Universidad Galileo
FISICC
Teoría de Sistemas II

Hoja de Trabajo 6 – Matriz de Transición y Respuesta del Sistema

Resuelva las siguientes cuestiones dejando constancia de todo procedimiento.

I. Responda las siguientes preguntas

1. ¿Qué representa la matriz de transferencia?
2. ¿Qué representa la matriz de transición?
3. ¿Cuáles son las propiedades que cumple la matriz de transición?
4. ¿Qué métodos existen para calcular la matriz de transición?
5. ¿Por qué es importante conocer la matriz de transición?

1) Representación en espacio de estado

2) Es la matriz que al multiplicar por el vector de estados en $t=0$ nos transiciona a un estado t

3)

1) $\Phi(0) = e^{A(0)} = I$
2) $\Phi(-t) = e^{A(-t)} = e^{(At)^{-1}} = [\Phi(t)]^{-1}$
3) $\Phi(t_1)\,\Phi(t_2) = e^{At_1}e^{At_2} = e^{A(t_1+t_2)} = \Phi(t_1+t_2)$
4) Si A es diagonal entonces $\Phi(t)$ es diagonal

4)

- Expansión por serie de Taylor.
- Transformada de Laplace inversa
- Diagonalización de A (matriz modal)

5) Nos permite saver la salida del sistema en el dominio del tiempo.

II. Encuentre la matriz de transición utilizando el método de descomposición de la matriz modal

1. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} u; y = [0\ 0\ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & -4 \end{bmatrix}$ $\quad \det(A-\lambda I) = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & -\lambda & 3 \\ 0 & 1 & -4-\lambda \end{bmatrix} \begin{matrix} -\lambda & 0 \\ 1 & -\lambda \\ 0 & 1 \end{matrix}$

$= -(4\lambda^2 + \lambda^3) - 2 + 3\lambda$

$= -\lambda^3 - 4\lambda^2 + 3\lambda - 2$

$\lambda_1 = -4.72 \quad \lambda_{2,3} = 0.36 \pm 0.54i$

$\lambda_1 = -\frac{118}{25} \quad \begin{bmatrix} 118/25 & 0 & -2 \\ 1 & 118/25 & 3 \\ 0 & 1 & 18/25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{118 V_1}{25} - 2V_3 = 0$

$V_1 + \frac{118}{25} V_2 + 3V_3 = 0$

$V_2 + \frac{18}{25} V_3 = 0$

$V_1 = \frac{50}{118} V_3$

$V_2 = \frac{18}{25} V_3 \qquad V_1 = \begin{bmatrix} 0.42 \\ -0.72 \\ 1 \end{bmatrix}$

$V_3 = V_3$

$\lambda_2 = 0.36 + 0.54i$

$\begin{bmatrix} -0.36-0.54i & 0 & -2 \\ 1 & -0.36-0.54i & 3 \\ 0 & 1 & -4.36-0.54i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$(-0.36 - 0.54i) V_1 - 2V_3 = 0$

$V_1 + (-0.36 - 0.54i) V_2 + 3V_3 = 0$

$V_2 + (-4.36 - 0.54i) V_3 = 0$

$V_1 = (-1.1 + 2.56i) V_3$

$V_2 = (4.36 + 0.54i) V_3 \qquad V_2 = \begin{bmatrix} -1.7 + 2.56i \\ 4.36 + 0.54i \\ 1 \end{bmatrix}$

$V_3 = V_3$

$$\lambda_3 = -0.36 - 0.54i$$

$$\begin{bmatrix} -0.36+0.54i & 0 & -2 \\ 0 & -0.36+0.54i & 3 \\ 1 & 1 & -4.36-0.54i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} -1.7 - 2.54i \\ 4.36 - 0.54i \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.42 & -1.7+2.56i & -1.7-2.54i \\ -0.72 & 4.36+0.54i & 4.36-0.54i \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -4.72 & 0 & 0 \\ 0 & 0.36+0.54i & 0 \\ 0 & 0 & 0.36-0.54i \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0.038 & -0.18 & 0.85 \\ -0.019-0.18i & 0.09-0.07i & 0.07+0.02i \\ -0.02+0.18i & 0.09+0.07i & 0.07-0.02i \end{bmatrix}$$

$$e^{At} = M e^{Dt} M^{-1}$$

2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 5 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u; y = [2\ 3\ 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 5 & -7\end{bmatrix} \quad \det(A-\lambda I) = \begin{bmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ -2 & 5 & -7-\lambda\end{bmatrix}$$

$$\lambda^2(-7-\lambda) - 2 + 5\lambda = 0$$
$$-\lambda^3 - 7\lambda^2 + 5\lambda - 2 = 0$$

$$\lambda_1 = -7.68 \qquad \lambda_{2,3} = 0.34 \pm 0.38i$$

$$\lambda_1 = -7.68$$

$$\begin{bmatrix}7.68 & 1 & 0\\ 0 & 7.68 & 1\\ -2 & 5 & 0.68\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad \begin{aligned}&7.68V_1 + V_2 = 0\\ &7.68V_2 + V_3 = 0\\ &-2V_1 + 5V_2 + 0.68V_3 = 0\end{aligned}$$

$V_1 = -25V_2/192$

$V_3 = -7.68V_2$

$V_2 = V_2$

$$V1 = \begin{bmatrix}-0.13\\ 1\\ -7.68\end{bmatrix}$$

$$\lambda_2 = 0.34 - 0.38i$$

$$\begin{bmatrix}-0.34+0.38i & 1 & 0\\ 0 & -0.34+0.38i & 1\\ -2 & 5 & -7.34+0.38i\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$(-0.34+0.38i)V_1 + V_2 = 0$$
$$(-0.34+0.38i)V_2 + V_3 = 0$$
$$-2V_1 + 5V_2 + (-7.34+0.38i)V_3 = 0$$

$V_1 = (1.3+1.46i)V_2$

$V_2 = V_2$

$V_3 = (0.34-0.38i)V_2$

$$V_2 = \begin{bmatrix}1.3+1.46i\\ 1\\ 0.34-0.38i\end{bmatrix}$$

$$\lambda_3 = 0.34 + 0.38i$$

$$\begin{bmatrix} -0.34-0.38i & 1 & 0 \\ 0 & -0.34-0.38i & 1 \\ -2 & 5 & -7.34-0.38i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-0.34-0.38i)V_1 + V_2 = 0$$
$$(-0.34-0.38i)V_2 + V_3 = 0$$
$$-2V_1 + 5V_2 + (-7.34-0.38i)V_3 = 0$$

$$V_1 = (1.3-1.46i)V_2$$
$$V_2 = V_2$$
$$V_3 = (0.34+0.38i)V_2$$

$$V_3 = \begin{bmatrix} 1.3-1.46i \\ 1 \\ 0.34+0.38i \end{bmatrix}$$

$$M = \begin{bmatrix} -0.13 & 1.3+1.46i & 1.3-1.46i \\ 1 & 1 & 1 \\ -7.66 & 0.34-0.38i & 0.34+0.38i \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -0.03 & 0.08 & -0.12 \\ 0.015-0.32i & 0.45+0.4i & 0.059+0.06i \\ 0.015+0.32i & 0.45-0.4i & 0.059-0.06i \end{bmatrix}$$

$$D = \begin{bmatrix} -7.66 & 0 & 0 \\ 0 & 0.34-0.38i & 0 \\ 0 & 0 & 0.34+0.38i \end{bmatrix}$$

$$e^{At} = M e^{Dt} M^{-1}$$

III. Encuentre la matriz de transición utilizando el método de Cayley Hamilton

1. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} u; \quad y = [0\ 0\ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\det(I\lambda - A) = \begin{bmatrix} \lambda-1 & 0 & -4 \\ -1 & \lambda-1 & -2 \\ 0 & -1 & \lambda+2 \end{bmatrix}$$

$$(\lambda-1)(\lambda-1)(\lambda+2) + 4 - 2(\lambda-1)$$

$$\lambda^3 - 5\lambda + 8$$

$$\lambda_1 = -2.8 \qquad \lambda_2 = 1.4 + 0.94i \qquad \lambda_3 = 1.4 - 0.94i$$

$$\left.\begin{aligned} e^{\lambda_1 t} &= \alpha_0 + \alpha_1(\lambda_1) + \alpha_2(\lambda_1)^2 \\ e^{\lambda_2 t} &= \alpha_0 + \alpha_1(\lambda_2) + \alpha_2(\lambda_2)^2 \\ e^{\lambda_3 t} &= \alpha_0 + \alpha_1(\lambda_3) + \alpha_2(\lambda_3)^2 \end{aligned}\right\} \Rightarrow \begin{aligned} e^{-2.8t} &= \alpha_0 - 2.8\alpha_1 + 7.84\alpha_2 \\ e^{(1.4+0.94i)t} &= \alpha_0 + \alpha_1(1.4+0.94i) + \alpha_2(1.07+2.6i) \\ e^{(1.4-0.94i)t} &= \alpha_0 + \alpha_1(1.4-0.94i) + \alpha_2(1.07-2.6i) \end{aligned}$$

$$\begin{aligned} e^{-2.8t} &= \alpha_0 - 2.8\alpha_1 + 7.84\alpha_2 \\ A &= \alpha_0 + \alpha_1(1.4+0.94i) + \alpha_2(1.07+2.6i) \\ B &= \alpha_0 + \alpha_1(1.4-0.94i) + \alpha_2(1.07-2.6i) \end{aligned}$$

$$\left\{\begin{aligned} e^{-2.8t} &= \alpha_0 - 2.8\alpha_1 + 7.84\alpha_2 \\ A &= e^{-2.8t} + 2.8\alpha_1 - 7.84\alpha_2 + \alpha_1(1.4+0.94i) + \alpha_2(1.07+2.6i) \\ A &= e^{-2.8t} + \alpha_1(4.2+0.94i) + \alpha_2(-6.77+2.6i) \end{aligned}\right.$$

$$\alpha_1 = \frac{A - e^{-2.8t}}{4.2+0.94i} - \alpha_2(-1.4+0.93i)$$

$$\alpha_0 = e^{-2.8t} + A(0.63-0.14i) - e^{-2.8t}(0.63-0.14i) - \alpha_2(-3.92+2.63i)$$

2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 5 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}u; y = [2\ 3\ 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

$$A = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 5 & -3\end{bmatrix} \qquad \det(\lambda I - A) = \begin{bmatrix}\lambda-1 & -1 & 0\\ 0 & \lambda & -1\\ 0 & -5 & \lambda+3\end{bmatrix}$$

$$(\lambda-1)\lambda(\lambda+3) - (\lambda-1)5 \qquad \lambda_1 = -4.2$$

$$\lambda_2 = 1.2$$

$$= \lambda^3 + 2\lambda^2 - 6\lambda + 5 \qquad \lambda_3 = 1$$

$$\left.\begin{aligned} e^{\lambda_1 t} &= \alpha_0 + \alpha_1(\lambda_1) + \alpha_2(\lambda_1)^2 \\ e^{\lambda_2 t} &= \alpha_0 + \alpha_1(\lambda_2) + \alpha_2(\lambda_2)^2 \\ e^{\lambda_3 t} &= \alpha_0 + \alpha_1(\lambda_3) + \alpha_2(\lambda_3)^2 \end{aligned}\right\} \begin{aligned} e^{-4.2t} &= \alpha_0 - 4.2\alpha_1 + 17.64\alpha_2 \\ e^{1.2t} &= \alpha_0 + 1.2\alpha_1 + 1.44\alpha_2 \\ e^{t} &= \alpha_0 + \alpha_1 + \alpha_2 \end{aligned}$$

$$\alpha_0 = e^t - \alpha_1 - \alpha_2$$

$$\alpha_0 = 6e^t - 5e^{1.2t} + 1.2\alpha_2$$

$$e^{1.2t} = e^t - \alpha_1 - \alpha_2 + 1.2\alpha_1 + 1.44\alpha_2$$

$$\alpha_1 = 5e^{1.2t} - 5e^t - 2.2\alpha_2$$

$$e^{-4.2t} = 6e^t - 5e^{1.2t} + 1.2\alpha_2 - 21e^{1.2t} + 21e^t + 9.24\alpha_2$$

$$e^{4.2t} = 27e^t - 26e^{1.2t} + 10.44\alpha_2$$

$$\underline{\alpha_2 = 0.095e^{4.2t} - 2.59e^t + 2.5e^{1.2t}}$$

$$\alpha_1 = e^t - 0.2e^{4.2t} - 0.5e^{1.2t}$$

$$\alpha_0 = 2.89e^t + 0.11e^{-4.2t} + 2.48e^{1.2t}$$

$$e^{At} = (2.89e^t + 0.11e^{-4.2t} + 2.48e^{1.2t})\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + (e^t - 0.2e^{-4.2t} - 0.5e^{1.2t})\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 5 & -3\end{bmatrix} +$$

$$(0.095e^{-4.2t} - 2.59e^t + 2.5e^{1.2t})\begin{bmatrix}1 & 1 & 1\\ 0 & 5 & -3\\ 0 & -15 & 14\end{bmatrix}$$