3D MAGNETIC MODELLING FOR ELIPSOIDS

Diego Takahashi Tomazella[1] and Vanderlei Coelho Oliveira Junior[1]

[1]Department of Geophysics, Observatorio Nacional, Rio de Janeiro, Brazil

Correspondence to: DIEGO TAKAHASHI TOMAZELLA (diego.takahashi@gmail.com)

Abstract. In this work we will present results from a numerical modeling of the magnetic field and total field anomaly, represented by triaxial and prolate elipsoidal sources. Such approach provide analytical results for anisotropy of magnetic susceptibility as well as for self-demagnetization effects, which can be easily adapted for distinctive geologic structures - hence being an useful tool for educational (e.g., potential methods and rock magnetism) and applied geophysics (e.g., characterization of high magnetic susceptibility, mineralized bodies) purposes. Numerical tests by means of a Python code (currently under development) allowed us to compared the effects of different geometries (ellipsoidal sources, spheres, dipole lines and elliptic cylinders), which were used to validate our computational implementation. This code will be freely available to the scientific community by the end of the year.

1 Introduction

In the 19th century Dirichlet (1839), manage to solve the problem of the potential of a homogeneous ellipsoid with semi-axes $a \geq b \geq c$ by using the first and second Legendre's normal elliptic integrals.

In the late 19th century Maxwell (1881), showed that only bodies of second degree can be uniformly magnetized when placed in a uniform field. He also presented the formulae of the demagnetizing field in a uniformly magnetized ellipsoid of revolution.

Osborn (1945), published formulas for the demagnetizing factors of the general ellipsoid alongside with a table of values for different proportions of semi-axis

Stoner (1945), presented a more simple and friendly formulae of the demagnetizing factors for an ellipsoid, with step-by-step deduction. This work includes formulae for different parametrizations for the ellipsoid (triaxial, prolate, oblate).

Peake and Davy (1953) published the formulas for the external field of ellipsoid of revolution in terms of elliptical integrals for external fields.

Chang (1961) published the formulas for the external field of triaxial, prolate and oblate ellipsoids in terms of elliptical integrals for external fields.

In the decade of 1970 Farrar (1979) showed in his work the value of the elipsoidal model in geophysic exploration, where it was used for analysis of magnetic anomalies in Tennant Creeks's gold mine in Australia. The elipsoid was the most correct geometry to model bodies of lenticular pipe shape and the most appropriate to handle self-desmagnetization of bodies of high susceptibilty - magnetic susceptibility higher than 0.1 SI (Clark, 2014).

The magnetic field resultant of an ellipsoid was presented the first time in full modeling in the space domain by Emerson et al. (1985), along a compilation of several others geometric shapes. Through a consistent notation this work tapped a hole in the literature that missed the ellipsoidal model. This model, however, consists in a division between prolate and oblate ellipsoids, with a few differences in the algorithm. A generalized model for triaxials ellipsoids was published by Clark et al. (1986). Both in Emerson et al. and Clark et al. works, they were the firsts to include the calculations of the magnetic field in a different coordinate system other than with the position $(0,0,0)$ in the center of the ellipsoid (through a matrix of rotation). However, the codes presented for HP-41C are outdated and doesn't have a detailed descrition of the algorithm.

Tejedor et al. (1995), presented a work where they calculate the external field generated by ellipsoids using a variational method, in contrast to the usual method of Dirichlet.

Guo et al. (1998), showed three case studies of the importance of self-demagnetization of high susceptibilties bodies when modelling magnetic sources. A magnetic-iron deposit, volcanic-hosted iron deposit and an ultramafic-hoste nickel andcopper deposits.

Guo et al. (2001), published another paper analysing the errors of what an uncorrected by demagnetization data can affect on anmalies interpretation.

2 Methodology

The total-field anomaly can be described as a difference between the total field vector and the induced magnetization of the crust of the Earth, that includes any anomalous distribuition of magnetization that can occur by magnetizied bodies in subsurface. This difference can be written as:

$$\Delta T_i^0 = ||T_i|| - ||F_i||, \tag{1}$$

where ΔT_i^0 is the observed vector of total-field anomaly in the i-th position (x_i, y_i, z_i), $i = 1, ...N$, F_i is the geomagnetic field and $||.||$ is the Euclidian norm. The total-field vector T_i is:

$$T_i = F_i + B_i, \tag{2}$$

and B_i is the total induced magnetization vector produced by all sources of anomalous susceptibility distribuition. Considering F_i a constant vector, F_0 for local ou regional scale, and that $||F_0|| \gg ||B_0||$, since, B_i is a small pertubation of the magnetic field F_i, we can approximate the euclidian norm of the vector T_i by a Taylor's expansion of first order:

$$||T_i|| \approx ||F_0 + B_i|| \approx ||F_0|| + F^T B_i \tag{3}$$

where T indicates transposition and

$$\hat{F} = \frac{F_o}{||F_0||} \tag{4}$$

is a unit vector that represents the gradient of the function $||T_i||$ in relation to the vector's components T_i. This way, we can approximate the Eq.(1) of total-field anomaly to:

$$\Delta T \approx \hat{F}^T B_i, \quad i = 1, ..., N \tag{5}$$

2.1 The Forward model of a sphere

2.2 The Forward model of a triaxial elipsoid

The implementation of the forward problem of a triaxial elipsoid's magnetic field (three semi-axis $a > b > c$) is done in a new coordinate system, where its origin is the center of this elipsoidal body.

This new coordinate system (x_1, x_2, x_3) is defined by the unit vectors $\hat{v}_h (h = 1, 2, 3)$ with the respect with the geographic axis x, y e z:

$$\hat{v_1} = (l_1, m_1, n_1) = (-\cos\alpha\cos\delta, -\sin\alpha\cos\delta, -\sin\delta) \tag{6}$$

$$\begin{aligned}\hat{v_2} = (l_2, m_2, n_2) = (&\cos\alpha\cos\gamma\sin\delta + \sin\alpha\sin\gamma,\\ &\sin\alpha\cos\gamma\sin\delta - \cos\alpha\sin\gamma, -\cos\gamma\cos\delta)\end{aligned} \tag{7}$$

$$\begin{aligned}\hat{v_3} = (l_3, m_3, n_3) = (&\sin\alpha\cos\gamma - \cos\alpha\sin\gamma\sin\delta,\\ &-\cos\alpha\cos\gamma - \sin\alpha\sin\gamma\sin\delta, \sin\gamma\cos\delta)\end{aligned} \tag{8}$$

The angles referring to the unit vectors are determined by the orientations of the elipsoid's semi-axis. The angle α is the azimuth of semi-major axe (a) plus 180°. While δ is the inclination of semi-major axe (a) in relation to the geographic plane. Lastly, γ is the angle between the semi-mid axe (b) and the vertical projection of the elipsoid's center with the geographic plane.

Thus, the coordinates of the body's semi-axis are given by:

$$x_h = (x - xc)l_h + (y - yc)m_h + (z - zc)n_h \quad (h = 1, 2, 3) \tag{9}$$

Where x_c, y_c e z_c are the coordinates of the elipsoid's center in the geographic system x, y e z.

For an elipsoid of semi-axis $a > b > c$, the equation that defines your surface is:

$$\frac{x_1^2}{(a^2 + s)} + \frac{x_2^2}{(b^2 + s)} + \frac{x_3^2}{(c^2 + s)} = 1 \tag{10}$$

The parameter s controls the elipsoid form. When s gets close to ∞ the equation (10) tends to the sphere equation of radius $r = \sqrt{\lambda}$. When $s = -c^2$, the last term of the elipsoid's equation is less than zero and it becomes the equation of a circle.

There is, however, a set of values for $s(\lambda, \mu, \nu)$, which are roots of the cubic equation:

$$s^3 + p_2 s^2 + p_1 s + p_0 = 0 \tag{11}$$

This set of roots, called elipsoidal coordinates, correspond to the parameters of a point (x_1, x_2, x_3) which are under the intersection of three ortogonal surfaces related to the body coordinates. Their expressions are:

$$\lambda = 2\sqrt{\left(\frac{-p}{3}\right)}\cos\left(\frac{\theta}{3}\right) - \frac{p_2}{3} \tag{12}$$

$$\mu = -2\sqrt{\left(\frac{-p}{3}\right)}\cos\left(\frac{\theta}{3} + \frac{\pi}{3}\right) - \frac{p_2}{3} \tag{13}$$

$$\mu = -2\sqrt{\left(\frac{-p}{3}\right)}\cos\left(\frac{\theta}{3} - \frac{\pi}{3}\right) - \frac{p_2}{3} \tag{14}$$

Where:

$$p_0 = a^2 b^2 c^2 - b^2 c^2 x_1^2 - c^2 a^2 x_2^2 - a^2 b^2 x_3^2 \tag{15}$$

$$p_1 = a^2 b^2 + b^2 c^2 + c^2 a^2 - (b^2 + c^2) x_1^2 - (c^2 + a^2) x_2^2 - (a^2 + b^2) x_3^2 \tag{16}$$

$$p_2 = a^2 + b^2 + c^2 - x_1^2 - x_2^2 - x_3^2 \tag{17}$$

$$p = p_1 - \frac{p_2^2}{3} \tag{18}$$

$$q = p_0 - \frac{p_1 p_2}{3} + 2\left(\frac{p_2}{3}\right)^3 \tag{19}$$

$$\theta = \cos^{-1}\left[\frac{-q}{2}\sqrt{\left(\frac{-p}{3}\right)^{3}}\right] \tag{20}$$

The calculation of the largest root λ of the equation (11) is essential, since the magnetic field depends on the spatial derivatives of the equation 10, where s admits the value of λ.

$$\frac{\partial\lambda}{\partial x_1} = \frac{2x_1/(a^2+\lambda)}{\left(\frac{x_1}{a^2+\lambda}\right)^2 + \left(\frac{x_2}{b^2+\lambda}\right)^2 + \left(\frac{x_3}{c^2+\lambda}\right)^2} \tag{21}$$

$$\frac{\partial\lambda}{\partial x_2} = \frac{2x_2/(b^2+\lambda)}{\left(\frac{x_1}{a^2+\lambda}\right)^2 + \left(\frac{x_2}{b^2+\lambda}\right)^2 + \left(\frac{x_3}{c^2+\lambda}\right)^2} \tag{22}$$

$$\frac{\partial\lambda}{\partial x_3} = \frac{2x_3/(c^2+\lambda)}{\left(\frac{x_1}{a^2+\lambda}\right)^2 + \left(\frac{x_2}{b^2+\lambda}\right)^2 + \left(\frac{x_3}{c^2+\lambda}\right)^2} \tag{23}$$

The desmagnetization factors are given by:

$$N_1 = \frac{4\pi abc}{(a^2-b^2)\sqrt{(a^2-c^2)}}[F(\theta,k) - E(\theta,k)] \tag{24}$$

$$N_2 = \frac{4\pi abc\sqrt{(a^2-c^2)}}{(a^2-b^2)(b^2-c^2)}\left[E(\theta,k) - \left(\frac{b^2-c^2}{a^2-c^2}\right)F(\theta,k) - \frac{c(a^2-b^2)}{ab\sqrt{(a^2-c^2)}}\right] \tag{25}$$

$$N_3 = \frac{4\pi abc}{(b^2-c^2)\sqrt{(a^2-c^2)}}\left[\frac{b\sqrt{(a^2-c^2)}}{ac} - E(\theta,k)\right] \tag{26}$$

Where $F(\theta,k)$ e $E(\theta,k)$ are first and second Legendre's normal eliptical integrals, respectively. To calculate k and θ we have the following expressions:

$$k = \sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)} \tag{27}$$

$$\theta = \cos^{-1}(c/a) \quad (0 \leq \theta \leq \pi/2) \tag{28}$$

The susceptibility tensor matrix is:

$$k_{ij} = \sum_r k_r (L_r l_i + M_r m_i + N_r n_i)(L_r l_j + M_r m_j + N_r n_j) \quad (r = 1,2,3) \tag{29}$$

The Earth's field vector, F, and the remanent magnetization ,d, also must be converted to the body's coordinates:

$$F_h = F(l\,l_h + m\,m_h + n\,n_h) \tag{30}$$

$$(J_N)_h = J_N(l_N l_h + m_n m_h + n_N n_h) \tag{31}$$

Note that L_r, M_r, N_r $(r = 1,2,3)$, l, m, n e l_N, m_N, n_N depends on their respective vectors of inclinations and declinations:

$$L_R, l, l_N = \cos(dec)\cos(inc) \tag{32}$$

$$M_R, m, m_N = \sin(dec)\cos(inc) \tag{33}$$

$$N_R, n, n_N = \sin(inc) \tag{34}$$

In the case that the body has a very low susceptibility ($\chi < 0.1$ SI) the self-demagnatization is negligible and the resultant magnetic vector is given by:

$$\tilde{J}_R = K\tilde{F} + \tilde{J}_{NRM} \tag{35}$$

For values bigger than 0.1 SI the resultant magnetic vector is:

$$\tilde{J}_{Rc} = A^{-1}\tilde{J}_R \tag{36}$$

Where:

$$A = I + KN = \begin{bmatrix} 1+k_{11}N_1 & k_{12}N_2 & k_{13}N_3 \\ \\ k_{21}N_1 & 1+k_{22}N_2 & k_{23}N_3 \\ \\ k_{31}N_1 & k_{23}N_2 & 1+k_{33}N_3 \end{bmatrix} \tag{37}$$

This way, the components of the magnetic field produced by a triaxial elipsoid in the body's coordinates outside of it is:

$$\Delta B_1 = f_1 \frac{\partial \lambda}{\partial x_1} - 2\pi abc\, J_1\, A(\lambda) \tag{38}$$

$$\Delta B_2 = f_1 \frac{\partial \lambda}{\partial x_2} - 2\pi abc\, J_2\, B(\lambda) \tag{39}$$

$$\Delta B_3 = f_1 \frac{\partial \lambda}{\partial x_3} - 2\pi abc\, J_3\, C(\lambda) \tag{40}$$

where:

$$f_1 = \frac{2\pi abc}{\sqrt{[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]}} \left[\frac{J_1 x_1}{a^2+\lambda} + \frac{J_2 x_2}{b^2+\lambda} + \frac{J_3 x_3}{c^2+\lambda} \right] \tag{41}$$

$$A(\lambda) = \frac{2}{(a^2-b^2)\sqrt{(a^2-c^2)}} [F(\theta^{'}, k) - E(\theta^{'}, k)] \tag{42}$$

$$B(\lambda) = \frac{2\sqrt{(a^2-c^2)}}{(a^2-b^2)(b^2-c^2)} \left[E(\theta^{'}, k) - \left(\frac{b^2-c^2}{a^2-c^2} \right) F(\theta^{'}, k) - \frac{k^2 \sin\theta^{'} \cos\theta^{'}}{\sqrt{(1-k^2\sin^2\theta')}} \right] \tag{43}$$

$$C(\lambda) = \frac{2}{(b^2-c^2)\sqrt{(a^2-c^2)}} \left[\frac{\sin\theta^{'} \sqrt{(1-k^2\sin^2\theta')}}{\cos\theta'} - E(\theta^{'}, k) \right] \tag{44}$$

And:

$$\theta^{'} = \sin^{-1}\left(\frac{a^2-c^2}{a^2+\lambda} \right)^{0.5} \quad (0 \leq \theta^{'} \leq \pi/2) \tag{45}$$

Both $F(\theta^{'},k)$ as $E(\theta^{'},k)$ are again the first and second noraml Legendre's eliptical integrals. While $A(\lambda), B(\lambda), C(\lambda)$ are analytic solutions of the integrals of the potential equation of an elipsoid. This problem was solved by Dirichlet in 1839 (Clark et al., 1986) for the gravitational potential given by:

$$U_i(x_1,x_2,x_3) = \pi abcG\rho \int_0^{\infty} \left[1 - \frac{x_1^2}{a^2+u} - \frac{x_2^2}{b^2+u} - \frac{x_3^2}{c^2+u}\right] \frac{du}{R(u)} \tag{46}$$

That can be rewritten as:

$$U_i(x_1,x_2,x_3) = \pi abcG\rho \int_0^{\infty} [D(\lambda) - A(\lambda) - B(\lambda) - C(\lambda)] \frac{du}{R(u)} \tag{47}$$

Where G is the gravitational constant and ρ the body's density.

Using the equations from (38) to (44), we can rearrange them in a matrix format that will be more computationally efficient. Thus, the magnetic field generated by the elipsoid is:

$$b_i^j = 2\pi a_j b_j c_j \times [M_i^j - D_i^j] \times J^j \quad j = 1,...,L \tag{48}$$

With:

$$M_i^j = \frac{1}{\sqrt{[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]}} \begin{bmatrix} \frac{\partial\lambda}{\partial x_1}\frac{x_1}{a^2+\lambda} & \frac{\partial\lambda}{\partial x_1}\frac{x_2}{b^2+\lambda} & \frac{\partial\lambda}{\partial x_1}\frac{x_3}{c^2+\lambda} \\ \frac{\partial\lambda}{\partial x_2}\frac{x_1}{a^2+\lambda} & \frac{\partial\lambda}{\partial x_2}\frac{x_2}{b^2+\lambda} & \frac{\partial\lambda}{\partial x_2}\frac{x_3}{c^2+\lambda} \\ \frac{\partial\lambda}{\partial x_3}\frac{x_1}{a^2+\lambda} & \frac{\partial\lambda}{\partial x_3}\frac{x_2}{b^2+\lambda} & \frac{\partial\lambda}{\partial x_3}\frac{x_3}{c^2+\lambda} \end{bmatrix}_{3\times3} \tag{49}$$

$$D_i^j = \begin{bmatrix} A(\lambda) & 0 & 0 \\ 0 & B(\lambda) & 0 \\ 0 & 0 & C(\lambda) \end{bmatrix} \tag{50}$$

For $j = 1,...,L$, the number of elipsoids been modeled and $i = 1,...,N$ for the ith element of the calculated field. The total magnetic field then is:

$$B_i = \sum_{j=1}^{L} b_i^j, \quad i = 1,...,N \tag{51}$$

We must remember that every calculation so far was done using the body's coordinates, however the notation in equation (48) the magnetic field is already in geographics coordinates. To calculate each component of the magnetic field's vector back:

$$\Delta B_x = \Delta B_1 l_1 + \Delta B_2 l_2 + \Delta B_3 l_3 \tag{52}$$

$$\Delta B_y = \Delta B_1 m_1 + \Delta B_2 m_2 + \Delta B_3 m_3 \tag{53}$$

$$\Delta B_z = \Delta B_1 n_1 + \Delta B_2 n_2 + \Delta B_3 n_3 \tag{54}$$

2.2.1 HEADING

TEXT

3 Conclusions

TEXT

Appendix A

A1

Author contributions. TEXT

Acknowledgements. TEXT

References

Chang, H.: Fields external to open-structure magnetic devices represented by ellipsoid or spheroid, British Journal of Applied Physics, 12, 160, 1961.

Clark, D., Saul, S., and Emerson, D.: Magnetic and gravity anomalies of a triaxial ellipsoid, Exploration Geophysics, 17, 189–200, 1986.

Clark, D. A.: Methods for determining remanent and total magnetisations of magnetic sources–a review, Exploration Geophysics, 45, 271–304, 2014.

Dirichlet, P. G. L.: Sur un nouvelle methode pour la determination des integrales multiples, Comp. Rend. Acad. Science, 8, 156–160, 1839.

Emerson, D. W., Clark, D., and Saul, S.: Magnetic exploration models incorporating remanence, demagnetization and anisotropy: HP 41C handheld computer algorithms, Exploration Geophysics, 16, 1–122, 1985.

Farrar, L.: Some comments on detailed magnetic investigations of ellipsoidal bodies at Tennant Creek, Exploration Geophysics, 10, 26–33, 1979.

Guo, W., Dentith, M. C., Li, Z., and Powell, C. M.: Self demagnetisation corrections in magnetic modelling: some examples, Exploration Geophysics, 29, 396–401, 1998.

Guo, W., Dentith, M. C., Bird, R. T., and Clark, D. A.: Systematic error analysis of demagnetization and implications for magnetic interpretation, Geophysics, 66, 562–570, 2001.

Maxwell, J. C.: A treatise on electricity and magnetism, vol. 1, Clarendon press, 1881.

Osborn, J.: Demagnetizing factors of the general ellipsoid, Physical review, 67, 351, 1945.

Peake, H. and Davy, N.: The magnetic fields produced by uniformly magnetized ellipsoids of revolution, British Journal of Applied Physics, 4, 207, 1953.

Stoner, E. C.: XCVII. The demagnetizing factors for ellipsoids, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 36, 803–821, 1945.

Tejedor, M., Rubio, H., Elbaile, L., and Iglesias, R.: External fields created by uniformly magnetized ellipsoids and spheroids, IEEE transactions on magnetics, 31, 830–836, 1995.