

The computation aspects of the equivalent-layer technique: review and perspective

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2 ABSTRACT

3 Equivalent-layer technique is a powerful tool for processing potential-field data in the space
4 domain. However, the greatest hindrance for using the equivalent-layer technique is its high
5 computational cost for processing massive data sets. The large amount of computer memory
6 usage to store the full sensitivity matrix combined with the computational time required for
7 matrix-vector multiplications and to solve the resulting linear system, are the main drawbacks
8 that made unfeasible the use of the equivalent-layer technique for a long time. More recently, the
9 advances in computational power propelled the development of methods to overcome the heavy
10 computational cost associated with the equivalent-layer technique. We present a comprehensive
11 review of the computation aspects concerning the equivalent-layer technique addressing how
12 previous works have been dealt with the computational cost of this technique. Historically,
13 the high computational cost of the equivalent-layer technique has been overcome by using a
14 variety of strategies such as: moving data-window scheme, column- and row-action updates of
15 the sensitivity matrix, reparametrization, wavelet compression, iterative methods using the full
16 sensitivity matrix, iterative deconvolution by using the concept of block-Toeplitz Toeplitz-block
17 (BTTB) matrices and direct deconvolution. We compute the number of floating-point operations of
18 some of these strategies adopted in the equivalent-layer technique to show their effectiveness in
19 reducing the computational demand. Numerically, we also address the stability of some of these
20 strategies used in the equivalent-layer technique by comparing with the stability via the classic
21 equivalent-layer technique with the zeroth-order Tikhonov regularization. We show that even for
22 the most computationally efficient methods, which can save up to 10^9 flops, the stability of the
23 linear system is maintained. The two most efficient strategies, iterative and direct deconvolutions,
24 can process large datasets quickly and yield good results. However, direct deconvolution has
25 some drawbacks. Real data from Carajás Mineral Province, Brazil, is also used to validate the
26 results showing a potential field transformation.

27 **Keywords:** equivalent layer, gravimetry, fast algorithms, computational cost, stability analysis

1 INTRODUCTION

The equivalent-layer technique has been used by exploration geophysicists for processing potential-field data since the late 1960s (Dampney, 1969). This technique is based on a widely accepted principle, which states that a discrete set of observed potential-field data due to 3D sources can be approximated by that due to a discrete set of virtual sources (such as point masses, dipoles, prisms, doublets). From a theoretical point of view, the equivalent-layer technique is grounded on potential theory (Kellogg, 1967) and consists in considering that the potential field data can be approximated by a linear combination of harmonic functions describing the potential field due to the virtual sources. These sources, commonly called equivalent sources, are arranged on a layer with finite horizontal dimensions and located below the observations. In the classical approach, a linear inverse problem is solved to estimate the physical property of each equivalent source subject to fit the observations. Then, the estimated physical-property distribution on the equivalent layer is used to accomplish the desired potential-field transformation (e.g., interpolation, upward/downward continuation, reduction to the pole). The later step is done by multiplying the estimated physical-property distribution by the matrix of Green's functions associated with the desired potential-field transformation.

Because the linear inverse problem to be solved in the equivalent-layer technique is set up with a full sensitivity matrix, its computational cost strongly depends on the number of potential-field observations and can be very inefficient for dealing with massive data sets. To overcome this problem, computationally efficient methods based on equivalent-layer technique have arose in the late 1980s. To our knowledge, the first method towards improving the efficiency was proposed by Leão and Silva (1989), who used an overlapping moving-window scheme spanning the data set. The strategy adopted in Leão and Silva (1989) involves solving several smaller, regularized linear inverse problems instead of one large problem. This strategy uses a small data window and distributes equivalent sources on a small regular grid at a constant depth below the data surface, with the sources' window extending beyond the boundaries of the data window. Because of the spatial layouts of observed data and equivalent sources in Leão and Silva (1989), the small sensitivity submatrix containing the coordinates of the data and equivalent sources within a window remains constant for all data windows. This holds true regardless of the specific locations of the data and equivalent sources within each window. For each position of the data window, this scheme consists in computing the processed field at the center of the data window only, and the next estimates of the processed field are obtained by shifting the data window across the entire dataset. More recently, Soler and Uieda (2021) extended the method introduced by Leão and Silva (1989) to accommodate irregularly spaced data collected on a non-flat surface. Unlike Leão and Silva (1989), in the generalization proposed by Soler and Uieda (2021), the sensitivity submatrix that includes the coordinates of the data and equivalent sources needs to be computed for each window. Soler and Uieda (2021) developed a computational approach to further enhance the efficiency of the equivalent-layer technique by combining two strategies. The first one — the block-averaging source locations — reduces the number of model parameters and the second strategy — the gradient-boosted algorithm — reduces the size of the linear system to be solved by iteratively fitting the equivalent source model along overlapping windows. It is worth noting that the equivalent-layer strategy of using a moving-window scheme either in Leão and Silva (1989) or in Soler and Uieda (2021) is similar to discrete convolution.

As another strategy to reduce the computational workload of the equivalent-layer technique, some authors have employed column- and row-action updates, which are commonly applied to image reconstruction methods (e.g., Elfving et al., 2017). These methods involve iterative calculations of a single column and a single row of the sensitivity matrix, respectively. Following the strategy column-action update,

71 Cordell (1992) proposed a computational method in which a single equivalent source positioned below a
72 measurement station is iteratively used to compute both the predicted data and residual data for all stations.
73 In Cordell's method, a single column of the sensitivity matrix is calculated per iteration, meaning that
74 a single equivalent source contributes to data fitting in each iteration. Guspí and Novara (2009) further
75 extended Cordell's method by applying it to scattered magnetic observations. Following the strategy of
76 column-action update, Mendonça and Silva (1994) developed an iterative procedure where one data point
77 is incorporated at a time, and a single row of the sensitivity matrix is calculated per iteration. This strategy
78 adopted by Mendonça and Silva (1994) is known as *equivalent data concept*. This concept is based on
79 the principle that certain data points within a dataset are redundant and, as a result, do not contribute to
80 the final solution. On the other hand, there is a subset of observations known as equivalent data, which
81 effectively contributes to the final solution and fits the remaining redundant data. In their work, Mendonça
82 and Silva (1994) adopted an iterative approach to select a substantially smaller subset of equivalent data
83 from the original dataset.

84 The next strategy involves reparametrizing the equivalent layer with the objective of solving a smaller
85 linear inverse problem by reducing the dimension of the model space. Oliveira Jr. et al. (2013) reduced
86 the model parameters by approximating the equivalent-source layer by a piecewise-polynomial function
87 defined on a set of user-defined small equivalent-source windows. The estimated parameters are the
88 polynomial coefficients for each window and they are much smaller than the original number of equivalent
89 sources. By using the subspace method, Mendonça (2020) reparametrizes the equivalent layer, which
90 involves reducing the dimension of the linear system from the original parameter-model space to a lower-
91 dimensional subspace. The subspace bases span the parameter-model space and they are constructed by
92 applying the singular value decomposition to the matrix containing the gridded data.

93 Following the strategy of sparsity induction, Li and Oldenburg (2010) transformed the full sensitivity
94 matrix into a sparse one using orthonormal compactly supported wavelets. Barnes and Lumley (2011)
95 proposed an alternative approach to introduce sparsity based on the use of quadtree discretization to group
96 equivalent sources far from the computation points. Those authors explore the induced sparsity by using
97 specific iterative methods to solve the linear system.

98 The strategy named iterative methods estimates iteratively the parameter vector that represents a
99 distribution over an equivalent layer. Xia and Sprowl (1991) and Xia et al. (1993) have developed efficient
100 iterative algorithms for updating the distribution of physical properties within the equivalent layer in the
101 wavenumber and space domains, respectively. Specifically, in Xia and Sprowl's (1991) method the physical-
102 property distribution is updated by using the ratio between the squared depth to the equivalent source and
103 the gravitational constant multiplied by the residual between the observed and predicted observation at the
104 measurement station. Siqueira et al. (2017) developed an iterative solution where the sensitivity matrix
105 is transformed into a diagonal matrix with constant terms through the use of the *excess mass criterion*
106 and of the positive correlation between the observed gravity data and the masses on the equivalent layer.
107 The fundamentals of the Siqueira et al.'s method is based on the Gauss' theorem (e.g., Kellogg, 1967,
108 p. 43) and the total excess of mass (e.g., Blakely, 1996, p. 60). All these iterative methods use the full
109 and dense sensitivity matrix to calculate the predicted data and residual data in the whole survey data per
110 iteration. Hence, the iterative methods proposed by Xia and Sprowl (1991), Xia et al. (1993) and Siqueira
111 et al. (2017) neither compress nor reparametrize the sensitivity matrix. Jirigalatu and Ebbing (2019) also
112 proposed an iterative equivalent layer that uses the full and dense sensitivity matrix. However, in their
113 approach, Jirigalatu and Ebbing (2019) efficiently compute the predicted data and residual data for the
114 entire survey per iteration in the wavenumber domain.

115 Following the strategy of the iterative deconvolution, Takahashi et al. (2020, 2022) developed fast and
116 effective equivalent-layer techniques for processing, respectively, gravity and magnetic data by modifying
117 the forward modeling to estimate the physical-property distribution over the equivalent layer through a
118 2D discrete fast convolution. These methods took advantage of the Block-Toeplitz Toeplitz-block (BTTB)
119 structure of the sensitivity matrices, allowing them to be calculated by using only their first column. In
120 practice, the forward modeling uses a single equivalent source, which significantly reduces the required
121 RAM memory.

122 The method introduced by Takahashi et al. (2020, 2022) can be reformulated to eliminate the need for
123 conjugate gradient iterations. This reformulation involves employing a *direct deconvolution* approach (e.g.,
124 Aster et al., 2019, p. 220) with *Wiener filter* (e.g., Gonzalez and Woods, 2002, p. 263).

125 Here, we present a comprehensive review of diverse strategies to solve the linear system of the equivalent
126 layer alongside an analysis of the computational cost and stability of these strategies. To do this analysis,
127 we are using the floating-point operations count to evaluate the performance of a selected set of methods.
128 To test the stability, we are using the linear system sensitivity to noise as a comparison parameter for the
129 fastest of these methods alongside the classical normal equations. A potential-field transformation will also
130 be used to evaluate the quality of the equivalent sources estimation results using both synthetic and real
131 data from Carajás Mineral Province, Brazil.

2 FUNDAMENTALS

132 Let \mathbf{d} be a $D \times 1$ vector, whose i -th element d_i is the observed potential field at the position (x_i, y_i, z_i) ,
 133 $i \in \{1 : D\}$, of a topocentric Cartesian system with x , y and z axes pointing to north, east and down,
 134 respectively. Consider that d_i can be satisfactorily approximated by a harmonic function

$$f_i = \sum_{j=1}^P g_{ij} p_j, \quad i \in \{1 : D\}, \quad (1)$$

135 where, p_j represents the scalar physical property of a virtual source (i.e., monopole, dipole, prism) located
 136 at (x_j, y_j, z_j) , $j \in \{1 : P\}$ and

$$g_{ij} \equiv g(x_i - x_j, y_i - y_j, z_i - z_j), \quad z_i < \min\{z_j\}, \quad \forall i \in \{1 : D\}, \quad (2)$$

137 is a harmonic function, where $\min\{z_j\}$ denotes the minimum z_j , or the vertical coordinate of the shallowest
 138 virtual source. These virtual sources are called *equivalent sources* and they form an *equivalent layer*. In
 139 matrix notation, the potential field produced by all equivalent sources at all points (x_i, y_i, z_i) , $i \in \{1 : D\}$,
 140 is given by:

$$\mathbf{f} = \mathbf{G}\mathbf{p}, \quad (3)$$

141 where \mathbf{p} is a $P \times 1$ vector with j -th element p_j representing the scalar physical property of the j -th
 142 equivalent source and \mathbf{G} is a $D \times P$ matrix with element g_{ij} given by equation 2.

143 The equivalent-layer technique consists in solving a linear inverse problem to determine a parameter
 144 vector \mathbf{p} leading to a predicted data vector \mathbf{f} (equation 3) *sufficiently close to* the observed data vector \mathbf{d} ,
 145 whose i -th element d_i is the observed potential field at (x_i, y_i, z_i) . The notion of *closeness* is intrinsically
 146 related to the concept of *vector norm* (e.g., Golub and Van Loan, 2013, p. 68) or *measure of length* (e.g.,
 147 Menke, 2018, p. 41). Because of that, almost all methods for determining \mathbf{p} actually estimate a parameter
 148 vector $\tilde{\mathbf{p}}$ minimizing a length measure of the difference between \mathbf{f} and \mathbf{d} (see subsection 3.1). Given an
 149 estimate $\tilde{\mathbf{p}}$, it is then possible to compute a potential field transformation

$$\mathbf{t} = \mathbf{A}\tilde{\mathbf{p}}, \quad (4)$$

150 where \mathbf{t} is a $T \times 1$ vector with k -th element t_k representing the transformed potential field at the position
 151 (x_k, y_k, z_k) , $k \in \{1 : T\}$, and

$$a_{kj} \equiv a(x_k - x_j, y_k - y_j, z_k - z_j), \quad z_k < \min\{z_j\}, \quad \forall k \in \{1 : T\}, \quad (5)$$

152 is a harmonic function representing the kj -th element of the $T \times P$ matrix \mathbf{A} .

153 2.1 Spatial distribution and total number of equivalent sources

154 There is no well-established criteria to define the optimum number P or the spatial distribution of the
 155 equivalent sources. We know that setting an equivalent layer with more (less) sources than potential-field
 156 data usually leads to an underdetermined (overdetermined) inverse problem (e.g., Menke, 2018, p. 52–53).
 157 Concerning the spatial distribution of the equivalent sources, the only condition is that they must rely on a
 158 surface that is located below and does not cross that containing the potential field data. Soler and Uieda
 159 (2021) present a practical discussion about this topic.

160 From a theoretical point of view, the equivalent layer reproducing a given potential field data set cannot
 161 cross the true gravity or magnetic sources. This condition is a consequence of recognizing that the equivalent
 162 layer is essentially an indirect solution of a boundary value problem of potential theory (e.g., Roy, 1962;
 163 Zidarov, 1965; Dampney, 1969; Li et al., 2014; Reis et al., 2020). In practical applications, however, there
 164 is no guarantee that this condition is satisfied. Actually, it is widely known from practical experience (e.g.,
 165 Gonzalez et al., 2022) that the equivalent-layer technique works even for the case in which the layer cross
 166 the true sources.

167 Regarding the depth of the equivalent layer, Dampney (1969) proposed a criterion based on horizontal data
 168 sampling, suggesting that the equivalent-layer depth should be between two and six times the horizontal grid
 169 spacing, considering evenly spaced data. However, when dealing with a survey pattern that has unevenly
 170 spaced data, Reis et al. (2020) adopted an alternative empirical criterion. According to their proposal,
 171 the depth of the equivalent layer should range from two to three times the spacing between adjacent
 172 flight lines. The criteria of Dampney (1969) and Reis et al. (2020) are valid for planar equivalent layers.
 173 Cordell (1992) have proposed and an alternative criterion for scattered data that leads to an undulating
 174 equivalent layer. This criterion have been slightly modified by Guspí et al. (2004), Guspí and Novara
 175 (2009) and Soler and Uieda (2021), for example, and consists in setting one equivalent source below
 176 each datum at a depth proportional to the horizontal distance to the nearest neighboring data points. Soler
 177 and Uieda (2021) have compared different strategies for defining the equivalent sources depth for the
 178 specific problem of interpolating gravity data, but they have not found significant differences between them.
 179 Regarding the horizontal layout, Soler and Uieda (2021) proposed the block-averaged sources locations
 180 in which the survey area is divided into horizontal blocks and one single equivalent source is assigned
 181 to each block. The horizontal coordinates of the single source in a given block is defined by the average
 182 horizontal coordinates of the observation points at the block. According to Soler and Uieda (2021), this
 183 block-averaged layout may prevent aliasing of the interpolated values, specially when the observations
 184 are unevenly sampled. This strategy also reduces the number of equivalent sources without affecting the
 185 accuracy of the potential-field interpolation. Besides, it reduces the computational load for estimating the
 186 physical property on the equivalent layer.

187 2.2 Matrix G

188 Generally, the harmonic function g_{ij} (equation 2) is defined in terms of the inverse distance between the
 189 observation point (x_i, y_i, z_i) and the j -th equivalent source at (x_j, y_j, z_j) ,

$$\frac{1}{r_{ij}} \equiv \frac{1}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}}, \quad (6)$$

190 or by its partial derivatives of first and second orders, respectively given by

$$\partial_\alpha \frac{1}{r_{ij}} \equiv \frac{-(\alpha_i - \alpha_j)}{r_{ij}^3}, \quad \alpha \in \{x, y, z\}, \quad (7)$$

191 and

$$\partial_{\alpha\beta} \frac{1}{r_{ij}} \equiv \begin{cases} \frac{3(\alpha_i - \alpha_j)^2}{r_{ij}^5}, & \alpha = \beta, \\ \frac{3(\alpha_i - \alpha_j)(\beta_i - \beta_j)}{r_{ij}^5} - \frac{1}{r_{ij}^3}, & \alpha \neq \beta, \end{cases} \quad \alpha, \beta \in \{x, y, z\}. \quad (8)$$

In this case, the equivalent layer is formed by punctual sources representing monopoles or dipoles (e.g., Dampney, 1969; Emilia, 1973; Leão and Silva, 1989; Cordell, 1992; Oliveira Jr. et al., 2013; Siqueira et al., 2017; Reis et al., 2020; Takahashi et al., 2020; Soler and Uieda, 2021; Takahashi et al., 2022). Another common approach consists in not defining g_{ij} by using equations 6–8, but other harmonic functions obtained by integrating them over the volume of regular prisms (e.g., Li and Oldenburg, 2010; Barnes and Lumley, 2011; Li et al., 2014; Jirgalatu and Ebbing, 2019). There are also some less common approaches defining the harmonic function g_{ij} (equation 2) as the potential field due to plane faces with constant physical property (Hansen and Miyazaki, 1984), doublets (Silva, 1986) or by computing the double integration of the inverse distance function with respect to z (Guspí and Novara, 2009).

A common assumption for most of the equivalent-layer methods is that the harmonic function g_{ij} (equation 2) is independent on the actual physical relationship between the observed potential field and their true sources (e.g., Cordell, 1992; Guspí and Novara, 2009; Li et al., 2014). Hence, g_{ij} can be defined according to the problem. The only condition imposed to this function is that it decays to zero as the observation point (x_i, y_i, z_i) goes away from the position (x_j, y_j, z_j) of the j -th equivalent source. However, several methods use a function g_{ij} that preserves the physical relationship between the observed potential field and their true sources. For the case in which the observed potential field is gravity data, g_{ij} is commonly defined as a component of the gravitational field produced at (x_i, y_i, z_i) by a point mass or prism located at (x_j, y_j, z_j) , with unit density. On the other hand, g_{ij} is commonly defined as a component of the magnetic induction field produced at (x_i, y_i, z_i) by a dipole or prism located at (x_j, y_j, z_j) , with unit magnetization intensity, when the observed potential field is magnetic data.

The main challenge in the equivalent-layer technique is the computational complexity associated with handling large datasets. This complexity arises because the sensitivity matrix \mathbf{G} (equation 3) is dense regardless of the harmonic function g_{ij} (equation 2) employed. In the case of scattered potential-field data, the structure of \mathbf{G} is not well-defined, regardless of the spatial distribution of the equivalent sources. However, in a specific scenario where (i) each potential-field datum is directly associated with a single equivalent source located directly below it, and (ii) both the data and sources are based on planar and regularly spaced grids, Takahashi et al. (2020, 2022) demonstrate that \mathbf{G} exhibits a block-Toeplitz block (BTTB) structure. In such cases, the product of \mathbf{G} and an arbitrary vector can be efficiently computed using a 2D fast Fourier transform as a discrete convolution.

3 LINEAR INVERSE PROBLEM OF EQUIVALENT-LAYER TECHNIQUE

3.1 General formulation

A general formulation for almost all equivalent-layer methods can be achieved by first considering that the $P \times 1$ parameter vector \mathbf{p} (equation 3) can be reparameterized into a $Q \times 1$ vector \mathbf{q} according to:

$$\mathbf{p} = \mathbf{H} \mathbf{q}, \quad (9)$$

where \mathbf{H} is a $P \times Q$ matrix. The predicted data vector \mathbf{f} (equation 3) can then be rewritten as follows:

$$\mathbf{f} = \mathbf{G} \mathbf{H} \mathbf{q}. \quad (10)$$

Note that the original parameter vector \mathbf{p} is defined in a P -dimensional space whereas the reparameterized parameter vector \mathbf{q} (equation 9) lies in a Q -dimensional space. For convenience, we use the terms P -space and Q -space to designate them.

228 In this case, the problem of estimating a parameter vector $\tilde{\mathbf{p}}$ minimizing a length measure of the difference
 229 between \mathbf{f} (equation 3) and \mathbf{d} is replaced by that of estimating an auxiliary vector $\tilde{\mathbf{q}}$ minimizing the goal
 230 function

$$\Gamma(\mathbf{q}) = \Phi(\mathbf{q}) + \mu \Theta(\mathbf{q}), \quad (11)$$

231 which is a combination of particular measures of length given by

$$\Phi(\mathbf{q}) = (\mathbf{d} - \mathbf{f})^\top \mathbf{W}_d (\mathbf{d} - \mathbf{f}), \quad (12)$$

232 and

$$\Theta(\mathbf{q}) = (\mathbf{q} - \bar{\mathbf{q}})^\top \mathbf{W}_q (\mathbf{q} - \bar{\mathbf{q}}), \quad (13)$$

233 where the regularization parameter μ is a positive scalar controlling the trade-off between the data-misfit
 234 function $\Phi(\mathbf{q})$ and the regularization function $\Theta(\mathbf{q})$; \mathbf{W}_d is a $D \times D$ symmetric matrix defining the relative
 235 importance of each observed datum d_i ; \mathbf{W}_q is a $Q \times Q$ symmetric matrix imposing prior information on \mathbf{q} ;
 236 and $\bar{\mathbf{q}}$ is a $Q \times 1$ vector of reference values for \mathbf{q} that satisfies

$$\bar{\mathbf{p}} = \mathbf{H} \bar{\mathbf{q}}, \quad (14)$$

237 where $\bar{\mathbf{p}}$ is a $P \times 1$ vector containing reference values for the original parameter vector \mathbf{p} .

238 After obtaining an estimate $\tilde{\mathbf{q}}$ for the reparameterized parameter vector \mathbf{q} (equation 9), the estimate $\tilde{\mathbf{p}}$ for
 239 the original parameter vector (equation 3) is computed by

$$\tilde{\mathbf{p}} = \mathbf{H} \tilde{\mathbf{q}}. \quad (15)$$

240 The reparameterized vector $\tilde{\mathbf{q}}$ is obtained by first computing the gradient of $\Gamma(\mathbf{q})$,

$$\nabla \Gamma(\mathbf{q}) = -2 \mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d (\mathbf{d} - \mathbf{f}) + 2\mu \mathbf{W}_q (\mathbf{q} - \bar{\mathbf{q}}). \quad (16)$$

241 Then, by considering that $\nabla \Gamma(\tilde{\mathbf{q}}) = \mathbf{0}$ (equation 16), where $\mathbf{0}$ is a vector of zeros, as well as adding and
 242 subtracting the term $(\mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d \mathbf{G} \mathbf{H}) \bar{\mathbf{q}}$, we obtain

$$\tilde{\boldsymbol{\delta}}_q = \mathbf{B} \boldsymbol{\delta}_d, \quad (17)$$

243 where

$$\tilde{\mathbf{q}} = \tilde{\boldsymbol{\delta}}_q + \bar{\mathbf{q}}, \quad (18)$$

$$\boldsymbol{\delta}_d = \mathbf{d} - \mathbf{G} \mathbf{H} \bar{\mathbf{q}}, \quad (19)$$

$$\mathbf{B} = (\mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d \mathbf{G} \mathbf{H} + \mu \mathbf{W}_q)^{-1} \mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d, \quad (20)$$

246 or, equivalently (Menke, 2018, p. 62),

$$\mathbf{B} = \mathbf{W}_q^{-1} \mathbf{H}^\top \mathbf{G}^\top (\mathbf{G} \mathbf{H} \mathbf{W}_q^{-1} \mathbf{H}^\top \mathbf{G}^\top + \mu \mathbf{W}_d^{-1})^{-1}. \quad (21)$$

247 Evidently, we have considered that all inverses exist in equations 20 and 21.

248 The $Q \times D$ matrix \mathbf{B} defined by equation 20 is commonly used for the case in which $D > Q$, i.e., when
 249 there are more data than parameters (overdetermined problems). In this case, we consider that the estimate

250 $\tilde{\mathbf{q}}$ is obtained by solving the following linear system for $\tilde{\boldsymbol{\delta}}_q$ (equation 18):

$$\left(\mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d \mathbf{G} \mathbf{H} + \mu \mathbf{W}_q \right) \tilde{\boldsymbol{\delta}}_q = \mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d \boldsymbol{\delta}_d . \quad (22)$$

251 On the other hand, for the cases in which $D < Q$ (underdetermined problems), matrix \mathbf{B} is usually defined
252 according to equation 21. In this case, the general approach involves estimating $\tilde{\mathbf{q}}$ in two steps. The first
253 consists in solving a linear system for a dummy vector, which is subsequently used to compute $\tilde{\mathbf{q}}$ by a
254 matrix-vector product as follows:

$$\begin{aligned} \left(\mathbf{G} \mathbf{H} \mathbf{W}_q^{-1} \mathbf{H}^\top \mathbf{G}^\top + \mu \mathbf{W}_d^{-1} \right) \mathbf{u} &= \boldsymbol{\delta}_d , \\ \tilde{\boldsymbol{\delta}}_q &= \mathbf{W}_q^{-1} \mathbf{H}^\top \mathbf{G}^\top \mathbf{u} \end{aligned} \quad (23)$$

255 where \mathbf{u} is a dummy vector. After obtaining $\tilde{\boldsymbol{\delta}}_q$ (equations 22 and 23), the estimate $\tilde{\mathbf{q}}$ is computed with
256 equation 18.

257 3.2 Formulation without reparameterization

258 Note that, for the particular case in which $\mathbf{H} = \mathbf{I}_P$ (equation 9), where \mathbf{I}_P is the identity of order P ,
259 $P = Q$, $\mathbf{p} = \mathbf{q}$, $\bar{\mathbf{p}} = \bar{\mathbf{q}}$ (equation 14) and $\tilde{\mathbf{p}} = \tilde{\mathbf{q}}$ (equation 15). In this case, the linear system (equations 22
260 and 23) is directly solved for

$$\tilde{\boldsymbol{\delta}}_p = \tilde{\mathbf{p}} - \bar{\mathbf{p}} , \quad (24)$$

261 instead of $\tilde{\boldsymbol{\delta}}_q$ (equation 18).

262 3.3 Linear system solvers

263 According to their properties, the linear systems associated with over and underdetermined problems
264 (equations 22 and 23) can be solved by using *direct methods* such as LU, Cholesky or QR factorization, for
265 example (Golub and Van Loan, 2013, sections 3.2, 4.2 and 5.2). These methods involve factorizing the
266 linear system matrix in a product of “simple” matrices (i.e., triangular, diagonal or orthogonal). Here, we
267 consider the *Cholesky factorization*, (Golub and Van Loan, 2013, p. 163).

268 Let us consider a real linear system $\mathbf{M} \mathbf{x} = \mathbf{y}$, where \mathbf{M} is a symmetric and positive definite matrix
269 (Golub and Van Loan, 2013, p. 159). In this case, the Cholesky factorization consists in computing

$$\mathbf{M} = \mathcal{G} \mathcal{G}^\top , \quad (25)$$

270 where \mathcal{G} is a lower triangular matrix called *Cholesky factor* and having positive diagonal entries. Given \mathcal{G} ,
271 the original linear system is replaced by two triangular systems, as follows:

$$\begin{aligned} \mathcal{G} \mathbf{s} &= \mathbf{y} \\ \mathcal{G}^\top \mathbf{x} &= \mathbf{s} \end{aligned} \quad (26)$$

272 where \mathbf{s} is a dummy vector. For the overdetermined problem (equation 22), $\mathbf{M} =$
273 $(\mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d \mathbf{G} \mathbf{H} + \mu \mathbf{W}_q)$, $\mathbf{x} = \tilde{\boldsymbol{\delta}}_q$ and $\mathbf{y} = (\mathbf{H}^\top \mathbf{G}^\top \mathbf{W}_d \boldsymbol{\delta}_d)$. For the underdetermined problem
274 (equation 23), $\mathbf{M} = (\mathbf{G} \mathbf{H} \mathbf{W}_q^{-1} \mathbf{H}^\top \mathbf{G}^\top + \mu \mathbf{W}_d^{-1})$, $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = \boldsymbol{\delta}_d$.

275 The use of direct methods for solving large linear systems may be problematic due to computer (i) storage
 276 of large matrices and (ii) time to perform matrix operations. This problem may be specially complicated in
 277 equivalent-layer technique for the cases in which the sensitivity matrix \mathbf{G} does not have a well-defined
 278 structure (sec. 2.2)

279 These problems can be overcome by solving the linear system using an iterative method. These methods
 280 produce a sequence of vectors that typically converge to the solution at a reasonable rate. The main
 281 computational cost associated with these methods is usually some matrix-vector products per iteration. The
 282 *conjugate gradient* (CG) is a very popular iterative method for solving linear systems in equivalent-layer
 283 methods. This method was originally developed to solve systems having a square and positive definite
 284 matrix. There are two adapted versions of the CG method. The first is called *conjugate gradient normal*
 285 *equation residual* (CGNR) Golub and Van Loan (2013, sec. 11.3) or *conjugate gradient least squares*
 286 (CGLS) (Aster et al., 2019, p. 165) and is used to solve overdetermined problems (equation 22). The second
 287 is called *conjugate gradient normal equation error* (CGNE) method Golub and Van Loan (2013, sec. 11.3)
 288 and is used to solve the underdetermined problems (equation 23). Algorithm 1 outlines the CGLS method
 289 applied to the overdetermined problem (equation 22).

4 FLOATING-POINT OPERATIONS

290 Two important factors affecting the efficiency of a given matrix algorithm are the storage and amount of
 291 required arithmetic. Here, we quantify this last factor associated with different computational strategies to
 292 solve the linear system of the equivalent-layer technique (section 7). To do it, we opted by counting *flops*,
 293 which are floating point additions, subtractions, multiplications or divisions (Golub and Van Loan, 2013,
 294 p. 12–14). This is a non-hardware dependent approach that allows us to do direct comparison between
 295 different equivalent-layer methods. Most of the flops count used here can be found in Golub and Van Loan
 296 (2013, p. 12, 106, 107 and 164).

297 Let us consider the case in which the overdetermined problem (equation 22) is solved by Cholesky
 298 factorization (equations 25 and 26) directly for the parameter vector $\tilde{\mathbf{p}}$ by considering the particular case in
 299 which $\mathbf{H} = \mathbf{I}_P$ (equation 9 and subsection 3.2), $\mu = 0$ (equation 11), $\mathbf{W}_d = \mathbf{I}_D$ (equation 12) and $\bar{\mathbf{p}} = \mathbf{0}$
 300 (equation 14), where \mathbf{I}_P and \mathbf{I}_D are the identities of order P and D , respectively. Based on the information
 301 provided in table 1, the total number of flops can be determined by aggregating the flops required for
 302 various computations. These computations include the matrix-matrix and matrix-vector products $\mathbf{G}^\top \mathbf{G}$
 303 and $\mathbf{G}^\top \mathbf{d}$, the Cholesky factor \mathcal{G} , and the solution of triangular systems. Thus, we can express the total
 304 number of flops as follows:

$$f_{\text{Cholesky}} = 1/3D^3 + 2D^2 + 2(P^2 + P)D. \quad (27)$$

305 The same particular overdetermined problem can be solved by using the CGLS method (Algorithm 1).
 306 In this case, we use table 1 again to combine the total number of flops associated with the matrix-vector
 307 and inner products defined in line 3, before starting the iteration, and the 3 saxpys, 2 inner products and 2
 308 matrix-vector products per iteration (lines 7 – 12). By considering a maximum number of iterations ITMAX,
 309 we obtain

$$f_{\text{CGLS}} = 2P(D + 1) + \text{ITMAX} [2P(2D + 3) + 4D]. \quad (28)$$

310 The same approach used to deduce equations 27 and 28 is applied to compute the total number of flops for
 311 the selected equivalent-layer methods discussed in section 7.

312 To simplify our analysis, we do not consider the number of flops required to compute the sensitivity
 313 matrix \mathbf{G} (equation 3) or the matrix \mathbf{A} associated with a given potential-field transformation (equation 4)
 314 because they depend on the specific harmonic functions g_{ij} and a_{ij} (equations 2 and 5). We also neglect
 315 the required flops to compute \mathbf{H} , \mathbf{W}_d , \mathbf{W}_q (equations 9, 12 and 13), \bar{p} (equation 14), retrieve $\tilde{\mathbf{q}}$ from $\tilde{\delta}_q$
 316 (equation 18) and computing δ_d (equation 19).

5 NUMERICAL STABILITY

317 All equivalent-layer methods aim at obtaining an estimate $\tilde{\mathbf{p}}$ for the parameter vector \mathbf{p} (equation 3), which
 318 contains the physical property of the equivalent sources. Some methods do it by first obtaining an estimate
 319 $\tilde{\mathbf{q}}$ for the reparameterized parameter vector \mathbf{q} (equation 9) and then using it to obtain $\tilde{\mathbf{p}}$ (equation 15).
 320 The stability of a solution $\tilde{\mathbf{p}}$ against noise in the observed data is rarely addressed. Here, we follow the
 321 numerical stability analysis presented in Siqueira et al. (2017).

322 For a given equivalent-layer method (section 7), we obtain an estimate $\tilde{\mathbf{p}}$ assuming noise-free potential-
 323 field data \mathbf{d} . Then, we create L different noise-corrupted data \mathbf{d}^ℓ , $\ell \in \{1 : L\}$, by adding L different
 324 sequences of pseudorandom Gaussian noise to \mathbf{d} , all of them having zero mean. From each \mathbf{d}^ℓ , we obtain
 325 an estimate $\tilde{\mathbf{p}}^\ell$. Regardless of the particular equivalent-layer method used, the following inequality (Aster
 326 et al., 2019, p. 66) holds true:

$$\Delta p^\ell \leq \kappa \Delta d^\ell, \quad \ell \in \{1 : L\}, \quad (29)$$

327 where κ is the constant of proportionality between the model perturbation

$$\Delta p^\ell = \frac{\|\tilde{\mathbf{p}}^\ell - \tilde{\mathbf{p}}\|}{\|\tilde{\mathbf{p}}\|}, \quad \ell \in \{1 : L\}, \quad (30)$$

328 and the data perturbation

$$\Delta d^\ell = \frac{\|\mathbf{d}^\ell - \mathbf{d}\|}{\|\mathbf{d}\|}, \quad \ell \in \{1 : L\}, \quad (31)$$

329 with $\|\cdot\|$ representing the Euclidean norm. The constant κ acts as the condition number associated with the
 330 pseudo-inverse in a given linear inversion. The larger (smaller) the value of κ , the more unstable (stable) is
 331 the estimated solution. Equation 29 shows a linear relationship between the model perturbation Δp^ℓ and
 332 the data perturbation Δd^ℓ (equations 30 and 31). We estimate the κ (equation 29) associated with a given
 333 equivalent-layer method as the slope of the straight line fitted to the L points $(\Delta p^\ell, \Delta d^\ell)$.

6 NOTATION FOR SUBVECTORS AND SUBMATRICES

Here, we use a notation inspired on that presented by Van Loan (1992, p. 4) to represent subvectors and submatrices. Subvectors of \mathbf{d} , for example, are specified by $\mathbf{d}[\mathbf{i}]$, where \mathbf{i} is a list of integer numbers that “pick out” the elements of \mathbf{d} forming the subvector $\mathbf{d}[\mathbf{i}]$. For example, $\mathbf{i} = (1, 6, 4, 6)$ gives the subvector $\mathbf{d}[\mathbf{i}] = [d_1 \ d_6 \ d_4 \ d_6]^\top$. Note that the list \mathbf{i} of indices may be sorted or not and it may also have repeated indices. For the particular case in which the list has a single element $\mathbf{i} = (i)$, then it can be used to extract the i -th element $d_i \equiv \mathbf{d}[i]$ of \mathbf{d} . Sequential lists can be represented by using the colon notation. We consider two types of sequential lists. The first has starting index is smaller than the final index and increment of 1.

The second has starting index is greater than the final index and increment of -1 . For example,

$$\begin{aligned}\mathbf{i} = (3 : 8) &\Leftrightarrow \mathbf{d}[\mathbf{i}] = [d_3 \ d_4 \ \dots \ d_8]^\top \\ \mathbf{i} = (8 : 3) &\Leftrightarrow \mathbf{d}[\mathbf{i}] = [d_8 \ d_7 \ \dots \ d_3]^\top \\ \mathbf{i} = (: 8) &\Leftrightarrow \mathbf{d}[\mathbf{i}] = [d_1 \ d_2 \ \dots \ d_8]^\top \\ \mathbf{i} = (3 :) &\Leftrightarrow \mathbf{d}[\mathbf{i}] = [d_3 \ d_4 \ \dots \ d_D]^\top\end{aligned}$$

334 where D is the number of elements forming \mathbf{d} .

The notation above can also be used to define submatrices of a $D \times P$ matrix \mathbf{G} . For example, $\mathbf{i} = (2, 7, 4, 6)$ and $\mathbf{j} = (1, 3, 8)$ lead to the submatrix

$$\mathbf{G}[\mathbf{i}, \mathbf{j}] = \begin{bmatrix} g_{21} & g_{23} & g_{28} \\ g_{71} & g_{73} & g_{78} \\ g_{41} & g_{43} & g_{48} \\ g_{61} & g_{63} & g_{68} \end{bmatrix}.$$

Note that, in this case, the lists \mathbf{i} and \mathbf{j} “pick out”, respectively, the rows and columns of \mathbf{G} that form the submatrix $\mathbf{G}[\mathbf{i}, \mathbf{j}]$. The i -th row of \mathbf{G} is given by the $1 \times P$ vector $\mathbf{G}[i, :]$. Similarly, the $D \times 1$ vector $\mathbf{G}[:, j]$ represents the j -th column. Finally, we may use the colon notation to define the following submatrix:

$$\mathbf{i} = (2 : 5), \mathbf{j} = (3 : 7) \Leftrightarrow \mathbf{G}[\mathbf{i}, \mathbf{j}] = \begin{bmatrix} g_{23} & g_{24} & g_{25} & g_{26} & g_{27} \\ g_{33} & g_{34} & g_{35} & g_{36} & g_{37} \\ g_{43} & g_{44} & g_{45} & g_{46} & g_{47} \\ g_{53} & g_{54} & g_{55} & g_{56} & g_{57} \end{bmatrix},$$

335 which contains the contiguous elements of \mathbf{G} from rows 2 to 5 and from columns 3 to 7.

7 COMPUTATIONAL STRATEGIES

336 The linear inverse problem of the equivalent-layer technique (section 3) for the case in which there are
337 large volumes of potential-field data requires dealing with:

- 338 (i) the large computer memory to store large and full matrices;
- 339 (ii) the long computation time to multiply a matrix by a vector; and
- 340 (iii) the long computation time to solve a large linear system of equations.

341 Here, we review some strategies aiming at reducing the computational cost of the equivalent-layer technique.
342 We quantify the computational cost by using flops (section 4) and compare the results with those obtained
343 for Cholesky factorization and CGLS (equations 27 and 28). We focus on the overall strategies used by the
344 selected methods.

345 7.1 Moving window

346 The initial approach to enhance the computational efficiency of the equivalent-layer technique is
347 commonly denoted *moving window* and involves first splitting the observed data $d_i, i \in \{1 : D\}$, into
348 M overlapping subsets (or data windows) formed by D^m data each, $m \in \{1 : M\}$. The data inside the

349 m -th window are usually adjacent to each other and have indices defined by an integer list \mathbf{i}^m having
 350 D^m elements. The number of data D^m forming the data windows are not necessarily equal to each other.
 351 Each data window has a $D^m \times 1$ observed data vector $\mathbf{d}^m \equiv \mathbf{d}[\mathbf{i}^m]$. The second step consists in defining
 352 a set of P equivalent sources with scalar physical property p_j , $j \in \{1 : P\}$, and also split them into M
 353 overlapping subsets (or source windows) formed by P^m data each, $m \in \{1 : M\}$. The sources inside the
 354 m -th window have indices defined by an integer list \mathbf{j}^m having P^m elements. Each source window has a
 355 $P^m \times 1$ parameter vector \mathbf{p}^m and is located right below the corresponding m -th data window. Then, each
 356 $\mathbf{d}^m \equiv \mathbf{d}[\mathbf{i}^m]$ is approximated by

$$\mathbf{f}^m = \mathbf{G}^m \mathbf{p}^m, \quad (32)$$

357 where $\mathbf{G}^m \equiv \mathbf{G}[\mathbf{i}^m, \mathbf{j}^m]$ is a submatrix of \mathbf{G} (equation 3) formed by the elements computed with equation
 358 2 using only the data and equivalent sources located inside the window m -th. The main idea of the moving-
 359 window approach is using the $\tilde{\mathbf{p}}^m$ estimated for each window to obtain (i) an estimate $\tilde{\mathbf{p}}$ of the parameter
 360 vector for the entire equivalent layer or (ii) a given potential-field transformation \mathbf{t} (equation 4). The main
 361 advantages of this approach is that (i) the estimated parameter vector $\tilde{\mathbf{p}}$ or transformed potential field are
 362 not obtained by solving the full, but smaller linear systems and (ii) the full matrix \mathbf{G} (equation 3) is never
 363 stored.

364 Leão and Silva (1989) presented a pioneer work using the moving-window approach. Their method
 365 requires a regularly-spaced grid of observed data on a horizontal plane z_0 . The data windows are defined by
 366 square local grids of $\sqrt{D'} \times \sqrt{D'}$ adjacent points, all of them having the same number of points D' . The
 367 equivalent sources in the m -th data window are located below the observation plane, at a constant vertical
 368 distance Δz_0 . They are arranged on a regular grid of $\sqrt{P'} \times \sqrt{P'}$ adjacent points following the same
 369 grid pattern of the observed data. The local grid of sources for all data windows have the same number
 370 of elements P' . Besides, they are vertically aligned, but expands the limits of their corresponding data
 371 windows, so that $D' < P'$. Because of this spatial configuration of observed data and equivalent sources,
 372 we have that $\mathbf{G}^m = \mathbf{G}'$ (equation 32) for all data windows (i.e., $\forall m \in \{1 : M\}$), where \mathbf{G}' is a $D' \times P'$
 373 constant matrix.

374 By omitting the normalization strategy used by Leão and Silva (1989), their method consists in directly
 375 computing the transformed potential field t_c^m at the central point $(x_c^m, y_c^m, z_0 + \Delta z_0)$ of each data window
 376 as follows:

$$t_c^m = (\mathbf{a}')^\top \mathbf{B}' \mathbf{d}^m, \quad m \in \{1 : M\}, \quad (33)$$

377 where \mathbf{a}' is a $P' \times 1$ vector with elements computed by equation 5 by using all equivalent sources in the
 378 m -th window and only the coordinate of the central point in the m -th data window and

$$\mathbf{B}' = (\mathbf{G}')^\top \left[\mathbf{G}' (\mathbf{G}')^\top + \mu \mathbf{I}_{D'} \right]^{-1} \quad (34)$$

379 is a particular case of matrix \mathbf{B} associated with underdetermined problems (equation 21) for the particular
 380 case in which $\mathbf{H} = \mathbf{W}_q = \mathbf{I}_{P'}$ (equations 9 and 13), $\mathbf{W}_d = \mathbf{I}_{D'}$ (equation 12), $\bar{\mathbf{p}} = \mathbf{0}$ (equation 14), where
 381 $\mathbf{I}_{P'}$ and $\mathbf{I}_{D'}$ are identity matrices of order P' and D' , respectively, and $\mathbf{0}$ is a vector of zeros. Due to the
 382 presumed spatial configuration of the observed data and equivalent sources, \mathbf{a}' and \mathbf{G}' are the same for all
 383 data windows. Hence, only the data vector \mathbf{d}^m is modified according to the position of the data window.
 384 Note that equation 33 combines the potential-field transformation (equation 4) with the solution of the
 385 undetermined problem (equation 23).

386 The method proposed by Leão and Silva (1989) can be outlined by the Algorithm 2. Note that Leão and
 387 Silva (1989) directly compute the transformed potential t_c^m at the central point of each data window without
 388 explicitly computing and storing an estimated for \mathbf{p}^m (equation 32). It means that their method allows
 389 computing a single potential-field transformation. A different transformation or the same one evaluated at
 390 different points require running their moving-data window method again.

391 The total number of flops in Algorithm 2 depends on computing the $P' \times D'$ matrix \mathbf{B}' (equation 34) in
 392 line 6 and use it to define the $1 \times P'$ vector $(\mathbf{a}')^\top \mathbf{B}'$ (line 7) before starting the iterations and computing
 393 an inner product (equation 33) per iteration. We consider that the total number of flops associated with \mathbf{B}'
 394 is obtained by the matrix-matrix product $\mathbf{G}' (\mathbf{G}')^\top$, its inverse and then the premultiplication by $(\mathbf{G}')^\top$. By
 395 using table 1 and considering that inverse is computed via Cholesky factorization, we obtain that the total
 396 number of flops for lines 6 and 7 is $2(D')^2 P' + 7(D')^3/6 + 2(D')^2 P'$. Then, the total number of flops for
 397 Algorithm 2 is

$$f_{LS89} = 7/6(D')^3 + 4P'(D')^2 + M 2P'. \quad (35)$$

398 Soler and Uieda (2021) generalized the method proposed by Leão and Silva (1989) for irregularly spaced
 399 data on an undulating surface. A direct consequence of this generalization is that a different submatrix
 400 $\mathbf{G}^m \equiv \mathbf{G}[\mathbf{i}^m, \mathbf{j}^m]$ (equation 32) must be computed for each window. Differently from Leão and Silva
 401 (1989), Soler and Uieda (2021) store the computed $\tilde{\mathbf{p}}^m$ for all windows and subsequently use them to obtain
 402 a desired potential-field transformation (equation 4) as the superposed effect of all windows. The estimated
 403 $\tilde{\mathbf{p}}^m$ for all windows are combined to form a single $P \times 1$ vector $\tilde{\mathbf{p}}$, which is an estimate for original
 404 parameter vector \mathbf{p} (equation 3). For each data window, Soler and Uieda (2021) solve an overdetermined
 405 problem (equation 22) for $\tilde{\mathbf{p}}^m$ by using $\mathbf{H} = \mathbf{W}_q = \mathbf{I}_{P^m}$ (equations 9 and 13), \mathbf{W}_d^m (equation 12) equal to
 406 a diagonal matrix of weights for the data inside the m -th window and $\bar{\mathbf{p}} = \mathbf{0}$ (equation 14), so that

$$[(\mathbf{G}^m)^\top \mathbf{W}_d^m \mathbf{G}^m + \mu \mathbf{I}_{P'}] \tilde{\mathbf{p}}^m = (\mathbf{G}^m)^\top \mathbf{W}_d^m \mathbf{d}^m. \quad (36)$$

407 Unlike Leão and Silva (1989), Soler and Uieda (2021) do not adopt a sequential order of the data windows;
 408 rather, they adopt a randomized order of windows in their iterations. The overall steps of the method
 409 proposed by Soler and Uieda (2021) are defined by the Algorithm 3. For convenience, we have omitted the
 410 details about the randomized window order, normalization strategy employed and block-averaged sources
 411 layout proposed by those authors (see subsection 2.1). Note that this algorithm starts with a residuals vector
 412 \mathbf{r} that is iteratively updated. The iterative algorithm in Soler and Uieda (2021) estimates a solution ($\tilde{\mathbf{p}}^m$ in
 413 equation 36) using the data and the equivalent sources that fall within a moving-data window; however, it
 414 calculates the predicted data and the residual data in the whole survey data. Next, the residual data that fall
 415 within a new position of the data window is used as input data to estimate a new solution within the data
 416 window which, in turn, is used to calculate a new predicted data and a new residual data in the whole
 417 survey data.

418 The computational cost of Algorithm 3 can be defined in terms of the linear system (equation 36) to be
 419 solved for each window (line 10) and the subsequent updates in lines 11 and 12. We consider that the linear
 420 system cost can be quantified by the matrix-matrix and matrix-vector products $(\mathbf{G}^m)^\top \mathbf{G}^m$ and $(\mathbf{G}^m)^\top \mathbf{d}^m$,
 421 respectively, and solution of the linear system (line 10) via Cholesky factorization (equations 25 and 26).
 422 The following updates represent a saxpy without scalar-vector product (line 11) and a matrix-vector product
 423 (line 12). In this case, according to table 1, the total number of flops associated with Algorithm 3 is given

424 by:

$$f_{\text{SU21}} = M \left[\frac{1}{3}(P')^3 + 2(D' + 1)(P')^2 + (4D' + 1)P' \right], \quad (37)$$

425 where P' and D' represent, respectively, the average number of equivalent sources and data at each window.426 **7.2 Column-action update**427 We call the computational strategy *column-action update* because a single source is used to calculate the
428 predicted data and the residual data in the whole survey data. Hence, a single column of the sensitivity
429 matrix \mathbf{G} (equation 3) is calculated iteratively.430 Cordell (1992) proposed a computational strategy that was later used by Guspí and Novara (2009) and
431 relies on first defining one equivalent source located right below each observed data d_i , $i \in \{1 : D\}$, at
432 a vertical coordinate $z_i + \Delta z_i$, where Δz_i is proportional to the distance from the i -th observation point
433 (x_i, y_i, z_i) to its closest neighbor. The second step consists in updating the physical property p_j of a single
434 equivalent source, $j \in \{1 : D\}$ and remove its predicted potential field from the observed data vector \mathbf{d} ,
435 producing a residuals vector \mathbf{r} . At each iteration, the single equivalent source is the one located vertically
436 beneath the observation station of the maximum data residual. Next, the predicted data produced by this
437 single source is calculated over all of the observation points and a new data residual \mathbf{r} and the $D \times 1$
438 parameter vector \mathbf{p} containing the physical property of all equivalent sources are updated iteratively. During
439 each subsequent iteration, Cordell's method either incorporates a single equivalent source or adjusts an
440 existing equivalent source to match the maximum amplitude of the current residual field. The convergence
441 occurs when all of the residuals are bounded by an envelope of prespecified expected error. At the end, the
442 algorithm produces an estimate $\tilde{\mathbf{p}}$ for the parameter vector yielding a predicted potential field \mathbf{f} (equation
443 3) satisfactorily fitting the observed data \mathbf{d} according to a given criterion. Note that the method proposed
444 by Cordell (1992) iteratively solves the linear $\mathbf{G}\tilde{\mathbf{p}} \approx \mathbf{d}$ with a $D \times D$ matrix \mathbf{G} . At each iteration, only a
445 single column of \mathbf{G} (equation 3) is used. An advantage of this *column-action update approach* is that the
446 full matrix \mathbf{G} is never stored.447 Algorithm 4 delineates the Cordell's method. Note that a single column $\mathbf{G}[:, i_{\max}]$ of the $D \times D$ matrix \mathbf{G}
448 (equation 3) is used per iteration, where i_{\max} is the index of the maximum absolute value in \mathbf{r} . As pointed out
449 by Cordell (1992), the method does not necessarily decrease monotonically along the iterations. Besides,
450 the method may not converge depending on how the vertical distances Δz_i , $i \in \{1 : D\}$, controlling the
451 depths of the equivalent sources are set. According to Cordell (1992), the maximum absolute value r_{\max}
452 in \mathbf{r} decreases robustly at the beginning and oscillates within a narrowing envelope for the subsequent
453 iterations.454 Guspí and Novara (2009) generalized Cordell's method to perform reduction to the pole and other
455 transformations on scattered magnetic observations by using two steps. The first step involves computing
456 the vertical component of the observed field using equivalent sources while preserving the magnetization
457 direction. In the second step, the vertical observation direction is maintained, but the magnetization
458 direction is shifted to the vertical. The main idea employed by both Cordell (1992) and Guspí and Novara
459 (2009) is an iterative scheme that uses a single equivalent source positioned below a measurement station
460 to compute both the predicted data and residual data for all stations. This approach entails a computational
461 strategy where a single column of the sensitivity matrix \mathbf{G} (equation 3) is calculated per iteration.462 The total number of flops in Algorithm 4 consists in finding the maximum absolute value in vector \mathbf{r}
463 (line 6) before the while loop. Per iteration, there is a saxpy (line 11) and another search for the maximum
464 absolute value in vector \mathbf{r} (line 12). By considering that selecting the maximum absolute value in a $D \times 1$

465 vector is a $D \log_2(D)$ operation (e.g., Press et al., 2007, p. 420), the total number of flops in Algorithm 38
 466 is given by:

$$f_{C92} = D \log(D) + \text{ITMAX} [2D + D \log_2(D)] . \quad (38)$$

467 **7.3 Row-action update**

468 We call the computational strategy *row-action update* because a single row of the sensitivity matrix
 469 \mathbf{G} (equation 3) is calculated iteratively. Hence, the equivalent-layer solution is updated by processing a
 470 new datum (one matrix row) at each iteration. To reduce the total processing time and memory usage of
 471 equivalent-layer technique, Mendonça and Silva (1994) proposed a strategy called *equivalent data concept*.
 472 The equivalent data concept is grounded on the principle that there is a subset of redundant data that does
 473 not contribute to the final solution and thus can be dispensed. Conversely, there is a subset of observations,
 474 called equivalent data, that contributes effectively to the final solution and fits the remaining observations
 475 (redundant data). Iteratively, Mendonça and Silva (1994) selected the subset of equivalent data that is
 476 substantially smaller than the original dataset. This selection is carried out by incorporating one data point
 477 at a time.

478 Mendonça and Silva (1994) proposes an algebraic reconstruction technique (ART) (e.g., van der Sluis
 479 and van der Vorst, 1987, p. 58) to estimate a parameter vector $\tilde{\mathbf{p}}$ for a regular grid of P equivalent sources
 480 on a horizontal plane z_0 . Such methods iterate on the linear system rows to estimate corrections for the
 481 parameter vector, which may substantially save computer time and memory required to compute and store
 482 the full linear system matrix along the iterations. The convergence of such *row-update methods* depends
 483 on the linear system condition. The main advantage of such methods is not computing and storing the
 484 full linear system matrix, but iteratively using its rows. In contrast to ART-type algorithms, the rows in
 485 Mendonça and Silva (1994) are not processed sequentially. Instead, in Mendonça and Silva (1994), the
 486 rows are introduced according to their residual magnitudes (maximum absolute value in \mathbf{r}), which are
 487 computed based on the estimate over the equivalent layer from the previous iteration. The particular ART
 488 method proposed by Mendonça and Silva (1994) considers that

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_e \\ \mathbf{d}_r \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_e \\ \mathbf{R}_r \end{bmatrix}, \quad (39)$$

489 where \mathbf{d}_e and \mathbf{d}_r are $D_e \times 1$ and $D_r \times 1$ vectors and \mathbf{G}_e and \mathbf{G}_r are $D_e \times P$ and $D_r \times P$ matrices,
 490 respectively. Mendonça and Silva (1994) designate \mathbf{d}_e and \mathbf{d}_r as, respectively, *equivalent* and *redundant*
 491 data. With the exception of a normalization strategy, Mendonça and Silva (1994) calculate a $P \times 1$ estimated
 492 parameter vector $\tilde{\mathbf{p}}$ by solving an underdetermined problem (equation 23) involving only the equivalent
 493 data \mathbf{d}_e (equation 39) for the particular case in which $\mathbf{H} = \mathbf{W}_p = \mathbf{I}_P$ (equations 9 and 13), $\mathbf{W}_d = \mathbf{I}_{D_e}$
 494 (equation 12) and $\bar{\mathbf{p}} = \mathbf{0}$ (equation 14), which results in

$$\begin{aligned} (\mathbf{F} + \mu \mathbf{I}_{D_e}) \mathbf{u} &= \mathbf{d}_e \\ \tilde{\mathbf{p}} &= \mathbf{G}_e^\top \mathbf{u} , \end{aligned} \quad (40)$$

495 where \mathbf{F} is a computationally-efficient $D_e \times D_e$ matrix that approximates $\mathbf{G}_e \mathbf{G}_e^\top$. Mendonça and Silva
 496 (1994) presume that the estimated parameter vector $\tilde{\mathbf{p}}$ obtained from equation 40 leads to a $D_r \times 1$ residuals
 497 vector

$$\mathbf{r} = \mathbf{d}_r - \mathbf{G}_r \tilde{\mathbf{p}} \quad (41)$$

498 having a maximum absolute value $r_{\max} \leq \epsilon$, where ϵ is a predefined tolerance.

499 The overall method of Mendonça and Silva (1994) is defined by Algorithm 5. It is important noting
500 that the number D_e of equivalent data in \mathbf{d}_e increases by one per iteration, which means that the order
501 of the linear system in equation 40 also increases by one at each iteration. Those authors also propose
502 a computational strategy based on Cholesky factorization (e.g., Golub and Van Loan, 2013, p. 163) for
503 efficiently updating $(\mathbf{F} + \mu \mathbf{I}_{D_e})$ at a given iteration (line 16 in Algorithm 5) by computing only its new
504 elements with respect to those computed in the previous iteration.

505 7.4 Reparameterization

506 Another approach for improving the computational performance of equivalent-layer technique consists
507 in setting a $P \times Q$ reparameterization matrix \mathbf{H} (equation 9) with $Q \ll P$. This strategy has been used
508 in applied geophysics for decades (e.g., Skilling and Bryan, 1984; Kennett et al., 1988; Oldenburg et al.,
509 1993; Barbosa et al., 1997) and is known as *subspace method*. The main idea relies in reducing the linear
510 system dimension from the original P -space to a lower-dimensional subspace (the Q -space). An estimate
511 $\tilde{\mathbf{q}}$ for the reparameterized parameter vector \mathbf{q} is obtained in the Q -space and subsequently used to obtain
512 an estimate $\tilde{\mathbf{p}}$ for the parameter vector \mathbf{p} (equation 3) in the P -space by using equation 9. Hence, the key
513 aspect of this *reparameterization approach* is solving an appreciably smaller linear inverse problem for $\tilde{\mathbf{q}}$
514 than that for the original parameter vector $\tilde{\mathbf{p}}$ (equation 3).

515 Oliveira Jr. et al. (2013) have used this approach to describe the physical property distribution on the
516 equivalent layer in terms of piecewise bivariate polynomials. Specifically, their method consists in splitting
517 a regular grid of equivalent sources into source windows inside which the physical-property distribution
518 is described by bivariate polynomial functions. The key aspect of their method relies on the fact that the
519 total number of coefficients required to define the bivariate polynomials is considerably smaller than the
520 original number of equivalent sources. Hence, they formulate a linear inverse problem for estimating the
521 polynomial coefficients and use them later to compute the physical property distribution on the equivalent
522 layer.

523 The method proposed by Oliveira Jr. et al. (2013) consists in solving an overdetermined problem (equation
524 22) for estimating the polynomial coefficients $\tilde{\mathbf{q}}$ with $\mathbf{W}_d = \mathbf{I}_D$ (equation 12) and $\bar{\mathbf{q}} = \mathbf{0}$ (equation 14), so
525 that

$$(\mathbf{H}^\top \mathbf{G}^\top \mathbf{G} \mathbf{H} + \mu \mathbf{W}_q) \tilde{\mathbf{q}} = \mathbf{H}^\top \mathbf{G}^\top \mathbf{d}, \quad (42)$$

526 where $\mathbf{W}_q = \mathbf{H}^\top \mathbf{W}_p \mathbf{H}$ is defined by a matrix \mathbf{W}_p representing the zeroth- and first-order Tikhonov
527 regularization (e.g., Aster et al., 2019, p. 103). Note that, in this case, the prior information is defined in the
528 P -space for the original parameter vector \mathbf{p} and then transformed to the Q -space. Another characteristic of
529 their method is that it is valid for processing irregularly-spaced data on an undulating surface.

530 Mendonça (2020) also proposed a reparameterization approach for the equivalent-layer technique. Their
531 approach, however, consists in setting \mathbf{H} as a truncated singular value decomposition (SVD) (e.g., Aster
532 et al., 2019, p. 55) of the observed potential field. Differently from Oliveira Jr. et al. (2013), however, the
533 method of Mendonça (2020) requires a regular grid of potential-field data on horizontal plane. Another
534 difference is that these authors uses $\mathbf{W}_q = \mathbf{I}_Q$ (equation 13), which means that the regularization is defined
535 directly in the Q -space.

536 We consider an algorithm (not shown) that solves the overdetermined problem (equation 22) by combining
537 the reparameterization with CGLS method (Algorithm 1). It starts with a reparameterization step defined

538 by defining a matrix $\mathbf{C} = \mathbf{G} \mathbf{H}$ (equation 10). Then, the CGLS (Algorithm 1) is applied by replacing \mathbf{G}
 539 with \mathbf{C} . In this case, the linear system is solved by the reparameterized parameter vector $\tilde{\mathbf{q}}$ instead of $\tilde{\mathbf{p}}$.
 540 At the end, the estimated $\tilde{\mathbf{q}}$ is transformed into $\tilde{\mathbf{p}}$ (equation 15). Compared to the original CGLS shown
 541 in Algorithm 1, the algorithm discussed here has the additional flops associated with the matrix-matrix
 542 product to compute \mathbf{C} and the matrix-vector product of equation 15 outside the while loop. Then, according
 543 to table 1, the total number of flops given by:

$$f_{\text{reparam.}} = 2Q(DP + D + 1) + 2PQ + \text{ITMAX} [2Q(2D + 3) + 4D] . \quad (43)$$

544 The important aspect of this approach is that, for the case in which $Q \ll P$ (equation 9), the number of
 545 flops per iteration can be substantially decreased with respect to those associated with Algorithm 1. In this
 546 case, the flops decrease per iteration compensates the additional flops required to compute \mathbf{C} and obtain $\tilde{\mathbf{p}}$
 547 from $\tilde{\mathbf{q}}$ (equation 15).

548 7.5 Sparsity induction

549 Li and Oldenburg (2010) proposed a method that applies the discrete wavelet transform to introduce
 550 sparsity into the original dense matrix \mathbf{G} (equation 3). Those authors approximate a planar grid of potential-
 551 field data by a regularly-spaced grid of equivalent sources, so that the number of data D and sources P is
 552 the same, i.e., $D = P$. Specifically, Li and Oldenburg (2010) proposed a method that applies the wavelet
 553 transform to the original dense matrix \mathbf{G} and sets to zero the small coefficients that are below a given
 554 threshold, which results in an approximating sparse representation of \mathbf{G} in the wavelet domain. They first
 555 consider the following approximation

$$\mathbf{d}_w \approx \mathbf{G}_s \mathbf{p}_w , \quad (44)$$

556 where

$$\mathbf{d}_w = \mathcal{W} \mathbf{d} , \quad \mathbf{p}_w = \mathcal{W} \mathbf{p} , \quad (45)$$

557 are the observed data and parameter vector in the wavelet domain; \mathcal{W} is a $D \times D$ orthogonal matrix
 558 defining a discrete wavelet transform; and \mathbf{G}_s is a sparse matrix obtained by setting to zero the elements of

$$\mathbf{G}_w = \mathcal{W} \mathbf{G} \mathcal{W}^\top \quad (46)$$

559 with absolute value smaller than a given threshold.

560 Li and Oldenburg (2010) solve a normalized inverse problem in the wavelet domain. Specifically, they
 561 first define a matrix

$$\mathbf{G}_L = \mathbf{G}_s \mathbf{L}^{-1} \quad (47)$$

562 and a normalized parameter vector

$$\mathbf{p}_L = \mathbf{L} \mathbf{p}_w , \quad (48)$$

563 where \mathbf{L} is a diagonal and invertible matrix representing an approximation of the first-order Tikhonov
 564 regularization in the wavelet domain. Then they solve an overdetermined problem (equation 22) to obtain
 565 an estimate $\tilde{\mathbf{p}}_L$ for \mathbf{p}_L (equation 48), with \mathbf{G}_L (equation 47), $\mathbf{H} = \mathbf{I}_P$ (equations 9), $\mu = 0$ (equation 11),
 566 $\mathbf{W}_d = \mathbf{I}_D$ (equation 12) and $\bar{\mathbf{p}} = \mathbf{0}$ (equation 14) via conjugate-gradient method (e.g., Golub and Van
 567 Loan, 2013, sec. 11.3). Finally, Li and Oldenburg (2010) compute an estimate $\tilde{\mathbf{p}}$ for the original parameter
 568 vector given by

$$\tilde{\mathbf{p}} = \mathcal{W}^\top (\mathbf{L}^{-1} \tilde{\mathbf{p}}_L) , \quad (49)$$

569 where the term within parenthesis is an estimate $\tilde{\mathbf{p}}_w$ of the parameter vector \mathbf{p}_w (equation 45) in the wavelet
 570 domain and matrix \mathcal{W}^\top represents an inverse wavelet transform.

571 Barnes and Lumley (2011) also proposed a computationally efficient method for equivalent-layer
 572 technique by inducing sparsity into the original sensitivity matrix \mathbf{G} (equation 3). Their approach consists
 573 in setting a $P \times Q$ reparameterization matrix \mathbf{H} (equation 9) with $Q \approx 1.7 P$. Note that, differently from
 574 Oliveira Jr. et al. (2013) and Mendonça (2020), Barnes and Lumley (2011) do not use the reparameterization
 575 with the purpose of reducing the number of the parameters. Instead, they use a reparameterization scheme
 576 that groups distant equivalent sources into blocks by using a bisection process. This scheme leads to
 577 a quadtree representation of the physical-property distribution on the equivalent layer, so that matrix
 578 \mathbf{GH} (equation 10) is notably sparse. Barnes and Lumley (2011) explore this sparsity in solving the
 579 overdetermined problem for $\tilde{\mathbf{q}}$ (equation 42) via conjugate-gradient method (e.g., Golub and Van Loan,
 580 2013, sec. 11.3).

581 It is difficult to predict the exact sparsity obtained from the methods proposed by Li and Oldenburg (2010)
 582 and Barnes and Lumley (2011) because it depends on several factors, including the observed potential-field
 583 data. According to Li and Oldenburg (2010), their wavelet approach results in a sparse matrix having $\approx 2\%$
 584 of the elements in \mathbf{G}_w (equation 46). The reparameterization proposed by Barnes and Lumley (2011) leads
 585 to a sparse matrix \mathbf{GH} (equation 10) with only $\approx 1\%$ of non-zero elements. These sparsity patterns can be
 586 efficiently explored, for example, in computing the required matrix-vector products along the iterations of
 587 the CGLS method (Algorithm 1).

588 7.6 Iterative methods using the full matrix \mathbf{G}

589 Xia and Sprowl (1991) introduced an iterative method for estimating the parameter vector $\tilde{\mathbf{p}}$ (equation 3),
 590 which was subsequently adapted to the Fourier domain by Xia et al. (1993). Their method uses the full
 591 and dense sensitivity matrix \mathbf{G} (equation 3) (without applying any compression or reparameterization, for
 592 example) to compute the predicted data at all observation points per iteration. More than two decades later,
 593 Siqueira et al. (2017) have proposed an iterative method similar to that presented by Xia and Sprowl (1991).
 594 The difference is that Siqueira et al.'s algorithm was deduced from the *Gauss' theorem* (e.g., Kellogg, 1967,
 595 p. 43) and the *total excess of mass* (e.g., Blakely, 1996, p. 60). Besides, Siqueira et al. (2017) have included
 596 a numerical analysis showing that their method produces very stable solutions, even for noise-corrupted
 597 potential-field data.

598 The iterative method proposed by Siqueira et al. (2017) is outlined in Algorithm 6, presumes an equivalent
 599 layer formed by monopoles (point masses) and can be applied to irregularly-spaced data on an undulating
 600 surface. Note that the residuals \mathbf{r} are used to compute a correction $\Delta\mathbf{p}$ for the parameter vector at each
 601 iteration (line 11), which requires a matrix-vector product involving the full matrix \mathbf{G} . Interestingly, this
 602 approach for estimating the physical property distribution on an equivalent layer is the same originally
 603 proposed by Bott (1960) for estimating the basement relief under sedimentary basins. The methods of Xia
 604 and Sprowl (1991) and Siqueira et al. (2017) were originally proposed for processing gravity data, but can
 605 be potentially applied to any harmonic function because they actually represent iterative solutions of the
 606 classical *Dirichlet's problem* or the *first boundary value problem of potential theory* (Kellogg, 1967, p.
 607 236) on a plane.

608 Recently, Jirigalatu and Ebbing (2019) presented another iterative method for estimating a parameter
 609 vector $\tilde{\mathbf{p}}$ (equation 3). With the purpose of combining different potential-field data, their method basically
 610 modifies that shown in Algorithm 6 by changing the initial approximation and the iterative correction for
 611 the parameter vector. Specifically, Jirigalatu and Ebbing (2019) replace line 5 by $\tilde{\mathbf{p}} = \mathbf{0}$, where $\mathbf{0}$ is a vector

of zeros, and line 11 by $\Delta\mathbf{p} = \omega \mathbf{G}^\top \mathbf{r}$, where ω is a positive scalar defined by trial and error. Note that this modified approach requires two matrix-vector products involving the full matrix \mathbf{G} per iteration. To overcome the high computational cost of these two products, Jirigalatu and Ebbing (2019) set an equivalent layer formed by prisms and compute their predicted potential field in the wavenumber domain by using the Gauss-FFT technique Zhao et al. (2018).

The iterative method proposed by Siqueira et al. (2017) (Algorithm 6) requires one entrywise product in line 5 and a matrix-vector followed by subtraction in line 7 before the while loop. At each iteration, there is another entrywise product (line 11), a half saxpy (line 12) and a saxpy (lines 11 and 12). Then, we get from table 1 that the total number of flops is given by:

$$f_{SOB17} = 2D^2 + 2D + \text{ITMAX} (2D^2 + 3D) . \quad (50)$$

Note that the number of flops per iteration in f_{SOB17} (equation 50) has the same order of magnitude, but is smaller than that in f_{CGLS} (equation 28).

7.7 Iterative deconvolution

Recently, Takahashi et al. (2020, 2022) proposed the *convolutional equivalent-layer method*, which explores the structure of the sensitivity matrix \mathbf{G} (equation 3) for the particular case in which (i) there is a single equivalent source right below each potential-field datum and (ii) both data and sources rely on planar and regularly spaced grids. Specifically, they consider a regular grid of D potential-field data at points (x_i, y_i, z_0) , $i \in \{1 : D\}$, on a horizontal plane z_0 . The data indices i may be ordered along the x - or y -direction, which results in an x - or y -oriented grid, respectively. They also consider a single equivalent source located right below each datum, at a constant vertical coordinate $z_0 + \Delta z$, $\Delta z > 0$. In this case, the number of data and equivalent sources are equal to each other (i.e., $D = P$) and \mathbf{G} (equation 3) assumes a *doubly block Toeplitz* (Jain, 1989, p. 28) or *block-Toeplitz-Toeplitz-block* (BTTB) (Chan and Jin, 2007, p. 67) structure formed by $N_B \times N_B$ blocks, where each block has $N_b \times N_b$ elements, with $D = N_B N_b$. This particular structure allows formulating the product of \mathbf{G} and an arbitrary vector as a *fast discrete convolution* via *Fast Fourier Transform* (FFT) (Van Loan, 1992, section 4.2).

Consider, for example, the particular case in which $N_B = 4$, $N_b = 3$ and $D = 12$. In this case, \mathbf{G} (equation 3) is a 12×12 block matrix given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}^0 & \mathbf{G}^1 & \mathbf{G}^2 & \mathbf{G}^3 \\ \mathbf{G}^{-1} & \mathbf{G}^0 & \mathbf{G}^1 & \mathbf{G}^2 \\ \mathbf{G}^{-2} & \mathbf{G}^{-1} & \mathbf{G}^0 & \mathbf{G}^1 \\ \mathbf{G}^{-3} & \mathbf{G}^{-2} & \mathbf{G}^{-1} & \mathbf{G}^0 \end{bmatrix}_{D \times D}, \quad (51)$$

where each block \mathbf{G}^n , $n \in \{(1 - N_B) : (N_B - 1)\}$, is a 3×3 Toeplitz matrix. Takahashi et al. (2020, 2022) have deduced the specific relationship between blocks \mathbf{G}^n and \mathbf{G}^{-n} and also between a given block \mathbf{G}^n and its transposed $(\mathbf{G}^n)^\top$ according to the harmonic function g_{ij} (equation 2) defining the element ij of the sensitivity matrix \mathbf{G} (equation 3) and the orientation of the data grid.

Consider the matrix-vector products

$$\mathbf{G} \mathbf{v} = \mathbf{w} \quad (52)$$

and

$$\mathbf{G}^\top \mathbf{v} = \mathbf{w} , \quad (53)$$

644 involving a $D \times D$ sensitivity matrix \mathbf{G} (equation 3) defined in terms of a given harmonic function g_{ij}
 645 (equation 2), where

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}^0 \\ \vdots \\ \mathbf{v}^{N_B-1} \end{bmatrix}_{D \times 1}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}^0 \\ \vdots \\ \mathbf{w}^{N_B-1} \end{bmatrix}_{D \times 1}, \quad (54)$$

646 are arbitrary partitioned vectors formed by N_B sub-vectors \mathbf{v}^n and \mathbf{w}^n , $n \in \{0 : (N_B - 1)\}$, all of them
 647 having N_b elements. Equations 52 and 53 can be computed in terms of an auxiliary matrix-vector product

$$\mathbf{G}_c \mathbf{v}_c = \mathbf{w}_c, \quad (55)$$

648 where

$$\mathbf{v}_c = \begin{bmatrix} \mathbf{v}_c^0 \\ \vdots \\ \mathbf{v}_c^{N_B-1} \\ \mathbf{0} \end{bmatrix}_{4D \times 1}, \quad \mathbf{w}_c = \begin{bmatrix} \mathbf{w}_c^0 \\ \vdots \\ \mathbf{w}_c^{N_B-1} \\ \mathbf{0} \end{bmatrix}_{4D \times 1}, \quad (56)$$

649 are partitioned vectors formed by $2N_b \times 1$ sub-vectors

$$\mathbf{v}_c^n = \begin{bmatrix} \mathbf{v}_c^n \\ \mathbf{0} \end{bmatrix}_{2N_b \times 1}, \quad \mathbf{w}_c^n = \begin{bmatrix} \mathbf{w}_c^n \\ \mathbf{0} \end{bmatrix}_{2N_b \times 1}, \quad (57)$$

650 and \mathbf{G}_c is a $4D \times 4D$ *doubly block circulant* (Jain, 1989, p. 28) or *block-circulant circulant-block* (BCCB)
 651 (Chan and Jin, 2007, p. 76) matrix. What follows aims at explaining how the original matrix-vector products
 652 defined by equations 52 and 53, involving a $D \times D$ BTTB matrix \mathbf{G} exemplified by equation 51, can be
 653 efficiently computed in terms of the auxiliary matrix-vector product given by equation 55, which has a
 654 $4D \times 4D$ BCCB matrix \mathbf{G}_c .

655 Matrix \mathbf{G}_c (equation 55) is formed by $2N_B \times 2N_B$ blocks, where each block \mathbf{G}_c^n , $n \in \{(1 - N_B) : (N_B - 1)\}$ is a $2N_b \times 2N_b$ circulant matrix. For the case in which the original matrix-vector product is that
 656 defined by equation 52, the first column of blocks forming the BCCB matrix \mathbf{G}_c is given by
 657

$$\mathbf{G}_c[:, :2N_b] = \begin{bmatrix} \mathbf{G}_c^0 \\ \mathbf{G}_c^{-1} \\ \vdots \\ \mathbf{G}_c^{1-N_B} \\ \mathbf{0} \\ \mathbf{G}_c^{N_B-1} \\ \vdots \\ \mathbf{G}_c^1 \end{bmatrix}_{4D \times 2N_b}, \quad (58)$$

658 with blocks \mathbf{G}_c^n having the first column given by

$$\mathbf{G}_c^n[:, 1] = \begin{bmatrix} \mathbf{G}^n[:, 1] \\ 0 \\ (\mathbf{G}^n[1, N_b : 2])^\top \end{bmatrix}_{2N_b \times 2N_b}, \quad n \in \{(1 - N_B) : (N_B - 1)\}, \quad (59)$$

659 where \mathbf{G}^n are the blocks forming the BTTB matrix \mathbf{G} (equation 51). For the case in which the original
 660 matrix-vector product is that defined by equation 53, the first column of blocks forming the BCCB matrix
 661 \mathbf{G}_c is given by

$$\mathbf{G}_c[:, : 2N_b] = \begin{bmatrix} \mathbf{G}_c^0 \\ \mathbf{G}_c^1 \\ \vdots \\ \mathbf{G}_c^{N_B-1} \\ \mathbf{0} \\ \mathbf{G}_c^{1-N_B} \\ \vdots \\ \mathbf{G}_c^{-1} \end{bmatrix}_{4D \times 2N_b}, \quad (60)$$

662 with blocks \mathbf{G}_c^n having the first column given by

$$\mathbf{G}_c^n[:, 1] = \begin{bmatrix} (\mathbf{G}^n[1, :])^\top \\ 0 \\ \mathbf{G}^n[N_b : 2, 1] \end{bmatrix}_{2N_b \times 2N_b}, \quad n \in \{(1 - N_B) : (N_B - 1)\}. \quad (61)$$

663 The complete matrix \mathbf{G}_c (equation 55) is obtained by properly downshifting the block columns $\mathbf{G}_c[:, : 2N_b]$
 664 defined by equation 58 or 60. Similarly, the n -th block \mathbf{G}_c^n of \mathbf{G}_c is obtained by properly downshifting
 665 the first columns $\mathbf{G}_c^\ell[:, 1]$ defined by equation 59 or 61.

666 Note that \mathbf{G}_c (equation 55) is a $4D \times 4D$ matrix and \mathbf{G} (equation 51) is a $D \times D$ matrix. It seems weird
 667 to say that computing $\mathbf{G}_c \mathbf{v}_c$ is more efficient than directly computing $\mathbf{G} \mathbf{v}$. To understand this, we need first
 668 to use the fact that BCCB matrices are diagonalized by the 2D unitary discrete Fourier transform (DFT)
 669 (e.g., Davis, 1979, p. 31). Because of that, \mathbf{G}_c can be written as

$$\mathbf{G}_c = (\mathcal{F}_{2N_B} \otimes \mathcal{F}_{2N_b})^* \Lambda (\mathcal{F}_{2N_B} \otimes \mathcal{F}_{2N_b}), \quad (62)$$

670 where the symbol “ \otimes ” denotes the Kronecker product (e.g., Horn and Johnson, 1991, p. 243), \mathcal{F}_{2N_B} and
 671 \mathcal{F}_{2N_b} are the $2N_B \times 2N_B$ and $2N_b \times 2N_b$ unitary DFT matrices (e.g., Davis, 1979, p. 31), respectively,
 672 the superscript “ $*$ ” denotes the complex conjugate and Λ is a $4D \times 4D$ diagonal matrix containing the
 673 eigenvalues of \mathbf{G}_c . Due to the diagonalization of the matrix \mathbf{G}_c , equation 55 can be rewritten by using
 674 equation 62 and premultiplying both sides of the result by $(\mathcal{F}_{2N_B} \otimes \mathcal{F}_{2N_b})$, i.e.,

$$\Lambda (\mathcal{F}_{2N_B} \otimes \mathcal{F}_{2N_b}) \mathbf{v}_c = (\mathcal{F}_{2N_B} \otimes \mathcal{F}_{2N_b}) \mathbf{w}_c. \quad (63)$$

675 By following Takahashi et al. (2020), we rearrange equation 63 as follows

$$\mathcal{L} \circ (\mathcal{F}_{2N_B} \mathbf{v}_c \mathcal{F}_{2N_b}) = \mathcal{F}_{2N_B} \mathbf{W}_c \mathcal{F}_{2N_b} \quad (64)$$

676 where “ \circ ” denotes the Hadamard product (e.g., Horn and Johnson, 1991, p. 298) and \mathcal{L} , \mathbf{V}_c and \mathbf{W}_c are
 677 $2N_B \times 2N_b$ matrices obtained by rearranging, along their rows, the elements forming the diagonal of Λ
 678 (equation 62), vector \mathbf{v}_c and vector \mathbf{w}_c (equation 56), respectively. Then, by premultiplying both sides of
 679 equation 64 by $\mathcal{F}_{2N_B}^*$ and then postmultiplying both sides by $\mathcal{F}_{2N_b}^*$, we obtain

$$\mathcal{F}_{2N_B}^* [\mathcal{L} \circ (\mathcal{F}_{2N_B} \mathbf{v}_c \mathcal{F}_{2N_b})] \mathcal{F}_{2N_b}^* = \mathbf{W}_c. \quad (65)$$

680 Finally, we get from equation 62 that matrix \mathcal{L} can be computed by using only the first column $\mathbf{G}_c[:, 1]$ of
 681 the BCCB matrix \mathbf{G}_c (equation 55) according to (Takahashi et al., 2020)

$$\mathcal{L} = \sqrt{4D} \mathcal{F}_{2N_B} \mathcal{C} \mathcal{F}_{2N_b}, \quad (66)$$

682 where \mathcal{C} is a $2N_B \times 2N_b$ matrix obtained by rearranging, along its rows, the elements of $\mathbf{G}_c[:, 1]$ (equation
 683 55). It is important noting that the matrices \mathcal{C} and \mathcal{L} (equation 66) associated with the BTTB matrix \mathbf{G}
 684 (equation 51) are different from those associated with \mathbf{G}^\top .

685 The whole procedure to compute the original matrix-vector products $\mathbf{G}\mathbf{v}$ (equation 52) and $\mathbf{G}^\top\mathbf{v}$
 686 (equation 53) consists in (i) rearranging the elements of the vector \mathbf{v} and the first column $\mathbf{G}[:, 1]$ of matrix
 687 \mathbf{G} into the matrices \mathcal{V}_c and \mathcal{C} (equations 65 and 66), respectively; (ii) computing terms $\mathcal{F}_{2N_B} \mathcal{A} \mathcal{F}_{2N_b}$ and
 688 $\mathcal{F}_{2N_B}^* \mathcal{A} \mathcal{F}_{2N_b}^*$, where \mathcal{A} is a given matrix, and a Hadamard product to obtain \mathcal{W}_c (equation 65); and (iii)
 689 retrieve the elements of vector \mathbf{w} (equation 52) from \mathcal{W}_c (equation 65). It is important noting that the steps
 690 (i) and (iii) do not have any computational cost because they involve only reorganizing elements of vectors
 691 and matrices. Besides, the terms $\mathcal{F}_{2N_B} \mathcal{A} \mathcal{F}_{2N_b}$ and $\mathcal{F}_{2N_B}^* \mathcal{A} \mathcal{F}_{2N_b}^*$ in step (ii) represent, respectively, the
 692 2D Discrete Fourier Transform (2D-DFT) and the 2D Inverse Discrete Fourier Transform (2D-IDFT) of \mathcal{A} .
 693 These transforms can be efficiently computed by using the 2D Fast Fourier Transform (2D-FFT). Hence,
 694 the original matrix-vector products $\mathbf{G}\mathbf{v}$ (equation 52) and $\mathbf{G}^\top\mathbf{v}$ (equation 53) can be efficiently computed
 695 by using the 2D-FFT.

696 Algorithms 7 and 8 show pseudo-codes for the convolutional equivalent-layer method proposed by
 697 Takahashi et al. (2020, 2022). Note that those authors formulate the overdetermined problem (equation
 698 22) of obtaining an estimate $\tilde{\mathbf{p}}$ for the parameter vector \mathbf{p} (equation 3) as an *iterative deconvolution* via
 699 *conjugate gradient normal equation residual* (CGNR) Golub and Van Loan (2013, sec. 11.3) or *conjugate*
 700 *gradient least squares* (CGLS) (Aster et al., 2019, p. 165) method. They consider $\mathbf{H} = \mathbf{I}_P$ (equation 9),
 701 $\mu = 0$ (equation 11), $\mathbf{W}_d = \mathbf{W}_q = \mathbf{I}_P$ (equations 12 and 13) and $\bar{\mathbf{p}} = \mathbf{0}$ (equation 14). As shown by
 702 Takahashi et al. (2020, 2022), the CGLS produces stable estimates $\tilde{\mathbf{p}}$ for the parameter vector \mathbf{p} (equation
 703 3) in the presence of noisy potential-field data \mathbf{d} . This is a well-known property of the CGLS method (e.g.,
 704 Aster et al., 2019, p. 166).

705 The key aspect of Algorithm 7 is replacing the matrix-vector products of CGLS (Algorithm 1) by fast
 706 convolutions (Algorithm 8). A fast convolution requires one 2D-DFT, one 2D-IDFT and an entrywise
 707 product of matrices. We consider that the 2D-DFT/IDFT are computed with 2D-FFT and requires
 708 $\kappa(4D) \log_2(4D)$ flops, where $\kappa = 5$ is compatible with a radix-2 FFT (Van Loan, 1992, p. 16), and
 709 the entrywise product $24D$ flops because it involves two complex matrices having $4D$ elements (Golub
 710 and Van Loan, 2013, p. 36). Hence, Algorithm 8 requires $\kappa(16D) \log_2(4D) + 26D$ flops, whereas a
 711 conventional matrix-vector multiplication involving a $D \times D$ matrix requires $2D^2$ (table 1). Finally,
 712 Algorithm 7 requires two 2D-FFTs (lines 4 and 5), one fast convolution and an inner product (line 8)
 713 previously to the while loop. Per iteration, there are three saxpys (lines 12, 15 and 16), two inner products
 714 (lines 14 and 17) and two fast convolutions (lines 13 and 17), so that:

$$f_{\text{T0B20}} = \kappa(16D) \log_2(4D) + 26D + \text{ITMAX} [\kappa(16D) \log_2(4D) + 58D]. \quad (67)$$

715 7.8 Direct deconvolution

716 The method proposed by Takahashi et al. (2020, 2022) can be reformulated to avoid the iterations of the
 717 conjugate gradient method. This alternative formulation consists in considering that $\mathbf{v} = \mathbf{p}$ and $\mathbf{w} = \mathbf{d}$ in

718 equation 52, where \mathbf{p} is the parameter vector (equation 3) and \mathbf{d} the observed data vector. In this case, the
 719 equality “=” in equation 52 becomes an approximation “ \approx ”. Then, equation 64 is manipulated to obtain

$$\mathcal{V}_c \approx \mathcal{F}_{2N_B}^* \left[(\mathcal{F}_{2N_B} \mathcal{W}_c \mathcal{F}_{2N_b}) \circ \check{\mathcal{L}} \right] \mathcal{F}_{2N_b}^*, \quad (68)$$

720 where

$$\check{\mathcal{L}} = \mathcal{L}^* \oslash (\mathcal{L} \circ \mathcal{L}^* + \zeta \mathbf{1}), \quad (69)$$

721 $\mathbf{1}$ is a $4D \times 4D$ matrix of ones, “ \oslash ” denotes entrywise division and ζ is a positive scalar. Note that $\zeta = 0$
 722 leads to $\mathbf{1} \oslash \mathcal{L}$. In this case, the entrywise division may be problematic due to the elements of \mathcal{L} having
 723 absolute value equal or close to zero. So, a small ζ is set to avoid this problem in equation 69. Next, we use
 724 $\check{\mathcal{L}}$ to obtain a matrix \mathcal{V}_c from equation 68. Finally, the elements of the estimated parameter vector $\tilde{\mathbf{p}}$ are
 725 retrieved from the first quadrant of \mathcal{V}_c . This procedure represents a *direct deconvolution* (e.g., Aster et al.,
 726 2019, p. 220) using a *Wiener filter* (e.g., Gonzalez and Woods, 2002, p. 263).

727 The required total number of flops associated with the direct deconvolution aggregates one 2D-FFT
 728 to compute matrix \mathcal{L} (equation 66), one entrywise product $\mathcal{L} \circ \mathcal{L}^*$ involving complex matrices and one
 729 entrywise division to compute $\check{\mathcal{L}}$ (equation 69) and a fast convolution (Algorithm 8) to evaluate equation
 730 68, which results in:

$$f_{\text{deconv.}} = \kappa (12D) \log_2(4D) + 72D. \quad (70)$$

731 Differently from the convolutional equivalent-layer method proposed by Takahashi et al. (2020, 2022), the
 732 alternative direct deconvolution presented here produces an estimated parameter vector $\tilde{\mathbf{p}}$ directly from
 733 the observed data \mathbf{d} , in a single step, avoiding the conjugate gradient iterations. On the other hand, the
 734 alternative method presented here requires estimating a set of tentative parameter vectors $\tilde{\mathbf{p}}$ for different
 735 predefined ζ . Besides, there must be criterion to chose the best $\tilde{\mathbf{p}}$ from this tentative set. This can be
 736 made, for example, by using the well-known *L-curve* (Hansen, 1992). From a computational point of view,
 737 the number of CGLS iterations in the method proposed by Takahashi et al. (2020, 2022) is equivalent to
 738 the number of tentative estimated parameter vectors required to form the L-curve in the proposed direct
 739 deconvolution.

8 SYNTHETIC DATA SIMULATIONS

For all applications, we generate a model composed by two spheres and a polygonal prism in a regular spaced grid of 50×50 . The upper left sphere has a density contrast of 600 kg/m^3 , the right upper sphere a negative contrast of -500 kg/m^3 and the bottom prism is equal to 550 kg/m^3 . To generate the magnetic data, the bodies are in the same position and all of them have the same magnetization intensity and direction (3.46 A/m intensity, 35.26° inclination and 45.0° declination) within a simulated geomagnetic field direction of 20.0° inclination and 35.0° declination. These synthetic datas are shown in figures 3 and 7, respectively.

8.1 Stability analysis

For the stability analysis we show the comparison of the Cholesky factorization (equations 25 and 26) to solve overdetermined problems (equation 22), the iterative deconvolutional method (algorithms 7 and 8) and the deconvolutional method with different values for the Wiener stabilization (equation 69). We create 21 data sets, for both gravity and magnetic data, adding a crescent pseudo-random noise to the original data, which varies from 0% to 10% of the maximum anomaly value in intervals of 0.5%. These noises has mean equal to zero and a Gaussian distribution. These synthetic datas are shown in figures 3 and 7, where panel (A) of each figure represents the noise free data and panel (B) is the maximum noised data for gravity and magnetic, respectively.

Figure 2 shows how the euclidian norm of the equivalent sources residuals varies as the level of the noise is increased for the gravimetric data. We can see that for all methods, a linear tendency can be observed as it is expected. The inclination of the straight line is a indicative of the stability of each method. As show in the graph the deconvolutional method is very unstable and it is really necessary to use a stabilization method to have a good parameter estimative. In contrast, a correct value of the stabilization parameter is necessary to not overshoot the smoothness of the solution as it is the case for the zeroth-order Tikhonov regularization as well. Using this gravimetric data, the optimal value for the Wiener stabilization parameter is $\mu = 10^{-20}$.

Figure 4 shows the comparison of the predicted data for each method with the original data (figure 3) using the most noised-corrupted data from the set of the stability analysis. The classical with zeroth-order Tikhonov regularization and the convolutional methods (figures 4(A) and 4(B)) yield very similar results for the predicted data confirming its similarities with the stabilization despite the bid difference in floating-point operations. Figure 4(C) shows the deconvolutional method without a stabilization and demonstrates the necessity to use it for this method. Figure 4(D) shows the deconvolutional method with Wiener stabilization $\mu = 10^{-15}$ which is too high, demonstrating the over smoothness of the predicted data. Figures 4(E) and 4(F) shows the predicted data for an optimal value of the Wiener parameter $\mu = 10^{-20}$ and a low value $\mu = 10^{-25}$, respectively.

The upward continuation is a processing technique to visualize the data in a higher altitude. In practice is expected a lower amplitude signal and a smoother data as the high frequency anomalies tends to disappear. Figure 5(A) shows the true modeled upward data at an height of -500 m . Figures 5(B), (C), (D) and (E) show the result of the upward processing for the classical, convolutional, deconvolutional and the deconvolutional with Wiener parameter $\mu = 10^{-20}$, respectively. It is clear that all methods seems to predict the upward data very reasonable, except the deconvolutional method without stabilization.

For the magnetic data, figure 6 shows a very similar behavior of the stability as the previous case. The Wiener parameter seems to have the best solution for $\mu = 10^{-13}$. For both types of data the best Wiener

781 parameter seems to be one that produces a low slope for the straight line in the stability analysis, discordant
782 from the classical and convolutional methods.

783 Figure 8 shows the comparison of the predicted data for each method with the original magnetic data in
784 figure 7 using the most noised-corrupted data modeled from the stability analysis. As the previous case the
785 classical (figure 8(A)) and the convolutional (figure 8(B)) methods have very similar predicted data but
786 estimated with less orders of magnitude in floating-point operations. The deconvolutional (figure 8(C)) have
787 have a strong disagreement with the observed data showing the need for a stabilization method. Figure
788 8(D) has a value of $\mu = 10^{-10}$ and the predicted data became to smooth by it. The optimal value of the
789 Wiener parameter is shown in figure 8(E) with $\mu = 10^{-13}$ and figure 8(F) shows a predicted data with a
790 low stabilization value with $\mu = 10^{-16}$.

791 Figure 9(A) shows the true modeled upward data at an height of -1400 m. Figures 9(B), (C), (D) and
792 (E) show the result of the upward processing for the classical, convolutional, deconvolutional and the
793 deconvolutional with Wiener parameter $\mu = 10^{-13}$, respectively. As in the gravimetric case, all methods
794 seems to predict the upward data, except the deconvolutional method without stabilization.

9 REAL DATA RESULTS

795 In this section, we show the applications of the convolutional and the deconvolutional strategies in a real
796 data set from the North of Brazil. The region is located in the Carajás Mineral Province (CMP) in the
797 Amazon craton (Moroni et al., 2001; Villas and Santos, 2001). This area is known for its intensive mineral
798 exploration such as iron, copper, gold, manganese, and, recently, bauxite.

799 9.1 Geological setting

800 The Amazon craton is one of the largest and least-known Archean-Proterozoic areas in the world,
801 comprehending a region with a thousand square kilometers. It is one of the main tectonic units in South
802 America, which is covered by five Phanerozoic basins: Maranhão (Northeast), Amazon (Central), Xingu-
803 Alto Tapajós (South), Parecis (Southwest), and Solimões (West). The craton is limited by the Andean
804 Orogenic Belt to the west and the by Araguaia Fold Belt to the east and southeast. The Amazon craton has
805 been subdivided into provinces according to two models, one geochronological and the other geophysical-
806 structural (Amaral, 1974; Teixeira et al., 1989; Tassinari and Macambira, 1999). Thus, seven geological
807 provinces with distinctive ages, evolution, and structural patterns can be observed, namely : (i) Carajás with
808 two domains - the Mesoarchean Rio Maria and Neoarchean Carajás; (ii) Archean-Paleoproterozoic Central
809 Amazon, with Iriri-Xingu and Curuá-Mapuera domains; (ii) Trans-Amazonian (Ryacian), with the Amapá
810 and Bacajá domains; (iv) the Orosinian Tapajós-Parima, with Peixoto de Azevedo, Tapajós, Uaimiri, and
811 Parima domains; (v) Rondônia-Juruena (Statherian), with Jamari, Juruena, and Jauru domains; (vi) The
812 Statherian Rio Negro, with Rio Negro and Imeri domains; and (vii) Sunsás (Meso-Neoproterozoic), with
813 Santa Helena and Nova Brasilândia domains (Santos et al., 2000). Nevertheless, we focus this work only
814 on the Carajás Province.

815 The Carajás Mineral Province (CMP) is located in the east-southeast region of the craton, within an old
816 tectonically stable nucleus in the South American Plate that became tectonically stable at the beginning of
817 Neoproterozoic (Salomao et al., 2019). This area has been the target of intensive exploration at least since
818 the final of the '60s, after the discovery of large iron ore deposits. There are several greenstone belts in the
819 region, among them are the Andorinhas, Inajá, Cumaru, Carajás, Serra Leste, Serra Pelada, and Sapucaia
820 (Santos et al., 2000). The mineralogic and petrologic studies in granite stocks show a variety of minerals
821 found in the province, such as amphibole, plagioclase, biotite, ilmenite, and magnetite (Cunha et al., 2016).
822 These two latter minerals contribute to the high magnetic response in the CMP area. This fact opens the
823 opportunity for potential field applications for the geophysical description of the area.

824 9.2 Potential field data applications

825 Here we compare the performance of the convolutional and deconvolutional algorithms in a real potential
826 field data set. We focus the application on a region in the Southeast of the State of Pará. The aeromagnetic
827 data were acquired by the Geological Survey of Brazil-CPRM. The survey area covers $\approx 58000 \text{ km}^2$ with
828 high-resolution gravity and magnetic data. The flight and the tie lines were acquired and spaced at 3 km
829 and 12 km oriented in the directions $N - S$ and $E - W$, respectively, with a mean flight height of 900 m
830 above the ground. For both applications, we interpolated gravity and magnetic anomalies data, calculating
831 the data set in a grid of 1000×500 ($N = 500000$ observation points) at the same mean flight height. About
832 the computational resources, we processed both data on an Intel Core i7 7700HQ@2.8 GHz processor and
833 16GB RAM. We show in Figure 10 and Figure 12 the interpolated aerogravimetric and aeromagnetic data,
834 respectively. We also use the same equivalent layer grid configuration in gravity and magnetic applications.

835 This setup is composed by a grid of 1000×500 equivalent sources (a total number of $M = 500000$ points)
836 positioned below the observation plane, but a different depth in each application.

837 We apply both strategies to the gravimetric case. We set a depth for the equivalent layer equal to 1200 m
838 below the observation plane. Figure 11A and Figure 11C show the predicted data for convolutional and
839 deconvolutional strategies. The residual maps (the difference between the observed and predicted data)
840 are show in figures 11B and 11D for the convolutional and deconvolutional equivalent-layer technique,
841 respectively. For the convolutional case, the mean residual and standard deviation values are $\approx 0.00\text{ mGal}$
842 and $\approx 0.15\text{ mGal}$, respectively. For the deconvolutional case, the mean residual and standard deviation
843 values are $\approx 0.46\text{ mGal}$ and $\approx 1.23\text{ mGal}$, respectively. These last results show that the estimated density
844 distributions (not shown) fit the observed data for both applications. To show the performance of the
845 algorithms, we performed an upward continuation by using the estimated density distributions (figures 11E
846 and 11F). There is a little difference on the processing time between both strategies. The convolutional
847 algorithm took $\approx 9.18\text{s}$ and the deconvolutional algorithm took $\approx 0.53\text{s}$. We conclude that both strategies
848 are capable of processing gravimetric observations from large areas with dense coverage data. Despite a
849 little difference in processing time, the deconvolutional equivalent-layer technique proved to be faster than
850 the convolutional strategy.

851 Finally, we test the convolutional and deconvolutional algorithms for processing total-field anomalies.
852 We stress that the Carajás area is very large and the main field direction varies significantly. For this reason,
853 we consider a mean direction for the main field equal to -19.865° and -7.43915° (the same as the mid
854 location of the area) for the inclination and declination, respectively. Furthermore, we are not considering
855 the knowledge about the magnetization direction of the sources, and choose a magnetization direction for
856 the equivalent layer equal to the main field direction. For this application, we set a depth of 900 m (below
857 the observation plane) for the equivalent layer. Figure 12A and Figure 12C show the predicted data for
858 convolutional and deconvolutional algorithms. The residual maps (the difference between the observed and
859 predicted data) are show in figures 12B and 12D for the convolutional and deconvolutional techniques,
860 respectively. The convolutional equivalent layer produced a mean residual and standard deviation values
861 of $\approx 0.06\text{ nT}$ and $\approx 1.97\text{ nT}$, respectively. The deconvolutional algorithm produced a mean residual
862 and standard deviation values of $\approx 18.99\text{ nT}$ and $\approx 33.64\text{ nT}$, respectively. To show the performance
863 of the algorithms, we performed an upward continuation (figures 12E and 12F) by using the estimated
864 magnetic-moment distributions (not shown). Similarly to the gravity application, the deconvolutional
865 equivalent layer presents faster results than the convolutional algorithm. The deconvolutional and the
866 convolutional approaches took $\approx 0.89\text{s}$ and $\approx 82.08\text{s}$, respectively. Despite the difference between the
867 processing time of both strategies and considering the mean value of residuals and standard deviations, we
868 conclude that the convolutional strategy fits the observation data better than the deconvolutional approach.

10 DISCUSSION AND CONCLUSION

869 We have presented a review of the strategies used to overcome the intensive computational cost of the
870 equivalent-layer technique for processing potential-field data. Each of these strategies is rarely used
871 individually; rather, some developed equivalent-layer methods combine more than one strategy to make
872 them computationally efficient in handling large-scale data sets. This comprehensive review addresses the
873 following specific strategies for reducing the computational cost of equivalent-layer technique.

874 The first one is the moving data-window scheme spanning the data set. This strategy solves several much
875 smaller, regularized linear inverse problems instead of a single large one. Each linear inversion is solved
876 using the potential-field observations and equivalent sources within a given moving window and can be
877 applied to both regularly or irregularly spaced data sets. If the data and the sources are distributed on planar
878 and regularly spaced grids, this strategy offers a significant advantage because the sensitivity submatrix of
879 a given moving window remains the same for all windows. Otherwise, the computational efficiency of the
880 equivalent-layer technique using the moving-window strategy decreases because the sensitivity submatrix
881 for each window must be computed.

882 The second and third strategies, referred to as the column-action and row-action updates, involve
883 iteratively calculating a single column and a single row of the sensitivity matrix, respectively. By following
884 the column-action update strategy, a single column of the sensitivity matrix is calculated during each
885 iteration. This implies that a single equivalent source contributes to the fitting of data in each iteration.
886 Conversely, in the row-action update strategy, a single row of the sensitivity matrix is calculated per
887 iteration, which means that one potential-field observation is incorporated in each iteration, forming a new
888 subset of equivalent data much smaller than the original data. Both strategies (column- and row-action
889 updates) have a great advantage because a single column or a single row of the sensitivity matrix is
890 calculated iteratively. However, to our knowledge, the strategy of the column-action update presents some
891 issues related to convergence, and the strategy of the row-action update can also have issues if the number
892 of equivalent data is not significantly smaller than the original number of data points.

893 The fourth strategy is the wavelet compression which consists in transforming a large and full sensitivity
894 matrix into a sparse one, with a few nonzero elements, by using a fast wavelet transform. By using the sparse
895 sensitivity matrix, the inverse problem is solved in the wavelet domain without an explicit regularization
896 parameter. A regularized solution is obtained by using a conjugate gradient least squares where the number
897 of iterations performed acts as a regularization. Computationally, the significant advantage of the wavelet-
898 compression strategy is its ability to handle a sparse sensitivity matrix, which enables a rapid iteration
899 of the conjugate gradient (CG) algorithm. However, we understand that this strategy requires computing
900 the full and dense sensitivity matrix, which can be considered a drawback when processing large-scale
901 potential-field data.

902 The fifth strategy is the reparametrization of the original parameters to be estimated in the equivalent-layer
903 technique. In this strategy, the developed equivalent-layer methods reduce the dimension of the linear
904 system of equations to be solved by estimating a lower-dimensional parameter vector. We highlight three
905 methods that used the reparametrization strategy: i) the quadtree discretization of the equivalent layer; ii)
906 the polynomial equivalent layer (PEL) and; ii) the lower-dimensional subspace of the equivalent layer.
907 In the quadtree discretization, the equivalent sources located far from the observation point are grouped
908 together to create a larger averaged source implying in a reduction in the number of parameters to be
909 estimated. In the PEL, there is an explicit reparametrization of the equivalent layer by representing the
910 unknown distribution over the equivalent layer as a set of piecewise-polynomial functions defined on a set of

911 equivalent-source windows. The PEL method estimates the polynomial coefficients of all equivalent-source
912 windows. Hence, PEL reduces the dimension of the linear system of equations to be solved because the
913 polynomial coefficients within all equivalent-source windows are much smaller than both the number of
914 equivalent sources and the number of data points. In the lower-dimensional subspace of the equivalent layer,
915 there is an implicit reparametrization of the equivalent layer by reducing the linear system dimension
916 from the original and large-model space to a lower-dimensional subspace. The lower-dimensional subspace
917 is grounded on eigenvectors of the matrix composed by the gridded data set. The main advantage of the
918 reparametrization of the equivalent layer is to deal with lower-dimensional linear system of equations.
919 However, we acknowledge that this strategy may impose an undesirable smoothing effect on both the
920 estimated parameters over the equivalent layer and the predicted data.

921 The sixth strategy involves an iterative scheme in which the estimated distribution over the equivalent
922 layer is updated iteratively. Following this strategy, the developed equivalent-layer methods differ either in
923 terms of the expression used for the estimated parameter correction or the domain utilized (wavenumber or
924 space domains). The iterative estimated correction may have a physical meaning, such as the excess mass
925 constraint. All the iterative methods are efficient as they can handle irregularly spaced data on an undulating
926 surface, and the updated corrections for the parameter vector at each iteration are straightforward, involving
927 the addition of a quantity proportional to the data residual. However, they have a disadvantage because the
928 iterative strategy requires computing the full and dense sensitivity matrix to compute the predicted and
929 residual data in all observation stations per iteration.

930 The seventh strategy is called iterative deconvolutional of the equivalent layer. This strategy deals with
931 regularly spaced grids of data stations and equivalent sources which are located at a constant height and
932 depth, respectively. Specifically, one source is placed directly below each observation station, which results
933 in sensitivity matrices with a BTTB (Block-Toeplitz Toeplitz-Block) structure. It is possible to embed the
934 BTTB matrix into a matrix of Block-Circulant Circulant-Block (BCCB) structure, which requires only
935 one equivalent source. This allows for fast matrix-vector product using a 2D fast Fourier transform (2D
936 FFT). As a result, the potential-field forward modeling can be calculated using a 2D FFT with only one
937 equivalent source required. The main advantages of this strategy are that the entire sensitivity matrices
938 do not need to be formed or stored; only their first columns are required. Additionally, it allows for a
939 highly efficient iteration of the CG algorithm. However, the iterative deconvolutional of the equivalent
940 layer requires observations and equivalent sources aligned on a horizontal and regularly-spaced grid.

941 The eighth strategy is a direct deconvolution method, which is a mathematical process very common in
942 geophysics. However, to our knowledge, direct deconvolution has never been used to solve the inverse
943 problem associated with the equivalent-layer technique. From the mathematical expressions in the iterative
944 deconvolutional equivalent layer with BTTB matrices, direct deconvolution arises naturally since it is an
945 operation inverse to convolution. The main advantage of applying the direct deconvolution strategy in
946 the equivalent layer is that it avoids, for example, the iterations of the CG algorithm. However, the direct
947 deconvolution is known to be an unstable operation. To mitigate this instability, the Wiener deconvolution
948 method can be adopted.

949 ***** PAREI AQUI *****

950 ***** O TEXTO ABAIXO PERTENCE A VERSAO ANTERIOR DA CONCLUSAO *****

951 From the above-mentioned strategies, further classifications may be identified for reducing the
952 computational cost of equivalent-layer technique. For example, taking into account the mathematical
953 bases, we identify four groups: i) the reduction of the dimensionality of the linear system of equations to

be solved; ii) the generation of a sparse linear system of equations to be solved; iii) the explicit iterative method without solving a linear system of equations and; iv) the convolution (deconvolution). The first mathematical basis reduces the linear system of equations to be solved for estimating the distribution over the equivalent layer. This is achieved by using different strategies: a) the moving data-window scheme spanning the data set by setting a small moving-data window; b) the reparametrization of the dataset by selecting a subset of observations much smaller than the original data; c) the explicit reparametrization of the model (the equivalent layer) by using quadtree discretization or piecewise-polynomial functions defined on a set of equivalent-source windows and; d) the implicit reparametrization of the model (the equivalent layer) by using the subspace method. The second mathematical basis generates a sparse linear system of equations by transforming the full sensitivity matrix into a sparse one. This is achieved by using different strategies: a) the compression of the coefficient of the sensitivity matrix via wavelet transforms and; b) the grouping of equivalent sources distant from an observation point to form a larger equivalent source via quadtree discretization of the equivalent layer; The third mathematical basis does not solve a linear system of equations for estimating the distribution over the equivalent layer. This is grounded on gradient method as an optimization algorithm that iteratively updates the parameter without calculating a full Hessian matrix and solving linear systems. This is achieved by using different strategies: a) the conjugate gradient least-squares regularized by the number of iterations with the sparse wavelet compression of the coefficient matrix; b) the iterative Landweber algorithm with forward modelling of potential-field data with Gauss-FFT; c) the gradient-boosting algorithm operating on overlapping windows with the block-averaged sources reducing the number of the sources; and d) The explicit iterative method with (or not) physical reasoning without calculating a full Hessian matrix and solving linear systems The fourth mathematical basis for estimating the distribution over the equivalent layer is based on FFT convolution with the sensitivity matrices exhibiting BTTB structure when the observations and equivalent sources are aligned on a horizontal and regularly-spaced grid.

We would like to draw the readers' attention to the possibility of grouping the aforementioned strategies for reducing the computational cost of the equivalent-layer technique according to different mathematical bases. An example of the mathematical basis might be the reduction of the processing time spent on the forward modelling accounts.

CONFLICT OF INTEREST STATEMENT

982 The authors declare that the research was conducted in the absence of any commercial or financial
983 relationships that could be construed as a potential conflict of interest.

AUTHOR CONTRIBUTIONS

984 The Author Contributions section is mandatory for all articles, including articles by sole authors. If an
985 appropriate statement is not provided on submission, a standard one will be inserted during the production
986 process. The Author Contributions statement must describe the contributions of individual authors referred
987 to by their initials and, in doing so, all authors agree to be accountable for the content of the work. Please
988 see here for full authorship criteria.

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DATA AVAILABILITY STATEMENT

995 The datasets generated for this study can be found in the frontiers-paper Github repository link:
996 <https://github.com/DiegoTaka/frontiers-paper>.

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11 ALGORITHMS

Algorithm 1: Generic pseudo-code for the CGLS applied to the overdetermined problem (equation 22) for the particular case in which $\mathbf{H} = \mathbf{I}_P$ (equation 9 and subsection 3.2), $\mu = 0$ (equation 11), $\mathbf{W}_d = \mathbf{I}_D$ (equation 12) and $\bar{\mathbf{p}} = \mathbf{0}$ (equation 14), where \mathbf{I}_P and \mathbf{I}_D are the identities of order P and D , respectively.

Initialization :

```

1 Compute  $\mathbf{G}$ ;
2 Set  $\mathbf{r} = \mathbf{d}$  and compute  $\delta = \|\mathbf{r}\|/D$  ;
3 Compute  $\boldsymbol{\vartheta} = \mathbf{G}^\top \mathbf{r}$  and  $\rho_0 = \boldsymbol{\vartheta}^\top \boldsymbol{\vartheta}$  ;
4 Set  $\tilde{\mathbf{p}} = \mathbf{0}$ ,  $\tau = 0$  and  $\boldsymbol{\eta} = \mathbf{0}$  ;
5  $m = 1$  ;
6 while ( $\delta > \epsilon$ ) and ( $m < \text{ITMAX}$ ) do
7   Update  $\boldsymbol{\eta} \leftarrow \boldsymbol{\vartheta} + \tau \boldsymbol{\eta}$  ;
8   Compute  $\boldsymbol{\nu} = \mathbf{G} \boldsymbol{\eta}$  ;
9   Compute  $v = \rho_0 / (\boldsymbol{\nu}^\top \boldsymbol{\nu})$  ;
10  Update  $\tilde{\mathbf{p}} \leftarrow \tilde{\mathbf{p}} + v \boldsymbol{\eta}$  ;
11  Update  $\mathbf{r} \leftarrow \mathbf{r} - v \boldsymbol{\nu}$  and  $\delta \leftarrow \|v \boldsymbol{\nu}\|/D$  ;
12  Compute  $\boldsymbol{\vartheta} = \mathbf{G}^\top \mathbf{r}$  and  $\rho = \boldsymbol{\vartheta}^\top \boldsymbol{\vartheta}$  ;
13  Compute  $\tau = \rho / \rho_0$  ;
14  Update  $\rho_0 \leftarrow \rho$  ;
15   $m \leftarrow m + 1$  ;
16 end
```

Algorithm 2: Generic pseudo-code for the method proposed by Leão and Silva (1989).

Initialization :

```

1 Set the indices  $\mathbf{i}^m$  for each data window,  $m \in \{1 : M\}$  ;
2 Set the indices  $\mathbf{j}^m$  for each source window,  $m \in \{1 : M\}$  ;
3 Set the constant depth  $z_0 + \Delta z_0$  for all equivalent sources ;
4 Compute the vector  $\mathbf{a}'$  associated with the desired potential-field transformation ;
5 Compute the matrix  $\mathbf{G}'$  ;
6 Compute the matrix  $\mathbf{B}'$  (equation 34) ;
7 Compute the vector  $(\mathbf{a}')^\top \mathbf{B}'$  ;
8  $m = 1$  ;
9 while  $m < M$  do
10   Compute  $t_c^m$  (equation 33) ;
11    $m \leftarrow m + 1$  ;
12 end
```

Algorithm 3: Generic pseudo-code for the method proposed by Soler and Uieda (2021).

Initialization :

```

1 Set the indices  $\mathbf{i}^m$  for each data window,  $m \in \{1 : M\}$  ;
2 Set the indices  $\mathbf{j}^m$  for each source window,  $m \in \{1 : M\}$  ;
3 Set the depth of all equivalent sources ;
4 Set a  $D \times 1$  residuals vector  $\mathbf{r} = \mathbf{d}$  ;
5 Set a  $P \times 1$  vector  $\tilde{\mathbf{p}} = \mathbf{0}$  ;
6  $m = 1$  ;
7 while  $m < M$  do
8   | Set the matrix  $\mathbf{W}_d^m$  ;
9   | Compute the matrix  $\mathbf{G}^m$  ;
10  | Compute  $\tilde{\mathbf{p}}^m$  (equation 36) ;
11  |  $\tilde{\mathbf{p}}[\mathbf{j}^m] \leftarrow \tilde{\mathbf{p}}[\mathbf{j}^m] + \tilde{\mathbf{p}}^m$  ;
12  |  $\mathbf{r} \leftarrow \mathbf{r} - \mathbf{G}[:, \mathbf{j}^m] \tilde{\mathbf{p}}^m$  ;
13  |  $m \leftarrow m + 1$  ;
14 end

```

Algorithm 4: Generic pseudo-code for the method proposed by Cordell (1992).

Initialization :

```

1 Compute a  $D \times 1$  vector  $\Delta\mathbf{z}$  whose  $i$ -th element  $\Delta z_i$  is a vertical distance controlling the depth of
  the  $i$ -th equivalent source,  $i \in \{1 : D\}$  ;
2 Set a tolerance  $\epsilon$  ;
3 Set a maximum number of iterations ITMAX ;
4 Set a  $D \times 1$  residuals vector  $\mathbf{r} = \mathbf{d}$  ;
5 Set a  $D \times 1$  vector  $\tilde{\mathbf{p}} = \mathbf{0}$  ;
6 Define the maximum absolute value  $r_{\max}$  in  $\mathbf{r}$  ;
7  $m = 1$  ;
8 while ( $r_{\max} > \epsilon$ ) and ( $m < \text{ITMAX}$ ) do
9   | Define the coordinates  $(x_{\max}, y_{\max}, z_{\max})$  and index  $i_{\max}$  of the observation point associated with
      $r_{\max}$  ;
10  |  $\tilde{\mathbf{p}}[i_{\max}] \leftarrow \tilde{\mathbf{p}}[i_{\max}] + (r_{\max} \Delta\mathbf{z}[i_{\max}])$  ;
11  |  $\mathbf{r} \leftarrow \mathbf{r} - (\mathbf{G}[:, i_{\max}] \tilde{\mathbf{p}}[i_{\max}])$  ;
12  | Define the new  $r_{\max}$  in  $\mathbf{r}$  ;
13  |  $m \leftarrow m + 1$  ;
14 end

```

Algorithm 5: Generic pseudo-code for the method proposed by Mendonça and Silva (1994).

Initialization :

- 1 Set a regular grid of P equivalent sources at a horizontal plane z_0 ;
- 2 Set a tolerance ϵ ;
- 3 Set a $D \times 1$ residuals vector $\mathbf{r} = \mathbf{d}$;
- 4 Define the maximum absolute value r_{\max} in \mathbf{r} ;
- 5 Define the index i_{\max} of r_{\max} ;
- 6 Define the list of indices \mathbf{i}_r of the remaining data in \mathbf{r} ;
- 7 Define $\mathbf{d}_e = \mathbf{d}[i_{\max}]$;
- 8 Compute $(\mathbf{F} + \mu \mathbf{I}_{D_e})$ and \mathbf{G}_e ;
- 9 Compute $\tilde{\mathbf{p}}$ (equation 40) ;
- 10 Compute $\mathbf{r} = \mathbf{d}[\mathbf{i}_r] - \mathbf{G}[\mathbf{i}_r, :] \tilde{\mathbf{p}}$;
- 11 Define the maximum absolute value r_{\max} in \mathbf{r} ;
- 12 **while** ($r_{\max} > \epsilon$) **do**
- 13 Define the index i_{\max} of r_{\max} ;
- 14 Define the list of indices \mathbf{i}_r of the remaining elements in \mathbf{r} ;
- 15 $\mathbf{d}_e \leftarrow \begin{bmatrix} \mathbf{d}_e \\ \mathbf{d}[i_{\max}] \end{bmatrix}$;
- 16 Update $(\mathbf{F} + \mu \mathbf{I}_{D_e})$ and \mathbf{G}_e ;
- 17 Update $\tilde{\mathbf{p}}$ (equation 40) ;
- 18 Update $\mathbf{r} = \mathbf{d}[\mathbf{i}_r] - \mathbf{G}[\mathbf{i}_r, :] \tilde{\mathbf{p}}$;
- 19 Define the maximum absolute value r_{\max} in \mathbf{r} ;
- 20 **end**

Algorithm 6: Generic pseudo-code for the iterative method proposed by Siqueira et al. (2017). The symbol “ \circ ” denotes the entrywise or Hadamard product (e.g., Horn and Johnson, 1991, p. 298) and σ is a $P \times 1$ vector whose j -th element is the ratio of a predefined element of area centered at the j -th equivalent source and the term $2\pi\gamma$, where γ is the gravitational constant.

Initialization :

- 1 Set P equivalent sources on a horizontal plane z_0 ;
- 2 Set a tolerance ϵ ;
- 3 Set a maximum number of iterations ITMAX ;
- 4 Set an auxiliary vector σ ;
- 5 Compute $\tilde{\mathbf{p}} = \sigma \circ \mathbf{d}$;
- 6 Compute \mathbf{G} (equation 3) ;
- 7 Compute $\mathbf{r} = \mathbf{d} - \mathbf{G} \tilde{\mathbf{p}}$;
- 8 Compute $\delta = \|\mathbf{r}\|/D$;
- 9 $m = 1$;
- 10 **while** ($\delta > \epsilon$) **and** ($m < \text{ITMAX}$) **do**
- 11 Compute $\Delta\mathbf{p} = \sigma \circ \mathbf{r}$;
- 12 Update $\tilde{\mathbf{p}} \leftarrow \tilde{\mathbf{p}} + \Delta\mathbf{p}$;
- 13 Compute $\nu = \mathbf{G} \Delta\mathbf{p}$;
- 14 Update $\mathbf{r} \leftarrow \mathbf{r} - \nu$;
- 15 Compute $\delta = \|\nu\|/D$;
- 16 $m \leftarrow m + 1$;
- 17 **end**

Algorithm 7: Generic pseudo-code for the convolutional equivalent-layer method proposed by Takahashi et al. (2020, 2022).

Initialization :

- 1 Set the regular grid of P equivalent sources on a horizontal plane z_0 ;
 - 2 Set a tolerance ϵ and a maximum number of iterations ITMAX ;
 - 3 Compute the first column $\mathbf{G}[:, 1]$ and row $\mathbf{G}[1, :]$ of the sensitivity matrix \mathbf{G} (equation 3) for the particular case in which it has a BTTB structure (equation 51);
 - 4 Rearrange the elements of $\mathbf{G}[:, 1]$ into matrix \mathcal{C} , compute its 2D-DFT via 2D-FFT and multiply by $\sqrt{4D}$ to obtain a matrix \mathcal{L}' (equation 66);
 - 5 Rearrange the elements of $\mathbf{G}[1, :]$ into matrix \mathcal{C} , compute its 2D-DFT via 2D-FFT and multiply by $\sqrt{4D}$ to obtain a matrix \mathcal{L}'' (equation 66);
 - 6 Set $\tilde{\mathbf{p}} = \mathbf{0}$;
 - 7 Set $\mathbf{r} = \mathbf{d}$ and compute $\delta = \|\mathbf{r}\|/D$;
 - 8 Compute $\vartheta = \mathbf{G}^\top \mathbf{r}$ (Algorithm 8) and $\rho_0 = \vartheta^\top \vartheta$;
 - 9 Set $\tau = 0$ and $\eta = \mathbf{0}$;
 - 10 $m = 1$;
 - 11 **while** ($\delta > \epsilon$) **and** ($m < \text{ITMAX}$) **do**
 - 12 Update $\eta \leftarrow \vartheta + \tau \eta$;
 - 13 Compute $\nu = \mathbf{G} \eta$ (Algorithm 8);
 - 14 Compute $v = \rho_0 / (\nu^\top \nu)$;
 - 15 Update $\tilde{\mathbf{p}} \leftarrow \tilde{\mathbf{p}} + v \eta$;
 - 16 Update $\mathbf{r} \leftarrow \mathbf{r} - v \nu$ and $\delta \leftarrow \|v \nu\|/D$;
 - 17 Compute $\vartheta = \mathbf{G}^\top \mathbf{r}$ (Algorithm 8) and $\rho = \vartheta^\top \vartheta$;
 - 18 Compute $\tau = \rho / \rho_0$;
 - 19 Update $\rho_0 \leftarrow \rho$;
 - 20 $m \leftarrow m + 1$;
 - 21 **end**
-

Algorithm 8: Pseudo-code for computing the generic matrix-vector products given by equations 52 and 53 via fast 2D discrete convolution for a given vector \mathbf{v} (equation 54) and matrix \mathcal{L} (equation 66).

- 1 Rearrange the elements of \mathbf{v} (equations 52 and 54) into the matrix \mathcal{V}_c (equation 65);
 - 2 Compute $\mathcal{F}_{2N_B} \mathcal{V}_c \mathcal{F}_{2N_b}$ via 2D-FFT;
 - 3 Compute the Hadamard product with matrix \mathcal{L} (equation 66);
 - 4 Compute 2D-IDFT via 2D-FFT to obtain matrix \mathcal{W}_c (65);
 - 5 Retrieve \mathbf{w} (equations 52 and 54) from \mathbf{w}_c (equations 55–57);
-

12 TABLES

Reference	Term	flops
eq. 10	$\mathbf{G} \mathbf{H}$	$2DQP$
eq. 15	$\mathbf{H} \tilde{\mathbf{q}}$	$2PQ$
eq. 22	$(\mathbf{G} \mathbf{H})^\top (\mathbf{G} \mathbf{H})$	$2Q^2D$
eq. 22	$(\mathbf{G} \mathbf{H})^\top \boldsymbol{\delta}_d$	$2QD$
eq. 23	$(\mathbf{G} \mathbf{H}) (\mathbf{G} \mathbf{H})^\top$	$2D^2Q$
eq. 23	$(\mathbf{G} \mathbf{H})^\top \mathbf{u}$	$2QD$
eq. 25	lower triangle of \mathcal{G}	$D^3/3$ or $Q^3/3$
eq. 26	solve triangular systems	$2D^2$ or $2Q^2$
Alg. 1	$\boldsymbol{\eta} \leftarrow \boldsymbol{\vartheta} + \tau \boldsymbol{\eta}$	$2Q$
Alg. 1	$\boldsymbol{\vartheta}^\top \boldsymbol{\vartheta}$	$2Q$
Alg. 6	$\boldsymbol{\sigma} \circ \mathbf{d}$	D

Table 1. Total number of flops associated with some useful terms according to Golub and Van Loan (2013, p. 12). The flops associated with equations 25 and 26 depends if the problem is over or underdetermined. Note that $P = Q$ for the case in which $\mathbf{H} = \mathbf{I}_P$ (subsection 3.2). The term associated with Algorithm 1 is a vector update called *saxpy* (Golub and Van Loan, 2013, p. 4). The terms defined here are references to compute the total number of flops throughout the manuscript.

13 FIGURES

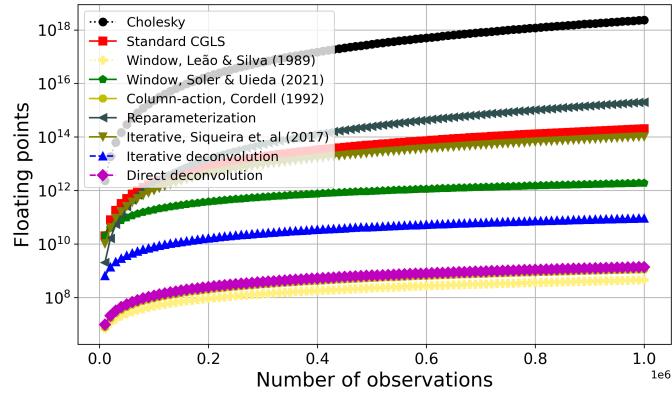


Figure 1. Number of *flops* for many of the methods described in this work to estimate the equivalent sources using gravity data. The range of observations varies from 10,000 to 1,000,000.

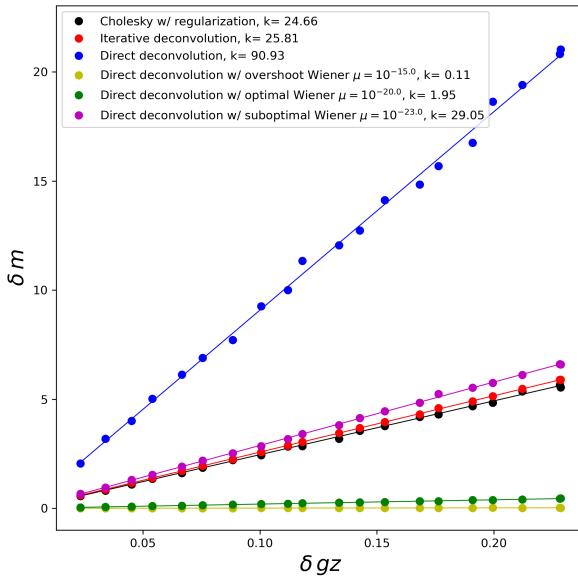


Figure 2. Stability analysis of some of the equivalent layer methods of the gravimetric case.

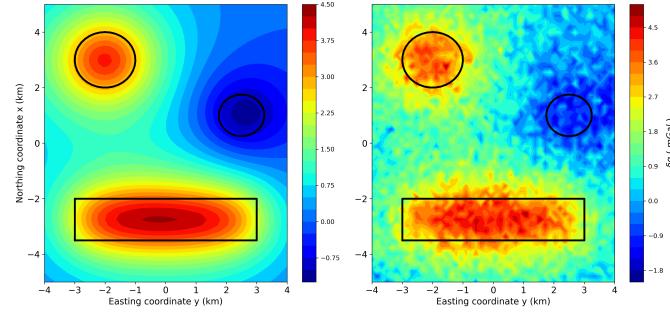


Figure 3. Synthetic data of the gravimetric case. The observations points are placed in a regular grid of 50×50 . Panel (A) shows the noise-free data and panel (B) shows the maximum noised data (10%).

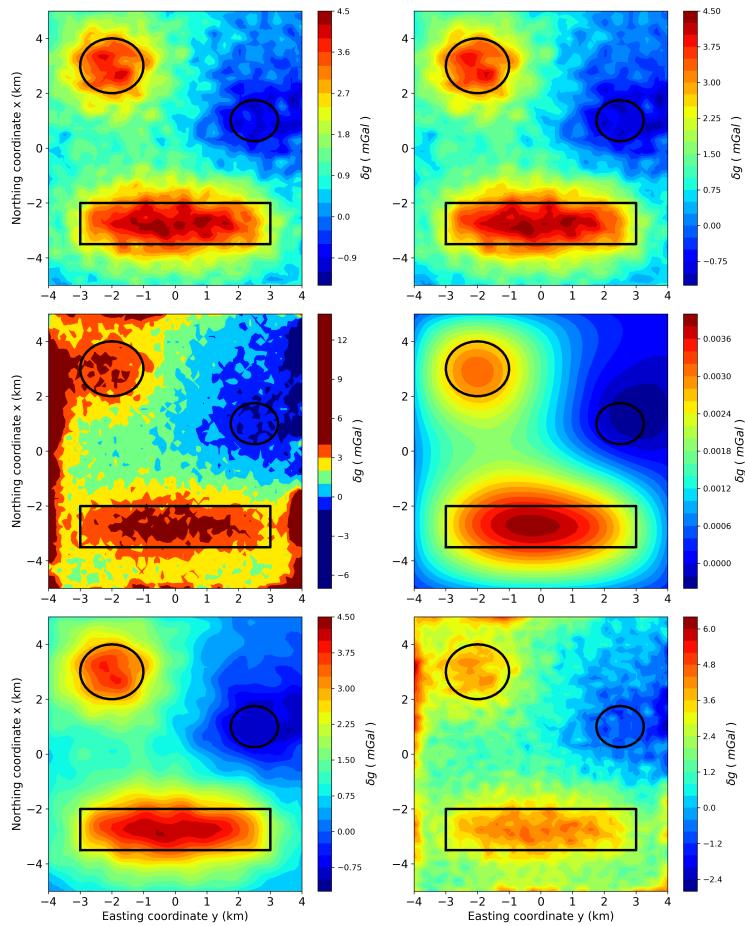


Figure 4. Predicted gravity data for different methods of the equivalent layer with maximum level of noise. Panel (A) is the classical method, (B) is the convolutional, (C) is the deconvolutional, (D) is the deconvolutional method using Wiener stabilization with a too high value for μ , (E) is the deconvolutional method using Wiener stabilization with a optimal value for μ and (F) is the deconvolutional method using Wiener stabilization with a too low value for μ .

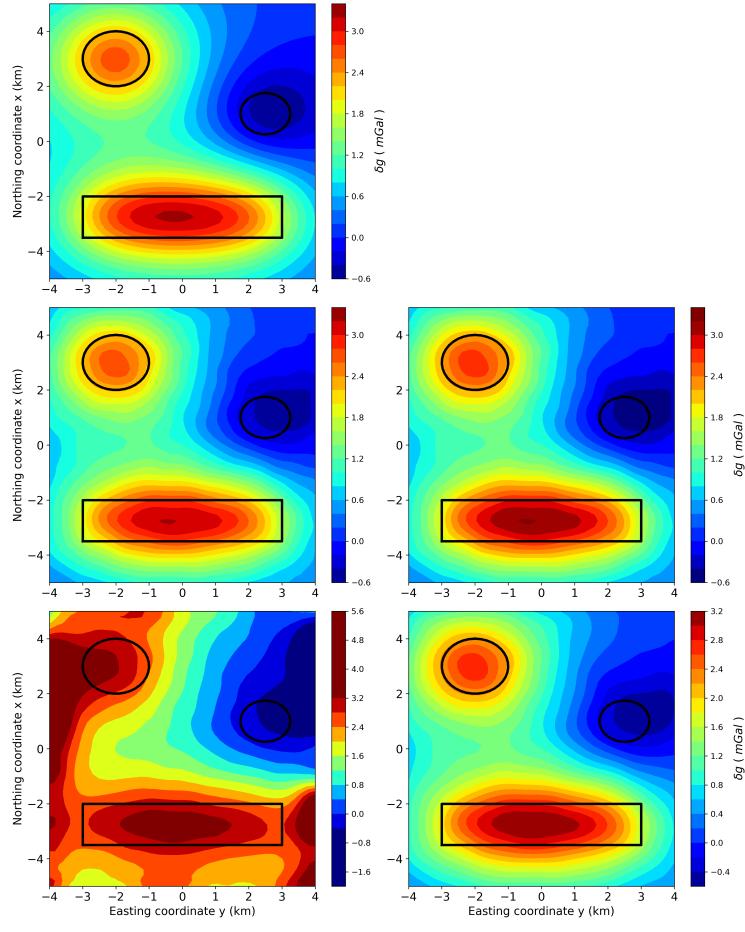


Figure 5. True noiseless upward gravimetric data at $z_i = -500$ m height and predicted data for different methods of the equivalent layer with maximum level of noise. Panel **(A)** is the true upward gravity data, Panel **(B)** is the classical method, **(C)** is the convolutional, **(D)** is the deconvolutional, **(E)** is the deconvolutional method using Wiener stabilization with a optimal value for $\mu = 10^{-20}$.

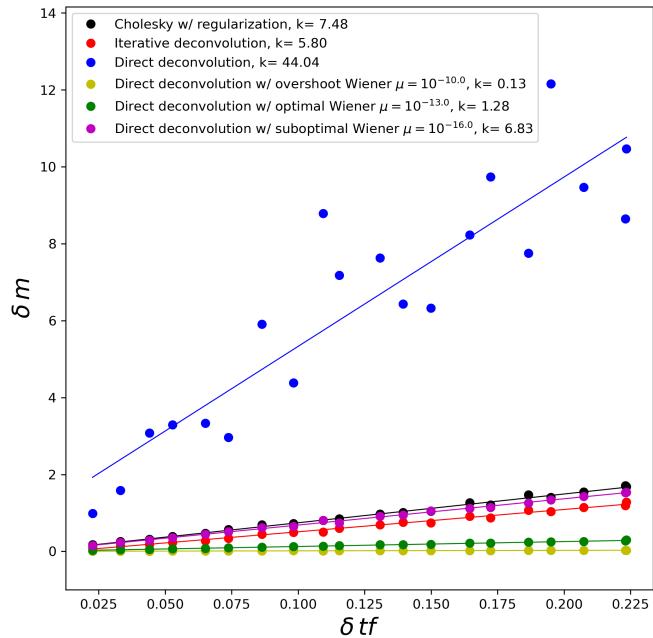


Figure 6. Stability analysis of some of the equivalent layer methods of the magnetic case.

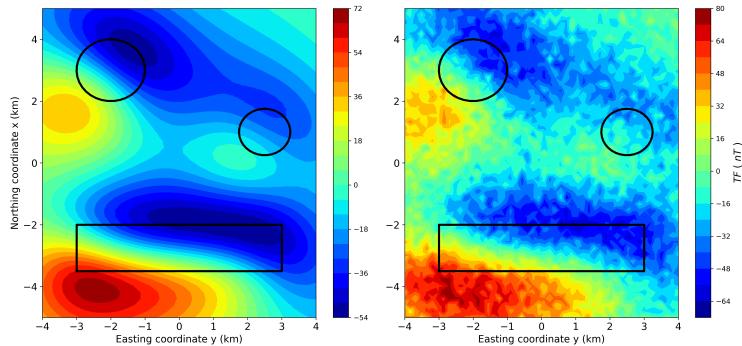


Figure 7. Synthetic data of the magnetic case. The observations points are placed in a regular grid of 50×50 . Panel (A) shows the noise-free data and panel (B) shows the maximum noised data (10%).

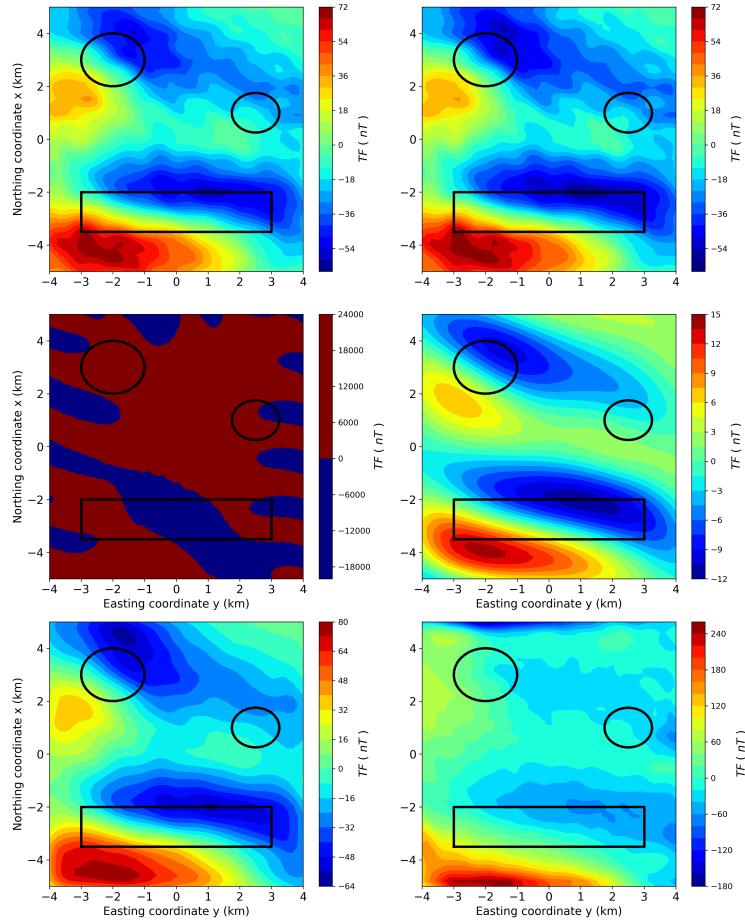


Figure 8. Predicted magnetic data for different methods of the equivalent layer with maximum level of noise. Panel **(A)** is the classical method, **(B)** is the convolutional, **(C)** is the deconvolutional, **(D)** is the deconvolutional method using Wiener stabilization with a too high value for μ , **(E)** is the deconvolutional method using Wiener stabilization with a optimal value for μ and **(F)** is the deconvolutional method using Wiener stabilization with a too low value for μ .

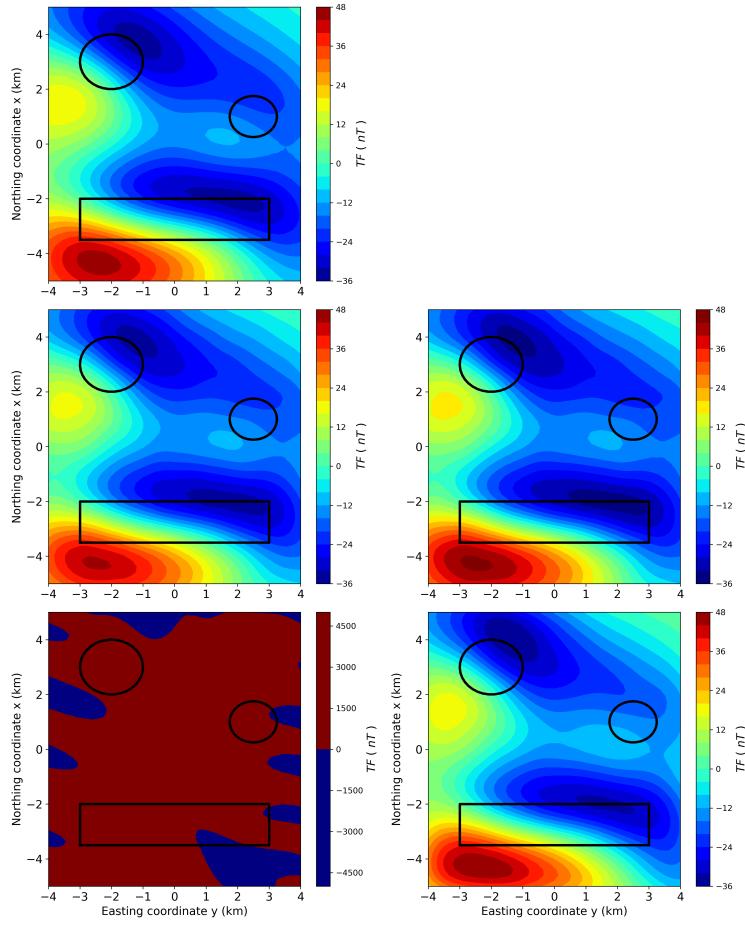


Figure 9. True noiseless upward magnetic data at $z_i = -1400$ m height and predicted data for different methods of the equivalent layer with maximum level of noise. Panel **(A)** is the true upward magnetic data, Panel **(B)** is the classical method, **(C)** is the convolutional, **(D)** is the deconvolutional, **(E)** is the deconvolutional method using Wiener stabilization with a optimal value for $\mu = 10^{-13}$.

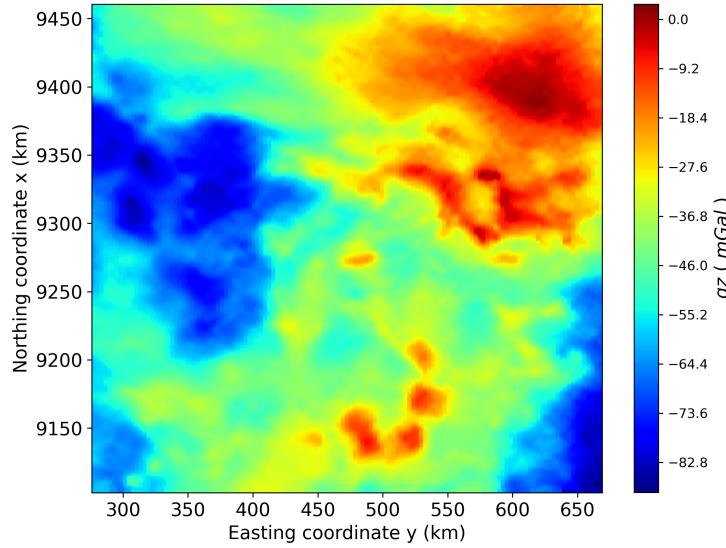


Figure 10. Gridded real aerogravimetric data from Carajás, Brazil. A regular grid of $1,000 \times 500$ is being used, totalizing $N, M = 500,000$ obsevation points and equivalent sources.

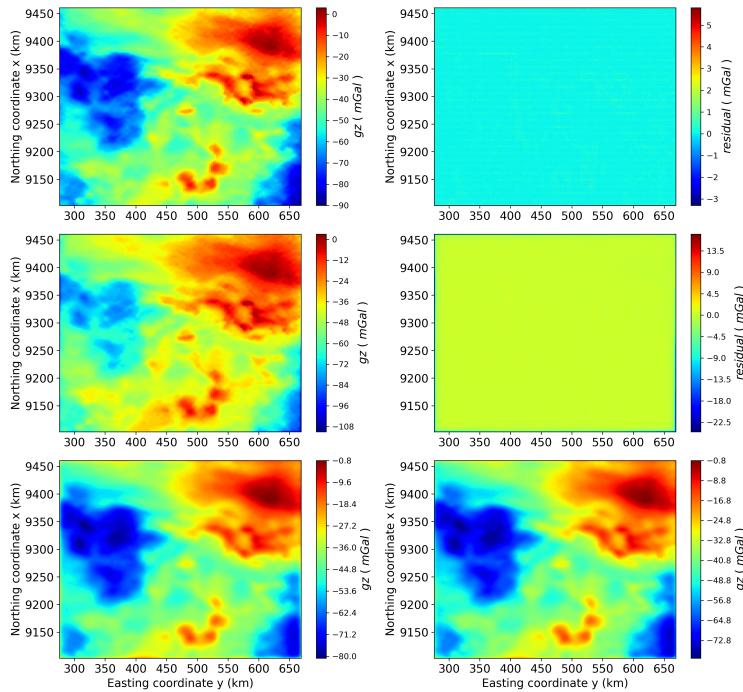


Figure 11. Panel (A) shows the Carajás predicted gravimetric data from convolutional equivalent layer method. Panel (B) shows the residual from the convolutional equivalent layer method. Panel (C) shows the predicted data from deconvolutional equivalent layer method. Panel (D) shows the residual from the deconvolutional equivalent layer method. Panel (E) shows the upward continuation at $z_i = -3500$ m for the convolutional method and Panel (F) shows the upward continuation at $z_i = -3500$ m for the deconvolutional method.

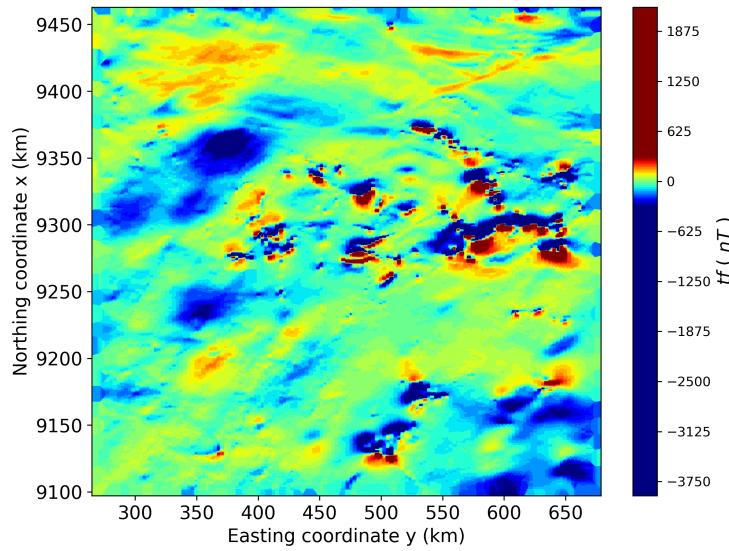


Figure 12. Gridded real aeromagnetic data from Carajás, Brazil. A regular grid of $1,000 \times 500$ is being used, totalizing $N, M = 500,000$ obsevation points and equivalent sources.

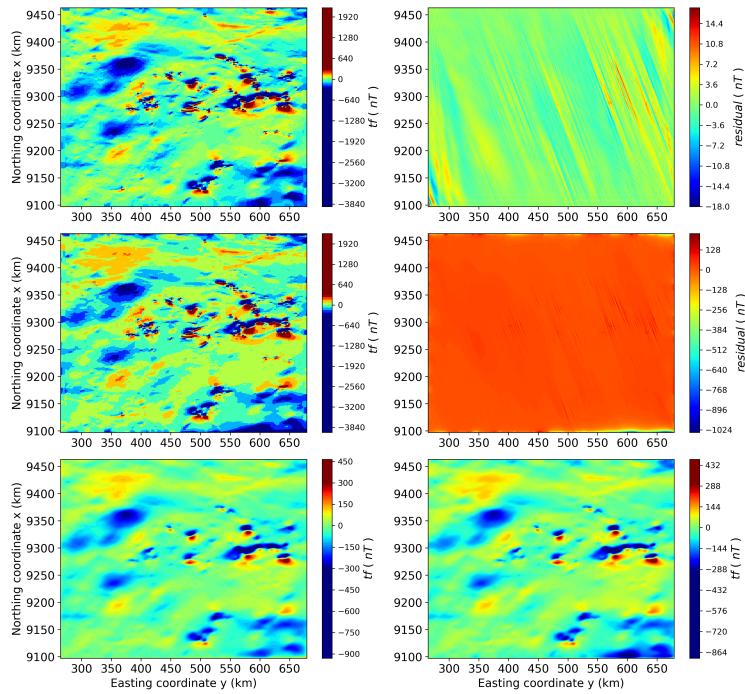


Figure 13. Panel (A) shows the Carajás predicted magnetic data from convolutional equivalent layer method. Panel (B) shows the residual from the convolutional equivalent layer method. Panel (C) shows the predicted data from deconvolutional equivalent layer method. Panel (D) shows the residual from the deconvolutional equivalent layer method. Panel (E) shows the upward continuation at $z_i = -3500$ m for the convolutional method and Panel (F) shows the upward continuation at $z_i = -3500$ m for the deconvolutional method.