Bidimensional Solid Analysis using the Positional Finite Element Method

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Course: SET5884 - Introduction to the Positional Finite Element Method

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Topics

Methodology

Software used

AcadMesh
- Modeling
- Meshing

Fortran + SparseSET + OpenMP
- Implementation

Python
- Post-processing

Acadview
- Visualization

- Principle of **Stationarity of Mechanical Energy**:

$$\delta\Pi = \frac{\partial \Pi}{\partial \vec{Y}} \cdot \delta\vec{Y} = 0 \quad \Rightarrow \quad \delta\vec{Y} \quad \text{Arbitrary} \quad \Rightarrow \quad \frac{\partial \Pi}{\partial \vec{Y}} = 0 \tag{1}$$

- **Nature of Equilibrium** (Curvature of the Π surface):

$$H_{ij} = \frac{\partial}{\partial \vec{Y}}\left(\frac{\partial \Pi}{\partial \vec{Y}}\right) = \frac{\partial^2 \Pi}{\partial \vec{Y} \otimes \partial \vec{Y}} \tag{2}$$

- $H > 0$
- $H = 0$
- $H < 0$

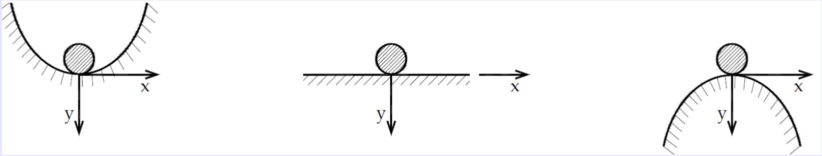

Figure: Nature of Equilibrium

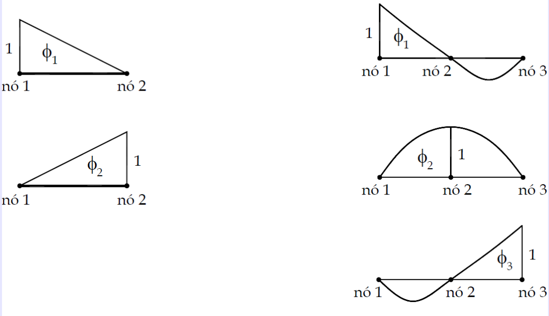

Figure: Lagrange Polynomials

- **Lagrange Polynomials:**
  - Continuous Domain $\rightarrow$ Discrete Domain
  - Interpolation of Nodal Quantities in the Discrete Domain

- **Partition of Unity:**

$$\sum_{i=1}^{n} \phi_i\left(\vec{\xi}\right) = 1 \tag{3}$$

- Shape Functions

$$\phi_i = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \tag{4}$$

- **Integral in $\vec{\xi}$:**

$$\int_{\Omega} f(\vec{x}) d\Omega = \int_{\Omega} f\left(\vec{\xi}\right) J_0 d\vec{\xi} \tag{5}$$

- **Triangular-based domain:** Hammer Quadrature

$$\int_{\Omega} f\left(\vec{x}\right) d\Omega = \sum_{ih=1}^{nh} f\left(\vec{\xi}\right) J_0 w_{ih} \tag{6}$$

- **Mapping:** Coordinates $\vec{X}$ and $\vec{Y}$ $\underbrace{\Longleftrightarrow}_{\text{Jacobian}}$ $\vec{\xi}$

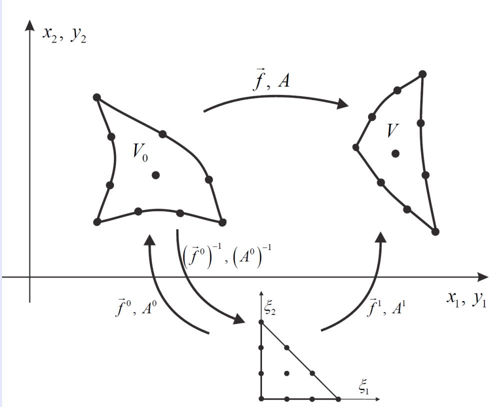

Figure: Mapping

- **Zero kinetic energy:** $\mathbb{K} = 0 \quad \Rightarrow \quad \mathit{\Pi} = \mathbb{U} + \mathbb{P} + \mathbb{K} = \mathbb{U} + \mathbb{P}$

$$\delta \mathit{\Pi} = -\int\limits_{V_0} \left(b_i^0 \delta y_i\right) dV^0 - \int\limits_{A_0} \left(p_i^0 \delta y_i\right) dA^0 + \int\limits_{V_0} \left(S_{ij} \delta E_{ij}\right) dV^0 = 0 \tag{10}$$

$$\delta y_i(\xi) = \phi_i(\xi)\delta Y_i \quad \text{and} \quad y_i(\xi) = \phi_i(\xi) Y_i \quad \text{and} \quad p_i^0 = \phi_i(\xi) P_i^0 \quad \text{and} \quad b_i^0 = \phi_i(\xi) B_i^0 \tag{11}$$

- **Arbitrariness of δY_i:**

$$\underbrace{-B_i^m \int\limits_{V_0} \left(\phi^m \phi^\ell\right) dV^0 - Q_i^m \int\limits_{A_0} \left(\phi^m \phi^\ell\right) dA^0 - F_i^\ell}_{\text{External Forces}} + \underbrace{\int\limits_{V_0} \left(S_{ij} \frac{\partial E_{ij}}{\partial Y_i^\ell}\right) dV^0}_{\text{Internal Forces}} = 0 \tag{12}$$

$$(F_i^\ell)^{ext} - (F_i^\ell)^{int} = 0 \tag{13}$$

- **Body Loads:**

$$(F_i^\ell)^{vol} = -B_i^m \int\limits_{V_0} \left(\phi^\ell \phi^m\right) dV^0 = \sum_{ih=1}^{nh} \left(B_i^m \phi^\ell \phi^m\right) J_0 w_{ih} \qquad (14)$$

- Matrix $\left[\phi^\ell \phi^m\right]_{ij} \quad i,j \in [0,n]\,, n = dof \cdot nnp$:

$$\phi^\ell \phi^m = \begin{pmatrix} \phi_1\phi_1 & 0 & \phi_1\phi_2 & 0 & \phi_1\phi_3 & 0 \\ 0 & \phi_1\phi_1 & 0 & \phi_1\phi_2 & 0 & \phi_1\phi_3 \\ \phi_2\phi_1 & 0 & \phi_2\phi_2 & 0 & \phi_2\phi_3 & 0 \\ 0 & \phi_2\phi_1 & 0 & \phi_2\phi_2 & 0 & \phi_2\phi_3 \\ \phi_3\phi_1 & 0 & \phi_3\phi_2 & 0 & \phi_3\phi_3 & 0 \\ 0 & \phi_3\phi_1 & 0 & \phi_3\phi_2 & 0 & \phi_3\phi_3 \end{pmatrix} \qquad (15)$$

- **Surface Loads:** (2D $\Rightarrow$ Line and 3D $\Rightarrow$ Area)
  - Similar procedure, but with load lines
  - Shape functions with one dimension less

- Recalling the internal forces equation:

$$\left(F_i^{\ell}\right)^{int} = \int\limits_{V^{el}} \left(\frac{\partial \Psi(Y_k^m)}{\partial E_{km}} \frac{\partial E_{km}}{\partial Y_i^{\ell}} \right) d\Omega = \int\limits_{V^{el}} \left(S_{km} \frac{\partial E_{km}}{\partial Y_i^{\ell}} \right) d\Omega \tag{16}$$

- Using numerical integration and rewriting $\frac{\partial E_{km}}{\partial Y_i^{\ell}}$ in terms of direction α and node β:

$$\left(F_{\alpha}^{\beta}\right)^{int} = \sum_{ih=1}^{nh} \left(S_{ij} \frac{\partial E_{ij}}{\partial Y_{\alpha}^{\beta}} \right) \vec{\xi}(ih) \tag{17}$$

- Recalling the definition of E_{ij}:

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{A}^T\mathbf{A} - \mathbf{I}\right) \tag{18}$$

- Thus, $\frac{\partial E_{ij}}{\partial Y_\alpha^\beta}$ takes the form:

$$DE_{\alpha\beta} = \frac{1}{2}\left[\left(A^0\right)^{-t}\left(DA_{\alpha\beta}\right)^t A^1\left(A^0\right)^{-1} + \left(A^0\right)^{-t}\left(A^1\right)^t DA_{\alpha\beta}\left(A^0\right)^{-1}\right] \tag{19}$$

$$DA^1_{1\beta} = \begin{pmatrix} \phi_{\beta,1} & \phi_{\beta,2} \\ 0 & 0 \end{pmatrix} \quad , \quad DA^1_{2\beta} = \begin{pmatrix} 0 & 0 \\ \phi_{\beta,1} & \phi_{\beta,2} \end{pmatrix} \tag{20}$$

- In other words: $(\alpha \Rightarrow \text{direction}, \quad \beta \Rightarrow \text{node})$

$$DA^1_{\alpha\beta} = \begin{pmatrix} \frac{\partial\phi_\beta}{\partial\xi}(2-\alpha) & \frac{\partial\phi_\beta}{\partial\eta}(2-\alpha) \\ \frac{\partial\phi_\beta}{\partial\xi}(\alpha-1) & \frac{\partial\phi_\beta}{\partial\eta}(\alpha-1) \end{pmatrix} \tag{21}$$

- **Hessian:** Also called *Tangent Stiffness Matrix*, is defined as:

$$\frac{\partial^2 U}{\partial Y_\gamma^z \partial Y_\alpha^\beta} = \frac{\partial \left(F_\alpha^\beta\right)^{int}}{\partial Y_\gamma^z} = \int_{V^{el}} \frac{\partial}{\partial Y_\gamma^z} \left(\frac{\partial \Psi(Y_k^m)}{\partial E_{km}} \frac{\partial E_{km}}{\partial Y_\gamma^\beta} \right) dV_0^{el} \tag{22}$$

$$H_{\alpha\beta\gamma z}^{el} = \sum_{ih=1}^{nh} \underbrace{\left[h_{\alpha\beta\gamma z} \left(\xi(\vec{ih}) \right) \right] \vec{\xi}(ih) \omega_{ih}}_{\text{Domain } \vec{\xi}} \underbrace{J_0 \left(\vec{\xi}(ih) \right)}_{\det[J]_0 = \det \mathbf{A}^0} \tag{23}$$

- Where:

$$h_{\alpha\beta\gamma z} = \frac{\partial}{\partial Y_\gamma^z} \left(\frac{\partial \Psi(Y_k^m)}{\partial E_{km}} \frac{\partial E_{km}}{\partial Y_\gamma^\beta} \right) \tag{24}$$

$$h_{\alpha\beta\gamma z} = \frac{\partial}{\partial Y_\gamma^z} \underbrace{\left(\frac{\partial \Psi}{\partial E} : \frac{\partial E}{\partial Y_\gamma^\beta} \right)}_{\left(F_\gamma^\beta\right)^{int}} \tag{25}$$

$$h_{\alpha\beta\gamma z} = \frac{\partial E}{\partial Y_\gamma^\beta} : \underbrace{\frac{\partial^2 \Psi}{\partial E \otimes \partial E}}_{\mathbb{C}} : \frac{\partial E}{\partial Y_\alpha^\beta} + \underbrace{\frac{\partial \Psi}{\partial E}}_{\mathbf{S}} : \frac{\partial^2 E}{\partial Y_\gamma^z \partial Y_\alpha^\beta} \tag{26}$$

$$h_{\alpha\beta\gamma z} = \frac{\partial E}{\partial Y_\gamma^z} : \underbrace{\mathbb{C} : \frac{\partial E}{\partial Y_\alpha^\beta}}_{\frac{\partial S}{\partial Y_\alpha^\beta}} + \mathbf{S} : \frac{\partial^2 E}{\partial Y_\gamma^z \partial Y_\alpha^\beta} \tag{27}$$

$$h_{\alpha\beta\gamma z} = \frac{\partial E}{\partial Y_\gamma^z} : \frac{\partial S}{\partial Y_\alpha^\beta} + \mathbf{S} : \frac{\partial^2 E}{\partial Y_\gamma^z \partial Y_\alpha^\beta} \tag{28}$$

- Developing the terms $\frac{\partial S}{\partial Y_\alpha^\beta}$ and $\frac{\partial^2 E}{\partial Y_\gamma^z \partial Y_\alpha^\beta}$:

$$DS_{\alpha\beta} = \frac{\partial S}{\partial Y_\alpha^\beta} = \mathbb{C} : \underbrace{\frac{\partial E}{\partial Y_\alpha^\beta}}_{DE_{\alpha\beta}} \tag{29}$$

$$D2E_{\alpha\beta\gamma z} = \frac{\partial^2 E}{\partial Y_\gamma^z \partial Y_\alpha^\beta} \tag{30}$$

$$D2E_{\alpha\beta\gamma z} = \frac{1}{2}\left[\left(A^0\right)^{-t}\left(DA^1_{\alpha\beta}\right)^t\left(DA^1_{\gamma z}\right)\left(A^0\right)^{-1} + \left(A^0\right)^{-t}\left(DA^1_{\gamma z}\right)\left(DA^1_{\alpha\beta}\right)^t\left(A^0\right)^{-1}\right] \tag{31}$$

$$h_{\alpha\beta\gamma z} = DE_{\gamma z} : DS_{\alpha\beta} + S : D2E_{\alpha\beta\gamma z} \tag{32}$$

$$\underbrace{\vec{g}}_{\text{Mechanical Imbalance}} = \vec{F}_{\text{int}} - \vec{F}_{\text{ext}} \quad \Rightarrow \quad \vec{g} = \frac{\partial \Pi}{\partial \vec{Y}} \tag{33}$$

- Taylor series expansion around $\vec{Y}_0$ (trial position):

$$\vec{g}\underbrace{\left(\vec{Y}\right)}_{\text{Solution}} = \vec{g}\underbrace{\left(\vec{Y}_0\right)}_{\text{Trial}} + \underbrace{\left.\frac{\partial \vec{g}}{\partial \vec{Y}}\right|_{\vec{Y}_0}}_{\left.\frac{\partial^2 \Pi}{\partial \vec{Y}^2}\right|_{\vec{Y}_0} = \mathbf{H}} \Delta\vec{Y} + \underbrace{\frac{1}{2}\left.\frac{\partial^2 \vec{g}}{\partial \vec{Y}^2}\right|_{\vec{Y}_0} \Delta\vec{Y}^2}_{\approx \vec{0}} \tag{34}$$

$$\vec{g}\left(\vec{Y}\right) \approx \vec{0} \quad \Rightarrow \quad \underbrace{\vec{g}\left(\vec{Y}_0\right) + \mathbf{H}|_{\vec{Y}_0} \Delta\vec{Y} = \vec{0}}_{\text{Linear System}} \tag{35}$$

$$\Delta\vec{Y} = -\left[\mathbf{H}|_{\vec{Y}_0}\right]^{-1} \vec{g}\left(\vec{Y}_0\right) \tag{36}$$

Algorithm 1 Pseudo-Algorithm

First trial solution $\vec{Y}^0 = \vec{X}$
for Load Steps **do**
  $\vec{F}^{ext} \leftarrow \vec{F}^{ext} + \Delta\vec{F}^{ext}$ (Prescribed load) $\vec{Y} \leftarrow \vec{Y} + \Delta\vec{Y}$ (Prescribed position)
  while $\mathrm{RES} > \mathrm{TOL}$ **do**
    for each Element **do**
      Compute $\left[(\vec{F}_{\alpha}^{\beta})_{\text{int}}\right]^{el}$ and $[\mathbf{H}_{\alpha\beta\gamma z}]_{el}$
    end for
    $\left(\vec{F}_{int}\right)^{\text{global}} \leftarrow \left(\vec{F}_{int}\right)^{\text{global}} + \left(\vec{F}_{int}\right)^{\text{el}}$ and $\mathbf{H}^{\text{global}} \leftarrow \mathbf{H}^{\text{global}} + \mathbf{H}^{\text{el}}$
    $\vec{g} \leftarrow \vec{F}_{\text{int}} - \vec{F}_{\text{ext}}$
    Boundary conditions on $\mathbf{H}$ and $\vec{g}$
    $\mathbf{H} \cdot \Delta\vec{Y} = -\vec{g}$
    $\vec{Y} \leftarrow \vec{Y} + \Delta\vec{Y}$
    $\mathrm{RES} = \dfrac{|\Delta\vec{Y}|}{|\vec{X}|}$
  end while
end for

- Considering $\mathbb{K} \quad \Rightarrow \quad \mathit{\Pi} = \mathbb{U} + \mathbb{P} + \mathbb{K}$

$$\mathbb{K} = \frac{1}{2} \int\limits_{V_0} \left(\rho^0 \dot{y}_i \dot{y}_i\right) dV^0 \quad \Rightarrow \quad \delta\mathbb{K} = \frac{\delta\mathbb{K}}{dt}\delta t = \overbrace{\int\limits_{V_0} \left(\rho^0 \ddot{y}_i \delta y_i\right) dV^0}^{\vec{F}^{inercial}} \tag{37}$$

$$\delta\mathbb{K} = \int\limits_{V_0} \left(\rho^0 \phi_I \phi_\alpha \ddot{Y}_i^\alpha\right) dV^0 \delta Y_i^\ell \tag{38}$$

- Equilibrium: $\delta\mathit{\Pi} = \frac{\partial \mathit{\Pi}}{\partial Y_i^\ell} \delta Y_i^\ell$
- Arbitrariness of $\delta Y_i^\ell \quad \Rightarrow \quad \delta\mathit{\Pi} = \frac{\partial \mathit{\Pi}}{\partial Y_i^\ell} = \frac{\partial \mathbb{U}}{\partial Y_i^\ell} + \frac{\partial \mathbb{P}}{\partial Y_i^\ell} + \frac{\partial \mathbb{K}}{\partial Y_i^\ell} = 0$

$$\frac{\partial \mathbb{K}}{\partial Y_i^\ell} = \int\limits_{V_0} \left(\rho^0 \phi_I \phi_\alpha \ddot{Y}_i^\alpha\right) dV^0 = \mathbf{M}\ddot{\vec{Y}} \tag{39}$$

- **Solution technique:** *Newton-Raphson + Newmark*

$$\vec{g} = \vec{F}_{\text{int}}\left(\vec{Y}\right) + \mathbf{M}\ddot{\vec{Y}} - \vec{F}_{\text{ext}}\left(\vec{Y}\right) = 0 \quad , \quad \mathbf{H}^{din} = \mathbf{H}^{est} + \frac{\mathbf{M}}{\gamma \Delta t^2} + \frac{\mathbf{C}}{\beta \Delta t} \quad (40)$$

- Considering damping **C** proportional to the mass matrix **M**
- Newmark time integrator:

$$\vec{Y}^{t+1} = \vec{Y}^{t} + \Delta t \dot{\vec{Y}}^{t} + \left[\left(\frac{1}{2} - \beta\right)\ddot{\vec{Y}}^{t} + \beta \ddot{\vec{Y}}^{t+1}\right]\Delta t^2$$

$$\ddot{\vec{Y}}^{t+1} = \frac{\vec{Y}^{t+1}}{\beta \Delta t^2} - \vec{Q}^{t} \quad \text{and} \quad \dot{\vec{Y}}^{t+1} = \frac{\gamma}{\beta \Delta t}\vec{Y}^{t+1} + \vec{R}^{t} - \gamma \Delta t \vec{Q}^{t}$$

$$\vec{Q}^{t} = \frac{\vec{Y}^{t}}{\beta \Delta t^2} + \frac{\dot{\vec{Y}}^{t}}{\beta \Delta t} + \left(\frac{1}{2\beta} - 1\right)\ddot{\vec{Y}}^{t} \quad \text{and} \quad \vec{R}^{t} = \left(\dot{\vec{Y}}^{t} + (1-\gamma)\Delta t \ddot{\vec{Y}}^{t}\right)$$

Algorithm 2 Pseudo-Algorithm

First trial solution $\vec{Y}^0 = \vec{X} \quad , \quad \dot{\vec{Y}}^0 = \vec{0} \quad , \quad \ddot{\vec{Y}}^0 = \vec{0}$
for Time Steps **do**
  $\vec{F}^{ext} \leftarrow \vec{F}^{ext} + \Delta\vec{F}^{ext}$ (Prescribed load) $\vec{Y} \leftarrow \vec{Y} + \Delta\vec{Y}$ (Prescribed position)
  $\vec{Q}$ and $\vec{R}$
  while $\mathrm{RES} > \mathrm{TOL}$ **do**
    for each Element **do**
      Compute $\left[(\vec{F}_{\alpha}^{\beta})_{\text{int}}\right]^{el}$ and $[\mathbf{H}_{\alpha\beta\gamma z}]_{el}$
    end for
    $\left(\vec{F}_{int}\right)^{\text{global}} \leftarrow \left(\vec{F}_{int}\right)^{\text{global}} + \left(\vec{F}_{int}\right)^{\text{el}}$ and $\mathbf{H}^{\text{global}} \leftarrow \mathbf{H}^{\text{global}} + \mathbf{H}^{\text{el}}$
    $\vec{g} \leftarrow \vec{F}_{\text{int}} - \vec{F}_{\text{ext}}$
    Boundary conditions on $\mathbf{H}$ and $\vec{g}$
    $\mathbf{H} \cdot \Delta\vec{Y} = -\vec{g}$
    $\vec{Y} \leftarrow \vec{Y} + \Delta\vec{Y}$
    Update: $\dot{\vec{Y}} \quad , \quad \ddot{\vec{Y}}$
    $\mathrm{RES} = \dfrac{|\Delta\vec{Y}|}{|\vec{X}|}$
  end while
end for

Static Code Validation

Marques (2006)

Example 4: Pillar instability under compression
Example 5: Simply Supported / Fixed Beam

Kishino (2022)

Example 1: Cantilever Beam

Static Code Validation

Example 4 – Marques (2006)

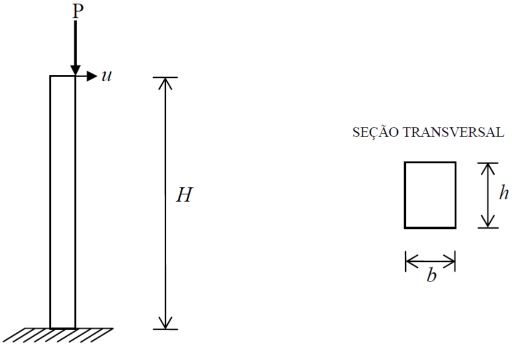

Figure: Pillar instability under compression

- **H:** 2 m
- **h:** 0.0663 m
- **b:** 1 m
- **E:** $210 \cdot 10^9 \, Pa$
- ν **:** 0.0

Results:

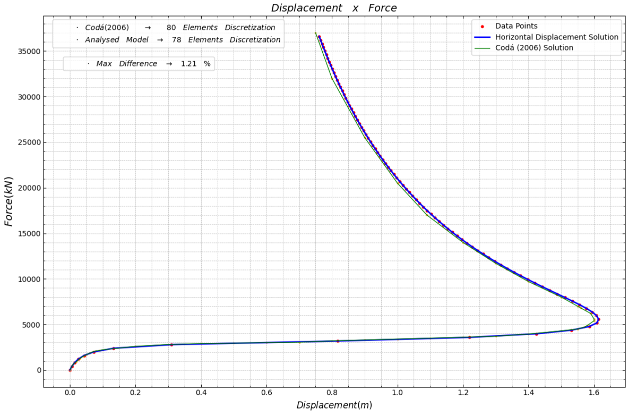

Figure: Horizontal displacement

Dynamic Code Validation

Marques (2006)

Example 2: Cantilever Beam - Sudden Loading
Example 4: Cantilever Beam - Increasing-Decreasing Loading

Kishino (2022)

Example 1: Cantilever Beam - Increasing-Constant Loading

Dynamic Code Validation

Example 2 – Marques (2006)

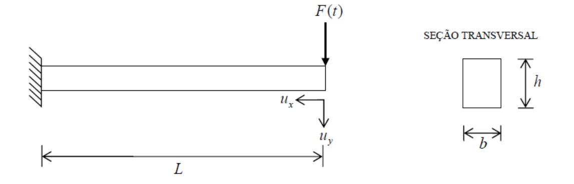

Figure: Cantilever Beam

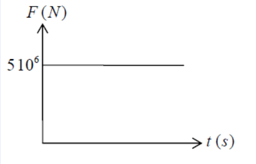

Figure: Loading

- **L** $= 1.2$ m
- **h** $= 0.1856$ m
- **b** $= 1.0$ m

- **E** $= 210 \cdot 10^9 Pa$
- $\boldsymbol{\nu} = 0.0$
- $\rho = 1691.81 \frac{N \cdot s^2}{m^4}$
- **P** $= 5 \cdot 10^6$ N

- **Elements** $=$ 78 T10
- $\Delta t = 2.5 \cdot 10^{-4}$ **s**
- $t_{tot} = 10^{-6}$

Results

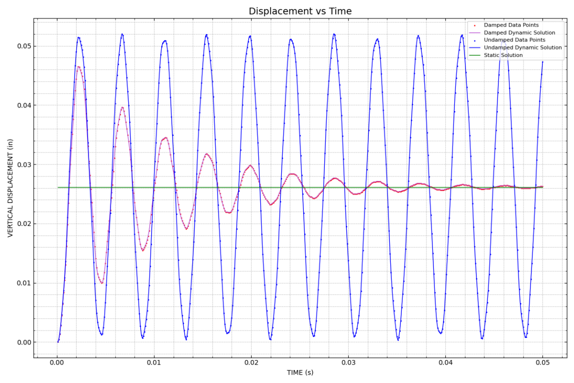

Figure: Cantilever Beam

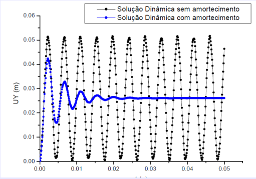

Figure: Marques (2006)

- **Griffith's Functional Griffith and Taylor (1921)**:

$$\mathit{\Pi}(\varepsilon, \mathit{\Gamma}) = \int_{\Omega} \psi_0 dV + \underbrace{G_c \int_{\mathit{\Gamma}} dS}_{\text{Fracture Dissipation}} - \Pi_{ext} \quad (41)$$

- Regularization:

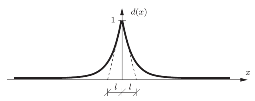

Figure: Variable d

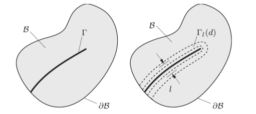

Figure: Regularization

$$\mathit{\Pi}(\varepsilon, \mathit{\Gamma}) = \underbrace{\int_{\Omega} \omega(d)\psi_0(\varepsilon)\, d\Omega}_{\text{Internal}} + \underbrace{G_c \int_{\Omega} \left(w(d) + c_d |\nabla d|^2\right) d\Omega}_{\text{Dissipation}} - \underbrace{\Pi_{ext}}_{\text{External Potential}} \quad (42)$$

- **Internal Energy Degradation Function:** $\omega = (1 - d)^2$
- **Functions Defining the Dissipated Energy:**

$$w(d) \begin{cases} \dfrac{3}{8\ell_0} d & \text{For the AT1 model} \\ \dfrac{1}{2\ell_0} d^2 & \text{For the AT2 model} \end{cases} \qquad c_d \begin{cases} \dfrac{3}{8}\ell_0 & \text{For the AT1 model} \\ \dfrac{1}{2}\ell_0 & \text{For the AT2 model} \end{cases}$$

Damage Model

Phase-Field

- **Asymmetric energy degradation:**

$$\psi\left(\varepsilon, d\right) = \left(1-d\right)^2 \psi_0^+\left(\varepsilon\right) + \psi_0^-\left(\varepsilon\right) \tag{43}$$

$$\sigma_{ij} = \frac{\partial\psi}{\partial\varepsilon_{ij}} = \left(1-d\right)^2 \left(2\mu\varepsilon_{ij}^D + \kappa\varepsilon_{ij}^{\upsilon+}\mathbf{I}\right) + \kappa\varepsilon_{ij}^{\upsilon-}\mathbf{I} \tag{44}$$

- **Functional Minimization with respect to $\vec{u}$:**

$$\delta_{\boldsymbol{u}} \Pi = \int_{\Omega} \partial_{\boldsymbol{u}} \Psi \left(\varepsilon, d \right) \left[\delta \boldsymbol{u} \right] d\Omega - \int_{\Omega} \left(\boldsymbol{f} \cdot \delta_{\boldsymbol{u}} \right) d\Omega - \boldsymbol{F} \cdot \delta_{\boldsymbol{u}} \quad . \tag{45}$$

- **Functional Minimization with respect to d:**

$$\delta_d \Pi = \int_{\Omega} \partial_d \Psi \left(\varepsilon, d \right) \left[\delta d \right] d\Omega + \frac{G_c}{l_0} \int_{\Omega} d \left(\delta d \right) + l_0^2 \nabla d \cdot \nabla \left(\delta d \right) d\Omega \tag{46}$$

$$\underbrace{\delta_d \Pi \geqslant 0 \quad \forall \delta d \geqslant 0 \quad \text{where} \quad \delta d = d^{t_i+1} - d^{t_i}}_{\text{Irreversibility}} . \tag{47}$$

- **Writing in Incremental Form:**
  - Taylor series expansion around d^{t_i}

$$\varPi(u_{t_i}, d_{t_i} + \Delta d_{t_i+1}) = \varPi(u_{t_i}, d_{t_i}) + \underbrace{\nabla_d \varPi|_{d_{t_i}} \Delta d_{t_i+1} + \Delta d_{t_i+1}^T \left[\frac{1}{2}\left(\nabla_{dd}^2 \varPi\right)|_{d_{t_i}}\right] \Delta d_{t_i+1}}_{\text{Term to be minimized}} \tag{48}$$

- **Defining:**:
  - $\mathbf{Q} = \nabla_{dd}^2 \varPi \left(u_{t_i}, d_n\right)$
  - $\vec{q} = \nabla_d \varPi \left(u_{t_i}, d_n\right)$
- Reformulate $\varPi$ as Marengo *et al.* (2021)

$$\varPi_{t_i} = \frac{1}{2}\Delta d_{t_i+1}^T \mathbf{Q} \Delta d_{t_i+1} + \vec{q}\Delta d_{t_i+1} + \varPi_{t_i} \tag{49}$$

- **Definition of Q and $\vec{q}$:**

$$\mathbf{Q} = \Psi(u_{t_i}) + G_c \Phi \qquad \text{and} \qquad \vec{q} = \mathbf{Q}\vec{d}_{t_i} - \varphi(u_{t_i}) \tag{50}$$

$$\Phi = \int_{\Omega} \left(\ell_0^{-1} \mathbf{N}^T \mathbf{N} + \ell_0 \mathbf{B}^T \mathbf{B} \right) d\Omega \quad \text{where} \quad \mathbf{B} = \mathbf{J}_0^{-1} \frac{\partial \vec{\phi}}{\partial \vec{\xi}} \tag{51}$$

$$\Psi = \int_{\Omega} \left(2\psi_0^+ u_{t_i} \mathbf{N}^T \mathbf{N} \right) d\Omega \tag{52}$$

$$\varphi = \int_{\Omega} \left(\psi_0^+ u_{t_i} \mathbf{N}^T \right) d\Omega \tag{53}$$

- Finally, the solution of the incremental problem is given by the **restricted** minimization:

$$\Delta \boldsymbol{d} = \operatorname{argmin}\left\{\frac{1}{2}\Delta \boldsymbol{d}^T \mathbf{Q}\Delta \boldsymbol{d} + \vec{q}\Delta \boldsymbol{d} \quad : \quad \Delta \boldsymbol{d} \geqslant 0\right\} \tag{54}$$

- Which can be rewritten as:

$$\left(\mathbf{Q}\Delta\vec{d} + \vec{q}\right)^T \Delta\vec{d} = 0 \quad \text{and} \quad -\left(\mathbf{Q}\Delta\vec{d} + \vec{q}\right) \leqslant 0 \quad \text{and} \quad \Delta\vec{d} \geqslant 0 \tag{55}$$

- The problem is solved by the PSOR algorithm Marengo *et al.* (2021)

Algorithm 3 Pseudo-Algorithm

First trial solution $\vec{Y}^0 = \vec{X},\ \vec{d}^0 = \vec{0}$

for Load Steps **do**

  $\vec{F}^{ext} \leftarrow \vec{F}^{ext} + \Delta\vec{F}^{ext}$ ▷ Prescribed load

  $\vec{Y} \leftarrow \vec{Y} + \Delta\vec{Y}$ ▷ Prescribed position

  while $\mathrm{RES} > \mathrm{TOL}$ **do**

    $\delta_y\, \mathit{\Pi}\left(y^i, d^i\right) = 0 \quad \rightarrow \quad y^i$

    $\boldsymbol{u}^i = \boldsymbol{y}^i - \boldsymbol{y}^{i-1}$

    $\delta_d\, \mathit{\Pi}\left(u^i, d^n\right) \geq 0 \quad \rightarrow \quad d^i$

    $\mathrm{RES} = \left|\vec{F}^{int}\left(u^i, d^i\right)\right|$

  end while

  $\left(u^{n+1}, d^{n+1}\right) \leftarrow \left(u^i, d^i\right)$

end for

Phase-Field Validation

Ferreira, Marengo, and Perego (2024)

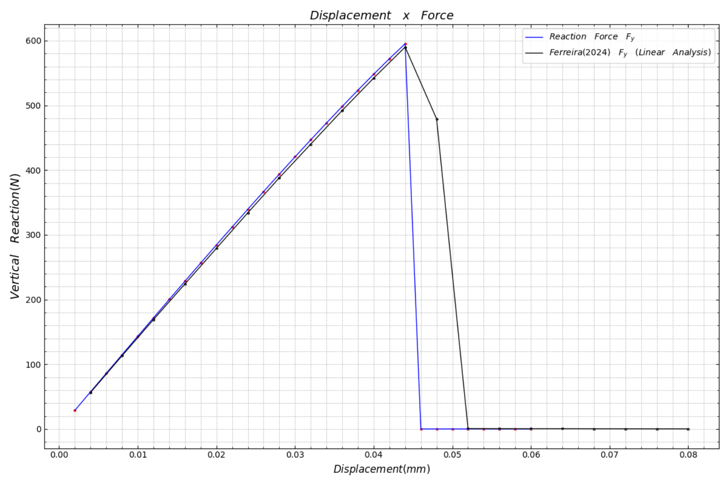

Figure: Reaction Force

Phase-Field Validation

Ferreira, Marengo, and Perego (2024)

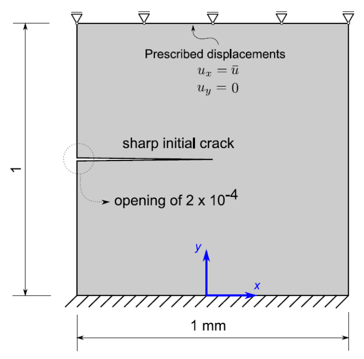

Figure: Analyzed Problem

- **E:** $210 \cdot 10^3 \frac{kN}{mm^2}$
- $\boldsymbol{\nu}$ **:** 0.3
- **h:** 1 mm
- $\overline{\boldsymbol{u}}$ 0.006 mm
- **Steps:** 20
- G_c : 2.7 $\frac{kN}{mm^2}$
- ℓ_0: 0.01mm

Phase-Field Validation

Ferreira, Marengo, and Perego (2024)

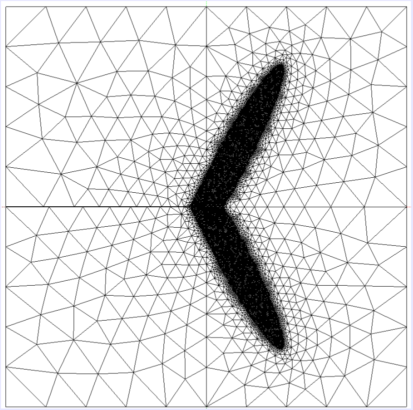

Figure: Problem Discretization

- **Nodes:** 16232
- **Elements:** 32357 T3
- **Equilibrium/Staggered Tolerance:** $10^{-6}/10^{-4}$

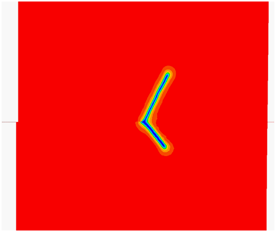

Figure: Symmetric Degradation

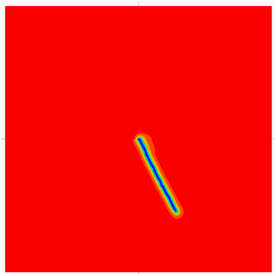

Figure: Asymmetric Degradation

Phase-Field Validation

Ferreira, Marengo, and Perego (2024)

- **Asymmetric Degradation**: Dissipation only under tension

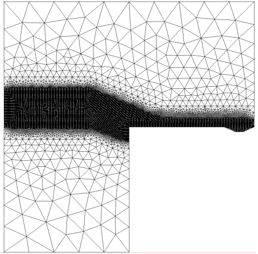

Figure: Problem Discretization

- **E** : $25.85 GPa$
- $\boldsymbol{\nu}$ **:** 0.18
- G_c : $0.095 \ \frac{N}{mm}$
- $\boldsymbol{\ell_0}$: 5.0 mm

- **Nodes:** 21212
- **Elements:** 42106 T3
- **Equilibrium/Staggered Tolerance:** $10^{-6}/10^{-4}$

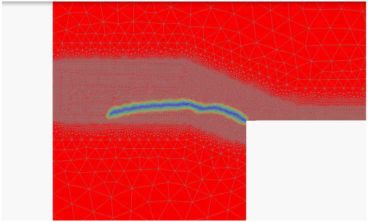

Figure: Damage

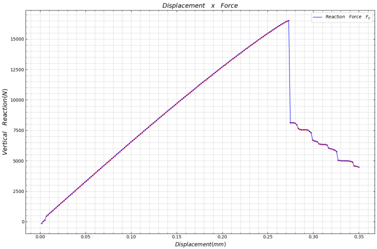

Figure: Vertical Reaction

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