3D Magnetic modelling of ellipsoidal bodies

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Abstract.

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1 Introduction

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2 Methodology

2.1 Geometrical parameters and coordinate systems

Let (x, y, z) be a point referred to a Cartesian coordinate system with axes x, y and z pointing to, respectively, North, East and down. For convenience, we denominate this coordinate system as *main coordinate system*. Let us consider an ellipsoidal body with centre at the point (x_c, y_c, z_c), semi-axes defined by positive constants a, b, c, where $a > b > c$, and orientation defined by three angles α, β, and γ. The points (x, y, z) located on the surface of this ellipsoidal body satisfy the following equation:

$$(\mathbf{r} - \mathbf{r}_c)^T \mathbf{A} (\mathbf{r} - \mathbf{r}_c) = 1 \, , \tag{1}$$

where $\mathbf{r} = [\begin{array}{ccc} x & y & z \end{array}]^\top$, $\mathbf{r}_c = [\begin{array}{ccc} x_c & y_c & z_c \end{array}]^\top$, $\mathbf{A}$ is a positive definite matrix given by

$$\mathbf{A} = \mathbf{V} \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{bmatrix} \mathbf{V}^\top \, , \tag{2}$$

and $\mathbf{V}$ is an orthogonal matrix whose columns are defined by unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ depend on the orientation angles α, β, γ and are defined as follows (Clark et al., 1986):

$$\mathbf{v}_1 = \begin{bmatrix} -\cos\alpha \, \cos\delta \\ -\sin\alpha \, \cos\delta \\ -\sin\delta \end{bmatrix} , \tag{3}$$

$$\mathbf{v}_2 = \begin{bmatrix} \cos\alpha\ \cos\gamma\ \sin\delta + \sin\alpha\ \sin\gamma \\ \sin\alpha\ \cos\gamma\ \sin\delta - \cos\alpha\ \sin\gamma \\ -\cos\gamma\ \cos\delta \end{bmatrix} , \tag{4}$$

$$\mathbf{v}_3 = \begin{bmatrix} \sin\alpha\ \cos\gamma - \cos\alpha\ \sin\gamma\ \sin\delta \\ -\cos\alpha\ \cos\gamma - \sin\alpha\ \sin\gamma\ \sin\delta \\ \sin\gamma\ \cos\delta \end{bmatrix} . \tag{5}$$

For triaxial ellipsoids (i.e., $a > b > c$), the orthogonal matrix $\mathbf{V}$ (equation 2) is calculated by using equations 3, 4, and 5 as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} . \tag{6}$$

Similarly, the matrix $\mathbf{V}$ (equation 2) for prolate ellipsoids (i.e., $a > b = c$) is calculated according to equation 6 by using equations 3, 4, and 5, but with $\gamma = 0°$ (Emerson et al., 1985). Finally, the matrix $\mathbf{V}$ (equation 2) for oblate ellipsoids (i.e., $a < b = c$) is calculated by using equations 3, 4, and 5, with $\gamma = 0°$, as follows (Emerson et al., 1985):

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_1 & -\mathbf{v}_3 \end{bmatrix} . \tag{7}$$

The orientation of the semi-axes a, b, and c are defined by the first, second, and third columns of the matrix $\mathbf{V}$ given by equation 6, in the case of a triaxial or prolate ellipsoid, or the matrix $\mathbf{V}$ given by equation 7, in the case of an oblate ellipsoid.

The magnetic modelling of an ellipsoidal body is commonly performed in a particular Cartesian coordinate system that is aligned with the body semi-axes and has the origin coincident with the body centre. For convenience, we denominate this particular coordinate system as *local coordinate system*. The relationship between the Cartesian coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ of a point in a local coordinate system and the Cartesian coordinates (x, y, z) of the same point in the main system is given by:

$$\tilde{\mathbf{r}} = \mathbf{V}^\top (\mathbf{r} - \mathbf{r}_c) , \tag{8}$$

where $\tilde{\mathbf{r}} = [\ \tilde{x} \quad \tilde{y} \quad \tilde{z}\ ]^\top$, $\mathbf{r}$ and $\mathbf{r}_c$ are defined in equation 1 and the matrix $\mathbf{V}$ is defined according to equations 6 or 7, depending on the ellipsoid type. Afterwards, we use the superscript "$\sim$" to implicitly define quantities referred to a local coordinate system.

2.2 Demagnetizing field within ellipsoids

Consider a magnetized ellipsoid immersed in a uniform magnetic field $\mathbf{H}_0$ (in Am^{-1}). The total magnetic intensity field $\mathbf{H}(\mathbf{r})$ at the position $\mathbf{r}$ (equations 2 and 8) of a point referred to the main coordinate system is defined as follows

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \nabla\phi(\mathbf{r}) , \tag{9}$$

where the second term, representing the induced magnetic field, is the negative gradient of the magnetic scalar potential $\phi(\mathbf{r})$ given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \iiint\limits_V \mathbf{J}(\mathbf{r}')^\top \left(\frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \right) dx' dy' dz' \,. \tag{10}$$

In this equation, $\mathbf{r}' = [\; x' \quad y' \quad z' \;]^\top$ is the position vector of a point located within the volume V, the integral is conducted over the variables x', y' and, z' representing the coordinates of a point located within the volume V of the ellipsoid, $\|\cdot\|$ denotes the Euclidean norm and $\mathbf{J}(\mathbf{r}')$ is the magnetization vector (in Am^{-1}). Equation 9 is valid anywhere, independently if the position vector $\mathbf{r}$ represents a point located inside or outside the magnetized body.

According to Maxwell (1873), the only finite bodies that can be uniformly magnetized in the presence of a uniform and static magnetic field are the ones bounded by surfaces of second degree, which are ellipsoids.

This property comes from the fact that the gravitational potential within an ellipsoid with uniform density is a quadratic function of the spatial coordinates.

Based on this, the magnetization vector $\mathbf{J}(\mathbf{r}')$ (equation 10) can be moved outside the integral, resulting that

2.3 Magnetic induction

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3 Conclusions

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Appendix A

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Author contributions. TEXT

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