3D Magnetic modelling of ellipsoidal bodies

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Abstract.

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1 Introduction

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2 Methodology

2.1 Geometrical parameters and coordinate systems

Let (x, y, z) be a point referred to a Cartesian coordinate system with axes x, y and z pointing to, respectively, North, East and down. For convenience, we denominate this coordinate system as *main coordinate system*. Let us consider an ellipsoidal body with centre at the point (x_c, y_c, z_c), semi-axes defined by positive constants a, b, c, where $a > b > c$, and orientation defined by three angles α, β, and γ. The points (x, y, z) located on the surface of this ellipsoidal body satisfy the following equation:

$$(\mathbf{r} - \mathbf{r}_c)^T \mathbf{A} (\mathbf{r} - \mathbf{r}_c) = 1 \, , \tag{1}$$

where $\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$, $\mathbf{r}_c = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^\top$, $\mathbf{A}$ is a positive definite matrix given by

$$\mathbf{A} = \mathbf{V} \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{bmatrix} \mathbf{V}^\top \, , \tag{2}$$

and $\mathbf{V}$ is an orthogonal matrix whose columns are defined by unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ depend on the orientation angles α, β, γ and are defined as follows (Clark et al., 1986):

$$\mathbf{v}_1 = \begin{bmatrix} -\cos\alpha \, \cos\delta \\ -\sin\alpha \, \cos\delta \\ -\sin\delta \end{bmatrix} , \tag{3}$$

$$\mathbf{v}_2 = \begin{bmatrix} \cos\alpha\ \cos\gamma\ \sin\delta + \sin\alpha\ \sin\gamma \\ \sin\alpha\ \cos\gamma\ \sin\delta - \cos\alpha\ \sin\gamma \\ -\cos\gamma\ \cos\delta \end{bmatrix}, \tag{4}$$

$$\mathbf{v}_3 = \begin{bmatrix} \sin\alpha\ \cos\gamma - \cos\alpha\ \sin\gamma\ \sin\delta \\ -\cos\alpha\ \cos\gamma - \sin\alpha\ \sin\gamma\ \sin\delta \\ \sin\gamma\ \cos\delta \end{bmatrix}. \tag{5}$$

For triaxial ellipsoids (i.e., $a > b > c$), the orthogonal matrix $\mathbf{V}$ (equation 2) is calculated by using equations 3, 4, and 5 as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}. \tag{6}$$

Similarly, the matrix $\mathbf{V}$ (equation 2) for prolate ellipsoids (i.e., $a > b = c$) is calculated according to equation 6 by using equations 3, 4, and 5, but with $\gamma = 0°$ (Emerson et al., 1985). Finally, the matrix $\mathbf{V}$ (equation 2) for oblate ellipsoids (i.e., $a < b = c$) is calculated by using equations 3, 4, and 5, with $\gamma = 0°$, as follows (Emerson et al., 1985):

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_1 & -\mathbf{v}_3 \end{bmatrix}. \tag{7}$$

The orientation of the semi-axes a, b, and c are defined by the first, second, and third columns of the matrix $\mathbf{V}$ given by equation 6, in the case of a triaxial or prolate ellipsoid, or the matrix $\mathbf{V}$ given by equation 7, in the case of an oblate ellipsoid.

The magnetic modelling of an ellipsoidal body is commonly performed in a particular Cartesian coordinate system that is aligned with the body semi-axes and has the origin coincident with the body centre. For convenience, we denominate this particular coordinate system as *local coordinate system*. The relationship between the Cartesian coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ of a point in a local coordinate system and the Cartesian coordinates (x, y, z) of the same point in the main system is given by:

$$\tilde{\mathbf{r}} = \mathbf{V}^\top (\mathbf{r} - \mathbf{r}_c), \tag{8}$$

where $\tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{bmatrix}^\top$, $\mathbf{r}$ and $\mathbf{r}_c$ are defined in equation 1 and the matrix $\mathbf{V}$ is defined according to equations 6 or 7, depending on the ellipsoid type. Similarly, the transformation of a generic linear system $\mathbf{Mp} = \mathbf{d}$ referred to the main coordinate system into a new linear system $\tilde{\mathbf{M}}\tilde{\mathbf{p}} = \tilde{\mathbf{d}}$ referred to a local coordinate system is given by

$$\tilde{\mathbf{M}} = \mathbf{V}^\top \mathbf{M} \mathbf{V}, \quad \tilde{\mathbf{p}} = \mathbf{V}^\top \mathbf{p}, \quad \tilde{\mathbf{d}} = \mathbf{V}^\top \mathbf{d}. \tag{9}$$

2.2 Magnetic field produced by ellipsoids

Based on the mathematical theory of the magnetic induction developed by **?**, Maxwell (1873) affirmed that, if V is the gravitational potential produced by any body with uniform density ρ and arbitrary shape at a point (x, y, z), then $-\frac{\partial V}{\partial x}$ is the magnetic scalar potential produced at the same point by the same body if it has a uniform magnetization oriented along x with intensity ρ. Maxwell (1873) generalized this idea as a way of determining the magnetic scalar potential produced by any body uniformly magnetized in a given direction. By presuming that this uniform magnetization is due to induction, he postulated that the resulting magnetic field (intensity) at all points within the body must also be uniform and parallel the magnetization, which results that the gravitational potential V at points within the body must be a quadratic function of the spatial coordinates. Apparently, Maxwell (1873) was the first one to affirm that the only finite bodies having a gravitational potential with this property and that, as a consequence, can be uniformly magnetized in the presence of a uniform and static magnetic field are the ones bounded by surfaces of second degree, which are ellipsoids.

Consider a magnetized ellipsoid immersed in a uniform magnetic field $\mathbf{H}_0$ (in Am^{-1}). In the absence of conduction currents, the total magnetic field $\mathbf{H}(\mathbf{r})$ at the position $\mathbf{r}$ (equations 2 and 8) of a point referred to the main coordinate system is defined as follows (Stratton, 2007):

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \nabla\phi(\mathbf{r})\,, \tag{10}$$

where the second term is the negative gradient of the magnetic scalar potential $\phi(\mathbf{r})$ given by:

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\iiint\limits_V \mathbf{M}(\mathbf{r}')^\top \nabla\left(\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\right) dx'dy'dz'\,. \tag{11}$$

In this equation, $\mathbf{r}' = \begin{bmatrix} x' & y' & z' \end{bmatrix}^\top$ is the position vector of a point located within the volume V, the integral is conducted over the variables x', y' and, z' representing the coordinates of a point located within the volume V of the ellipsoid, $\|\cdot\|$ denotes the Euclidean norm and $\mathbf{M}(\mathbf{r}')$ is the magnetization vector (in Am^{-1}). Equation 11 is valid anywhere, independently if the position vector $\mathbf{r}$ represents a point located inside or outside the magnetized body (DuBois, 1896).

Based on Maxwell's postulate, let us assume that the body has a uniform magnetization given by

$$\mathbf{M} = \mathbf{K}\,\mathbf{H}_i\,, \tag{12}$$

where $\mathbf{H}_i$ is the resultant uniform magnetic field at any point within the body and $\mathbf{K}$ is a constant and symmetrical 2nd-order tensor representing the magnetic susceptibility of the body. In this case, equation 10 can be rewritten as follows:

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \mathbf{N}(\mathbf{r})\,\mathbf{K}\,\mathbf{H}_i\,, \tag{13}$$

where $\mathbf{N}(\mathbf{r})$ is a 3×3 matrix whose $\alpha\beta$-element $n_{\alpha\beta}(\mathbf{r})$ is given by

$$n_{\alpha\beta}(\mathbf{r}) = -\frac{1}{4\pi}\frac{\partial^2}{\partial\alpha\,\partial\beta}\iiint\limits_V \frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\,dx'dy'dz'\,, \quad \alpha = x,y,z\,, \quad \beta = x,y,z\,. \tag{14}$$

It can be shown that the elements $n_{\alpha\beta}(\mathbf{r})$ are finite whether $\mathbf{r}$ is a point within or without the volume V (Peirce, 1902; Webster, 1904). The matrix $\mathbf{N}(\mathbf{r})$ (equation 13) is called *depolarization tensor* (Solivérez, 1981, 2008).

2.2.1 Internal magnetic field and magnetization

2.2.2 External magnetic induction

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2.3.1 TEXT

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3 Conclusions

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Appendix A

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Author contributions. TEXT

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