3D Magnetic modelling of ellipsoidal bodies

Diego Takahashi Tomazella[1] and Vanderlei C. Oliveira Jr[1]

[1]Department of Geophysics, Observatório Nacional, Rio de Janeiro, Brazil

Correspondence to: Diego Takahashi Tomazella (diego.takahashi@gmail.com)

Abstract.

TEXT

1 Introduction

TEXT

2 Methodology

2.1 Geometrical parameters and coordinate systems

Let (x, y, z) be a point referred to a Cartesian coordinate system with axes x, y and z pointing to, respectively, North, East and down. For convenience, we denominate this coordinate system as *main coordinate system*. Let us consider an ellipsoidal body with centre at the point (x_c, y_c, z_c), semi-axes defined by positive constants a, b, c, where $a > b > c$, and orientation defined by three angles α, β, and γ. The points (x, y, z) located on the surface of this ellipsoidal body satisfy the following equation:

$$(\mathbf{r} - \mathbf{r}_c)^T \mathbf{A} (\mathbf{r} - \mathbf{r}_c) = 1 \, , \tag{1}$$

where $\mathbf{r} = [\begin{array}{ccc} x & y & z \end{array}]^\top$, $\mathbf{r}_c = [\begin{array}{ccc} x_c & y_c & z_c \end{array}]^\top$, $\mathbf{A}$ is a positive definite matrix given by

$$\mathbf{A} = \mathbf{V} \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{bmatrix} \mathbf{V}^\top \, , \tag{2}$$

and $\mathbf{V}$ is an orthogonal matrix whose columns are defined by unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ depend on the orientation angles α, β, γ and are defined as follows (Clark et al., 1986):

$$\mathbf{v}_1 = \begin{bmatrix} -\cos\alpha \, \cos\delta \\ -\sin\alpha \, \cos\delta \\ -\sin\delta \end{bmatrix} \, , \tag{3}$$

$$\mathbf{v}_2 = \begin{bmatrix} \cos\alpha\,\cos\gamma\,\sin\delta + \sin\alpha\,\sin\gamma \\ \sin\alpha\,\cos\gamma\,\sin\delta - \cos\alpha\,\sin\gamma \\ -\cos\gamma\,\cos\delta \end{bmatrix}, \tag{4}$$

$$\mathbf{v}_3 = \begin{bmatrix} \sin\alpha\,\cos\gamma - \cos\alpha\,\sin\gamma\,\sin\delta \\ -\cos\alpha\,\cos\gamma - \sin\alpha\,\sin\gamma\,\sin\delta \\ \sin\gamma\,\cos\delta \end{bmatrix}. \tag{5}$$

For triaxial ellipsoids (i.e., $a > b > c$), the orthogonal matrix $\mathbf{V}$ (equation 2) is calculated by using equations 3, 4, and 5 as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}. \tag{6}$$

Similarly, the matrix $\mathbf{V}$ (equation 2) for prolate ellipsoids (i.e., $a > b = c$) is calculated according to equation 6 by using equations 3, 4, and 5, but with $\gamma = 0°$ (Emerson et al., 1985). Finally, the matrix $\mathbf{V}$ (equation 2) for oblate ellipsoids (i.e., $a < b = c$) is calculated by using equations 3, 4, and 5, with $\gamma = 0°$, as follows (Emerson et al., 1985):

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_1 & -\mathbf{v}_3 \end{bmatrix}. \tag{7}$$

The orientation of the semi-axes a, b, and c are defined by the first, second, and third columns of the matrix $\mathbf{V}$ given by equation 6, in the case of a triaxial or prolate ellipsoid, or the matrix $\mathbf{V}$ given by equation 7, in the case of an oblate ellipsoid.

The magnetic modelling of an ellipsoidal body is commonly performed in a particular Cartesian coordinate system that is aligned with the body semi-axes and has the origin coincident with the body centre. For convenience, we denominate this particular coordinate system as *local coordinate system*. The relationship between the Cartesian coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ of a point in a local coordinate system and the Cartesian coordinates (x, y, z) of the same point in the main system is given by:

$$\tilde{\mathbf{r}} = \mathbf{V}^\top (\mathbf{r} - \mathbf{r}_c), \tag{8}$$

where $\tilde{\mathbf{r}} = \begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{bmatrix}^\top$, $\mathbf{r}$ and $\mathbf{r}_c$ are defined in equation 1 and the matrix $\mathbf{V}$ is defined according to equations 6 or 7, depending on the ellipsoid type.

2.2 Theoretical background

Based on the mathematical theory of the magnetic induction developed by Poisson (1824), Maxwell (1873) affirmed that, if V is the gravitational potential produced by any body with uniform density ρ and arbitrary shape at a point (x, y, z), then $-\frac{\partial V}{\partial x}$ is the magnetic scalar potential produced at the same point by the same body if it has a uniform magnetization oriented along x with intensity ρ. Maxwell (1873) generalized this idea as a way of determining the magnetic scalar potential produced by any body uniformly magnetized in a given direction. By presuming that this uniform magnetization is due to induction, he postulated that the resulting magnetic field (intensity) at all points within the body must also be uniform and parallel the magnetization, which results that the gravitational potential V at points within the body must be a quadratic function of the spatial coordinates. Apparently, Maxwell (1873) was the first one to affirm that the only finite bodies having a gravitational potential with this property and that, as a consequence, can be uniformly magnetized in the presence of a uniform and static magnetic field are the ones bounded by surfaces of second degree, which are ellipsoids.

Consider a magnetized ellipsoid immersed in a uniform magnetic field $\mathbf{H}_0$ (in Am^{-1}). In the absence of conduction currents, the total magnetic field $\mathbf{H}(\mathbf{r})$ at the position $\mathbf{r}$ (equations 2 and 8) of a point referred to the main coordinate system is defined as follows (Stratton, 2007):

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \nabla\phi(\mathbf{r})\,, \tag{9}$$

where the second term is the negative gradient of the magnetic scalar potential $\phi(\mathbf{r})$ given by:

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\iiint\limits_{V} \mathbf{M}(\mathbf{r}')^{\top}\nabla\left(\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\right) dx'dy'dz'\,. \tag{10}$$

In this equation, $\mathbf{r}' = \begin{bmatrix} x' & y' & z' \end{bmatrix}^{\top}$ is the position vector of a point located within the volume V, the integral is conducted over the variables x', y' and, z' representing the coordinates of a point located within the volume V of the ellipsoid, $\|\cdot\|$ denotes the Euclidean norm and $\mathbf{M}(\mathbf{r}')$ is the magnetization vector (in Am^{-1}). Equation 10 is valid anywhere, independently if the position vector $\mathbf{r}$ represents a point located inside or outside the magnetized body (DuBois, 1896).

Based on Maxwell's postulate, let us assume that the body has a uniform magnetization given by

$$\mathbf{M} = \mathbf{K}\,\mathbf{H}_i\,, \tag{11}$$

where $\mathbf{H}_i$ is the resultant uniform magnetic field at any point within the body and $\mathbf{K}$ is a constant and symmetrical 2nd-order tensor representing the magnetic susceptibility of the body. In this case, equation 9 can be rewritten as follows:

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 - \mathbf{N}(\mathbf{r})\,\mathbf{K}\,\mathbf{H}_i\,, \tag{12}$$

where $\mathbf{N}(\mathbf{r})$ is a symmetrical matrix whose ij-element $n_{ij}(\mathbf{r})$ is given by

$$n_{ij}(\mathbf{r}) = -\frac{1}{4\pi}\frac{\partial^2 f(\mathbf{r})}{\partial r_i\,\partial r_j}\,, \quad i = 1,2,3\,, \quad j = 1,2,3\,, \tag{13}$$

$r_1 = x$, $r_2 = y$, $r_3 = z$ are the elements of the position vector $\mathbf{r}$ (equation 1), and

$$f(\mathbf{r}) = \iiint\limits_{V} \frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\, dx'dy'dz'\,. \tag{14}$$

Notice that the scalar function $f(\mathbf{r})$ (equation 14) is proportional to the gravitational potential that would be produced by the ellipsoidal body with volume V if it had a uniform density equal to the inverse of the gravitational constant. It can be shown that the elements $n_{ij}(\mathbf{r})$ are finite whether $\mathbf{r}$ is a point within or without the volume V (Peirce, 1902; Webster, 1904). The matrix $\mathbf{N}(\mathbf{r})$ (equation 12) is called *depolarization tensor* (Solivérez, 1981, 2008).

The following part of this paper moves on to describe the magnetic field $\mathbf{H}(\mathbf{r})$ (equation 12) at points located both within and without the volume V of the ellipsoidal body. However, the mathematical developments are conveniently done in the local coordinate system related to the respective ellipsoidal body.

2.3 Coordinate transformation

Let $\tilde{f}(\tilde{\mathbf{r}})$ be the scalar function obtained by transforming $f(\mathbf{r})$ (equation 14) from the main coordinate system to a local coordinate system. The function $\tilde{f}(\tilde{\mathbf{r}})$ was first presented by Dirichlet (1839) to describe the gravitational potential produced by homogeneous ellipsoids. Posteriorly, several authors also deduced and used this function for describing the magnetic and gravitational fields produced by triaxial, prolate, and/or oblate ellipsoids (Maxwell, 1873; Thomson and Tait, 1879; DuBois, 1896; Peirce, 1902; Webster, 1904; Kellogg, 1929; Stoner, 1945; Osborn, 1945; Lowes, 1974; Peake and Davy, 1953; Chang, 1961; Clark et al., 1986; Tejedor et al., 1995; Stratton, 2007). It is convenient to use $\tilde{f}_i(\tilde{\mathbf{r}})$ and $\tilde{f}_e(\tilde{\mathbf{r}})$ to define the function $\tilde{f}(\tilde{\mathbf{r}})$ evaluated, respectively, at points $\tilde{\mathbf{r}}$ inside and outside the volume V of the ellipsoidal body.

By following Webster (1904), $\tilde{f}_i(\tilde{\mathbf{r}})$ is given by

$$\tilde{f}_i(\tilde{\mathbf{r}}) = \pi\, abc \int\limits_0^{\infty} \left(1 - \frac{\tilde{x}^2}{a^2+u} - \frac{\tilde{y}^2}{b^2+u} - \frac{\tilde{z}^2}{c^2+u}\right) \frac{1}{R(u)}\, du\,, \quad \tilde{\mathbf{r}} \in V\,, \tag{15}$$

where

$$R(u) = \sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}\,. \tag{16}$$

This function represents the gravitational potential that would be produced by the ellipsoidal body at points located within its volume V if it had a uniform density equal to the inverse of the gravitational constant. Notice that, in this case, the gravitational potential is a quadratic function of the spatial coordinates $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$, which supported the Maxwell's (1873) postulate about uniformly magnetized ellipsoids.

In a similar way, the function $\tilde{f}_e(\tilde{\mathbf{r}})$ is given by (Webster, 1904)

$$\tilde{f}_e(\tilde{\mathbf{r}}) = \pi\, abc \int\limits_{\lambda}^{\infty} \left(1 - \frac{\tilde{x}^2}{a^2+u} - \frac{\tilde{y}^2}{b^2+u} - \frac{\tilde{z}^2}{c^2+u}\right) \frac{1}{R(u)}\, du\,, \quad \tilde{\mathbf{r}} \notin V\,, \tag{17}$$

where $R(u)$ is defined by equation 16 and the parameter λ is a constant defining the integral lower limit.

The parameter λ is defined according to the ellipsoid type. Details about the mathematical meaning of this parameter are given in Appendix B.

PAREI AQUI

By properly using the orthogonality of matrix $\mathbf{V}$ (equations 6 and 7), the magnetic field $\mathbf{H}(\mathbf{r})$ (equation 12) can be transformed from the main coordinate system to a local coordinate system as follows:

$$\underbrace{\mathbf{V}^{\top}\mathbf{H}(\mathbf{r})}_{\tilde{\mathbf{H}}(\tilde{\mathbf{r}})} = \underbrace{\mathbf{V}^{\top}\mathbf{H}_0}_{\tilde{\mathbf{H}}_0} - \underbrace{\mathbf{V}^{\top}\mathbf{N}(\mathbf{r})\mathbf{V}}_{\tilde{\mathbf{N}}(\tilde{\mathbf{r}})}\, \underbrace{\mathbf{V}^{\top}\mathbf{K}\mathbf{V}}_{\tilde{\mathbf{K}}}\, \underbrace{\mathbf{V}^{\top}\mathbf{H}_i}_{\tilde{\mathbf{H}}_i}\,, \tag{18}$$

where the superscript "$\sim$" denotes quantities referred to the respective local coordinate system.

In equation 18, the transformed depolarization tensor $\tilde{\mathbf{N}}(\tilde{\mathbf{r}})$ is calculated as a function of the original depolarization tensor $\mathbf{N}(\mathbf{r})$ (equation 12). In this case, the elements of $\tilde{\mathbf{N}}(\tilde{\mathbf{r}})$ are calculated as a function of the second derivatives of the function $f(\mathbf{r})$ (equation 14), which is defined in the main coordinate system. It can be shown (Appendix A), however, that the elements $\tilde{n}_{ij}(\tilde{\mathbf{r}})$ of $\tilde{\mathbf{N}}(\tilde{\mathbf{r}})$ can also be calculated as follows:

$$\tilde{n}_{ij}(\tilde{\mathbf{r}}) = -\frac{1}{4\pi}\frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_i \partial \tilde{r}_j}, \quad i = 1,2,3, \quad j = 1,2,3, \tag{19}$$

where $\tilde{r}_1 = \tilde{x}$, $\tilde{r}_2 = \tilde{y}$, and $\tilde{r}_3 = \tilde{z}$ are the elements of the transformed vector $\tilde{\mathbf{r}}$ (equation 8) and $\tilde{f}(\tilde{\mathbf{r}})$ is a scalar function obtained by transforming $f(\mathbf{r})$ (equation 14) from the main coordinate system to the respective local coordinate system.

2.3.1 Internal magnetic field and demagnetization

By considering $\mathbf{r}$ as a point internal to the volume V of the ellipsoid and using the Maxwell's postulate about the uniformity of the magnetic field $\mathbf{H}(\mathbf{r})$ inside ellipsoidal bodies, we can rewrite equation 18 as follows:

2.3.2 External magnetic induction

3 Conclusions

TEXT

Appendix A: Relationship between the derivatives of the functions $f(\mathbf{r})$ and $\tilde{f}(\tilde{\mathbf{r}})$

Let $\tilde{f}(\tilde{\mathbf{r}})$ the scalar function obtained by transforming $f(\mathbf{r})$ (equation 14) from the main coordinate system to a local coordinate system.

For convenience, let us rewrite equation 8 as follows:

$$\tilde{r}_k = v_{k1}\, r_1 + v_{k2}\, r_2 + v_{k3}\, r_3 + c_k\,, \tag{A1}$$

where r_j, $j = 1,2,3$, are the elements of the position vector $\mathbf{r}$ (equation 1), v_{kj}, $j = 1,2,3$, are the elements of the matrix $\mathbf{V}$ (equation 6 or 7), and c_k is a constant defined by the coordinates x_c, y_c, and z_c of the centre of the ellipsoid body.

By considering the functions $f(\mathbf{r})$ (equation 14) and $\tilde{f}(\tilde{\mathbf{r}})$ evaluated at the same point, but on different coordinate systems, we have:

$$\frac{\partial f(\mathbf{r})}{\partial r_j} = \frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_1}\frac{\partial \tilde{r}_1}{\partial r_j} + \frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_2}\frac{\partial \tilde{r}_2}{\partial r_j} + \frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_3}\frac{\partial \tilde{r}_3}{\partial r_j}, \quad j = 1,2,3,$$

which, from equation A1, can be given by

$$\frac{\partial f(\mathbf{r})}{\partial r_j} = v_{j1}\frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_1} + v_{j2}\frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_2} + v_{j3}\frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_3}, \quad j = 1,2,3. \tag{A2}$$

Now, by deriving $\frac{\partial f(\mathbf{r})}{\partial r_j}$ (equation A2) with respect to the ith element r_i of the position vector $\mathbf{r}$ (equation 1), we obtain:

$$\begin{aligned}
\frac{\partial^2 f(\mathbf{r})}{\partial r_i \, \partial r_j} &= v_{j1} \frac{\partial}{\partial r_i}\left(\frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_1}\right) + v_{j2} \frac{\partial}{\partial r_i}\left(\frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_2}\right) + v_{j3} \frac{\partial}{\partial r_i}\left(\frac{\partial \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_3}\right) \\
&= v_{j1}\left(\frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_1 \, \partial \tilde{r}_1} v_{i1} + \frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_2 \, \partial \tilde{r}_1} v_{i2} + \frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_3 \, \partial \tilde{r}_1} v_{i3}\right) + \\
&+ v_{j2}\left(\frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_1 \, \partial \tilde{r}_2} v_{i1} + \frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_2 \, \partial \tilde{r}_2} v_{i2} + \frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_3 \, \partial \tilde{r}_2} v_{i3}\right) + \\
&+ v_{j3}\left(\frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_1 \, \partial \tilde{r}_3} v_{i1} + \frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_2 \, \partial \tilde{r}_3} v_{i2} + \frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_3 \, \partial \tilde{r}_3} v_{i3}\right) \\
&= \begin{bmatrix} v_{j1} & v_{j2} & v_{j3} \end{bmatrix} \tilde{\mathbf{F}}(\tilde{\mathbf{r}}) \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix} \qquad , \qquad \text{(A3)}
\end{aligned}$$

where $\tilde{\mathbf{F}}(\tilde{\mathbf{r}})$ is a 3×3 matrix whose ij-th element is $\frac{\partial^2 \tilde{f}(\tilde{\mathbf{r}})}{\partial \tilde{r}_i \, \partial \tilde{r}_j}$. From equation A3, we obtain

$$\mathbf{F}(\mathbf{r}) = \mathbf{V} \, \tilde{\mathbf{F}}(\tilde{\mathbf{r}}) \, \mathbf{V}^{\top} \, , \qquad \text{(A4)}$$

where $\mathbf{F}(\mathbf{r})$ is a 3×3 matrix whose ij-th element is $\frac{\partial^2 f(\mathbf{r})}{\partial r_i \, \partial r_j}$ and $\mathbf{V}$ is defined by equations 6 or 7, depending on the ellipsoid type. As one may noticed, the matrices $\mathbf{F}(\mathbf{r})$ and $\tilde{\mathbf{F}}(\tilde{\mathbf{r}})$ represent the Hessians of the functions $f(\mathbf{r})$ (equation 14) and $\tilde{f}(\tilde{\mathbf{r}})$, respectively. Besides, the depolarization tensor $\mathbf{N}(\mathbf{r})$ (equation 12) can be rewritten by using the matrix $\mathbf{F}(\mathbf{r})$ as follows

$$\mathbf{N}(\mathbf{r}) = -\frac{1}{4\pi} \mathbf{F}(\mathbf{r}) \, . \qquad \text{(A5)}$$

By properly using the orthogonality of the matrix $\mathbf{V}$, we may rewrite equation A4 as follows:

$$\tilde{\mathbf{F}}(\tilde{\mathbf{r}}) = \mathbf{V}^{\top} \, \mathbf{F}(\mathbf{r}) \, \mathbf{V} \, . \qquad \text{(A6)}$$

Finally, by multiplying both sides of equation A6 by $-\frac{1}{4\pi}$ and using equations A5 and 18, we conclude that

$$\tilde{\mathbf{N}}(\tilde{\mathbf{r}}) = -\frac{1}{4\pi} \tilde{\mathbf{F}}(\tilde{\mathbf{r}}) \, . \qquad \text{(A7)}$$

Appendix B: Parameter λ and its spatial derivatives

Here, we follow the reasoning presented by Webster (1904) for analysing the parameter λ describing triaxial, prolate and oblate ellipsoids.

B1 Parameter λ defining triaxial ellipsoids

Let us consider an ellipsoid with semi-axes a, b, c oriented along the $\tilde{x}$-, $\tilde{y}$-, and $\tilde{z}$-axis, respectively, of its local coordinate system, where $a > b > c > 0$. This ellipsoid is defined by the following equation:

$$\frac{\tilde{x}^2}{a^2} + \frac{\tilde{y}^2}{b^2} + \frac{\tilde{z}^2}{c^2} = 1 \, . \qquad \text{(B1)}$$

A quadric surface which is confocal with the ellipsoid defined in equation B1 can be described as follows:

$$\frac{\tilde{x}^2}{a^2+u}+\frac{\tilde{y}^2}{b^2+u}+\frac{\tilde{z}^2}{c^2+u}=1\,, \tag{B2}$$

where u is a real number. We know that B2 represents an ellipsoid for u satisfying the condition

$$u+c^2>0\,. \tag{B3}$$

Given a, b, c, and a u satisfying B3, we may use B2 for determining a set of points (x,y,z) lying on the surface of an ellipsoid which is confocal with that one defined in equation B1. Now, consider the problem of determining the ellipsoid which is confocal with that one defined in B1 and pass through a particular point $(\tilde{x},\tilde{y},\tilde{z})$. This problem consists in determining the real number u that, given a, b, c, $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$, satisfies the condition expressed by equation B3. By rearranging equation B2, we obtain the following cubic equation for u:

$$p(u)=(a^2+u)(b^2+u)(c^2+u)-(b^2+u)(c^2+u)\,\tilde{x}^2-(a^2+u)(c^2+u)\,\tilde{y}^2-(a^2+u)(b^2+u)\,\tilde{z}^2\,. \tag{B4}$$

By analysing the signals of this cubic equation we conclude that:

$$u=\begin{cases} d\to\infty & , \quad p(u)>0 \\ -c^2 & , \quad p(u)<0 \\ -b^2 & , \quad p(u)>0 \\ -a^2 & , \quad p(u)<0 \end{cases}. \tag{B5}$$

Notice that, according to B5, the smaller, intermediate and largest roots of the cubic equation $p(u)$ (equation B4) are located, respectively, in the intervals $[-a^2,-b^2]$, $[-b^2,-c^2]$ and $[-c^2,\infty[$. Remember that we are interested in a u satisfying the condition expressed by equation B3. Consequently, according to the signal analysis shown in equation B5, we are interested in the largest root λ of the cubic equation $p(u)$ (equation B4).

From equation B4, we obtain a simpler one given by

$$p(u)=u^u+p_2\,u^2+p_1\,u+p_0\,, \tag{B6}$$

where

$$p_2=a^2+b^2+c^2-\tilde{x}^2-\tilde{y}^2-\tilde{z}^2\,, \tag{B7}$$

$$p_1=p_1=^2\;c^2+a^2\,c^2+a^2\,b^2-(b^2+c^2)\,\tilde{x}^2-(a^2+c^2)\,\tilde{y}^2-(a^2+b^2)\,\tilde{z}^2 \tag{B8}$$

and

$$p_0=a^2\,b^2\,c^2-b^2\,c^2\,\tilde{x}^2-a^2\,c^2\,\tilde{y}^2-a^2\,b^2\,\tilde{z}^2\,. \tag{B9}$$

Finally, from equations B7, B8 and B9, the largest root λ of $p(u)$ (equation B4) can be calculated as follows (Weisstein, 2017):

$$\lambda = 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) - \frac{p_2}{3}\,, \tag{B10}$$

where

$$\theta = \cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right)\,, \tag{B11}$$

$$Q = \frac{3p_1 - p_2^2}{9} \tag{B12}$$

and

$$R = \frac{9p_1p_2 - 27p_0 - 2p_2^3}{54}\,. \tag{B13}$$

B2 Parameter λ defining prolate and oblate ellipsoids

Let us now consider a prolate ellipsoid with semi-axes a, b, c oriented along the $\tilde{x}$-, $\tilde{y}$-, and $\tilde{z}$-axis, respectively, of its local coordinate system, where $a > b = c > 0$.

In this case,

Author contributions. TEXT

Acknowledgements. TEXT

References

Chang, H.: Fields external to open-structure magnetic devices represented by ellipsoid or spheroid, British Journal of Applied Physics, 12, 160, 1961.

Clark, D., Saul, S., and Emerson, D.: Magnetic and gravity anomalies of a triaxial ellipsoid, Exploration Geophysics, 17, 189–200, 1986.

Dirichlet, P. G. L.: Sur un nouvelle methode pour la determination des integrales multiples, Journal de mathématiques pures et appliquées, 1, 164–168, http://sites.mathdoc.fr/JMPA/PDF/JMPA_1839_1_4_A11_0.pdf, 1839.

DuBois, H.: The Magnetic Circuit in Theory and Practice, Longmans, Green, and Co., 1896.

Emerson, D. W., Clark, D., and Saul, S.: Magnetic exploration models incorporating remanence, demagnetization and anisotropy: HP 41C handheld computer algorithms, Exploration Geophysics, 16, 1–122, 1985.

Kellogg, O. D.: Foundations of Potential Theory, Frederick Ungar Publishing Company, 1929.

Lowes, F. J.: The Torque on a Magnet, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 337, 555–567, http://www.jstor.org/stable/78532, 1974.

Maxwell, J. C.: A treatise on electricity and magnetism, vol. 2, Clarendon press, 1873.

Osborn, J.: Demagnetizing factors of the general ellipsoid, Physical review, 67, 351, 1945.

Peake, H. J. and Davy, N.: The magnetic fields produced by uniformly magnetized ellipsoids of revolution, British Journal of Applied Physics, 4, 207, 1953.

Peirce, B.: Elements of the Theory of the Newtonian Potential Function, Cornell University Library historical math monographs, Ginn, 1902.

Poisson, S. D.: Second mémoire sur ta théorie du magnétism, Mémoires de l'Académie des sciences de l'Institut de France, pp. 488–533, 1824.

Solivérez, C. E.: Magnetostatics of anisotropic ellipsoidal bodies, IEEE Transactions on Magnetics, 17, 1363–1364, doi:10.1109/TMAG.1981.1061200, 1981.

Solivérez, C. E.: Campos eléctricos generados por elipsoides uniformemente polarizados, Revista mexicana de física E, 54, 203 – 207, 2008.

Stoner, E. C.: The demagnetizing factors for ellipsoids, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 36, 803–821, 1945.

Stratton, J. A.: Electromagnetic Theory, Wiley-IEEE Press, 2007.

Tejedor, M., Rubio, H., Elbaile, L., and Iglesias, R.: External fields created by uniformly magnetized ellipsoids and spheroids, IEEE transactions on magnetics, 31, 830–836, 1995.

Thomson, W. and Tait, P.: Treatise on Natural Philosophy, Part II," Cambridge University Press, Cambridge, 1879.

Webster, A.: The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies: Being Lectures on Mathematical Physics, B. G. Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, B.G. Teubner, 1904.

Weisstein, E. W.: Cubic Formula: From MathWorld – A Wolfram Web Resource, http://mathworld.wolfram.com/CubicFormula.html, [Online; accessed 2017-01-23], 2017.