3D MAGNETIC MODELLING FOR ELIPSOIDS

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Abstract. In this work we will present results from a numerical modeling of the magnetic field and total field anomaly, represented by triaxial and prolate elipsoidal sources. Such approach provide analytical results for anisotropy of magnetic susceptibility as well as for self-demagnetization effects, which can be easily adapted for distinctive geologic structures - hence being an useful tool for educational (e.g., potential methods and rock magnetism) and applied geophysics (e.g., characterization of high magnetic susceptibility, mineralized bodies) purposes. Numerical tests by means of a Python code (currently under development) allowed us to compared the effects of different geometries (ellipsoidal sources, spheres, dipole lines and elliptic cylinders), which were used to validate our computational implementation. This code will be freely available to the scientific community by the end of the year.

1 Introduction

The magnetic method is considered one of the oldest geophysics techniques. The greek philosofer Thales supposedly did the first obsevations on magnetism in the sixth century. The chineses in the XII century, already used magnetic compasses as an instrument for marine orientation (Nabighian et al., 2005). During the World War II there was a need for instruments of several magnitudes of precision higher than was available at the time for the detection of submarines trough aeromagnetic exploration. Thus, the fluxgate magnetometer was invented, expanding the use of this method for geophysic exploration (Reford and Sumner, 1964; Hanna, 1924). The diffusion of this method was quick and well accepted whereas the possiblity of covering large areas in a relatively short time (Blakely, 1996; Nabighian et al., 2005).

The applications of magnetic exploration includes several important study situations, as estimating the average of basement relief, mapping geologic structures as faults and lithologic contacts; and also for mineral exploration, identifying mineral deposits and mapping geologic traps for oil and gas (Oliveira Jr et al., 2015).

In the decade of 1970 Farrar (1979) showed in his work the value of the elipsoidal model in geophysic exploration, where it was used for analysis of magnetic anomalies in Tennant Creeks's gold mine in Australia. The elipsoid was the most correct geometry to model bodies of lenticular pipe shape and the most appropriate to handle self-desmagnetization of bodies of high susceptibilty - magnetic susceptibility higher than 0.1 SI (Clark, 2014).

In his book *A Treatise on Electricity and Magnetism*, Maxwell (1881) demonstrated that only shapes bounded by second degree surfaces are uniformly magnetized when placed in a uniform field. In the particular case of the elipsoidal geometry, the intern magnetic field is independent of spatial coordinates, therefore its magnetization is completely homogeneous, making it the only geometric shape that has a true analitic solution for self-desmagnetization (Clark et al., 1986).

The magnetic field resultant of an elipsoid was presented the first time in full modeling in the space domain by Emerson et al. (1985), along a compilation of several others geometric shapes. Through a consistent notation this work tapped a hole in the literature that missed the elipsoidal model. This model, however, consists in a division between prolate and oblate ellipsoids, with a few differences in the algorithm.

In this work, we will implement a generalized model for triaxial ellipsoids published by Clark et al. (1986). From the solution of gravitational potential for uniform elipsoids, formally solved by Dirichlet (1839) in his article *Sur un nouvelle methode pour la determination des integrales multiples*, that uses first and second Legendre's normal elliptic integrals, and using the Poisson's relation (Grant and West, 1965) it is possible to calculate the magnetic potential, and posteriorly the magnetic field generated by the body calculating the gradient of this potential.

2 Methodology

The total-field anomaly can be described as a difference between the total field vector and the induced magnetization of the crust of the Earth, that includes any anomalous distribuition of magnetization that can occur by magnetizied bodies in subsurface. This difference can be written as:

$$\Delta T_i^0 = ||T_i|| - ||F_i||, \tag{1}$$

where ΔT_i^0 is the observed vector of total-field anomaly in the i-th position (x_i, y_i, z_i), $i = 1, ...N$, F_i is the geomagnetic field and $||.||$ is the Euclidian norm. The total-field vector T_i is:

$$T_i = F_i + B_i, \tag{2}$$

and B_i is the total induced magnetization vector produced by all sources of anomalous susceptibility distribuition. Considering F_i a constant vector, F_0 for local ou regional scale, and that $||F_0|| \gg ||B_0||$, since, B_i is a small pertubation of the magnetic field F_i, we can approximate the euclidian norm of the vector T_i by a Taylor's expansion of first order:

$$||T_i|| \approx ||F_0 + B_i|| \approx ||F_0|| + F^T B_i \tag{3}$$

where T indicates transposition and

$$\hat{F} = \frac{F_o}{||F_0||} \tag{4}$$

is a unit vector that represents the gradient of the function $||T_i||$ in relation to the vector's components T_i. This way, we can approximate the Eq.(1) of total-field anomaly to:

$$\Delta T \approx \hat{F^T} B_i, \quad i = 1, ..., N \tag{5}$$

2.1 The Forward model of a sphere

2.2 The Forward model of a triaxial elipsoid

The implementation of the forward problem of a triaxial elipsoid's magnetic field (three semi-axis $a > b > c$) is done in a new coordinate system, where its origin is the center of this elipsoidal body.

This new coordinate system (x_1, x_2, x_3) is defined by the unit vectors $\hat{v}_h (h = 1, 2, 3)$ with the respect with the geographic axis x, y e z:

$$\hat{v_1} = (l_1, m_1, n_1) = (-\cos\alpha\cos\delta, -\sin\alpha\cos\delta, -\sin\delta) \tag{6}$$

$$\begin{aligned}\hat{v_2} = (l_2, m_2, n_2) = (&\cos\alpha\cos\gamma\sin\delta + \sin\alpha\sin\gamma,\\ &\sin\alpha\cos\gamma\sin\delta - \cos\alpha\sin\gamma, -\cos\gamma\cos\delta)\end{aligned} \tag{7}$$

$$\begin{aligned}\hat{v_3} = (l_3, m_3, n_3) = (&\sin\alpha\cos\gamma - \cos\alpha\sin\gamma\sin\delta,\\ &-\cos\alpha\cos\gamma - \sin\alpha\sin\gamma\sin\delta, \sin\gamma\cos\delta)\end{aligned} \tag{8}$$

The angles referring to the unit vectors are determined by the orientations of the elipsoid's semi-axis. The angle α is the azimuth of semi-major axe (a) plus 180°. While δ is the inclination of semi-major axe (a) in relation to the geographic plane. Lastly, γ is the angle between the semi-mid axe (b) and the vertical projection of the elipsoid's center with the geographic plane.

Thus, the coordinates of the body's semi-axis are given by:

$$x_h = (x - xc)l_h + (y - yc)m_h + (z - zc)n_h \quad (h = 1, 2, 3) \tag{9}$$

Where x_c, y_c e z_c are the coordinates of the elipsoid's center in the geographic system x, y e z.

For an elipsoid of semi-axis $a > b > c$, the equation that defines your surface is:

$$\frac{x_1^2}{(a^2 + s)} + \frac{x_2^2}{(b^2 + s)} + \frac{x_3^2}{(c^2 + s)} = 1 \tag{10}$$

The parameter s controls the elipsoid form. When s gets close to ∞ the equation (10) tends to the sphere equation of radius $r = \sqrt{\lambda}$. When $s = -c^2$, the last term of the elipsoid's equation is less than zero and it becomes the equation of a circle.

There is, however, a set of values for $s(\lambda, \mu, \nu)$, which are roots of the cubic equation:

$$s^3 + p_2 s^2 + p_1 s + p_0 = 0 \tag{11}$$

This set of roots, called elipsoidal coordinates, correspond to the parameters of a point (x_1, x_2, x_3) which are under the intersection of three ortogonal surfaces related to the body coordinates. Their expressions are:

$$\lambda = 2\sqrt{\left(\frac{-p}{3}\right)}\cos\left(\frac{\theta}{3}\right) - \frac{p_2}{3} \tag{12}$$

$$\mu = -2\sqrt{\left(\frac{-p}{3}\right)}\cos\left(\frac{\theta}{3} + \frac{\pi}{3}\right) - \frac{p_2}{3} \tag{13}$$

$$\mu = -2\sqrt{\left(\frac{-p}{3}\right)}\cos\left(\frac{\theta}{3} - \frac{\pi}{3}\right) - \frac{p_2}{3} \tag{14}$$

Where:

$$p_0 = a^2b^2c^2 - b^2c^2x_1^2 - c^2a^2x_2^2 - a^2b^2x_3^2 \tag{15}$$

$$p_1 = a^2b^2 + b^2c^2 + c^2a^2 - (b^2 + c^2)x_1^2 - (c^2 + a^2)x_2^2 - (a^2 + b^2)x_3^2 \tag{16}$$

$$p_2 = a^2 + b^2 + c^2 - x_1^2 - x_2^2 - x_3^2 \tag{17}$$

$$p = p_1 - \frac{p_2^2}{3} \tag{18}$$

$$q = p_0 - \frac{p_1p_2}{3} + 2\left(\frac{p_2}{3}\right)^3 \tag{19}$$

$$\theta = \cos^{-1}\left[\frac{-q}{2}\sqrt{\left(\frac{-p}{3}\right)^3}\right] \tag{20}$$

The calculation of the largest root λ of the equation (11) is essential, since the magnetic field depends on the spatial derivatives of the equation 10, where s admits the value of λ.

$$\frac{\partial\lambda}{\partial x_1} = \frac{2x_1/(a^2+\lambda)}{\left(\frac{x_1}{a^2+\lambda}\right)^2 + \left(\frac{x_2}{b^2+\lambda}\right)^2 + \left(\frac{x_3}{c^2+\lambda}\right)^2} \tag{21}$$

$$\frac{\partial\lambda}{\partial x_2} = \frac{2x_2/(b^2+\lambda)}{\left(\dfrac{x_1}{a^2+\lambda}\right)^2 + \left(\dfrac{x_2}{b^2+\lambda}\right)^2 + \left(\dfrac{x_3}{c^2+\lambda}\right)^2} \tag{22}$$

$$\frac{\partial\lambda}{\partial x_3} = \frac{2x_3/(c^2+\lambda)}{\left(\dfrac{x_1}{a^2+\lambda}\right)^2 + \left(\dfrac{x_2}{b^2+\lambda}\right)^2 + \left(\dfrac{x_3}{c^2+\lambda}\right)^2} \tag{23}$$

The desmagnetization factors are given by:

$$N_1 = \frac{4\pi abc}{(a^2-b^2)\sqrt{(a^2-c^2)}}[F(\theta,k) - E(\theta,k)] \tag{24}$$

$$N_2 = \frac{4\pi abc\sqrt{(a^2-c^2)}}{(a^2-b^2)(b^2-c^2)}\left[E(\theta,k) - \left(\frac{b^2-c^2}{a^2-c^2}\right)F(\theta,k) - \frac{c(a^2-b^2)}{ab\sqrt{(a^2-c^2)}}\right] \tag{25}$$

$$N_3 = \frac{4\pi abc}{(b^2-c^2)\sqrt{(a^2-c^2)}}\left[\frac{b\sqrt{(a^2-c^2)}}{ac} - E(\theta,k)\right] \tag{26}$$

Where $F(\theta,k)$ e $E(\theta,k)$ are first and second Legendre's normal eliptical integrals, respectively. To calculate k and θ we have the following expressions:

$$k = \sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)} \tag{27}$$

$$\theta = \cos^{-1}(c/a) \quad (0 \le \theta \le \pi/2) \tag{28}$$

The susceptibility tensor matrix is:

$$k_{ij} = \sum_r k_r(L_r l_i + M_r m_i + N_r n_i)(L_r l_j + M_r m_j + N_r n_j) \quad (r = 1,2,3) \tag{29}$$

The Earth's field vector, F, and the remanent magnetization ,d, also must be converted to the body's coordinates:

$$F_h = F(l\,l_h + m\,m_h + n\,n_h) \tag{30}$$

$$(J_N)_h = J_N(l_N l_h + m_n m_h + n_N n_h) \tag{31}$$

Note that L_r, M_r, N_r ($r = 1,2,3$), l,m,n e l_N, m_N, n_N depends on their respective vectors of inclinations and declinations:

$$L_R, l, l_N = \cos(dec)\cos(inc) \tag{32}$$

$$M_R, m, m_N = \sin(dec)\cos(inc) \tag{33}$$

$$N_R, n, n_N = \sin(inc) \tag{34}$$

In the case that the body has a very low susceptibility ($\chi < 0.1$ SI) the self-demagnatization is negligible and the resultant magnetic vector is given by:

$$\tilde{J}_R = K\tilde{F} + \tilde{J}_{NRM} \tag{35}$$

For values bigger than 0.1 SI the resultant magnetic vector is:

$$\tilde{J}_{Rc} = A^{-1}\tilde{J}_R \tag{36}$$

Where:

$$A = I + KN = \begin{bmatrix} 1 + k_{11}N_1 & k_{12}N_2 & k_{13}N_3 \\ k_{21}N_1 & 1 + k_{22}N_2 & k_{23}N_3 \\ k_{31}N_1 & k_{23}N_2 & 1 + k_{33}N_3 \end{bmatrix} \tag{37}$$

This way, the components of the magnetic field produced by a triaxial elipsoid in the body's coordinates outside of it is:

$$\Delta B_1 = f_1 \frac{\partial \lambda}{\partial x_1} - 2\pi abc\, J_1\, A(\lambda) \tag{38}$$

$$\Delta B_2 = f_1 \frac{\partial \lambda}{\partial x_2} - 2\pi abc\, J_2\, B(\lambda) \tag{39}$$

$$\Delta B_3 = f_1 \frac{\partial \lambda}{\partial x_3} - 2\pi abc\, J_3\, C(\lambda) \tag{40}$$

where:

$$f_1 = \frac{2\pi abc}{\sqrt{[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]}} \left[\frac{J_1 x_1}{a^2+\lambda} + \frac{J_2 x_2}{b^2+\lambda} + \frac{J_3 x_3}{c^2+\lambda} \right] \tag{41}$$

$$A(\lambda) = \frac{2}{(a^2-b^2)\sqrt{(a^2-c^2)}} [F(\theta^{'},k) - E(\theta^{'},k)] \tag{42}$$

$$B(\lambda) = \frac{2\sqrt{(a^2-c^2)}}{(a^2-b^2)(b^2-c^2)} \left[E(\theta^{'},k) - \left(\frac{b^2-c^2}{a^2-c^2} \right) F(\theta^{'},k) - \frac{k^2 \sin\theta^{'} \cos\theta^{'}}{\sqrt{(1-k^2\sin^2\theta^{'})}} \right] \tag{43}$$

$$C(\lambda) = \frac{2}{(b^2-c^2)\sqrt{(a^2-c^2)}} \left[\frac{\sin\theta^{'} \sqrt{(1-k^2\sin^2\theta^{'})}}{\cos\theta^{'}} - E(\theta^{'},k) \right] \tag{44}$$

And:

$$\theta^{'} = \sin^{-1}\left(\frac{a^2-c^2}{a^2+\lambda} \right)^{0.5} \quad (0 \le \theta^{'} \le \pi/2) \tag{45}$$

Both $F(\theta^{'},k)$ as $E(\theta^{'},k)$ are again the first and second noraml Legendre's eliptical integrals. While $A(\lambda), B(\lambda), C(\lambda)$ are analytic solutions of the integrals of the potential equation of an elipsoid. This problem was solved by Dirichlet in 1839 (Clark et al., 1986) for the gravitational potential given by:

$$U_i(x_1,x_2,x_3) = \pi abcG\rho \int_0^\infty \left[1 - \frac{x_1^2}{a^2+u} - \frac{x_2^2}{b^2+u} - \frac{x_3^2}{c^2+u} \right] \frac{du}{R(u)} \tag{46}$$

That can be rewritten as:

$$U_i(x_1,x_2,x_3) = \pi abcG\rho \int_0^\infty [D(\lambda) - A(\lambda) - B(\lambda) - C(\lambda)] \frac{du}{R(u)} \tag{47}$$

Where G is the gravitational constant and ρ the body's density.

Using the equations from (38) to (44), we can rearrange them in a matrix format that will be more computationally efficient. Thus, the magnetic field generated by the elipsoid is:

$$b_i^j = 2\pi a_j b_j c_j \times [M_i^j - D_i^j] \times J^j \quad j = 1, ..., L \tag{48}$$

With:

$$M_i^j = \frac{1}{\sqrt{[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]}} \begin{bmatrix} \frac{\partial\lambda}{\partial x_1}\frac{x_1}{a^2+\lambda} & \frac{\partial\lambda}{\partial x_1}\frac{x_2}{b^2+\lambda} & \frac{\partial\lambda}{\partial x_1}\frac{x_3}{c^2+\lambda} \\ \frac{\partial\lambda}{\partial x_2}\frac{x_1}{a^2+\lambda} & \frac{\partial\lambda}{\partial x_2}\frac{x_2}{b^2+\lambda} & \frac{\partial\lambda}{\partial x_2}\frac{x_3}{c^2+\lambda} \\ \frac{\partial\lambda}{\partial x_3}\frac{x_1}{a^2+\lambda} & \frac{\partial\lambda}{\partial x_3}\frac{x_2}{b^2+\lambda} & \frac{\partial\lambda}{\partial x_3}\frac{x_3}{c^2+\lambda} \end{bmatrix}_{3\times 3} \tag{49}$$

$$D_i^j = \begin{bmatrix} A(\lambda) & 0 & 0 \\ 0 & B(\lambda) & 0 \\ 0 & 0 & C(\lambda) \end{bmatrix} \tag{50}$$

For $j = 1, ..., L$, the number of elipsoids been modeled and $i = 1, ..., N$ for the ith element of the calculated field. The total magnetic field then is:

$$B_i = \sum_{j=1}^{L} b_i^j, \quad i = 1, ..., N \tag{51}$$

We must remember that every calculation so far was done using the body's coordinates, however the notation in equation (48) the magnetic field is already in geographics coordinates. To calculate each component of the magnetic field's vector back:

$$\Delta B_x = \Delta B_1 l_1 + \Delta B_2 l_2 + \Delta B_3 l_3 \tag{52}$$

$$\Delta B_y = \Delta B_1 m_1 + \Delta B_2 m_2 + \Delta B_3 m_3 \tag{53}$$

$$\Delta B_z = \Delta B_1 n_1 + \Delta B_2 n_2 + \Delta B_3 n_3 \tag{54}$$

2.2.1 HEADING

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3 Conclusions

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Appendix A

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Author contributions. TEXT

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