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Direct Calculation for Determinant Main Blocks of 2×2

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Abstract

A method for the calculation of determinants of 4×4 and 5×5 matrixes is developed by using a factoring into products of 2×2 determinants and if it is necessary with a 3×3 matrix factor which makes calculations more straightforward with respect to the *Laplace* Expansion. Some illustrative examples are provided.

Mathematics Subject Classification: 15A15, 15A23, 15A99

Keywords: The *Laplace* Expansion, Determinants

1 Introduction

The calculation of determinants for matrixes of any order is usually based on *Laplace* development better known as the cofactor method. Although this method is very simple to use, as the size of the matrix increases the calculation of its determinant is more and more exhaustive since the number of calculations is growing very rapidly. For example, the number of determinants of 2 2 matrices, which need to be calculated in the expansion of the Determinant of a 3 3 Matrix is only 3. by the way for this case there are some articles that provide some alternatives for its calculation, said works [3], are related to the famous *Sarrus* rule that is applied to this specific case. However, the number of determinants of 2 2 matrices increases to 12, when it comes to calculating the determinant of a 4 4 matrix, as well as 60 for a 5 5 matrix, and 360 determinants of 2 2 matrices For a 6 6 matrix. There are also other works that have proposed alternatives for the calculation of the determinants in general for any matrix size, as is the case in [8] and [4]. There is also the *Laplace* Expansion Theorem, which, although not well known, is a powerful tool for calculating determinants in general [2],[9],[5],[7],[6] and [1] etc. The development of the work is done as follows: in section 2 the *Laplace* Expansion is analyzed and an example is given, according to the book from which the information is taken [2]; in section 3, a method is developed in a very precise way about the calculation of determinants of a 4×4 matrix through products of blocks of determinants for 2×2 matrices, the rules of said method are proposed and an illustrative example is given;In section 4, the validity of the method is extended for higher order matrices where the case of determinants of 5×5 matrices is addressed and the rules to follow for the developed method are also exposed. It is illustrated with an example and at the end of this section a general rule is proposed for the calculation of higher order determinants; An application of the formula for the determinant of a 6×6 matrix is presented in an appendix; at the end the conclusions are presented.

2 The *Laplace* Expansion

There are few works in the literature where the generalization of *Laplace*'s cofactor method can be found. Let's start with the so-called *Laplace* expansion (where the cofactor method is a particular case)

$$|A| = a_1 A_{11} + a_{12}A_{12} + a_{13}A_{13} + ... + a_{1k}A_{1k} =$$

$$= \sum_{k=1}^{n} a_{1k}A_{1k}$$

The information presented below (through a Theorem) is taken from [2], where said result is specified (pages 514 and 515):

Theorem Let $n \in N$. Let $A \in K^{n \times n}$.For any subset $I$of $\{1, 2, 3..., n\}$.we let $\tilde{I}$ denote complement $1, 2, 3..., n$ I of I (For instance, if $n = 4$ and $I = \{1, 4\}$, then $\tilde{I} = \{2, 3\}$.)

a) For every subset P of $1, 2, 3, ..., n$ we have

$$detA = \sum_{\substack{Q \subseteq \{1,2..,n\}; \\ |Q| = |P|}} (-1)^{\sum P + \sum Q} det(sub^{w(Q)}_{w(P)} A) det(sub^{w(\tilde{Q})}_{w(\tilde{P})} A)$$

b) For every subset Q of $1, 2, 3, ..., n$ we have

$$detA = \sum_{\substack{P \subseteq \{1,2..,n\}; \\ |P| = |Q|}} (-1)^{\sum P + \sum Q} det(sub^{w(Q)}_{w(P)} A) det(sub^{w(\tilde{Q})}_{w(\tilde{P})} A)$$

An example is also shown in [2] for the case of the determinant of a 4×4 matrix. It should be noted that although the way in which the calculations of this 4×4 determinant are reduced is interesting, this book does not give examples for determinants of larger matrices and it also shows how a bit cumbersome the location of the signs for each product that appears as an addend in the given example. The proof of the theorem appears in [9] page 49. This expression also appears in a more simplified form [1] page 33. In this more "simple" expression, although unlike the previous reference, the quantity of products of determinants of 2 2 matrices involved is given there through of a combinatorial formula.

3 Calculation of the Determinant for a 4×4 matrix

In this section we develop a simple method for calculating the determinant of a 4×4 matrix, which allows us to decompose this determinant into a sum of products of determinants of 2×2 matrices. This method allows in section 3.1 to establish a series of rules which facilitate the understanding of it.

Applying the cofactors method to the 4×4 matrix, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} +$$

.

$$+a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

$$= a_{11}(a_{22} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - a_{23} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} + a_{24} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix})$$

$$-a_{12}(a_{21} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - a_{23} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + a_{24} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix})$$

$$+a_{13}(a_{21} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - a_{22} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + a_{24} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix})$$

$$-a_{14}(a_{21} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} - a_{22} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + a_{23} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix})$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - a_{11}a_{23} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} + a_{11}a_{24} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$

$$-a_{12}a_{21} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + a_{12}a_{23} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - a_{12}a_{24} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}$$

$$+a_{13}a_{21} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - a_{13}a_{22} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + a_{13}a_{24} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

$$-a_{14}a_{21} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + a_{14}a_{22} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} - a_{14}a_{23} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

This is the explicit expression for the 4×4 determinant. However if we are careful we can observe the following: there are determinants that are repeated in pairs. For this reason, we will "label" them.

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}^{I} - a_{11}a_{23} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}^{II} + a_{11}a_{24} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}^{III}$$

$$-a_{12}a_{21} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}^{I} + a_{12}a_{23} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}^{V} - a_{12}a_{24} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}^{IV}$$

$$+a_{13}a_{21} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}^{II} - a_{13}a_{22} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}^{V} + a_{13}a_{24} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}^{VI}$$

$$-a_{14}a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}^{III} + a_{14}a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}^{IV} - a_{14}a_{23}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}^{VI}$$

So when factoring each of these six main determinants, we have

$$= (a_{11}a_{22} - a_{12}a_{21})\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}^{I} + (a_{13}a_{21} - a_{11}a_{23})\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}^{II}$$

$$+(a_{11}a_{24} - a_{14}a_{21})\begin{vmatrix} +a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}^{III} + (a_{14}a_{22} - a_{12}a_{24})\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}^{IV}$$

$$+(a_{12}a_{23} - a_{13}a_{22})\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}^{V} + (a_{13}a_{24} - a_{14}a_{23})\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}^{VI}$$

That is,

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

It can also be expressed in the form

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$$

$$- \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

That is:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \sum_{k_1=2}^{4}(-1)^{k_1}\begin{vmatrix} a_{11} & a_{1k_1} \\ a_{21} & a_{2k_1} \end{vmatrix}|A_{k_1}|$$

$$+\sum_{k_2=3}^{4}(-1)^{k_2+1}\begin{vmatrix} a_{12} & a_{1k_2} \\ a_{22} & a_{2k_2} \end{vmatrix}|A_{k_2}| + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \qquad (1)$$

where $|A_{k_1}|$ are the determinants of the complementary matrices, for $k_1 = 2, 3, 4$, respectively:

$$\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$

As well as the $|A_{k_2}|$ are the determinants of the complementary matrices, for $k_2 = 3, 4$, respectively:

$$\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}$$

3.1 Rules for Calculating the Determinant of 4×4 Matrices

Given the 4×4 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

The calculation of its determinant applying our method is reduced to the following:

First.- The number of matrices given by (1), can be calculated from the formula of combinations $C_k^n = \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!}k!$, with $n = 4, k = 2$. We will have

$$C_2^4 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{4!}{(4-3)!2!} = \frac{(4)(3)2!}{2!2!} = (3)(2) = 6 \tag{2}$$

Second.- These are obtained in the following way.

The first upper principal semi-vector is defined as

$$C_1^{Sup} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

which is combined with the following higher semi-vectors

$$\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}$$

to form 2×2 matrices, whose determinant is multiplied by the determinants of their respective complementary matrices

$$\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}, \begin{pmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{pmatrix}, \begin{pmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{pmatrix}$$

That is, the different combinations of the products for this first semi-vector will be

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$

while this is how the second higher principal semi-vectors is defined:

$$C_1^{Sup} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

combines consecutively with the higher semi-vectors

$$\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}$$

to form the following 2 2 matrices, whose determinant is multiplied by the determinants of their respective complementary matrices

$$\begin{pmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{pmatrix}, \begin{pmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{pmatrix}$$

That is,

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}$$

finally the third higher principal semi-vector

$$C_2^{Sup} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

generates the product's only product by applying the same algorithm

$$\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

Third.- The signs in each addend (products of 2×2 matrices) generated by each of the main semi-vectors, will change in an iterative way, always starting with the plus sign, facilitating the calculation of the correct sign, in contrast to the *Laplace* expansion.

Fourth.- All main matrices will always be different from complementary matrices

Observations:

1.- The 6 products of 2×2 determinants are constructed more directly corresponding to the development of the determinant for the 4×4.
2.- With this method, unlike the *Laplace* expansion, the signs associated with each addend can be obtained very easily with the third rule established in this method.
3.- A simple path is given for the construction of the products of determinants of the involved 2×2 matrices.
4.- It is important to note that the first rule indicates how to find the total amount of products involved by means of the C_k^n combining formula. We obtained this same result on our own by doing a careful analysis when we developed this method.

It is very important to note that this developed method can be generalized to higher order matrices and thus reduce the number of calculations for the determinant of said matrix. In the following sections we will analyze the calculation for the specific case of a 5×5 matrix.

3.2 Example

Let's calculate the Determinant of the following Matrix

$$\begin{vmatrix} 2 & -3 & 6 & -4 \\ 5 & 7 & -8 & 11 \\ 9 & 12 & -13 & -1 \\ 10 & -6 & 3 & 5 \end{vmatrix}$$

Applying the mechanism generated in this section, we have

$$\begin{vmatrix} 2 & -3 & 6 & -4 \\ 5 & 7 & -8 & 11 \\ 9 & 12 & -13 & -1 \\ 10 & -6 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}\begin{vmatrix} -13 & -1 \\ 3 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 6 \\ 5 & -8 \end{vmatrix}\begin{vmatrix} 12 & -1 \\ -6 & 5 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & -4 \\ 5 & 11 \end{vmatrix}\begin{vmatrix} 12 & -13 \\ -6 & 3 \end{vmatrix} + \begin{vmatrix} -3 & 6 \\ 7 & -8 \end{vmatrix}\begin{vmatrix} 9 & -1 \\ 10 & 5 \end{vmatrix} - \begin{vmatrix} -3 & -4 \\ 7 & 11 \end{vmatrix}\begin{vmatrix} 9 & -13 \\ 10 & 3 \end{vmatrix}$$

$$-\begin{vmatrix} 6 & -4 \\ -8 & 11 \end{vmatrix}\begin{vmatrix} 9 & -12 \\ 10 & -6 \end{vmatrix} = [(2)(7)-(5)(-3)][(-13)(5)-(3)(-1)]$$

$$-[(2)(-8)-(6)(5)][(12)(5)-(-6)(-1)]+[(2)(11)-(5)(-4)][(12)(3)-(-6)(-13)]$$

$$+[(-3)(-8)-(7)(6)][(9)(5)-(10)(-1)]-[(-3)(11)-(7)(-4)][(9)(3)-(10)(-13)]$$

$$+[(6)(11)-(-8)(-4)][(9)(-6)-(10)(-12)] = -1798+2484-1764-990+$$

$$785-5916 = -7199$$

4 Application of the Method for the Determinant of a 5×5 matrix

As in the case of the previous section, it will be extended to the development of the method, but now applied to the calculation of the determinants of Matrices of 5×5. It should be noted that in this case the decomposition of the determinant of the 5×5 matrix, remains in terms of the sum of products of determinants of matrices of 2×2, by determinants of 3×3 matrices, which can be easily calculated by applying the *Sarrus* rule.

Let the matrix 5×5 be next

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} =$$

When applying the development of cofactors, one has to

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{53} & a_{54} & a_{55} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{54} & a_{55} \end{vmatrix} -$$

$$a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{55} \end{vmatrix} + a_{15}\begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix} =$$

$$=\mathrm{a}_{11}\left(a_{22}\begin{vmatrix} a_{33} & a_{34} & a_{35}\\ a_{43} & a_{44} & a_{45}\\ a_{53} & a_{54} & a_{55}\end{vmatrix} -a_{23}\begin{vmatrix} a_{32} & a_{34} & a_{35}\\ a_{42} & a_{44} & a_{45}\\ a_{52} & a_{54} & a_{55}\end{vmatrix} +a_{24}\begin{vmatrix} a_{32} & a_{33} & a_{35}\\ a_{42} & a_{43} & a_{45}\\ a_{52} & a_{53} & a_{55}\end{vmatrix} -a_{25}\begin{vmatrix} a_{32} & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\\ a_{52} & a_{53} & a_{54}\end{vmatrix}\right)$$

$$-\mathrm{a}_{12}\left(a_{21}\begin{vmatrix} a_{33} & a_{34} & a_{35}\\ a_{43} & a_{44} & a_{45}\\ a_{53} & a_{54} & a_{55}\end{vmatrix} -a_{23}\begin{vmatrix} a_{31} & a_{34} & a_{35}\\ a_{41} & a_{44} & a_{45}\\ a_{51} & a_{54} & a_{55}\end{vmatrix} +a_{24}\begin{vmatrix} a_{31} & a_{33} & a_{35}\\ a_{41} & a_{43} & a_{45}\\ a_{51} & a_{53} & a_{55}\end{vmatrix} -a_{25}\begin{vmatrix} a_{31} & a_{33} & a_{34}\\ a_{41} & a_{43} & a_{44}\\ a_{51} & a_{53} & a_{54}\end{vmatrix}\right)$$

$$+\,\mathrm{a}_{13}\left(a_{21}\begin{vmatrix} a_{32} & a_{34} & a_{35}\\ a_{42} & a_{44} & a_{45}\\ a_{52} & a_{54} & a_{55}\end{vmatrix} -a_{22}\begin{vmatrix} a_{31} & a_{34} & a_{35}\\ a_{41} & a_{44} & a_{45}\\ a_{51} & a_{54} & a_{55}\end{vmatrix} +a_{24}\begin{vmatrix} a_{31} & a_{32} & a_{35}\\ a_{41} & a_{42} & a_{45}\\ a_{51} & a_{52} & a_{55}\end{vmatrix} -a_{25}\begin{vmatrix} a_{31} & a_{32} & a_{34}\\ a_{41} & a_{42} & a_{44}\\ a_{51} & a_{52} & a_{54}\end{vmatrix}\right)$$

$$-\,\mathrm{a}_{14}\left(a_{21}\begin{vmatrix} a_{32} & a_{33} & a_{35}\\ a_{42} & a_{43} & a_{45}\\ a_{52} & a_{54} & a_{55}\end{vmatrix} -a_{22}\begin{vmatrix} a_{31} & a_{33} & a_{35}\\ a_{41} & a_{43} & a_{45}\\ a_{51} & a_{53} & a_{55}\end{vmatrix} +a_{23}\begin{vmatrix} a_{31} & a_{32} & a_{35}\\ a_{41} & a_{42} & a_{45}\\ a_{51} & a_{52} & a_{55}\end{vmatrix} -a_{25}\begin{vmatrix} a_{31} & a_{32} & a_{33}\\ a_{41} & a_{42} & a_{43}\\ a_{51} & a_{52} & a_{53}\end{vmatrix}\right)$$

$$+\,\mathrm{a}_{15}\left(a_{21}\begin{vmatrix} a_{32} & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\\ a_{52} & a_{54} & a_{54}\end{vmatrix} -a_{22}\begin{vmatrix} a_{31} & a_{33} & a_{34}\\ a_{41} & a_{43} & a_{44}\\ a_{51} & a_{53} & a_{54}\end{vmatrix} +a_{23}\begin{vmatrix} a_{31} & a_{32} & a_{34}\\ a_{41} & a_{42} & a_{44}\\ a_{51} & a_{52} & a_{54}\end{vmatrix} -a_{24}\begin{vmatrix} a_{31} & a_{32} & a_{33}\\ a_{41} & a_{42} & a_{43}\\ a_{51} & a_{52} & a_{53}\end{vmatrix}\right)$$

After some simple calculations and factorizations, it is not difficult to arrive at the following expression

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})\begin{vmatrix} a_{33} & a_{34} & a_{35}\\ a_{43} & a_{44} & a_{45}\\ a_{53} & a_{54} & a_{55}\end{vmatrix}$$

$$+(a_{13}a_{21} - a_{11}a_{23})\begin{vmatrix} a_{32} & a_{34} & a_{35} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix} + (a_{11}a_{24} - a_{14}a_{21})\begin{vmatrix} a_{32} & a_{33} & a_{35} \\ a_{42} & a_{43} & a_{45} \\ a_{52} & a_{53} & a_{55} \end{vmatrix} +$$

$$+(a_{15}a_{21} - a_{11}a_{25})\begin{vmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{52} & a_{53} & a_{54} \end{vmatrix} + (a_{14}a_{22} - a_{12}a_{24})\begin{vmatrix} a_{31} & a_{33} & a_{35} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{vmatrix} +$$

$$+(a_{12}a_{25} - a_{15}a_{22})\begin{vmatrix} a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix} + (a_{12}a_{23} - a_{13}a_{22})\begin{vmatrix} a_{31} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix} +$$

$$+(a_{13}a_{24} - a_{14}a_{23})\begin{vmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{vmatrix} + (a_{15}a_{23} - a_{13}a_{25})\begin{vmatrix} a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \\ a_{51} & a_{52} & a_{54} \end{vmatrix} +$$

$$+(a_{14}a_{25} - a_{15}a_{24})\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{vmatrix}$$

what is equivalent to

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} & a_{35} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{32} & a_{33} & a_{35} \\ a_{42} & a_{43} & a_{45} \\ a_{52} & a_{53} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{15} \\ a_{21} & a_{25} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} & a_{34} \\ a_{42} & a_{44} & a_{44} \\ a_{52} & a_{54} & a_{54} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} & a_{35} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{12} & a_{15} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{vmatrix} +$$

$$- \begin{vmatrix} a_{13} & a_{15} \\ a_{23} & a_{25} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \\ a_{51} & a_{52} & a_{54} \end{vmatrix} + \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{vmatrix}$$

that is, the calculation of the determinant of a matrix of 5×5, reduces to

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{vmatrix}
= \sum_{k_1=2}^{5} (-1)^{k_1} \begin{vmatrix} a_{11} & a_{1k_1} \\ a_{21} & a_{2k_1} \end{vmatrix} |A_{k_1}|
$$

$$
+ \sum_{k_2=3}^{5} (-1)^{k_2+1} \begin{vmatrix} a_{12} & a_{1k_2} \\ a_{22} & a_{2k_2} \end{vmatrix} |A_{k_2}| + \sum_{k_3=4}^{5} (-1)^{k_3} \begin{vmatrix} a_{13} & a_{1k_3} \\ a_{23} & a_{2k_3} \end{vmatrix} |A_{k_3}|
$$

$$
+ \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \tag{3}
$$

Where $|A_{k_1}|$, are the determinants of the complementary matrices, for $k_1 = 2, 3, 4, 5$, respectively:

$$
\begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix},
\begin{vmatrix} a_{32} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix},
\begin{vmatrix} a_{32} & a_{33} & a_{35} \\ a_{42} & a_{43} & a_{45} \\ a_{52} & a_{53} & a_{55} \end{vmatrix},
\begin{vmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{52} & a_{53} & a_{54} \end{vmatrix}
$$

as well as the $|A_{k_2}|$, are the determinants of the complementary matrices, for $k_2 = 3, 4, 5$, respectively:

$$
\begin{vmatrix} a_{31} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix},
\begin{vmatrix} a_{31} & a_{33} & a_{35} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{vmatrix},
\begin{vmatrix} a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix}
$$

and finally the $|A_{k_3}|$, are the determinants of the complementary matrices, for $k_2 = 4, 5$ respectively:

$$
\begin{vmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{vmatrix},
\begin{vmatrix} a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \\ a_{51} & a_{52} & a_{54} \end{vmatrix}
$$

4.1 Rules for calculating the determinant of 5×5 Matrices

The calculation of the determinant for the case of the 5×5 matrix, applying our method, is reduced to the following:

Given the matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}$$

First.- The number of 2×2 matrices to be constructed is given by the following relation

$$C_2^5 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{5!}{(5-2)!2!} = \frac{(5)(4)3!}{3!2!} = (5)(2) = 10 \tag{4}$$

Second.- These are obtained in the following way.

The first higher principal semi-vector is defined as

$$C_1^{Sup} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

which will be combined with the consecutive semi-vectors

$$\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}, \begin{pmatrix} a_{15} \\ a_{25} \end{pmatrix}$$

in pairs, to form the first 2×2 matrices, whose determinant must be multiplied by the respective determinants of the complementary matrices

$$= \begin{pmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{pmatrix}, \begin{pmatrix} a_{32} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{pmatrix}, \begin{pmatrix} a_{32} & a_{33} & a_{35} \\ a_{42} & a_{43} & a_{45} \\ a_{52} & a_{53} & a_{55} \end{pmatrix}$$

$$, \begin{pmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{52} & a_{53} & a_{54} \end{pmatrix}$$

for obtaining

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix} +$$

$$\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} & a_{35} \\ a_{42} & a_{43} & a_{45} \\ a_{52} & a_{53} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{15} \\ a_{21} & a_{25} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{52} & a_{53} & a_{54} \end{vmatrix}$$

now, the second higher principal semi-vector is defined as:

$$C_1^{Sup} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

which will be combined with the higher semi-vectors

$$\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}, \begin{pmatrix} a_{15} \\ a_{25} \end{pmatrix}$$

in pairs again to form the following 2 2 matrices, whose determinant must be multiplied by the respective determinants of the complementary matrices

$$\begin{pmatrix} a_{31} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{pmatrix}, \begin{pmatrix} a_{31} & a_{33} & a_{35} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{pmatrix}, \begin{pmatrix} a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{pmatrix}$$

for obtaining

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} & a_{35} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} & a_{35} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{15} \\ a_{22} & a_{25} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix}$$

Now, the third higher principal semi-vector is defined

$$C_2^{Sup} = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

to combine with the higher semi-vectors

$$\begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}, \begin{pmatrix} a_{15} \\ a_{25} \end{pmatrix}$$

whose determinant is multiplied by the determinants of the complementary matrices

$$\begin{pmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{pmatrix}, \begin{pmatrix} a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \\ a_{51} & a_{52} & a_{54} \end{pmatrix}$$

for obtaining

$$\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{13} & a_{15} \\ a_{23} & a_{25} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \\ a_{51} & a_{52} & a_{54} \end{vmatrix}$$

and finally the fourth higher principal semi-vector is defined

$$C_3^{Sup} = \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}$$

to obtain, according to the above

$$\begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{vmatrix}$$

Third.- The signs in each addend (products of 2×2 matrices) generated by each of the principal semi-vectors, will change in an iterative way, always starting with the plus sign.
Fourth.- All the main matrices will always be foreign to the complementary matrices.

In the same way as in the case of the calculation of the determinant of the 4×4 matrices, the simplification in the calculation of the sign is also highlighted, for the product of the determinants of the 2×2 matrices and the 10 products of the determinants of the 2×2 by 3×3 matrices are constructed in this way, which correspond to the determinant of the 5×5 matrix

4.2 Example

Next we will apply the technique to calculate the determinant of the following 5×5 matrix

$$\begin{vmatrix} -10 & 2 & -15 & 21 & -5 \\ -11 & 14 & -13 & 24 & 2 \\ -3 & 12 & 20 & -14 & 30 \\ 17 & 2 & 7 & 17 & 5 \\ 3 & 6 & -21 & -19 & 18 \end{vmatrix} \tag{5}$$

Then

$$\begin{vmatrix} -10 & 2 & -15 & 21 & -5 \\ -11 & 14 & -13 & 24 & 2 \\ -3 & 12 & 20 & -14 & 30 \\ 17 & 2 & 7 & 17 & 5 \\ 3 & 6 & -21 & -19 & 18 \end{vmatrix} = \begin{vmatrix} -10 & 2 \\ -11 & 14 \end{vmatrix} \begin{vmatrix} 20 & -14 & 30 \\ 7 & 17 & 5 \\ -21 & -19 & 18 \end{vmatrix}$$

$$- \begin{vmatrix} -10 & -15 \\ -11 & -13 \end{vmatrix} \begin{vmatrix} 12 & -14 & 30 \\ 2 & 17 & 5 \\ 6 & -19 & 18 \end{vmatrix} + \begin{vmatrix} -10 & 21 \\ -11 & 24 \end{vmatrix} \begin{vmatrix} 12 & 20 & 30 \\ 2 & 7 & 5 \\ 6 & -21 & 18 \end{vmatrix}$$

$$-\begin{vmatrix} -10 & -5 \\ -11 & 2 \end{vmatrix}\begin{vmatrix} 12 & 20 & -14 \\ 2 & 7 & 17 \\ 6 & -21 & -19 \end{vmatrix}+\begin{vmatrix} 2 & -15 \\ 14 & -13 \end{vmatrix}\begin{vmatrix} -3 & -14 & 30 \\ 17 & 17 & 5 \\ 3 & -19 & 18 \end{vmatrix}$$

$$-\begin{vmatrix} 2 & 21 \\ 14 & 24 \end{vmatrix}\begin{vmatrix} -3 & 20 & 30 \\ 17 & 7 & 5 \\ 3 & -21 & 18 \end{vmatrix}+\begin{vmatrix} 2 & -5 \\ 14 & 2 \end{vmatrix}\begin{vmatrix} -3 & 20 & -14 \\ 17 & 7 & 17 \\ 3 & -21 & -19 \end{vmatrix}$$

$$+\begin{vmatrix} -15 & 21 \\ -13 & 24 \end{vmatrix}\begin{vmatrix} -3 & 12 & 30 \\ 17 & 2 & 5 \\ 3 & 6 & 18 \end{vmatrix}-\begin{vmatrix} -15 & -5 \\ -13 & 2 \end{vmatrix}\begin{vmatrix} -3 & 12 & -14 \\ 17 & 2 & 17 \\ 3 & 6 & -19 \end{vmatrix}$$

$$+\begin{vmatrix} 21 & -5 \\ 24 & 2 \end{vmatrix}\begin{vmatrix} -3 & 12 & 20 \\ 17 & 2 & 7 \\ 3 & 6 & -21 \end{vmatrix}=(-118)(17974)+(184)(-8349)+(-87)(-630)$$

$$+(162)(6708)-(-75)(6664)-(-35)(696)+(-9)(132)-(-246)(-17853)+$$

$$(74)(12100)-(-95)(3564)=-5150528 \tag{6}$$

4.3 General Expression

The results obtained in the previous sections, where it is shown that the mechanism developed for the cases of the determinants 4×4 and 5×5, was also achieved for 6×6 and 7×7 (which are not presented in this work because the calculations are extensive), confirming that formulas (1) and (3) established in sections 3 and 4 are valid, for which we dare to propose a general formula.

Let A be an $n\times n$ matrix, then it is fulfilled that

$$|A|=\sum_{i_1=2}^{n}(-1)^i\begin{vmatrix} a_{1,1} & a_{1,i} \\ a_{2,1} & a_{2,i} \end{vmatrix}|A_{i_1}|+\sum_{i_2=3}^{n-1}(-1)^{i+1}\begin{vmatrix} a_{1,2} & a_{1,i} \\ a_{2,2} & a_{2,j} \end{vmatrix}|A_{i_2}|+\sum_{i_3=4}^{n-3}(-1)^i\begin{vmatrix} a_{1,3} & a_{1,i} \\ a_{2,3} & a_{2,i} \end{vmatrix}|A_{i_3}|$$

$$+\ldots\sum_{i_{n-2}=n-1}^{2}(-1)^{i+1}\begin{vmatrix} a_{1,n-3} & a_{1,i} \\ a_{2,n-3} & a_{2,i} \end{vmatrix}|A_{i_{n-1}}|+\begin{vmatrix} a_{1,n-1} & a_{1,n} \\ a_{2,n-1} & a_{2,n} \end{vmatrix}|A_{i_n}|$$

where $A_{i_1}, A_{i_2}, A_{i_3}, \ldots, A_{i_{n-1}}, A_{i_n}$ correspond to the complementary matrices for each product, respectively.

In the Appendix **A**, an example of the calculation for the determinant of 6×6 is given, as an application of this general rule.

5 Conclusion

Although *Laplace*'s expansion theorem provides a powerful mechanism for the computation of determinants of any size, According to our experience of many years teaching the course of Linear Algebra, the use of the formula itself could be confusing for an Engineering student. In addition to the fact that the determinants that students usually calculate in an 'algebra course are at most for 6×6 matrices, and in that sense we believe that student motivation with respect to this topic could be improved by using our method. On the other hand, as can be seen in this algorithm for the decomposition of the determinant, it can be generalized for an n matrix, as we propose in section 4.3 . In fact, in appendix A, the calculation is made for a concrete example for the case of a 6×6 matrix and this is achieved by applying the same rules described in sections 3.1 and 4.1. This result was verified in the Mathematica package verifying that the method works for higher order determinants, although in this article the proof was not made for the n case.

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References

[1] Frank Ayres, Jr. *Theory and Problemss of Matrices*, Schaums Outline Series, McGraw-Hill book company, Capitulo 4to, pag.33.

[2] Darij Grinberg, *Notes on the Combinatorial Fundamentals of Algebra*, May 25 2021, pags. 514-515.

[3] Dardan Hajrizaj, New Method Computhe the Determinant of a 3x3 Matrix, *International Journal of Algebra*, **3** (2009), no. 5, 2111-219.

[4] Yangkok Kim and Nirutt Pipattanajinda, New method for finding the determinant of a matrix, *Chamchuri Journal of Mathematics*, **9** (2017), 1-12.

[5] Viktor V. Prasolov, *Problems and Theorems in Linear Algebra*, Translations of Mathematical Monographs, Vol. 134, AMS 1994. https://doi.org/10.1090/mmono/134

[6] Karol Pak and Andrzej Trybulec, Laplace Expansion, *Formalized Mathematics*, **15** (2007), no. 3, 143150. https://doi.org/10.2478/v10037-007-0016-5

[7] Mirko Radic and Rene Susanj, On Determinants of rectangular matrices wichc have Laplaceś Expansion along rows, *Glasnik Mathematik*, **47** (2012), no. 67, 175-180. https://doi.org/10.3336/gm.47.1.15

[8] M.G.Sobamowo, On the extension of Sarrus'Rule to $nxn(n \geq 3)$ Matrices: Development of New Method for the Computation of the Determinant of 4x4 Matrix, *International Journal of Engineering Mathematics*, **2016** (2016), 1-14. https://doi.org/10.1155/2016/9382739

[9] S. Gill Williamson, *Matrix Canonical Forms, Notational Skills and Proof Techniques*, 2012, pag. 49.

A Appendix

Application example for the determinant of a 6×6 matrix. First, following the same mechanism, the number of 2×2 matrices that can be obtained in said factorization is

$$C_2^6 = \left(\begin{array}{c} 6 \\ 2 \end{array} \right) = \frac{6!}{(6-2)!2!} = \frac{(6)(5)4!}{4!2!} = \frac{30}{2} = 15$$

Let's calculate

$$\begin{vmatrix} 1 & -1 & 3 & -2 & 5 & -3 \\ 2 & 3 & 4 & 1 & 6 & 2 \\ -3 & 6 & 1 & 2 & -1 & 4 \\ 3 & 2 & -4 & 5 & 3 & -2 \\ 4 & 3 & 6 & -6 & 1 & 3 \\ 5 & 1 & -2 & 5 & 2 & 4 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \begin{vmatrix} 1 & 2 & -1 & 4 \\ -4 & 5 & 3 & -2 \\ 6 & -6 & 1 & 3 \\ -2 & 5 & 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 6 & 2 & -1 & 4 \\ 2 & 5 & 3 & -2 \\ 3 & -6 & 1 & 3 \\ 1 & 5 & 2 & 4 \end{vmatrix} +$$

$$\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 6 & 1 & -1 & 4 \\ 2 & -4 & 3 & -2 \\ 3 & 6 & 1 & 3 \\ 1 & -2 & 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} \begin{vmatrix} 6 & 1 & 2 & 4 \\ 2 & -4 & 5 & -2 \\ 3 & 6 & -6 & 3 \\ 1 & -2 & 5 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} 6 & 1 & 2 & -1 \\ 2 & -4 & 5 & 3 \\ 3 & 6 & -6 & 1 \\ 1 & -2 & 5 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} -3 & 2 & -1 & 4 \\ 3 & 5 & 3 & -2 \\ 4 & -6 & 1 & 3 \\ 5 & 5 & 2 & 4 \end{vmatrix}$$

$$
-\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}\begin{vmatrix} -3 & 1 & -1 & 4 \\ 3 & -4 & 3 & -2 \\ 4 & 6 & 1 & 3 \\ 5 & 2 & 2 & 4 \end{vmatrix}+\begin{vmatrix} -1 & 5 \\ 3 & 6 \end{vmatrix}\begin{vmatrix} -3 & 1 & 2 & 4 \\ 3 & -4 & 5 & -2 \\ 4 & 6 & -6 & 3 \\ 5 & 2 & 5 & 4 \end{vmatrix}
$$

$$
-\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix}\begin{vmatrix} -3 & 1 & 2 & -1 \\ 3 & -4 & 5 & 3 \\ 4 & 6 & -6 & 1 \\ 5 & 2 & 5 & 2 \end{vmatrix}+\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix}\begin{vmatrix} -3 & 6 & -1 & 4 \\ 3 & 2 & 3 & -2 \\ 4 & 3 & 1 & 3 \\ 5 & 1 & 2 & 4 \end{vmatrix}
$$

$$
-\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix}\begin{vmatrix} -3 & 6 & 2 & 4 \\ 3 & 2 & 5 & -2 \\ 4 & 3 & -6 & 3 \\ 5 & 1 & 5 & 4 \end{vmatrix}+\begin{vmatrix} 3 & -3 \\ 4 & 2 \end{vmatrix}\begin{vmatrix} -3 & 6 & 2 & -1 \\ 3 & 2 & 5 & 3 \\ 4 & 3 & -6 & 1 \\ 5 & 1 & 5 & 2 \end{vmatrix}
$$

$$
+\begin{vmatrix} -2 & 5 \\ 1 & 6 \end{vmatrix}\begin{vmatrix} -3 & 6 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 4 & 3 & 6 & 3 \\ 5 & 1 & 2 & 4 \end{vmatrix}-\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} -3 & 6 & 1 & -1 \\ 3 & 2 & -4 & 3 \\ 4 & 3 & 6 & 1 \\ 5 & 1 & 2 & 2 \end{vmatrix}
$$

$$
+\begin{vmatrix} 5 & -3 \\ 6 & 2 \end{vmatrix}\begin{vmatrix} -3 & 6 & 1 & 2 \\ 3 & 2 & -4 & 5 \\ 4 & 3 & 6 & -6 \\ 5 & 1 & 2 & 5 \end{vmatrix}
$$

$$
\begin{gathered}
= (5)(183) + (-13)(-507) + (11)(262) + (-17)(-1669) + (28)(-1824)- \\
(-2)(1077) + (5)(-787) - (-4)(-5) + (8)(392) - (5)(415) + (-21)(396)- \\
-(7)(-296) + (-2)(2289) - (18)(-656) - (-1)(384) = 1819 \qquad (7)
\end{gathered}
$$

Below we present the explicit development of only the first two addends, as an illustration of the determinant of the 4×4 matrices involved in the process of this calculation, and which also serves as an example of how the same mechanism generated in section 3.

$$
\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}\begin{vmatrix} 1 & 2 & -1 & 4 \\ -4 & 5 & 3 & -2 \\ 6 & -6 & 1 & 3 \\ -2 & 5 & 2 & 4 \end{vmatrix} =
$$

$$\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \left(\begin{array}{l} \begin{vmatrix} 1 & 2 \\ -4 & 5 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} \begin{vmatrix} 6 & 3 \\ -2 & 4 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} \begin{vmatrix} 6 & -6 \\ -2 & 5 \end{vmatrix} \\ - \begin{vmatrix} 1 & -1 \\ -4 & 3 \end{vmatrix} \begin{vmatrix} -6 & 3 \\ 5 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ -4 & -2 \end{vmatrix} \begin{vmatrix} -6 & 1 \\ 5 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} \begin{vmatrix} 6 & 1 \\ -2 & 2 \end{vmatrix} \end{array} \right)$$

$$= (5)(183) = 915$$

and

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 & -1 & 4 \\ -4 & 5 & 3 & -2 \\ 6 & -6 & 1 & 3 \\ -2 & 5 & 2 & 4 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \left(\begin{array}{l} \begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -1 \\ 2 & 3 \end{vmatrix} \begin{vmatrix} -6 & 3 \\ 5 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 4 \\ 2 & -2 \end{vmatrix} \begin{vmatrix} -6 & 1 \\ 5 & 2 \end{vmatrix} \\ - \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} \begin{vmatrix} 3 & -6 \\ 1 & 5 \end{vmatrix} \end{array} \right)$$

$$= (-13)(-507) = 6591$$

the result (7) was verified in the **Mathematica** package and confirmed.

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