Predictive Analytics

Module 11: Forecasting

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Discipline of Business Analytics, The University of Sydney Business School

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1. Problem definition
2. Time series patterns
3. Simple forecasting methods
4. Model diagnostics
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Time series

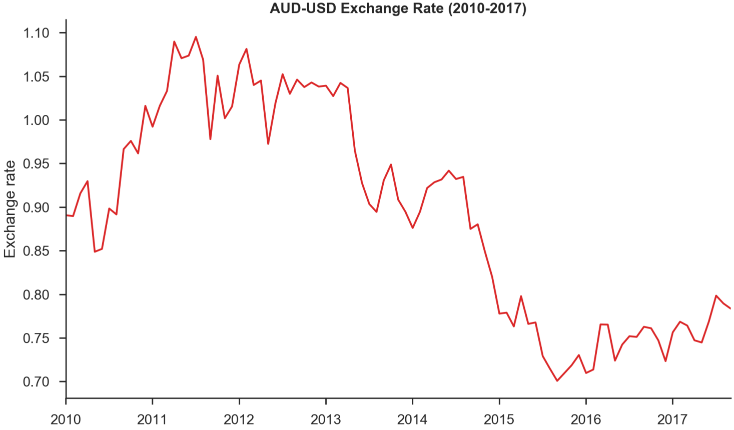
A **time series** is a set of observations $y_1, y_2, \ldots, y_t$ ordered in time.

Examples:

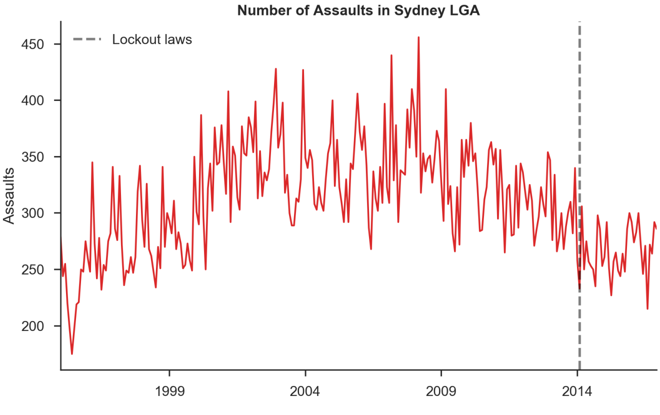
- Weekly unit sales of a product.
- Unemployment rate in Australia each quarter.
- Daily production levels of a product.
- Average annual temperature in Sydney.
- 5 minute prices for CBA stock on the ASX.

Example: visitor Arrivals in Australia

Example: AUD/USD exchange rate



Example: assaults in Sydney



Forecasting

A **forecast** is a prediction about future events and conditions given all current information, including historical data and knowledge of any future events that might impact these events.

The act of making such predictions is called **forecasting**. Forecasting informs business and economic decision making, planning, government policy, etc.

Examples

- Governments need to forecast unemployment, interest rates, expected revenues from income taxes to formulate policies.
- Retail stores need to forecast demand to control inventory levels, hire employees and provide training.
- Banks/investors/financial analysts need to forecast financial returns, risk or volatility, market 'timing'.
- University administrators need to forecast enrollments to plan for facilities and for faculty recruitment.
- Sports organisations need to project sports performance, crowd figures, club gear sales, revenues, etc. in the coming season.

Forecasting in business

Different problems lead to different approaches under the umbrella of forecasting.

- Quantitative (data based) forecasting (our focus in this unit).
- Qualitative (judgmental) forecasting.
- Prediction markets.
- Common approach: judgmentally adjusted statistical forecasting.

Problem definition

Forecasting

Our objective is to predict the value of a time indexed response variable at a future point $t+h$, given the observed series until the present point t. That is, we want to predict Y_{t+h} given $y_1, y_2, \ldots, y_t$, where h is the **forecast horizon**.

We can extend this setting to allow for the presence of predictors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t$, leading to a **dynamic regression** problem.

Decision theory

We denote a **point forecast** as $\widehat{Y}_t = f(Y_{1:t})$. As before, we assume a squared error loss function:

$$L(Y_{t+h}, f(Y_{1:t}) = (Y_{t+h} - f(Y_{1:t}))^2$$

We use the slice notation $Y_{1:t}$ as a compact way to write $Y_1, \ldots, Y_t$.

Point forecasting (key concept)

Using the arguments from earlier earlier in the unit, the optimal point forecast under the squared error loss is the conditional expectation:

$$f(Y_{1:t}) = E(Y_{t+h}|Y_{1:t})$$

Our objective is therefore to approximate the conditional expectation of Y_{t+h} given the historical data, possible for multiple values of h.

Interval forecasting (key concept)

Uncertainty quantification is an essential for business forecasting.

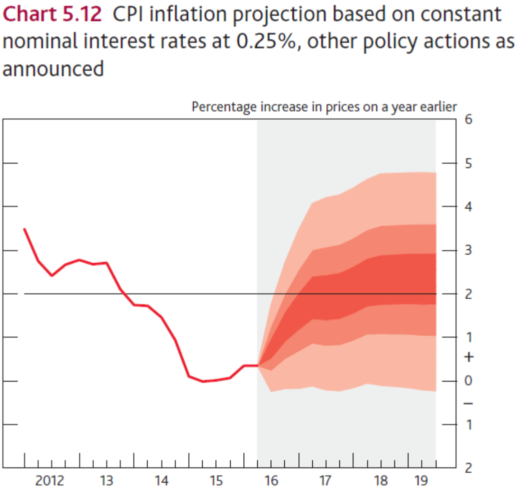
A **density forecast** $\widehat{p}(Y_{t+h}|y_1, \ldots, y_t)$ is an estimate of the entire conditional density $p(Y_{t+h}|y_1, \ldots, y_t)$.

An **interval forecast** is an interval $(\widehat{y}_{t+h,L}, \widehat{y}_{t+h,U})$ such that $\widehat{P}(\widehat{y}_{t+h,L} < Y_{t+h} < y_{t+h,U}) = 1 - \alpha$.

Fan chart (key concept)

- For consecutive forecast horizons, construct prediction intervals for different probability levels (say, 75%, 90%, and 99%) and plot the using different shades.
- The intervals typically get wider with the horizon, representing increasing uncertainty about future values.
- Fan charts are useful tools for presenting forecasts.

Example: fan chart



Time series patterns

Time series patterns (key concept)

We interpret a time series as

$$Y_t = f(T_t, S_t, C_t, E_t),$$

where T_t is the trend component, S_t is the seasonal component, C_t is the cyclic component, and E_t is an irregular or error component.

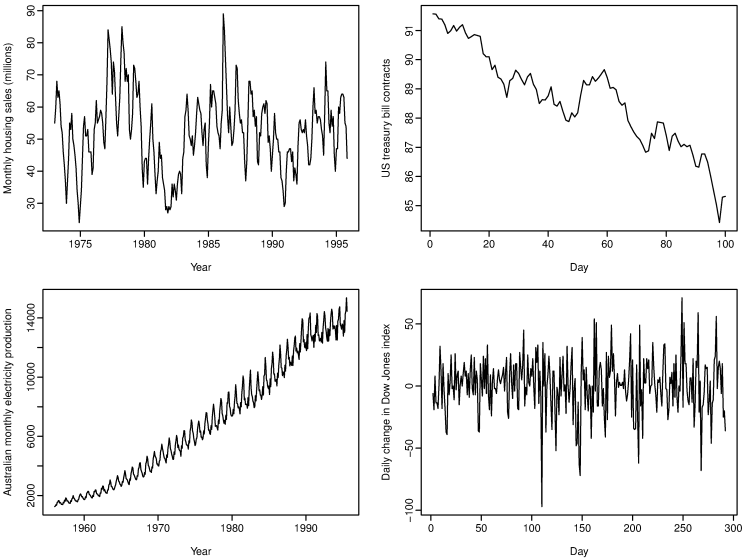
Trend. The systematic long term increase or decrease in the series.

Seasonal. A systematic change in the mean of the series due to seasonal factors (month, day of the week, etc).

Cyclic. A cyclic pattern exists when there are medium or long run fluctuations in the time series that are not of a fixed period.

Irregular. Short term fluctuations and noise.

Examples: time series patterns



Example: cyclic series

Time series models

Time series models can be additive or multiplicative.

$$\text{Additive: } Y_t = T_t + S_t + E_t$$

$$\text{Multiplicative: } Y_t = T_t \times S_t \times E_t$$

Log transformation

When the time series displays a multiplicative behaviour

$$Y_t = T_t \times S_t \times E_t,$$

we usually apply to log transformation to obtain a more convenient additive specification

$$\log Y_t = \log T_t + \log S_t + \log E_t.$$

Choosing an additive or multiplicative specification

Is the seasonal variation proportional to the trend?

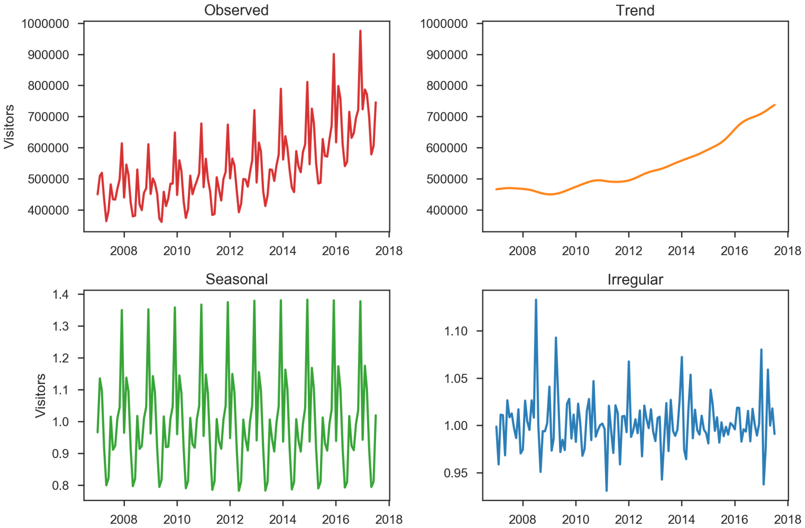
- If yes, a multiplicative model is more adequate.
- If not, we use an additive model.

Time series decomposition

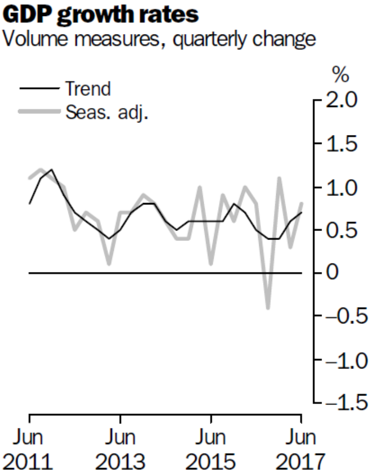
Time series decomposition methods are algorithms for splitting a time series into different components, typically for purposes of seasonal adjustment.

In the context of forecasting, decomposition methods are useful tools for exploratory data analysis, allowing us to visualise patterns in the data.

Time series decomposition: visitor arrivals



Example: seasonal adjustment and trend extraction



Simple forecasting methods

Random walk

The **random walk method** (called the naïve method in the book) forecasts the series using the value of the last available observation:

$$\widehat{y}_{t+h} = y_t$$

Seasonal random walk

For time series with seasonal patterns, we can extend the random walk method by forecasting the series with the value of the last available observation *in the same season*:

$$\widehat{y}_{t+h} = y_{t+h-m} \qquad \text{(if } h \leq m\text{)},$$

where m is the seasonal period. For example, $m = 12$ and $m = 4$ for monthly and quarterly data respectively.

The general formula is

$$\widehat{y}_{t+h} = y_{t+h-km}, \qquad k = \lfloor (h-1)/m + 1 \rfloor.$$

Drift method

The **drift method** forecasts the series as the sum of the most recent value (as in the naïve method) and the average change over time:

$$\widehat{y}_{t+1} = y_t + \sum_{i=2}^{t} \frac{y_i - y_{i-1}}{t-1}$$

$$\widehat{y}_{t+h} = y_t + h \times \sum_{i=2}^{t} \frac{y_i - y_{i-1}}{t-1}$$

Model diagnostics

Autocorrelation (key concept)

The **autocorrelation** of a time series process is

$$\rho_k = \frac{E\left[(Y_t - \mu)(Y_{t-k} - \mu)\right]}{\sigma^2} = \mathsf{Corr}(Y_t, Y_{t+k}),$$

where k is the lag, and μ and σ^2 are the mean and variance of the time series (assuming that they do not depend on t).

The **sample autocorrelation** is

$$r_k = \frac{\sum_{t=1}^{T-k}(y_{t+k} - \overline{y})(y_t - \overline{y})}{\sum_{t=1}^{T}(y - \overline{y})^2}.$$

The autocorrelation function (**ACF**) plot displays the autocorrelation for a range of lags.

White noise process (key concept)

A **white noise process** is a sequence of independently and identically distributed random variables with mean 0 and finite variance σ^2.

If a time series model is well specified, we expect the residual series of the fitted model to behave like a white noise process.

Model diagnostics (key concept)

Residual plot. The presence of patterns in the time series of residuals (such as non-constant variance over time) may suggest assumption violations and the need for alternative models.

Residual ACF plot. Well specified models should lead to small and insignificant sample autocorrelations, consistent with a white noise process.

Residual distribution plots (histogram, KDE, Q-Q plots, etc). Inspecting the distribution of the residuals will suggest the appropriate assumptions for interval forecasting.

Model validation

Training and validation sets

- We incorporate model validation into the forecasting process by setting aside a validation sample for estimating and comparing the performance of different models.
- We allocate the last part of the sample (typically 20-50% of the data) to the validation set.
- In time series, the training set is called "in-sample data" and the validation set the "out-of-sample data".
- Due to the dynamic nature of forecasting, there is no test set (though we may sometimes refer to model validation as forecast evaluation).

Real time forecasts (key concept)

We validate forecasts by following the “real time” approach: at every period t, we use all the available data at present to estimate the model and predict the future value of the series.

1. Starting at $t = n$, use the observations at times $1, 2, \ldots, t$ to estimate the forecasting model. Use the estimated model to forecast the observation at time $t + 1$.

2. Repeat the above step for $t = n + 1, \ldots, T - 1$.

3. Compute forecast accuracy measures based on the prediction errors $y_{n+1} - \widehat{y}_{n+1}, \ldots, y_T - \widehat{y}_T$.

We follow a similar procedure for multi-step forecasts.

Expanding and rolling windows

We can consider two schemes for updating the estimation sample at each validation period.

Expanding window. At each step, add the latest observation to the the estimation sample.

Rolling window. At each step, use only the most recent n observations for estimation. The rolling window scheme implicitly assumes that the dynamics of the series has a time changing nature, so that data far in the past are less relevant for estimation.

Measuring forecast accuracy

We typically assume the squared error loss and compute the out-of-sample MSE to measure forecast accuracy.

However, it is useful to be familiar with other measures that are common in business forecasting:

- Percentage errors.
- Scaled errors.

Percentage errors

- The percentage error is given by $p_t = 100 \times ((y_t - \widehat{y}_t)/y_t)$. It has the advantage of being scale-independent.

- The most commonly used measure is mean absolute percentage error

$$\text{MAPE} = \text{mean}(|p_t|).$$

- Measures based on percentage errors have the disadvantage of being infinite or undefined if $y_t = 0$ for any t in the period of interest, and having extreme values when any $y_t = 0$ is close to zero.

- Percentage errors are only valid under a meaningful zero.

Scaled errors

- Hyndman and Koehler (2006) proposed scaling the errors based on the training MAE (or MSE) from a benchmark method (typically a simple model).
- For a non-seasonal time series, a useful way to define a scaled error uses naïve forecasts:

$$q_t = \frac{y_t - \widehat{y}_t}{\dfrac{1}{T-1}\displaystyle\sum_{i=2}^{T} |y_i - y_{i-1}|}.$$

- Because the numerator and denominator both involve values on the scale of the original data, q_j is independent of the scale of the data.

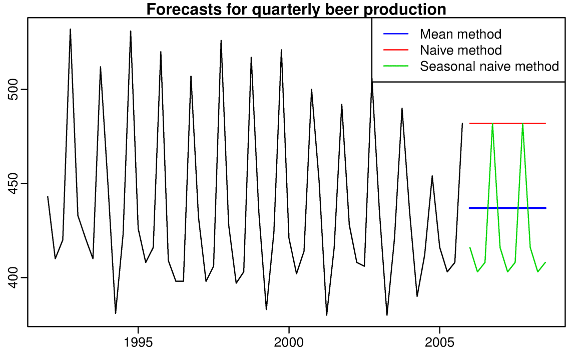
Mean absolute scaled error

The **mean absolute scaled error** is

$$\text{MASE} = \text{mean}(|q_t|).$$

A scaled error is less than one if it arises from a better set of forecasts than the random walk method evaluated on the training data.

Example: Quarterly Australian Beer Production



The figure shows shows three forecasting methods applied to the quarterly Australian beer production using data to the end of 2005. We compute the forecast accuracy measures for 2006-2008.

Example: Quarterly Australian Beer Production

Method	RMSE	MAE	MAPE	MASE
Mean method	38.01	33.78	8.17	2.30
Naïve method	70.91	63.91	15.88	4.35
Seasonal naïve method	**12.97**	**11.27**	**2.73**	**0.77**

It is clear from the graph that the seasonal naive method is best for the data, although it can still be improved.

Random walk model

Random walk model (key example)

In this section, we use the random walk method to illustrate how to obtain point and interval forecasts for multiple horizons based on a time series model.

We assume the model

$$Y_t = Y_{t-1} + \varepsilon_t,$$

where ε_t is i.i.d with constant variance σ^2.

Random walk model

Since $Y_t = Y_{t-1} + \varepsilon_t$, we can use back substitution to show that

$$Y_{t+1} = Y_t + \varepsilon_{t+1}$$

$$\begin{aligned} Y_{t+2} &= Y_{t+1} + \varepsilon_{t+2} \\ &= Y_t + \varepsilon_{t+1} + \varepsilon_{t+2} \\ &\vdots \\ Y_{t+h} &= Y_{t+h-1} + \varepsilon_{t+h} \\ &= Y_t + \varepsilon_{t+1} + \ldots + \varepsilon_{t+h} \end{aligned}$$

Point forecast

$$Y_{t+h} = Y_t + \sum_{i=1}^{h} \varepsilon_{t+i}$$

Therefore, we obtain the point forecast for any horizon as

$$\begin{aligned} \widehat{y}_{t+h} &= E(Y_{t+h} | y_{1:t}) \\ &= E\left(Y_t + \sum_{i=1}^{h} \varepsilon_{t+i} \middle| y_{1:t} \right) \\ &= y_t \end{aligned}$$

Forecast

The conditional variance is

$$\begin{aligned}\mathsf{Var}(Y_{t+h}|y_{1:t}) &= \mathsf{Var}(y_t + \sum_{i=1}^{h} \varepsilon_{t+i}|y_{1:t}) \\ &= h\sigma^2.\end{aligned}$$

For density forecasting, we need to make further assumptions about the errors. If we assume that $\varepsilon_t \sim N(0, \sigma^2)$,

$$Y_{t+h}|y_{1:t} \sim N\left(y_t, h\sigma^2\right).$$

Forecast interval

Under the Gaussian assumption,

$$Y_{t+h}|y_{1:t} \sim N\left(y_t, h\sigma^2\right),$$

leading to the forecast interval

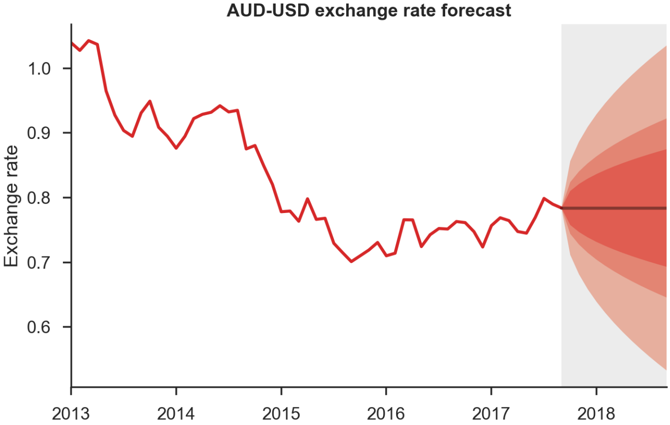
$$y_t \pm z_{\alpha/2} \times \sqrt{h\widehat{\sigma}^2},$$

where

$$\widehat{\sigma}^2 = \frac{\sum_{t=2}^{T}(y_t - y_{t-1})^2}{T-1},$$

and $z_{\alpha/2}$ is the appropriate critical value from the normal distribution.

Example: USD/AUD exchange rate



Forecast interval

Forecast interval based on the assumption of normal errors:

$$y_t \pm z_\alpha \times \sqrt{h\widehat{\sigma}^2}$$

- This forecast interval is based on the **plug-in method**, as we replace the unknown σ^2 with an estimate.
- The plug in method is a standard approach, but you should be aware that it ignores parameter uncertainty, leading to prediction intervals that are too narrow.
- If the errors are not Gaussian, you should use other methods such as the Bootstrap algorithm in the next slide.

Bootstrap forecast interval for the random walk model

Algorithm Bootstrap forecast interval (random walk)

Let $e_1, \ldots, e_t$ be the in-sample residuals.
If $h = 1$, the forecast interval is $(y_t + Q_1^*, y_t + Q_2^*)$, where Q_1^* and Q_2^* are the $\alpha/2$ and $1 - \alpha/2$ quantiles of the empirical distribution of the residuals.
If $h > 1$, set the number of replications S and proceed.

for $s = 1 : S$ **do**
    Sample h residuals with replacement to obtain $e_{s,1}^*, \ldots, e_{s,h}^*$.
    Compute the bootstrapped value $y_{s,t+h}^* = y_t + \sum_{i=1}^{h} e_{s,i}^*$.
end for

Compute the desired quantiles from the empirical distribution of $\{y_{s,t+h}^*\}_{s=1}^{S}$.

Forecast interval

- The assumption of i.i.d. errors is crucial for the Bootstrap algorithm in the previous slide.
- The residuals may be too small for complex models that are subject to large optimism. The prediction interval will be too narrow in this case.

Review questions

- What is point and interval forecasting?
- What are the four time series components?
- Which diagnostics do we use for univariate time series models, and why?
- How to we conduct model validation for forecasting?
- How do we compute forecasts and prediction intervals for the random walk model?