3D ANALYTICAL CALCULATION OF THE FORCES EXERTED BETWEEN TWO CUBOIDAL MAGNETS

GILLES AKOUN AND JEAN-PAUL YONNET

ABSTRACT

The interaction forces exerted between permanent magnets are used in many magneto-mechanical devices (magnetic bearings, couplings, etc...). By analytical calculation, 2D problems can be solved easily, when simple shaped magnets are used. Usually, the 3D problems are numerically computed, by using a finite element method for example.

This paper presents the 3D calculation of the interaction forces exerted between two cuboidal magnets, by analytical means only. The obtained expressions are rather complicated, but a pocket programmable calculator is sufficient to the force calculation. By derivation, the analytical expressions of the stiffnesses can be easily obtained. In addition, the 3D analytical calculation allows a simple optimization of the magnet dimensions.

I . INTRODUCTION

The interaction forces exerted between permanent magnets are used in many devices. In a permanent magnet coupling for example, the torque is transmitted by interaction between the magnets fixed on the two coupling halves. The transmitted torque can be calculated by the determination of the elementary interaction forces between the magnets.

The calculation of the forces interacting between two magnets can be applied to many other applications (magnetic bearings, attraction systems, etc...). For example the attraction force exerted between a magnet and a large iron piece, can be obtained by calculating the force between the magnet and its magnetic image. But the main difficulty, it is to calculate the interaction forces between two magnets.

This problem can be solved by numerical computation, by using a finite element method for example (1). By this method, complicated shaped magnets can be studied.

By analytical calculation, 2D problems can be solved easily, when simple shaped magnets are used. The forces and the stiffnesses exerted in a magnetic bearing made with permanent magnets only can be calculated by this means, because the cylindrical symmetry gives a 2D problem (2). In front of the difficulty of the 3D analytical calculation, the force determination is very often ended by a numerical integration made by a computer (3, 4).

Another solution is obtained by neglecting the edge effects in a first approximation, which leads to a 2D problem (5, 6).

The authors are with Laboratoire d'Electrotechnique de Grenoble (L.A. 355 C.N.R.S.), Boite Postale n° 46, 38402 SAINT-MARTIN D'HERES, FRANCE.

This article presents the calculation of the forces exerted between two cuboidal magnets, only with analytical means. The whole expressions are long, but the force can be easily calculated with a pocket programmable calculator. For simpler problems, the forces can be calculated with a small calculator, when two identical magnets are one on top of the other for example.

In addition to the force calculation, the 3D analytical expressions can be used to easily optimize the magnet dimensions. By deriving the force expressions, the stiffnesses can be easily obtained, which gives the system stability.

II . MATHEMATICAL MODEL

We study the interaction between two parallelepipedic magnets. Their sides are respectively parallel (see Figure 1). The magnetizations J and J' are supposed to be rigid and uniform in each magnet. Their direction is parallel to the 2c side. On the figure 1, J and J' have the same direction, but the calculation stays valid when they are in opposite direction. Only the analytical expression sign is reversed.

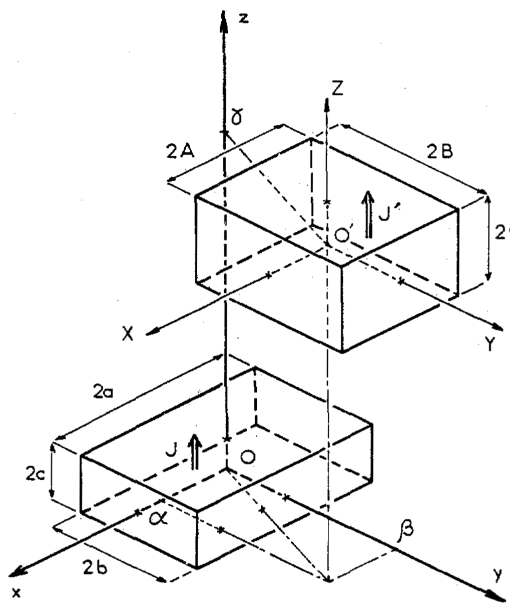

FIG.1 Magnet configuration

Axis	Ox	Oy	Oz
First magnet J	2a	2b	2c
Second magnet J'	2A	2B	2C
O O'	α	β	γ

TABLE 1. Magnet dimensions

0018-9464/84/0900-1962$01.00©1984 IEEE

The magnet centers are defined by O and O'. In the Oxyz reference, the O' co-ordonates are (α, β, γ). The magnet dimensions are given on the Table 1. The side 2a of the first magnet is parallel to the side 2A of the second magnet and parallel to the axis Ox. (Table 1)

The magnetizations J and J' are supposed rigid and uniform. They can be replaced by distributions of magnetic poles. Their density σ is defined by :

$\sigma = \vec{J} \cdot \vec{n}$

Since J is perpendicular to the 2a x 2b surfaces, these faces have the following density :

$\sigma = |\vec{J}|$

The magnetic field created by a uniformly charged rectangular surface can be easily obtained by analytical calculation (Appendix 1).

III . INTERACTION ENERGY BETWEEN THE TWO MAGNETS

If we consider two charged parallel surfaces, 2a x 2b and 2A x 2B, carrying the densities σ and σ', the interaction energy is expressed by :

$$W = \int_{-a}^{+a} dx \int_{-b}^{+b} dy \int_{-A}^{+A} dX \int_{-B}^{+B} dY \frac{\sigma\ \sigma'}{4\pi \mu_o r}$$

$$\text{with } r = \left[(\alpha + X - x)^2 + (\beta + Y - y)^2 + \gamma^2\right]^{1/2} \quad (1)$$

After four integrations, we have obtained the analytical expression of the interaction energy between two magnets. The steps are developed in Appendix 2.

This long expression has 256 terms :

$$W = \frac{J\ J'}{4\pi \mu_o} \sum\sum\sum\sum\sum\sum_{(i,j,k,l,p,q)=\{0,1\}^6} (-1)^{i+j+k+k+p+q} \psi(u_{ij}, v_{kl}, w_{pq}, r)$$

$$\text{with } \psi(u,v,w,r) = \frac{1}{2}u(v^2-w^2)\ln(r-u) + \frac{1}{2}v(u^2-w^2)\ln(r-v) + uvw\ \mathrm{tg}^{-1}\frac{u\ v}{r\ w} + \frac{r}{6}(u^2+v^2-2w^2).$$

by using

$$u_{ij} = \alpha + (-1)^j A - (-1)^i a$$
$$v_{kl} = \beta + (-1)^l B - (-1)^k b$$
$$w_{pq} = \gamma + (-1)^q C - (-1)^p c$$
$$r = (u_{ij}^2 + v_{kl}^2 + w_{pq}^2)^{1/2} \quad (2)$$

IV . FORCE CALCULATION

From the interaction energy, the three components F_x, F_y and F_z can be easily obtained by :

$\vec{F} = -\vec{\mathrm{grad}}\ W$

The force expressions are similar to the energy expression (256 terms).

$$F = \frac{J\ J'}{4\pi \mu_o} \sum\sum\sum\sum\sum\sum_{(i,j,k,l,p,q)=\{0,1\}^6} (-1)^{i+j+k+k+p+q} \phi(u_{ij}, v_{kl}, w_{pq}, r)$$

$$\text{For } F_x,\ \phi_x = \frac{1}{2}(v^2-w^2)\ln(r-u) + uv\ \ln(r-v) + vw\ \mathrm{tg}^{-1}\frac{uv}{rw} + \frac{1}{2}ru \quad (3)$$

$$\text{For } F_y,\ \phi_y = \frac{1}{2}(u^2-w^2)\ln(r-v) + uv\ \ln(r-u) + uw\ \mathrm{tg}^{-1}\frac{uv}{rw} + \frac{1}{2}rv \quad (4)$$

$$\text{For } F_z,\ \phi_z = -uw\ \ln(r-u) - vw\ \ln(r-v) + uv\ \mathrm{tg}^{-1}\frac{uv}{rw} - rw \quad (5)$$

by using always

$$u_{ij} = \alpha + (-1)^j A - (-1)^i a$$
$$v_{kl} = \beta + (-1)^l B - (-1)^k b$$
$$w_{pq} = \gamma + (-1)^q C - (-1)^p c$$
$$r = (u_{ij}^2 + v_{kl}^2 + w_{pq}^2)^{1/2}$$

The force variations can be deduced from these expressions. We can define the three stiffnesses K_x, K_y and K_z by :

$$K_x = -\frac{d\ F_x}{dx} \quad , \text{ etc...}$$

The sum of the stiffnesses gives :

$$K_x + K_y + K_z = 0 \quad (6)$$

This result agrees with Earnshaw's theorem. It is an analytical verification of the force expressions.

V . EXPERIMENTAL VERIFICATION

To prove the validity of the analytical force expressions, we have studied the forces exerted between two cuboidal ferrite magnets (J = 0,38 T). Their dimensions are :

2a = 20 mm	2A = 12 mm	α = -4 mm
2b = 12 mm	2B = 20 mm	β = -4 mm
2c = 6 mm	2C = 6 mm	γ = 8 mm

When the upper magnet moves in the Ox direction, we have measured the forces F_x and F_y (see Figure 2). The good agreement states the validy of the analytical expressions.

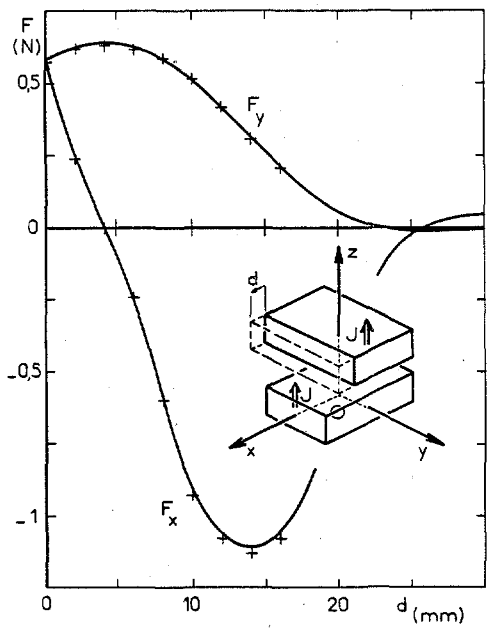

FIG.2 Experimental verification (analytically calculated curves and measured points).

VI . CONCLUSION

The interaction forces exterted between two cuboidal magnets have been calculated by analytical means only. This 3D problem has a rather complicated analytical solution. For the most general case, the obtained expressions has 256 terms, with ln and tg^{-1} functions.

From a pratical point of view, these expressions can be easily calculated with a pocket programmable calculator. In addition of the force calculation, by deriving the analytical expressions, the stiffnesses can be easily calculated.

These results can be applied to many magneto-mechanical applications : magnetic couplings, bearings, etc... The 3D analytical expressions allow a simple optimization of the magnet calculated.

APPENDIX 1

This analytical method can also be used for the field calculation. Let us consider a rectangular surface 2a x 2b having a uniform pole density σ (Figure 3). If we calculate the magnetic field at the point P(X, Y, Z), we obtain the following analytical expression:

$$H = \frac{\sigma}{4\pi\mu_o} \sum\sum_{(i,j)=\{0,1\}^2} (-1)^{i+j} \varepsilon(S_i, T_j, R)$$

For H_x, $\varepsilon_x = \ln(R-T)$

For H_y, $\varepsilon_y = \ln(R-S)$

For H_z, $\varepsilon_z = tg^{-1} \frac{S\ T}{R\ Z}$

by using $S_i = X - (-1)^i a$

$T_j = Y - (-1)^j b$

$R = (S_i^2 + T_j^2 + Z^2)^{½}$

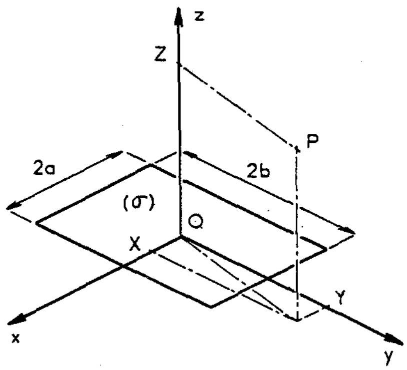

FIG.3 Geometrical disposition.

APPENDIX 2

Four integrations are necessary to calculate the analytical expression of the interaction energy. The calculation stages are obtained by using the four following integrals :

$$1.\quad \int \frac{dx}{\sqrt{x^2+\delta^2}} = \ln\left(\sqrt{x^2+\delta^2}+x\right) \tag{7}$$

$$2.\quad \int \ln\left(\sqrt{x^2+\delta^2}-\lambda\right)dx = x\{\ln\left(\sqrt{x^2+\delta^2}-\lambda\right) - \lambda \ln\left(\sqrt{x^2+\delta^2}+x\right) + z\ tg^{-1} \frac{\lambda\ x}{z\sqrt{x^2+\delta^2}} + z\ tg^{-1} \frac{x}{z} \tag{8}$$

$$3.\quad \int \{x \ln\left(\sqrt{x^2+\delta^2}-\lambda\right) + \lambda \ln\left(\sqrt{x^2+\delta^2}-x\right) + z\ tg^{-1} \frac{\lambda x}{z\sqrt{x^2+\delta^2}}\}dx$$
$$= \frac{1}{2}(x^2-z^2)\ln\left(\sqrt{x^2+\delta^2}-\lambda\right) + \lambda x \ln\left(\sqrt{x^2+\delta^2}-x\right) + zx\ tg^{-1} \frac{\lambda x}{z\sqrt{x^2+\delta^2}} + \frac{1}{2}\lambda\sqrt{x^2+\delta^2} - \frac{1}{4}\ x^2+z^2 \ln\sqrt{x^2+z^2} \tag{9}$$

$$4.\quad \int \{\frac{1}{2}(\lambda^2-z^2)\ln\left(\sqrt{x^2+\delta^2}-x\right) + \lambda x \ln\left(\sqrt{x^2+\delta^2}-\lambda\right) + \lambda z\ tg^{-1} \frac{\lambda x}{z\sqrt{x^2+\delta^2}} + \frac{1}{2}x\sqrt{x^2+\delta^2}\}dx =$$
$$\frac{1}{2}x(\lambda^2-z^2)\ln\left(\sqrt{x^2+\delta^2}-x\right) + \frac{1}{2}\lambda(x^2-z^2)\ln\left(\sqrt{x^2+\delta^2}-\lambda\right) + x\lambda z\ tg^{-1} \frac{\lambda x}{z\sqrt{x^2+\delta^2}} + \frac{1}{6}(x^2+\lambda^2-2z^2)\sqrt{x^2+\delta^2} + \frac{1}{4}\lambda x^2 - \lambda z \ln\sqrt{x^2+z^2} \tag{10}$$

REFERENCES

(1) . D. WEINMANN "Application of Rare Earth Magnets to Coaxial Couplings" Proceedings of the 3rd Int. workshop on RE-Co P.M., San Diego (Ca, USA), p.325-347, K.J. Strnat Ed., 1978

(2) . J-P. YONNET "Analytical Calculation of Magnetic Bearings" Proceedings of the 5th Int. workshop on RE-Co P.M., Roanoke (Va, USA), p.199-216, K.J. Strnat Ed., 1981

(3) . W. BARAN "Analytical Methods for the Design Calculation of P.M. Synchronous Couplings" Acta Polytechnica Scandinavica, Applied Physics, n°138, p.13-19, 1983

(4) . J.B.Y. TSUI and al. "The Effect of Intrinsic Magnetic Properties on Permanent Magnet Repulsion" IEEE Magnetics, MAG-8, n°2, p.188-194, 1972

(5) . R.M. HORNREICH and S. SHTRIKMAN "Optimal Design of Synchronous Torque Couplers" IEEE Magnetics, MAG-14, N°5, p.800-802, 1978

(6) . Y. TAWARA "Applications of Precipitation Hardened RE-Co Magnets to Torque Coupling Devices" Proceedings of the 2nd Int. workshop on RE-Co P.M., Dayton (USA), p.112-124, K.J. Strnat Ed., 1976.