3-D Analytical Calculation of the Torque and Force Exerted Between Two Cuboidal Magnets

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Most of the systems working by magnet interactions can be calculated by superposition of the interactions between parallelepiped elementary magnets. Each elementary magnet is submitted to a force and a torque. By 3-D fully analytical calculation, up to now, only the force components problem has been solved. It was published for the first time in 1984. Until now, the torque components have never been analytically calculated because of the angular derivation. We have solved it, so that now all the interactions (energy, three forces components, and three torque components) can be expressed by analytical formulations. The only hypothesis is that the magnetizations J and J' are supposed to be rigid and uniform in each magnet. The analytical calculation is made by replacing magnetizations by distributions of magnetic charges on the magnet poles. The torque is calculated for rotational movement of the second magnet around its center. The three components of the torque are written with functions based on logarithm and arc-tangent. The results have been verified and validated by comparison with finite-element calculation. Analytical calculation owns many advantages in comparison with other calculation methods. The 3-D analytical expressions are difficult to obtain, but the expressions of energy, force, and torque are very simple to use. For example, the analytical expression can be included in optimization software allowing to directly obtaining the shape optimization by a fast way.

Index Terms—**Analytical calculation, force, interaction energy, permanent magnet, torque.**

I. INTRODUCTION

THE analytical expressions are a very powerful, giving a very fast method to calculate magnetic interactions. It is why the analytical expressions of all the interactions, energy, forces, and torques between two cuboidal magnets are very important results. Many problems can be solved by the addition of element interactions. The simpler shape of elementary volume is the parallelepiped, with its cuboidal volume. It is why many 3-D calculations can be made by superposition of 3-D interactions between two elementary magnets of cuboidal shape.

Up to now, only the interaction energy and the force components between two magnets of the parallelepiped shape have been analytically solved. We have succeeded in a new result of first importance: the analytical calculation of the torque between two parallelepiped magnets. An example of direct analytical calculation and the comparison with computed results are presented.

II. 3-D ANALYTICAL CALCULATION BACKGROUND AND NEW RESULTS

The first 3-D fully analytical expressions of the energy and force were presented at the 1984 INTERMAG Conference, Hamburg, Germany [1]. The forces were analytically calculated for two cuboidal magnets with parallel magnetization directions.

New contributions on the 3-D calculation were presented later with the analytical calculation of multipolar magnetic couplings with discoidal shape [2], [3], but the last integration was made by a numerical way, or by a series calculation.

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When the two magnets are in angled position, the force calculus has been made by direct integration of magnetic induction [4], [5]. The results have been used to calculate magnetic couplings [6], [7].

Until now, all the analytical force calculation has been made for cuboidal magnets which magnetization is parallel to one of the edge of the magnet [8]. Recently, a new step has been made by the analytical energy and force calculation when the magnetization directions of the two magnets are perpendicular. By combination with the parallel case, the force for any magnetization direction and for any magnet position can be calculated by analytical expression [9]–[11].

The torque calculation is an innovative step. The three torque components between two magnets can be fully written with analytical expressions. Now all the interactions (interaction energy, force, and torque components) between two cuboidal magnets can be analytically calculated by relatively simple analytical expressions.

III. BASIC MATHEMATICAL MODEL

The interactions between two parallelepiped magnets are studied. Their edges are, respectively, parallel (see Fig. 1). The magnetizations J and J' are supposed to be rigid and uniform in each magnet. The dimensions of the first magnet are $2a \times 2b \times 2c$, and its polarization is J. Its center is O, the origin of the axes Oxyz. For the second magnet, the dimensions are $2A \times 2B \times 2C$, its polarization is J', and the coordinates of its center O' are (α, β, γ). The side 2a is parallel to the side 2A, and so on. The magnet dimensions are given in Table I. The magnetization directions shown in Fig. 1 correspond to the case when the polarizations J and J' have the same direction, parallel to the side 2c. Note that the calculation stays valid when they are in opposite direction; only the expression sign is reversed.

The polarizations J and J' are supposed to be rigid and uniform. They can be replaced by distributions of magnetic charges

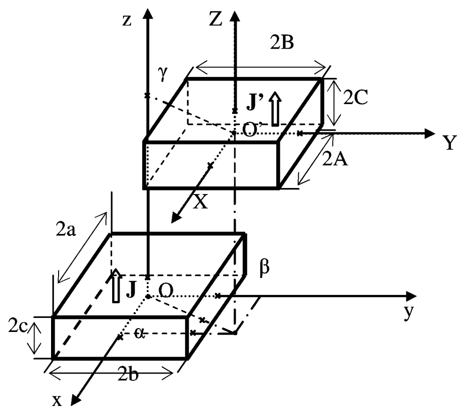

Fig. 1. Magnet configuration.

TABLE I
MAGNET DIMENSIONS AND POSITION

Axis	Ox	Oy	Oz
First Magnet (J)	2a	2b	2c
Second Magnet (J')	2A	2B	2C
Second Magnet Position O'	α	β	γ

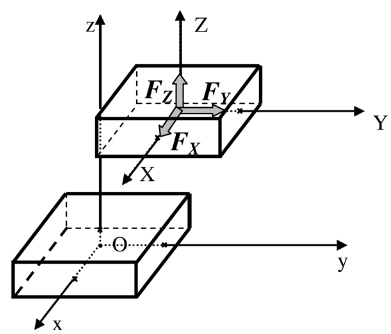

Fig. 2. Force component orientation.

on the poles. It is the Coulombian representation of the magnetization. Their density σ is defined by

$$\sigma = \vec{J} \cdot \vec{n}.$$

In the example of Fig. 1, since J is perpendicular to the surfaces 2a × 2b and oriented to the top, these polar faces wear the density $\sigma = +J$ on the upper face (North Pole), and $\sigma = -J$ on the lower face (South Pole).

All the analytical calculations have been made by successive integrals, to calculate the interaction energy for the two-magnets system. The forces and the torques can be obtained by linear and angular derivation. The analytical calculation of the interaction energy in 3-D is made by four successive integrations. The first one gives a logarithm function. In the second one, you have two logarithm and two arc-tangent functions, etc. The last one owns many complex functions based on logarithm and arc-tangent functions.

For all the calculations, the intermediary variable will always be

$$U_{ij} = \alpha + (-1)^j A - (-1)^i a$$
$$V_{kl} = \beta + (-1)^l B - (-1)^k b$$
$$W_{pq} = \gamma + (-1)^q C - (-1)^p c$$

and

$$r = \sqrt{U_{ij}^2 + V_{kl}^2 + W_{pq}^2}. \quad (1)$$

These lengths correspond to the distance between the cube corners and their projections on the axes. The parameters i, j, k, l, p, q are equal to 0 or 1 according to the considered corner; i, k, p for the first magnet and j, l, q for the second magnet.

IV. Analytical Calculation

A. Interaction Energy Calculation for Parallel Magnetization Directions

Direct calculation of the force components is possible. We have preferred to start by calculating the interaction energy because it is a scalar value, taking into account all the parameters. The oriented interactions like force and torque can be calculated from the interaction energy by derivation. The force F is obtained by linear derivation of the interaction energy (gradient function) and the torque T by angular derivation.

Intermediate steps of analytical calculation are developed in [1]. The interaction energy in a system of two magnets with parallel magnetization directions (Fig. 2) is given by

$$E = \frac{J \cdot J'}{4\pi\mu_0} \sum_{p=0}^{1} \sum_{q=0}^{1} (-1)^{p+q} \int_{-C}^{C} dY \int_{-A}^{A} dX \int_{-b}^{b} dy \int_{-a}^{a} \frac{1}{r} dx$$

with

$$r = \sqrt{(\alpha + X - x)^2 + (\beta + Y - y)^2 + (\gamma + (-1)^l C - (-1)^p c)^2}.$$

The obtained expressions of the interaction energy are

$$E = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{l=0}^{1} \sum_{p=0}^{1} \sum_{q=0}^{1} (-1)^{i+j+k+l+p+q} \cdot \psi(U_{ij}, V_{kl}, W_{pq}, r)$$

with

$$\begin{aligned} \psi(U, V, W, r) &= \frac{U(V^2 - W^2)}{2} \ln(r - U) + \frac{V(U^2 - W^2)}{2} \ln(r - V) \\ &\quad + UVW \cdot tg^{-1}\left(\frac{UV}{rW}\right) + \frac{r}{6}(U^2 + V^2 - 2W^2). \end{aligned}$$

It is a sum of 64 values of the function ψ. The function ψ itself is given by classical functions (logarithm and arc-tangent) of the geometrical parameters U, V, W, and r. The magnetizations J and J' are included in a multiplicative factor.

B. Force Calculation for Parallel Magnetization Directions

The first magnet of center O is supposed to be fixed. The second magnet is submitted to a force F, which components F_x, F_y, and F_z (Fig. 2). From the interaction energy, the force components can be obtained by $\vec{F} = -gr\vec{a}d \quad E$.

Consequently, the force components are

$$F = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{l=0}^{1} \sum_{p=0}^{1} \sum_{q=0}^{1} (-1)^{i+j+k+l+p+q} \cdot \phi(U_{ij}, V_{kl}, W_{pq}, r)$$

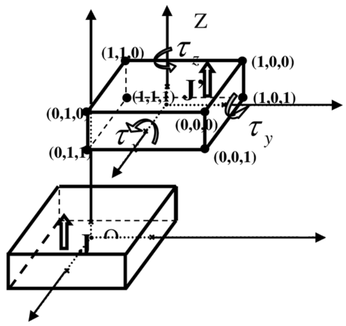

Fig. 3. Torque components and corner position.

with

$$
\begin{aligned}
F_x, \phi_x(U,V,W,r) &= \frac{(V^2-W^2)}{2}\ln(r-U) + UV\ln(r-V) \\
&\quad + VW\cdot \mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right) + \frac{1}{2}U\cdot r \\
F_y, \phi_y(U,V,W,r) &= \frac{(U^2-W^2)}{2}\ln(r-V) + UV\ln(r-U) \\
&\quad + UW\cdot \mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right) + \frac{1}{2}V\cdot r \\
F_z, \phi_z(U,V,W,r) &= -UW\ln(r-U) - VW\ln(r-V) \\
&\quad + UW\cdot \mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right) - W\cdot r.
\end{aligned}
$$

Like the energy, each component is obtained by a sum of 64 values of the function ϕ, calculated with the geometrical parameters.

C. Torque Component Calculation for Parallel Magnetization Directions

The first magnet of center O is supposed to be fixed. The second magnet is submitted to a torque T for a movement around its center O', which components are τ_x, τ_y, and τ_z (Fig. 3).

For the torque calculation, it is important to observe the energy expression. The interaction energy E depends on the two polarizations and on a geometrical analytical expression which is only a function of the magnet corner position [5]. Consequently, each corner wears a part of the global interaction energy. The scalar value of the energy of each corner E_c can be easily calculated; the global interaction energy on the magnet is the sum of the eight contributions of the magnet corners. In Fig. 3, each corner is defined by its coordinates for the E_c calculation, for example, $(1,0,0)$ means $j=1, l=0$, and $q=0$.

For an elementary displacement of the magnet, the variation of the energy of each corner gives a force on this corner F_c. When the displacement is a translation, all the corners are submitted to the same translation; the whole force exerted on the magnet is the sum of the contribution of the force F_c on the eight corners. When the magnet displacement is a rotation, all the local displacements of the magnet corners must be taken into account, and the sum of the elementary torque on the eight corners gives the global torque on the magnet.

The force and torque expressions are directly linked by the corner force F_c. The torque expressions are more complicated than the force expressions, but they are fully analytical with logarithm and arc-tangent functions.

The results of the torque calculation can be written either to be used alone, or by the relation with the force components. By starting directly from the magnet dimensions and positions, the torque is given by

$$
\begin{aligned}
T = \frac{J\cdot J'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q} \\
\cdot \tau(U_{ij}, V_{lk}, W_{pq}).
\end{aligned}
$$

For each torque component, the expression is given by

$$
\begin{aligned}
\boldsymbol{\tau}_X &= \frac{1}{2}(-(-1)^q C(U^2-W^2) + 2(-1)^l BVW)\ln(r-V) \\
&\quad - U((-1)^q CV + (-1)^l BW)\ln(r-U) \\
&\quad + U(-(-1)^q CW + (-1)^l BV)tg^{-1}\left(\frac{UV}{rW}\right) \\
&\quad + \frac{r}{2}(-(-1)^q CV - 2(-1)^l BW) \\
\boldsymbol{\tau}_Y &= \frac{1}{2}((-1)^q C\cdot(V^2-W^2) + 2(-1)^j AUW)\ln(r-U) \\
&\quad + V((-1)^q CU + (-1)^j AW)\ln(r-V) \\
&\quad + V((-1)^q CW - (-1)^j AU)tg^{-1}\left(\frac{UV}{W\cdot r}\right) \\
&\quad + \frac{r}{2}((-1)^q C\cdot U + 2(-1)^j AW) \\
\boldsymbol{\tau}_Z &= \frac{1}{2}(-(-1)^l B(V^2-W^2) + 2(-1)^j AUV)\ln(r-U) \\
&\quad - \frac{1}{2}((-1)^j A(U^2-W^2) \\
&\quad + (-1)^j A(U^2-W^2))\ln(r-V) \\
&\quad + (-(-1)^l BVW + (-1)^j UW)tg^{-1}\left(\frac{UV}{W\cdot r}\right) \\
&\quad - \frac{1}{2}((-1)^l B\cdot U + (-1)^j A\cdot V)\cdot r
\end{aligned}
$$

always with

$$
\begin{aligned}
U_{ij} &= \alpha + (-1)^j A - (-1)^i a \\
V_{kl} &= \beta + (-1)^l B - (-1)^k b \\
W_{pq} &= \gamma + (-1)^q C - (-1)^p c
\end{aligned}
$$

and

$$
r = \sqrt{U_{ij}^2 + V_{kl}^2 + W_{pq}^2}.
$$

The torque expressions can also be derived from the force components, exactly from the corner force F_c. The result can be written as a vector product

$$
\begin{bmatrix}\tau_X \\ \tau_Y \\ \tau_Z\end{bmatrix} = \sum_{j=0}^{1}\sum_{l=0}^{1}\sum_{q=0}^{1}\begin{pmatrix}A(-1)^j \\ B(-1)^l \\ C(-1)^q\end{pmatrix} \times \begin{pmatrix}F_X(j,l,q) \\ F_Y(j,l,q) \\ F_Z(j,l,q)\end{pmatrix}.
$$

The link between the force and the torque allows to easily verify the torque expressions. All the given expressions are valid for the basic case, i.e., parallel magnetization in parallelepiped magnets with parallel edges. The same approach by using the energy corner can be used for other cases. For example, for perpendicular magnetization directions, or for angled magnets,

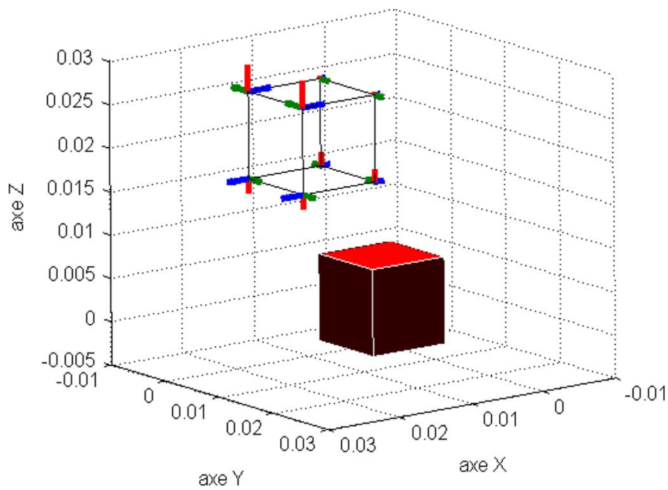

Fig. 4. Example of calculation. The magnets are identical: two cubes of 10-mm edge. The upper magnet moves in translation along the Ox axis above the lower fixed magnet at 10 mm. The segments attached to the magnet corners are proportional to the force components.

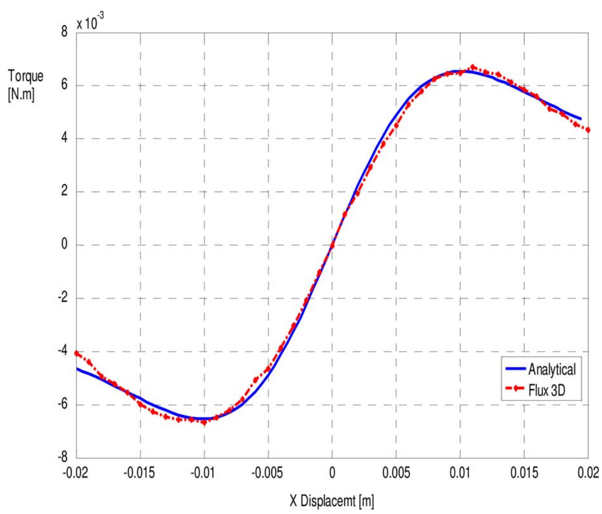

Fig. 5. Torque variation τ_y for a 20-mm displacement in the x-direction. Continuous lines are for the analytical calculation, and dots are computed points.

the interaction energy is only a function of the corner position [9]–[11]. It means that for all the cases (for any magnetization direction in the space, and for any relative position between the two magnets) the torque can be obtained by the same method. The only two hypotheses remain that the magnets own a cuboidal shape, and they are uniformly magnetized.

V. Example of Validation by Comparison With Numerical Simulation

The results have been verified by several ways. The following example presents the torque calculation between two magnets in interaction. These magnets are identical: two cubes of 10-mm edge (Fig. 4). Each cube wears a magnetization of 1 T. The upper magnet moves in translation along the Ox axis above the lower fixed magnet. The distance between the two magnets (airgap between the upper magnet and the fixed magnet) is 10 mm.

Fig. 5 shows the torque component calculation between the two magnets by two methods. For the first calculation method, the obtained analytical expressions are used; the second one is made by finite-element computation (FLUX3D software). The results are presented for a displacement in the x-direction.

The comparison between the results clearly shows that the two methods are in good agreement. The accuracy of the computed results seems to be lower than the accuracy of the analytical calculation; the numerical precision can probably be improved. The analytical results own more precision for ideal magnets, and can be calculated with a programmable pocket calculator. However, the analytical method is limited to simple cases with constant and rigid magnetization.

VI. Conclusion

All the analytical expressions of the interaction energy between two cuboidal magnets and all the force components and torque components are given in this paper for parallel magnetization directions. The only hypotheses are that the magnets own a parallelepiped shape with parallel edges, and that their magnetization directions are parallel. The torque component is an innovative contribution to the analytical calculation. Now, all the interactions between two elementary magnets represented by the six components (force F_x, F_y, F_z and torque τ_x, τ_y, and τ_z) can be calculated by simple analytical expressions.

Many problems can be solved by these results. The simpler shape of elementary volume is the parallelogram, with its cuboidal volume. By the superposition of 3-D interactions between elementary magnets, many 3-D calculations can be made. These results allow calculating not only the direct interaction of permanent magnets, but also many other devices including soft magnetic materials and currents. As an example, iron yokes can be taken into account by image effect.

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