

3D Magnetic modelling of ellipsoidal bodies

Diego Takahashi Tomazella¹ and Vanderlei C. Oliveira Jr¹

¹Department of Geophysics, Observatório Nacional, Rio de Janeiro, Brazil

Correspondence to: Diego Takahashi Tomazella (diego.takahashi@gmail.com)

Abstract.

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1 Introduction

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5 2 Methodology

2.1 Geometrical parameters and coordinate systems

Let (x, y, z) be a point referred to a Cartesian coordinate system with axes x , y and z pointing to, respectively, North, East and down. For convenience, we denominate this coordinate system as *main coordinate system*. Let us consider an ellipsoidal body with centre at the point (x_c, y_c, z_c) , semi-axes defined by positive constants a , b , c , where $a > b > c$, and orientation defined by
10 three angles α , β , and γ . The points (x, y, z) located on the surface of this ellipsoidal body satisfy the following equation:

$$(\mathbf{r} - \mathbf{r}_c)^T \mathbf{A} (\mathbf{r} - \mathbf{r}_c) = 1, \quad (1)$$

where $\mathbf{r} = [x \ y \ z]^T$, $\mathbf{r}_c = [x_c \ y_c \ z_c]^T$, \mathbf{A} is a positive definite matrix given by

$$\mathbf{A} = \mathbf{V} \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{bmatrix} \mathbf{V}^T, \quad (2)$$

and \mathbf{V} is an orthogonal matrix whose columns are defined by unit vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 .

15 The vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 depend on the orientation angles α , β , γ and are defined as follows (Clark et al., 1986):

$$\mathbf{v}_1 = \begin{bmatrix} -\cos \alpha \cos \delta \\ -\sin \alpha \cos \delta \\ -\sin \delta \end{bmatrix}, \quad (3)$$

$$\mathbf{v}_2 = \begin{bmatrix} \cos \alpha \cos \gamma \sin \delta + \sin \alpha \sin \gamma \\ \sin \alpha \cos \gamma \sin \delta - \cos \alpha \sin \gamma \\ -\cos \gamma \cos \delta \end{bmatrix}, \quad (4)$$

$$\mathbf{v}_3 = \begin{bmatrix} \sin \alpha \cos \gamma - \cos \alpha \sin \gamma \sin \delta \\ -\cos \alpha \cos \gamma - \sin \alpha \sin \gamma \sin \delta \\ \sin \gamma \cos \delta \end{bmatrix}. \quad (5)$$

- 5 For triaxial ellipsoids (i.e., $a > b > c$), the orthogonal matrix \mathbf{V} (equation 2) is calculated by using equations 3, 4, and 5 as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}. \quad (6)$$

- Similarly, the matrix \mathbf{V} (equation 2) for prolate ellipsoids (i.e., $a > b = c$) is calculated according to equation 6 by using equations 3, 4, and 5, but with $\gamma = 0^\circ$ (Emerson et al., 1985). Finally, the matrix \mathbf{V} (equation 2) for oblate ellipsoids (i.e.,
10 $a < b = c$) is calculated by using equations 3, 4, and 5, with $\gamma = 0^\circ$, as follows (Emerson et al., 1985):

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_1 & -\mathbf{v}_3 \end{bmatrix}. \quad (7)$$

The orientation of the semi-axes a , b , and c are defined by the first, second, and third columns of the matrix \mathbf{V} given by equation 6, in the case of a triaxial or prolate ellipsoid, or the matrix \mathbf{V} given by equation 7, in the case of an oblate ellipsoid.

- The magnetic modelling of an ellipsoidal body is commonly performed in a particular Cartesian coordinate system that is
15 aligned with the body semi-axes and has the origin coincident with the body centre. For convenience, we denominate this particular coordinate system as *local coordinate system*. The relationship between the Cartesian coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ of a point in a local coordinate system and the Cartesian coordinates (x, y, z) of the same point in the main system is given by:

$$\tilde{\mathbf{r}} = \mathbf{V}^\top (\mathbf{r} - \mathbf{r}_c), \quad (8)$$

- where $\tilde{\mathbf{r}} = [\tilde{x} \ \tilde{y} \ \tilde{z}]^\top$, \mathbf{r} and \mathbf{r}_c are defined in equation 1 and the matrix \mathbf{V} is defined according to equations 6 or 7, de-
20 pending on the ellipsoid type. Similarly, the transformation of a generic linear system $\mathbf{M}\mathbf{p} = \mathbf{d}$ referred to the main coordinate system into a new linear system $\tilde{\mathbf{M}}\tilde{\mathbf{p}} = \tilde{\mathbf{d}}$ referred to a local coordinate system is given by

$$\tilde{\mathbf{M}} = \mathbf{V}^\top \mathbf{M} \mathbf{V}, \quad \tilde{\mathbf{p}} = \mathbf{V}^\top \mathbf{p}, \quad \tilde{\mathbf{d}} = \mathbf{V}^\top \mathbf{d}. \quad (9)$$

2.2 Magnetic field produced by ellipsoids

- Based on the mathematical theory of the magnetic induction developed by ?, Maxwell (1873) affirmed that, if V is the gravita-
25 tional potential produced by any body with uniform density ρ and arbitrary shape at a point (x, y, z) , then $-\frac{\partial V}{\partial x}$ is the magnetic

scalar potential produced at the same point by the same body if it has a uniform magnetization oriented along x with intensity ρ . Maxwell (1873) generalized this idea as a way of determining the magnetic scalar potential produced by any body uniformly magnetized in a given direction. By presuming that this uniform magnetization is due to induction, he postulated that the resulting magnetic field intensity at all points within the body must also be uniform and parallel the magnetization, which results

5 that the gravitational potential V at points within the body must be a quadratic function of the spatial coordinates. Apparently, Maxwell (1873) was the first one to affirm that the only finite bodies having a gravitational potential with this property and that, as a consequence, can be uniformly magnetized in the presence of a uniform and static magnetic field are the ones bounded by surfaces of second degree, which are ellipsoids.

Let us start our description of the magnetic field produced by ellipsoids by considering a magnetized ellipsoid immersed in

10 a uniform magnetic field \mathbf{H}_0 (in Am^{-1}). The total magnetic intensity field $\mathbf{H}(\mathbf{r})$ at the position \mathbf{r} (equations 2 and 8) of a point referred to the main coordinate system is defined as follows

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \nabla\phi(\mathbf{r}), \quad (10)$$

where the second term is the negative gradient of the magnetic scalar potential $\phi(\mathbf{r})$ given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \iiint_V \mathbf{M}(\mathbf{r}')^\top \left(\frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \right) dx' dy' dz'. \quad (11)$$

15 In this equation, $\mathbf{r}' = [x' \ y' \ z']^\top$ is the position vector of a point located within the volume V , the integral is conducted over the variables x' , y' and z' representing the coordinates of a point located within the volume V of the ellipsoid, $\|\cdot\|$ denotes the Euclidean norm and $\mathbf{M}(\mathbf{r}')$ is the magnetization vector (in Am^{-1}). Equation 10 is valid anywhere, independently if the position vector \mathbf{r} represents a point located inside or outside the magnetized body.

Based on Maxwell's postulate, let us assume that the body has a uniform magnetization given by

$$20 \quad \mathbf{M} = \mathbf{K} \mathbf{H}_i, \quad (12)$$

where \mathbf{H}_i is the resultant uniform magnetic field intensity at any point within the body and \mathbf{K} is a constant and symmetrical 2nd-order tensor representing the magnetic susceptibility of the body. In this case, equation 10 can be rewritten as follows:

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \mathbf{N}(\mathbf{r}) \mathbf{K} \mathbf{H}_i, \quad (13)$$

where $\mathbf{N}(\mathbf{r})$ is a 3×3 matrix whose $\alpha\beta$ -element $n_{\alpha\beta}(\mathbf{r})$ is given by

$$25 \quad n_{\alpha\beta}(\mathbf{r}) = -\frac{1}{4\pi} \frac{\partial^2}{\partial\alpha\partial\beta} \iiint_V \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dx' dy' dz', \quad \alpha = x, y, z, \quad \beta = x, y, z. \quad (14)$$

2.3 Magnetic induction

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3 Conclusions

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Appendix A

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5 *Author contributions.* TEXT

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References

Clark, D., Saul, S., and Emerson, D.: Magnetic and gravity anomalies of a triaxial ellipsoid, *Exploration Geophysics*, 17, 189–200, 1986.

Emerson, D. W., Clark, D., and Saul, S.: Magnetic exploration models incorporating remanence, demagnetization and anisotropy: HP 41C handheld computer algorithms, *Exploration Geophysics*, 16, 1–122, 1985.

5 Maxwell, J. C.: A treatise on electricity and magnetism, vol. 2, Clarendon press, 1873.