = 1, ..., Q$ (equation 35) in the
294 subspace method is not strict. **?**, for example, chose the eigenvectors yielded by applying the singular value
295 decomposition of the matrix containing the gridded data set. The number of eigenvectors used to form
296 basis vectors will depend on the singular values.

297 The proposed subspace method for solving large-scale equivalent-layer problem by **?** was applied to
298 estimate the mass excess or deficiency caused by causative gravity sources.

3.0.3   The quadtree discretization

300 To make the equivalent-layer technique tractable, **?** also transformed the dense sensitivity matrix $\mathbf{A}$
301 (equation 3) into a sparse matrix. In **?**, a sparce version of the sensitivity matrix is achived by grouping
302 equivalent sources (e.g., they used prisms) distant from an observation point together to form a larger prism
303 or larger block. Each larger block has averaged physical properties and averaged top- and bottom-surfaces
304 of the grouped smaller prisms (equivalent sources) that are encompassed by the larger block. The authors
305 called it the 'larger averaged block' and the essence of their method is the reduction in the number of
306 equivalent sources, which means a reduction in the number of parameters to be estimated implying in
307 model dimension reduction.

308 The key of the **?**'s (**?**) method is the algorithm for deciding how to group the smaller prisms. In
309 practice, these authors used a recursive bisection process that results in a quadtree discretization of the
310 equivalent-layer model.

311 By using the quadtree discretization, **?** were able to jointly process multiple components of airborne
312 gravity-gradient data using a single layer of equivalent sources. To our knowledge, **?** are the pioneers on

313 processing full-tensor gravity-gradient data jointly. In addition to computational feasibility, **?**'s (**?**) method
314 reduces low-frequency noise and can also remove the drift in time-domain from the survey data. Those
315 authors stressed that the $G_{zz}-$component calculated through the single estimated equivalent-layer model
316 projected on a grid at a constant elevation by inverting full gravity-gradient data has the low-frequency error
317 reduced by a factor of 2.4 as compared to the inversion of an individual component of the gravity-gradient
318 data.

3.0.4   The reparametrization of the equivalent layer

320 Oliveira Jr. et al. (2013) reparametrized the whole equivalent-layer model by a piecewise bivariate-
321 polynomial function defined on a set of Q equivalent-source windows. In Oliveira Jr. et al.'s (2013)
322 approach, named polynomial equivalent layer (PEL), the parameter vector within the kth equivalent-source
323 window $\mathbf{p}^k$ can be written in matrix notation as

$$\mathbf{p}^k = \mathbf{B}^k \, \mathbf{c}^k \,, \qquad k \, = \, 1 \dots Q \,, \tag{37}$$

324 where $\mathbf{p}^k$ is an M_w-dimensional vector containing the physical-property distribution within the kth
325 equivalent-source window, $\mathbf{c}^k$ is a P-dimensional vector whose lth element is the lth coefficient of the
326 αth-order polynomial function and $\mathbf{B}^k$ is an $M_w \times P$ matrix containing the first-order derivative of the
327 αth-order polynomial function with respect to one of the P coefficients.

328 By using a regularized potential-field inversion, Oliveira Jr. et al. (2013) estimates the polynomial
329 coefficients for each equivalent-source window by solving the following linear system

$$\left(\mathbf{B}^\top \mathbf{A}^\top \mathbf{A} \mathbf{B} \, + \, \mu \mathbf{I} \right) \, \mathbf{c}^* \, = \, \mathbf{B}^\top \mathbf{A}^\top \mathbf{d}^o \,, \tag{38}$$

330 where μ is a regularizing parameter, $\mathbf{c}^*$ is an estimated H-dimensional vector containing all coefficients
331 describing all polynomial functions within all equivalent-source windows which compose the entire
332 equivalent layer, $\mathbf{I}$ is an identity matrix of order $H(H = P\dot{Q})$ and $\mathbf{B}$ is an $M \times H$ block diagonal matrix
333 such that the main-diagonal blocks are $\mathbf{B}^k$ matrices (equation 37) and all off-diagonal blocks are zero
334 matrices. For ease of the explanation of equation 38, we keep only the zeroth-order Tikhonov regularization
335 and omitting the first-order Tikhonov regularization (**?**) which was also used by Oliveira Jr. et al. (2013).

336 The main advantage of the PEL is solve H-dimensional system of equations (equation 38), where H
337 totalizes the number of polynomial coefficients composing all equivalent-source windows, requiring a
338 lower computational effort since $H <<< N$. To avoid the storage of matrices $\mathbf{A}$ and $\mathbf{B}$, Oliveira Jr. et al.
339 (2013) evaluate an element of the matrix $\mathbf{AB}$ by calculating the dot product between the row of matrix $\mathbf{A}$
340 and the column of the matrix $\mathbf{B}$. After estimating all polynomial coefficients of all windows, the estimated
341 coefficients ($\mathbf{c}^*$ in equation 38) are transformed into a single physical-property distribution encompassing
342 the entire equivalent layer.

343 As stated by Oliveira Jr. et al. (2013), the computational efficiency of PEL approach stems from the fact
344 that the total number of polynomial coefficients H required to depict the physical-property distribution
345 within the equivalent layer is generally much smaller than the number of equivalent sources. Consequently,
346 this leads to a considerably smaller linear system that needs to be solved. Hence, the main strategy of
347 polynomial equivalent layer is the model dimension reduction.

348 The polynomial equivalent layer was applied to perform upward continuations of gravity and magnetic
349 data and reduction to the pole of magnetic data.

350 ### 3.0.5 The iterative scheme without solving a linear system

351   There exists a class of methods that iteratively estimate the distribution of physical properties within an
352   equivalent layer without the need to solve linear systems. The method initially introduced by Cordell (1992)
353   and later expanded upon by Guspí and Novara (2009) updates the physical property of sources, located
354   beneath each potential-field data, by removing the maximum residual between the observed and fitted data.
355   In addition, Xia and Sprowl (1991) and Xia et al. (1993) have developed efficient iterative algorithms for
356   updating the distribution of physical properties within the equivalent layer in the wavenumber and space
357   domains, respectively. Specifically, in Xia and Sprowl's (1991) method the physical-property distribution is
358   updated by using the ratio between the squared depth to the equivalent source and the gravitational constant
359   multiplied by the residual between the observed and predicted observation at the measurement station.
360   Neither of these methods solve linear systems.

361   Following this class of methods of iterative equivalent-layer technique that does not solve linear systems,
362   Siqueira et al. (2017) developed a fast iterative equivalent-layer technique for processing gravity data in
363   which the sensitivity matrix $\mathbf{A}$ (equation 3) is replaced by a diagonal matrix $N \times N$, i.e.:

$$\tilde{\tilde{\mathbf{A}}} = 2\,\pi\,\gamma\,\mathbf{\Delta S^{-1}}\,, \tag{39}$$

364   where γ is Newton's gravitational constant and $\mathbf{\Delta S^{-1}}$ is a diagonal matrix of order N whose diagonal
365   elements Δs_i, $i = 1, ..., N$ are the element of area centered at the ith horizontal coordinates of the ith
366   observation point. The physical foundations of Siqueira et al.'s (2017) method rely on two constraints: i) the
367   excess of mass; and ii) the positive correlation between the gravity observations and the mass distribution
368   over the equivalent layer.

369   Although Siqueira et al.'s (2017) method does not solve any linear system of equations, it can be
370   theoretically explained by solving the following linear system at the kth iteration:

$$\tilde{\tilde{\mathbf{A}}}^{\top}\,\tilde{\tilde{\mathbf{A}}}\mathbf{\Delta}\,\hat{\mathbf{p}}^{k}\,=\,\tilde{\tilde{\mathbf{A}}}^{\top}\,\mathbf{r}^{k}\,, \tag{40}$$

371   where $\mathbf{r}^{k}$ is an N-dimensional residual vector whose ith element is calculated by subtracting the ith
372   observed data d_i^o from the ith fitted data d_i^k at the kth iteration, i.e.,

$$r_i^k = d_i^o \,-\, d_i^k\,. \tag{41}$$

373   and $\mathbf{\Delta}\,\hat{\mathbf{p}}^{k}$ is an estimated N-dimensional vector of parameter correction.

374   Because $\tilde{\tilde{\mathbf{A}}}$, in equation 40, is a diagonal matrix (equation 39), the parameter correction estimate is
375   directly calculated without solving system of linear equations, and thus, an ith element of $\mathbf{\Delta}\,\hat{\mathbf{p}}^{k}$ is directly
376   calculated by

$$\Delta\hat{p}^{k}{}_{i} = \frac{\Delta s_i\, r_i^k}{2\,\pi\,\gamma}\,. \tag{42}$$

377 The mass distribution over the equivalent layer is updated by:

$$\hat{p}_i^{k+1} = \hat{p}_i^{k} \,+\, \Delta\hat{p}^{k}{}_{i}\,. \tag{43}$$

378 Siqueira et al.'s (2017) method starts from a mass distribution on the equivalent layer, whose ith mass p_i^o is
379 proportional to the ith observed data d_i^o, i.e.,

$$p_i^o = \frac{\Delta s_i\, d_i^o}{2\,\pi\,\gamma}\,. \tag{44}$$

380 Siqueira et al. (2017) applied their fast iterative equivalent-layer technique to interpolate, calculate the
381 horizontal components, and continue upward (or downward) gravity data.

382 For jointly process two gravity gradient components, Jirigalatu and Ebbing (2019) used the Gauss-FFT
383 for forward calculation of potential fields in the wavenumber domain combined with Landweber's iteration
384 coupled with a mask matrix $\mathbf{M}$ to reduce the edge effects without increasing the computation cost. The
385 mask matrix $\mathbf{M}$ is defined in the following way: if the corresponding pixel does not contain the original
386 data, the element of $\mathbf{M}$ is set to zero; otherwise, it is set to one. The kth Landweber iteration is given by

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \omega \left[\mathbf{A_1}^\top (\mathbf{d_1} - \mathbf{M}\mathbf{A_1}\mathbf{p}_k) + \mathbf{A_2}^\top (\mathbf{d_2} - \mathbf{M}\mathbf{A_2}\mathbf{p}_k) \right]\,, \tag{45}$$

387 where ω is a relaxation factor, $\mathbf{d_1}$ and $\mathbf{d_2}$ are the two gravity gradient components and $\mathbf{A_1}$ and $\mathbf{A_2}$ are the
388 corresponding gravity gradient kernels. Jirigalatu and Ebbing (2019) applied their method for processing
389 two horizontal curvature components of Falcon airborne gravity gradient.

3.0.6 The convolutional equivalent layer with BTTB matrices

391 **?** (**?**, **?**) introduced the convolutional equivalent layer for gravimetric and magnetic data processing,
392 respectively.

393 **?** demonstrated that the sensitivity matrix $\mathbf{A}$ (equation 3) associated with a planar equivalent layer formed
394 by a set of point masses, each one directly beneath each observation point and considering a regular grid of
395 observation points at a constant height has a symmetric block-Toeplitz Toeplitz-block (BTTB) structure. A
396 symmetric BTTB matrix has, at least, two attractive properties. The first one is that it can be defined by
397 using only the elements forming its first column (or row). The second attractive property is that any BTTB
398 matrix can be embedded into a symmetric Block-Circulat Circulant-Block (BCCB) matrix. This means that
399 the full sensitivity matrix $\mathbf{A}$ (equation 3) can be completely reconstruct by using the first column of the
400 BCCB matrix only. In what follows, **?** computed the forward modeling by using only a single equivalent
401 source. Specifically, it is done by calculating the eigenvalues of the BCCB matrix that can be efficiently
402 computed by using only the first column of the BCCB matrix via 2D fast Fourier transform (2D FFT).
403 By comparing with the classic approach in the Fourier domain, the convolutional equivalent layer for
404 gravimetric data processing proposed by **?** performed upward- and downward-continue gravity data with a
405 very small border effects and noise amplification.

406 By using the original idea of the convolutional equivalent layer proposed by **?** for gravimetric data
407 processing, **?** developed the convolutional equivalent layer for magnetic data processing. By assuming
408 a regularly spaced grid of magnetic data at a constant height and a planar equivalent layer of dipoles, **?**
409 proved that the sensitivity matrix linked with this layer possess a BTTB structure in the specific scenario
410 where each dipole is exactly beneath each observed magnetic data point. **?** used a conjugate gradient
411 least-squares (CGLS) algorithm which does not require an inverse matrix or matrix-matrix multiplication.
412 Rather, it only requires matrix-vector multiplications per iteration, which can be effectively computed using
413 the 2D FFT as a discrete convolution. The matrix-vector product only uses the elements that constitute the

first column of the associated BTTB matrix, resulting in computational time and memory savings. **?** (**?**) showed the robustness of the convolutional equivalent layer in processing magnetic survey that violates the requirement of regular grids in the horizontal directions and flat observation surfaces.

The matrix-vector product in **?** (**?**, **?**) (e.g., $\mathbf{d} = \mathbf{A}\mathbf{p}$, such as in equation 3) is the main issue to be solved. To solve it efficiently, these authors involked the auxiliary linear system

$$\mathbf{w} = \mathbf{C}\mathbf{v}\,, \tag{46}$$

where $\mathbf{w}$ and $\mathbf{v}$ are, respectively, vectors of data and parameters completed by zeros and $\mathbf{C}$ is a BCCB matrix formed by $2Q \times 2Q$ blocks, where each block $\mathbf{C}_q$, $q = 0, \ldots, Q-1$, is a $2P \times 2P$ circulant matrix. The first column of $\mathbf{C}$ is obtained by rearranging the first column of the sensitivity matrix $\mathbf{A}$ (equation 3). Because a BCCB matrix is diagonalized by the 2D unitary discrete Fourier transform (DFT), $\mathbf{C}$ can be written as

$$\mathbf{C} = \left(\mathbf{F}_{2Q} \otimes \mathbf{F}_{2P}\right)^* \mathbf{\Lambda} \left(\mathbf{F}_{2Q} \otimes \mathbf{F}_{2P}\right)\,, \tag{47}$$

where the symbol "$\otimes$" denotes the Kronecker product (**?**), $\mathbf{F}_{2Q}$ and $\mathbf{F}_{2P}$ are the $2Q \times 2Q$ and $2P \times 2P$ unitary DFT matrices (**?**, p. 31), respectively, the superscritpt "$*$" denotes the complex conjugate and $\mathbf{\Lambda}$ is a $4QP \times 4QP$ diagonal matrix containing the eigenvalues of $\mathbf{C}$. Due to the diagonalization of the matrix $\mathbf{C}$, the auxiliary system (equation 46) can be rewritten by using equation 47 and premultiplying both sides of the result by $\left(\mathbf{F}_{2Q} \otimes \mathbf{F}_{2P}\right)$, i.e.,

$$\mathbf{\Lambda} \left(\mathbf{F}_{2Q} \otimes \mathbf{F}_{2P}\right) \mathbf{v} = \left(\mathbf{F}_{2Q} \otimes \mathbf{F}_{2P}\right) \mathbf{w}\,. \tag{48}$$

By applying the vec-operator (**?**) to both sides of equation 48, by premultiplying both sides of the result by $\mathbf{F}_{2Q}^*$ and then postmultiplying both sides of the result by $\mathbf{F}_{2P}^*$

$$\mathbf{F}_{2Q}^* \left[\mathbf{L} \circ \left(\mathbf{F}_{2Q}\,\mathbf{V}\,\mathbf{F}_{2P}\right)\right] \mathbf{F}_{2P}^* = \mathbf{W}\,, \tag{49}$$

where "$\circ$" denotes the Hadamard product (**?**, p. 298) and $\mathbf{L}$, $\mathbf{V}$ and $\mathbf{W}$ are $2Q \times 2P$ matrices obtained by rearranging, along their rows, the elements forming the diagonal of matrix $\mathbf{\Lambda}$, vector $\mathbf{v}$ and vector $\mathbf{w}$, respectively. The left side of equation 49 contains the 2D Inverse Discrete Fourier Transform (IDFT) of the term in brackets, which in turn represents the Hadamard product of matrix $\mathbf{L}$ and the 2D DFT of matrix $\mathbf{V}$. Matrix $\mathbf{L}$ contains the eigenvalues of $\mathbf{\Lambda}$ (equation 47) and can be efficiently computed by using only the first column of the BCCB matrix $\mathbf{C}$ (equation 46).

Actually, in **?** (**?**, **?**) a fast 2D discrete circular convolution (Van Loan, 1992) is used to process very large gravity and magnetic datasets efficiently. The convolutional equivalent layer was applied to perform upward continuation of large magnetic datasets. Compared to the classical Fourier approach, **?**'s (**?**) method produces smaller border effects without using any padding scheme.

Without taking advantage of the symmetric BTTB structure of the sensitivity matrix (**?**, **?**) that arises when gravimetric observations are measured on a horizontally regular grid, on a flat surface and considering a regular grid of equivalent sources whithin a horizontal layer, **?** explored the symmetry of the gravity kernel to reduce the number of forward model evaluations. By exploting the symmetries of the gravity kernels and redundancies in the forward model evaluations on a regular grid and combining the subspace solution based on eigenvectors of the gridded dataset, **?** estimated the mass excess or deficiency produced by anomalous sources with positive or negative density contrast.

448   3.0.7   The deconvolutional equivalent layer with BTTB matrices

449   To avoid the iterations of the conjugate gradient method in **?**, we can employ the deconvolution process.
450   Equation 49 shows that estimate the matrix $\mathbf{V}$, containing the elements of parameter vector $\mathbf{p}$, is a
451   inverse problem that could be solved by deconvolution. From equation 49, the matrix $\mathbf{V}$ can be obtain by
452   deconvolution, i.e.

$$\mathbf{V} \;=\; \mathbf{F}_{2Q}^{*} \left[\frac{\left(\mathbf{F}_{2Q} \mathbf{W} \, \mathbf{F}_{2P} \right)}{\mathbf{L}} \right] \mathbf{F}_{2P}^{*} . \tag{50}$$

453   Equation 50 shows that the parameter vector (in matrix $\mathbf{V}$) can be theoretically obtain by dividing each
454   potential-field observations (in matrix $\mathbf{W}$) by each eigenvalues (in matrix $\mathbf{L}$). Hence, the parameter vector
455   is constructed by element-by-element division of data by eigenvalues.

456   However, the deconvolution often is extremely unstable. This means that a small change in data can lead
457   to an enormous change in the estimated parameter. Hence, equation 50 requires regularization to be useful.
458   We usede wiener deconvolution to obtain a stable solution, i.e.,

$$\mathbf{V} \;=\; \mathbf{F}_{2Q}^{*} \left[\left(\mathbf{F}_{2Q} \mathbf{W} \, \mathbf{F}_{2P} \right) \frac{\mathbf{L}^{*}}{\left(\mathbf{L}\,\mathbf{L}^{*} + \mu \right)} \right] \mathbf{F}_{2P}^{*} , \tag{51}$$

459   where the matrix $\mathbf{L}^{*}$ contains the complex conjugate eigenvalues and μ is a parameter that controls the
460   degree of stabilization.

3.1   Solution stability

462   The solution stability of the equivalent-layer methods is rarely addressed. Here, we follow the numerical
463   stability analysis presented in Siqueira et al. (2017).

464   Let us assume noise-free potential-field data $\mathbf{d}$, we estimate a physical-property distribution $\mathbf{p}$ (estimated
465   solution) within the equivalent layer. Then, the noise-free data $\mathbf{d}$ are contaminated with additive D different
466   sequences of pseudorandom Gaussian noise, creating different noise-corrupted potential-field data $\mathbf{d}_{\ell}^{\mathrm{o}}$,
467   $\ell = 1, ..., D$. From each $\mathbf{d}_{\ell}^{\mathrm{o}}$, we estimate a physical-property distribution $\hat{\mathbf{p}}_{\ell}$ within the equivalent layer.

468   Next, for each noise-corrupted data $\mathbf{d}_{\ell}^{\mathrm{o}}$ and estimated solution $\hat{\mathbf{p}}_{\ell}$, the ℓth model perturbation δp_{ℓ} and the
469   ℓth data perturbation δd_{ℓ} are, respectively, evaluated by

$$\delta p_{\ell} = \frac{\| \, \hat{\mathbf{p}}_{\ell} - \mathbf{p} \, \|_{2}}{\| \, \mathbf{p} \, \|_{2}}, \quad \ell = 1, ..., D, \tag{52}$$

470   and

$$\delta d_{\ell} = \frac{\| \, \mathbf{d}_{\ell}^{\mathrm{o}} - \mathbf{d} \, \|_{2}}{\| \, \mathbf{d} \, \|_{2}}, \quad \ell = 1, ..., D. \tag{53}$$

471   Regardless of the particular method used, the following inequality (**?**, p. 66) is applicable:

$$\delta p_{\ell} \le \kappa \, \delta d_{\ell}, \quad \ell = 1, ..., D, \tag{54}$$

472   where κ is the constant of proportionality between the model perturbation δp_{ℓ} (equation 52) and the data
473   perturbation δd_{ℓ} (equation 53). The constant κ acts as the condition number of an invertible matrix in a
474   given inversion, and thus measures the instability of the solution. The larger (smaller) the value of κ the
475   more unstable (stable) is the estimated solution.

476   Equation 54 shows a linear relationship between the model perturbation and the data perturbation. By
477   plotting δp_ℓ (equation 52) against δd_ℓ (equation 53) produced by a set of D estimated solution obtained by
478   applying a given equivalent-layer method, we obtain a straight line behaviour described by equation 54.
479   By applying a linear regression, we obtain a fitted straight line whose estimated slope (κ in equation 54)
480   quantifies the solution stability.

481   Here, the analysis of solution stability is numerically conducted by applying the classical equivalent-
482   layer technique with zeroth-order Tikhonov regularization, the convolutional method for gravimetric and
483   magnetic data, the deconvolutional method (equation 50) and the deconvolutional method with different
484   values for the Wiener stabilization (equation 51).

4 NUMERICAL SIMULATIONS

485 We investigated different computational algorithms for inverting gravity disturbances and total-field
486 anomalies. To test the capability of the fast equivalent-layer technique for processing that potential field
487 data we measure of the computational effort by counting the number of floating-point operations (*flops*),
488 such as additions, subtractions, multiplications, and divisions (Golub and Van Loan, 2013) for different
489 number of observation points, ranging from $10,000$ up to $1,000,000$. The results generated when using
490 iterative methods are set to $it = 50$ for the number of iterations.

4.1 Floating-point operations calculation

492 To measure the computational effort of the different algorithms to solve the equivalent layer linear system,
493 a non-hardware dependent method can be useful because allow us to do direct comparison between them.
494 Counting the floating-point operations (*flops*), i.e., additions, subtractions, multiplications and divisions
495 is a good way to quantify the amount of work of a given algorithm (Golub and Van Loan, 2013). For
496 example, the number of *flops* necessary to multiply two vectors $\mathbb{R}^N$ is $2N$. A common matrix-vector
497 multiplication with dimension $\mathbb{R}^{N \times N}$ and $\mathbb{R}^N$, respectively, is $2N^2$ and a multiplication of two matrices
498 $\mathbb{R}^{N \times N}$ is $2N^3$. Figure **??** shows the total flops count for the different methods presented in this review
499 with a crescent number of data, ranging from $10,000$ to $1,000,000$ for the gravity equivalent layer and
500 figure **??** for magnetic data.

4.1.1 Normal equations using Cholesky decomposition

502 The equivalent sources can be estimated directly from solving the normal equations 3. In this work we
503 will use the Cholesky decompositions method to calculate the necessary *flops*. In this method it is calculated
504 the lower triangle of $\mathbf{A}^T\mathbf{A}$ $(1/2N^3)$, the Cholesky factor $(1/3N^3)$, a matrix-vector multiplication $(2N^2)$
505 and finally solving the triangular system $(2N^2)$, totalizing

$$f_{classical} = \frac{5}{6}N^3 + 4N^2 \tag{55}$$

4.1.2 Window method (Leão and Silva, 1989)

507 The moving data-window scheme (Leão and Silva, 1989) solve N linear systems with much smaller sizes
508 (equation 28b). For our results we are considering a data-window of the same size of wich the authors
509 presented in theirs work ($N_w = 49$) and the same number of equivalent sources ($M_w = 225$). We are doing
510 this process for all the other techniques to standardize the resolution of our problem. Using the Cholesky
511 decomposition with this method the *flops* are

$$f_{window} = N\frac{5}{6}M_wN_w^2 + 4N_wM_w \tag{56}$$

4.1.3 PEL method (Oliveira Jr. et al., 2013)

513 The polynomial equivalent layer uses a simliar approach od moving windows from Leão and Silva (1989).
514 For this operations calculation (equation 38) we used a first degree polynomial (two variables) and each
515 window contains $N_s = 1,000$ observed data and $M_s = 1,000$ equivalent sources. Following the steps
516 given in (Oliveira Jr. et al., 2013) the total *flops* becomes

$$f_{pel} = \frac{1}{3}H^3 + 2H^2 + 2NM_sH + H^2N + 2HN + 2NP \tag{57}$$

517 where H is the number of constant coefficients for the first degree polynomial ($P = 3$) times the number
518 of windows ($P \times N/N_s$).

4.1.4 Conjugate gradient least square (CGLS)

520 The CGLS method is a very stable and fast algorithm for solving linear systems iteratively. Its
521 computational complexity envolves a matrix-vector product outside the loop ($2N^2$), two matrix-vector
522 products inside the loop ($4N^2$) and six vector products inside the loop ($12N$) (**?**)

$$f_{cgls} = 2N^2 + it(4N^2 + 12N) \tag{58}$$

4.1.5 Wavelet compression method with CGLS (?)

524 For the wavelet method (equation 31) we have calculated a coompression rate of 98% ($C_r = 0.02$)
525 for the threshold as the authors used in **?** and the wavelet transformation requiring $\log_2(N)$ *flops* each
526 (equations 30 and 32c), with its inverse also using the same number of operations (equation 34). Combined
527 with the conjugate gradient least square necessary steps and iterations, the number of *flops* are

$$f_{wavelet} = 2NC_r + 4N\log_2(N) + it(4N\log_2(N) + 4NC_r + 12C_r) \tag{59}$$

4.1.6 Fast equivalent layer for gravity data (Siqueira et al., 2017)

529 The fast equivalent layer from Siqueira et al. (2017) solves the linear system in *it* iterations. The main
530 cost of this method (equations 40,41, 42 and 43)is the matrix-vector multiplication to asses the predicted
531 data ($2N^2$) and three simply element by element vector sum, subtraction and division ($3N$ total)

$$f_{siqueira} = it(3N + 2N^2) \tag{60}$$

4.1.7 Convolutional equivalent layer for gravity data (?)

533 This methods replaces the matrix-vector multiplication of the iterative fast-equivalent technique (Siqueira
534 et al., 2017) by three steps, involving a Fourier transform, an inverse Fourier transform, and a Hadamard
535 product of matrices (equation 49). Considering that the first column of our BCCB matrix has $4N$ elements,
536 the flops count of this method is

$$f_{convgrav} = \kappa 4N\log_2(4N) + it(27N + \kappa 8N\log_2(4N)) \tag{61}$$

537 In the resultant count we considered a *radix-2* algorithm for the fast Fourier transform and its inverse,
538 which has a κ equals to 5 and requires $\kappa 4N\log_2(4N)$ flops each. The Hadarmard product of two matrices
539 of $4N$ elements with complex numbers takes $24N$ flops. Note that equation 61 is different from the one
540 presented in **?** because we also added the flops necessary to calculate the eigenvalues in this form. It does
541 not differentiate much in order of magnitude because the iterative part is the most costful.

4.1.8 Convolutional equivalent layer for magnetic data (**?**)

The convolutional equivalent layer for magnetic data uses the same flops count of the main operations as in the gravimetric case (equation 49), the difference is the use of the conjugate gradient algorithm to solve the inverse problem. It requires a Hadamard product outside of the iterative loop and the matrix-vector and vector-vector multiplications inside the loop as seem in equation 58.

$$f_{convmag} = \kappa 16N \log_2(4N) + 24N + it(\kappa 16N \log_2(4N) + 60N) \tag{62}$$

4.1.9 Deconvolutional method

The deconvolution method does not require an iterative algorithm, rather it solves the estimative of the physical properties in a single step using the $4N$ eigenvalues of the BCCB matrix as in the convolutional method. From equation 50 it is possible to deduce this method requires two fast Fourier transform ($\kappa 4N \log_2(4N)$), one for the eigenvalues and another for the data transformation, a element by element division ($24N$) and finally, a fast inverse Fourier transform for the final estimative ($\kappa 4N \log_2(4N)$).

$$f_{deconv} = \kappa 12N \log_2(4N) + 24N \tag{63}$$

Using the deconvolutional method with a Wiener stabilization adds two multiplications of complex elements of the conjugates eigenvalues ($24N$ each) and the sum of $4N$ elements with the stabilization parameter μ as shown in equation 51

$$f_{deconvwiener} = \kappa 12N \log_2(4N) + 76N \tag{64}$$

CONFLICT OF INTEREST STATEMENT

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

AUTHOR CONTRIBUTIONS

The Author Contributions section is mandatory for all articles, including articles by sole authors. If an appropriate statement is not provided on submission, a standard one will be inserted during the production process. The Author Contributions statement must describe the contributions of individual authors referred to by their initials and, in doing so, all authors agree to be accountable for the content of the work. Please see here for full authorship criteria.

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DATA AVAILABILITY STATEMENT

The datasets generated for this study can be found in the frontiers-paper Github repository link: https://github.com/DiegoTaka/frontiers-paper.

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