Fórmulas de Límites

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Límites y límites laterales

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L \iff \lim_{x\to c} f(x) = L$$

$$\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x) \implies \lim_{x\to c} f(x) \text{ no existe}$$

Límites de funciones simples

$$\lim_{x\to c} a = a$$

$$\lim_{x\to c} x = c$$

$$\lim_{x\to c} ax + b = ac + b$$

$$\lim_{x\to c} x^r = c^r \qquad \text{si } r \text{ es entero positivo}$$

$$\lim_{x\to 0^+} \frac{1}{x^r} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x^r} = \begin{cases} -\infty, & \text{si } r \text{ es impar} \\ +\infty, & \text{si } r \text{ es par} \end{cases}$$

Hechos sobre $\pm\infty$

Si $a \neq 0$ y $a < \infty$:

$$0 + \infty = \infty$$

$$a + \infty = \infty$$

$$\frac{a}{\infty} = 0$$

$$\frac{a}{0} = \begin{cases} \infty, & a > 0 \\ -\infty, & a < 0 \end{cases}$$

$$a \cdot \infty = \begin{cases} \infty, & a > 0 \\ -\infty, & a < 0 \end{cases}$$

Hecho sobre funciones

$$\lim_{x\to 0} \sin(x) = \sin(0) = 0$$

$$\lim_{x\to 0} \cos(x) = \cos(0) = 1$$

$$\lim_{x\to a} \sin(x) = \sin(a)$$

$$\lim_{x\to a} \cos(x) = \cos(a)$$

$$\lim_{x\to 0} e^x = e^0 = 1$$

$$\lim_{x\to a} \log_a(x) = \log_a(a) = 1$$

Si $a > 1$:

$$\lim_{x\to 0^+} \log_a x = \lim_{x\to 0^+} \ln x = \lim_{x\to 0^+} \log_{10} x = -\infty$$

$$\lim_{x\to \infty} \log_a x = \lim_{x\to \infty} \ln x = \lim_{x\to \infty} \log_{10} x = \infty$$

Si $a < 1$:

$$\lim_{x\to 0^+} \log_a x = \infty$$

$$\lim_{x\to \infty} \log_a x = -\infty$$

Formas Indeterminadas

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0 \times \infty,\ 1^\infty,\ \infty - \infty,\ 0^0 \text{ y } \infty^0$$

Formas no Indeterminadas

Si $\lim_{x\to c} \frac{f(x)}{g(x)}$ tiene la forma $\left[\frac{1}{0}\right]$ entonces

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \begin{cases} -\infty, \\ +\infty, \\ \text{no existe} \end{cases}$$

Si $\lim_{x\to c} f(x)^{g(x)}$ tiene la forma $[0^\infty]$ entonces

$$\lim_{x\to c} f(x)^{g(x)} = 0$$

Límites cerca de Infinito

$$\lim_{x\to \infty} a/x = 0, \qquad \text{para todo real } a$$

$$\lim_{x\to \infty} \sqrt[x]{x} = 1$$

$$\lim_{x\to \infty} \sqrt[a]{x} = \infty \qquad \text{para todo } a > 0$$

$$\lim_{x\to \infty} x/a = \begin{cases} \infty, & a > 0 \\ \text{no existe}\ , & a = 0 \\ -\infty, & a < 0 \end{cases}$$

$$\lim_{x\to \infty} x^a = \begin{cases} \infty, & a > 0 \\ 1, & a = 0 \\ 0, & a < 0 \end{cases}$$

$$\lim_{x\to \infty} a^x = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \\ 0, & 0 < a < 1 \end{cases}$$

$$\lim_{x\to \infty} a^{-x} = \begin{cases} 0, & a > 1 \\ 1, & a = 1 \\ \infty, & 0 < a < 1 \end{cases}$$

Límites de Polinomios

$$\lim_{x\to\infty}[a_n x^n + ... + a_1] = \lim_{x\to\infty} a_n x^n \qquad \text{máxima potencia}$$

$$\lim_{x\to\infty}\frac{mx^a}{nx^b} = \begin{cases} 0, & a<b \\ \frac{m}{n}, & a=b \\ \infty, & a>b \end{cases}$$

Límites de funciones generales

Si $\lim_{x\to c} f(x) = F$ y $\lim_{x\to c} g(x) = G$ entonces

$$\lim_{x\to c}[f(x)\pm g(x)] = F \pm G$$

$$\lim_{x\to c}[a\cdot f(x)] = a\cdot F$$

$$\lim_{x\to c}[f(x)g(x)] = F\cdot G$$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{F}{G} \qquad \text{si } G\neq 0$$

$$\lim_{x\to c} f(x)^n = F^n \qquad \text{si } n \text{ es entero positivo}$$

$$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{F} \qquad \text{si } n \text{ es entero positivo,}$$

y si n es par, entonces $F>0$

Composición de funciones

Si $f(x)$ es continua $\lim_{x\to c} g(x) = G$ entonces

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(G)$$

Límites y Derivadas

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

$$\lim_{h\to 0}\sqrt[n]{\frac{f(x+h)}{f(x)}} = \exp\left(\frac{f'(x)}{f(x)}\right)$$

$$\lim_{h\to 0}\sqrt[n]{\frac{f(x+h\cdot x))}{f(x)}} = \exp\left(\frac{xf'(x)}{f(x)}\right)$$

Regla de L'Hopital

si $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ o

si $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \pm\infty$ entonces

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Aplicaciones de L'Hopital

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c}\exp[g(x)\cdot\ln(f(x))] =$$

$$\lim_{x\to c}\exp\left(\frac{\ln(f(x))}{1/g(x)}\right) = \exp\left(\lim_{x\to c}\frac{\ln(f(x))}{1/g(x)}\right)$$

luego aplicar L'Hopital

Transformaciones de otras formas indeterminadas a $\left[\frac{0}{0}\right]$, para aplicar L'Hopital

$$\infty/\infty \qquad \lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{1/g(x)}{1/f(x)}$$

$$0\cdot\infty \qquad \lim_{x\to c} f(x)g(x) = \lim_{x\to c}\frac{f(x)}{1/g(x)}$$

$$\infty-\infty \qquad \lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$$

$$0^0 \qquad \lim_{x\to c} f(x)^{g(x)} = \exp\left(\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}\right)$$

$$1^\infty \qquad \lim_{x\to c} f(x)^{g(x)} = \exp\left(\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}\right)$$

$$\infty^0 \qquad \lim_{x\to c} f(x)^{g(x)} = \exp\left(\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}\right)$$

Teorema de Sandwich

Si $f(x)\leq g(x)\leq h(x)$ para todo x en un intervalo abierto que contiene a, excepto posiblmemente en a, y

$$\lim_{x\to c} f(X) = \lim_{x\to c} h(x) = L, \qquad \text{entonces}$$

$$\lim_{x\to c} g(X) = L$$

Infinitésimos Equivalente

Estas funciones de la forma $\lim_{x\to c} f(x) = 0$ son infinitésimos equivalentes cuando $x\to c$. Si $\lim_{x\to c}\frac{f(x)}{g(x)}$ tiene la forma $\left[\frac{0}{0}\right]$ entonces son intercambiables:

$$x\sim\sin(x)$$

$$x\sim\arcsin(x)$$

$$x\sim\sinh(x)$$

$$x\sim\tan(x)$$

$$x\sim\arctan(x)$$

$$x\sim\ln(1+x)$$

$$1-\cos x\sim\frac{x^2}{2}$$

$$\cosh(x)-1\sim\frac{x^2}{2}$$

$$a^x-1\sim x\ln(a)$$

$$e^x-1\sim x$$

$$(1+x)^a-1\sim ax$$

$$\log_a(1+x)\sim\frac{x}{\ln a}$$

Funciones Trigonométricas

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

$$\lim_{x\to 0}\frac{\sin(ax)}{ax}=1 \qquad \text{para } a\neq 0$$

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}=0$$

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}$$

$$\lim_{x\to n^{\pm}}\tan\left(\pi x+\frac{\pi}{2}\right)=\mp\infty \qquad \text{para todo entero } n$$

$$\lim_{x\to 0}\frac{\sin(ax)}{x}=a$$

$$\lim_{x\to 0}\frac{\sin(ax)}{bx}=\frac{a}{b} \qquad \text{para } b\neq 0$$

Límites Especiales Notables

$$\lim_{x\to 0^+}x^x=1$$

$$\lim_{x\to 0}(1+x)^{1/x}=e$$

$$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e$$

$$\lim_{x\to +\infty}\left(1-\frac{1}{x}\right)^x=\frac{1}{e}$$

$$\lim_{x\to +\infty}\left(1+\frac{k}{x}\right)^{mx}=e^{mk}$$

$$\lim_{x\to +\infty}\left(\frac{x}{x+k}\right)^x=\frac{1}{e^k}$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a=log_e(a)$$

$$\lim_{x\to 0}\frac{c^{ax}-1}{bx}=\frac{a}{b}\ln c$$

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

$$\lim_{x\to 0}\frac{\tan(x)}{x}=1$$

$$\lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$$

$$\lim_{x\to 0}\frac{x^n-a^n}{x-a}=0$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$\lim_{x\to 0}\frac{e^{ax}-1}{bx}=\frac{a}{b}$$

$$\lim_{x\to 0}\left(1+a\left(e^{-x}-1\right)\right)^{-1/x}=e^a$$

Logaritmos y exponentes

$$\lim_{x\to\infty}xe^{-x}=0$$

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}=1$$

$$\lim_{x\to 0}\frac{\ln(x+1)}{x}=1$$

$$\lim_{x\to 0}\frac{\ln(1+ax)}{bx}=\frac{a}{b}$$

$$\lim_{x\to 0}\frac{\log_c(1+ax)}{bx}=\frac{a}{b\ln c}$$

$$\lim_{x\to 0}\frac{-\ln\left(1+a\cdot\left(e^{-x}-1\right)\right)}{x}=a$$

Ejemplos de Técnicas

Factorar y Cancelar

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}&=\lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}\\&=\lim_{x\to 2}\frac{x+6}{x}=\frac{8}{2}=4\end{aligned}$$

Racionalizar numerador/denominador

$$\begin{aligned}\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}&=\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\cdot\frac{3+\sqrt{x}}{3+\sqrt{x}}\\&=\lim_{x\to 9}\frac{9-x}{(x^2-81)\left(3+\sqrt{x}\right)}\\&=\lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}\\&=\frac{-1}{(18)(6)}=-\frac{1}{108}\end{aligned}$$

Combinar expresiones racionales

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right)&=\lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)\\&=\lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right)\\&=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}\end{aligned}$$

Polinomios al Infinito

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2-4}{5x-2x^2}&=\lim_{x\to\infty}\frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)}\\&=\lim_{x\to\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2}=-\frac{3}{2}\end{aligned}$$

Equivalencia de Infinitésimos

$$
\begin{aligned}
&\lim_{x\to 0} \frac{1}{x^3}\left[\left(\frac{2+\cos x}{3}\right)^x - 1\right] \\
&= \lim_{x\to 0} \frac{1}{x^3}\left[\exp\left(x \ln \frac{2+\cos x}{3}\right) - 1\right] && \leftarrow y^x = \exp(x \ln y) \\
&= \lim_{x\to 0} \frac{1}{x^3}\left[x \ln \frac{2+\cos x}{3}\right] && \leftarrow e^x - 1 \sim x \\
&= \lim_{x\to 0} \frac{1}{x^2} \ln\left(\frac{(3-1)+\cos x}{3}\right) \\
&= \lim_{x\to 0} \frac{1}{x^2} \ln\left(\left(\frac{3}{3}\right) + \frac{-1+\cos x}{3}\right) \\
&= \lim_{x\to 0} \frac{1}{x^2} \ln\left(1 + \frac{\cos(x)-1}{3}\right) \\
&= \lim_{x\to 0} \frac{\cos(x)-1}{3x^2} && \leftarrow x \sim \ln(1+x) \\
&= \lim_{x\to 0} \frac{-(1-\cos(x))}{3x^2} \\
&= \lim_{x\to 0} \frac{-x^2/2}{3x^2} && \leftarrow 1-\cos x \sim \frac{x^2}{2} \\
&= -\frac{1}{6}
\end{aligned}
$$

Equivalencia de Infinitésimos

$$
\begin{aligned}
&\lim_{x\to 1} \frac{\sin(x-1)}{x^4-1} \\
&= \lim_{t\to 0} \frac{\sin t}{(t+1)^4 - 1} && \leftarrow t = x-1,\ x\to 1 \Rightarrow t\to 0 \\
&= \lim_{t\to 0} \frac{\sin t}{(t^4+4t^3+6t^2+4t+1)-1} \\
&= \lim_{t\to 0} \frac{\sin t}{t^4+4t^3+6t^2+4t} \\
&= \lim_{t\to 0} \frac{t}{t\,(t^3+4t^2+6t+4)} && \leftarrow \sin t \sim t \\
&= \lim_{t\to 0} \frac{1}{(t^3+4t^2+6t+4)} = \frac{1}{4}
\end{aligned}
$$

Equivalencia de Infinitésimos

$$
\begin{aligned}
&\lim_{x\to e} \frac{\ln(\ln x)}{x-e} = \lim_{x\to e} \frac{\ln(\ln x + 1 - 1)}{x-e} \\
&= \lim_{x\to e} \frac{\ln[1+(\ln x - 1)]}{x-e} \\
&= \lim_{x\to e} \frac{\ln x - 1}{x-e} && \leftarrow ln(1+x) \sim x \\
&= \lim_{x\to e} \frac{\ln x - \ln e}{x-e} && \leftarrow 1 = \ln(e) \\
&= \lim_{x\to e} \frac{\ln\left(\frac{x}{e}\right)}{x-e} = \lim_{x\to e} \frac{\ln\left[1+\left(\frac{x}{e}-1\right)\right]}{x-e} \\
&= \lim_{x\to e} \frac{\frac{x}{e}-1}{x-e} && \leftarrow ln(1+x) \sim x \\
&= \lim_{x\to e} \frac{\frac{x-e}{e}}{x-e} = \lim_{x\to e} \left(\frac{1}{e}\right)\frac{x-e}{x-e} = \frac{1}{e}
\end{aligned}
$$