relationships

$$S_1 = \exp\left[\frac{x_1 + x_2}{2\sigma_2}\right] \quad \text{and} \quad S_2 = \exp\left[\frac{x_1 - x_2}{2\sigma_1}\right]$$

The procedure for rolling back through a three-dimensional tree to value a derivative is analogous to that for a two-dimensional tree.

Using a Nonrectangular Tree

Rubinstein has suggested a way of building a three-dimensional tree for two correlated stock prices by using a nonrectangular arrangement of the nodes.[24] From a node (S_1, S_2), where the first stock price is S_1 and the second stock price is S_2, there is a 0.25 chance of moving to each of the following:

$$(S_1u_1, S_2A), \quad (S_1u_1, S_2B), \quad (S_1d_1, S_2C), \quad (S_2d_1, S_2D)$$

where

$$u_1 = \exp[(r - q_1 - \sigma_1^2/2)\,\Delta t + \sigma_1\sqrt{\Delta t}\,]$$
$$d_1 = \exp[(r - q_1 - \sigma_1^2/2)\,\Delta t - \sigma_1\sqrt{\Delta t}\,]$$

and

$$A = \exp[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2\sqrt{\Delta t}\,(\rho + \sqrt{1-\rho^2}\,)]$$
$$B = \exp[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2\sqrt{\Delta t}\,(\rho - \sqrt{1-\rho^2}\,)]$$
$$C = \exp[(r - q_2 - \sigma_2^2/2)\Delta t - \sigma_2\sqrt{\Delta t}\,(\rho - \sqrt{1-\rho^2}\,)]$$
$$D = \exp[(r - q_2 - \sigma_2^2/2)\Delta t - \sigma_2\sqrt{\Delta t}\,(\rho + \sqrt{1-\rho^2}\,)]$$

When the correlation is zero, this method is equivalent to constructing separate trees for S_1 and S_2 using the alternative binomial tree construction method in Section 19.4.

Adjusting the Probabilities

A third approach to building a three-dimensional tree for S_1 and S_2 involves first assuming no correlation and then adjusting the probabilities at each node to reflect the correlation.[25] The alternative binomial tree construction method for each of S_1 and S_2 in Section 19.4 is used. This method has the property that all probabilities are 0.5. When the two binomial trees are combined on the assumption that there is no correlation, the probabilities are as shown in Table 26.1. When the probabilities are adjusted to reflect the correlation, they become those shown in Table 26.2.

26.8 MONTE CARLO SIMULATION AND AMERICAN OPTIONS

Monte Carlo simulation is well suited to valuing path-dependent options and options where there are many stochastic variables. Trees and finite difference methods are well

[24] See M. Rubinstein, "Return to Oz," *Risk*, November (1994): 67–70.

[25] This approach was suggested in the context of interest rate trees in J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, Winter (1994): 37–48.

Table 26.1 Combination of binomials assuming no correlation.

S_2-move	S_1-move Down	S_1-move Up
Up	0.25	0.25
Down	0.25	0.25

suited to valuing American-style options. What happens if an option is both path dependent and American? What happens if an American option depends on several stochastic variables? Section 26.5 explained a way in which the binomial tree approach can be modified to value path-dependent options in some situations. A number of researchers have adopted a different approach by searching for a way in which Monte Carlo simulation can be used to value American-style options.[26] This section explains two alternative ways of proceeding.

The Least-Squares Approach

In order to value an American-style option it is necessary to choose between exercising and continuing at each early exercise point. The value of exercising is normally easy to determine. A number of researchers including Longstaff and Schwartz provide a way of determining the value of continuing when Monte Carlo simulation is used.[27] Their approach involves using a least-squares analysis to determine the best-fit relationship between the value of continuing and the values of relevant variables at each time an early exercise decision has to be made. The approach is best illustrated with a numerical example. We use the one in the Longstaff–Schwartz paper.

Consider a 3-year American put option on a non-dividend-paying stock that can be exercised at the end of year 1, the end of year 2, and the end of year 3. The risk-free rate is 6% per annum (continuously compounded). The current stock price is 1.00 and the strike price is 1.10. Assume that the eight paths shown in Table 26.3 are sampled for the stock price. (This example is for illustration only; in practice many more paths would be

Table 26.2 Combination of binomials assuming correlation of ρ.

S_2-move	S_1-move Down	S_1-move Up
Up	$0.25(1-\rho)$	$0.25(1+\rho)$
Down	$0.25(1+\rho)$	$0.25(1-\rho)$

[26] Tilley was the first researcher to publish a solution to the problem. See J. A. Tilley, "Valuing American Options in a Path Simulation Model," *Transactions of the Society of Actuaries*, 45 (1993): 83–104.

[27] See F. A. Longstaff and E. S. Schwartz, "Valuing American Options by Simulation: A Simple Least-Squares Approach," *Review of Financial Studies*, 14, 1 (Spring 2001): 113–47.

Table 26.3 Sample paths for put option example.

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

sampled.) If the option can be exercised only at the 3-year point, it provides a cash flow equal to its intrinsic value at that point. This is shown in the last column of Table 26.4.

If the put option is in the money at the 2-year point, the option holder must decide whether to exercise. Table 26.3 shows that the option is in the money at the 2-year point for paths 1, 3, 4, 6, and 7. For these paths, we assume an approximate relationship:

$$V = a + bS + cS^2$$

where S is the stock price at the 2-year point and V is the value of continuing, discounted back to the 2-year point. Our five observations on S are: 1.08, 1.07, 0.97, 0.77, and 0.84. From Table 26.4 the corresponding values for V are: 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, $0.20e^{-0.06\times 1}$, and $0.09e^{-0.06\times 1}$. The values of a, b, and c that minimize

$$\sum_{i=1}^{5}(V_i - a - bS_i - cS_i^2)^2$$

where S_i and V_i are the ith observation on S and V, respectively, are $a = -1.070$, $b = 2.983$ and $c = -1.813$, so that the best-fit relationship is

$$V = -1.070 + 2.983S - 1.813S^2$$

Table 26.4 Cash flows if exercise only at the 3-year point.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.00	0.00	0.18
5	0.00	0.00	0.00
6	0.00	0.00	0.20
7	0.00	0.00	0.09
8	0.00	0.00	0.00

Table 26.5 Cash flows if exercise only possible at 2- and 3-year point.

Path	$t = 1$	$t = 2$	$t = 3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.00	0.13	0.00
5	0.00	0.00	0.00
6	0.00	0.33	0.00
7	0.00	0.26	0.00
8	0.00	0.00	0.00

This gives the value at the 2-year point of continuing for paths 1, 3, 4, 6, and 7 of 0.0369, 0.0461, 0.1176, 0.1520, and 0.1565, respectively. From Table 26.3 the value of exercising is 0.02, 0.03, 0.13, 0.33, and 0.26. This means that we should exercise at the 2-year point for paths 4, 6, and 7. Table 26.5 summarizes the cash flows assuming exercise at either the 2-year point or the 3-year point for the eight paths.

Consider next the paths that are in the money at the 1-year point. These are paths 1, 4, 6, 7, and 8. From Table 26.3 the values of S for the paths are 1.09, 0.93, 0.76, 0.92, and 0.88, respectively. From Table 26.5, the corresponding continuation values discounted back to $t = 1$ are 0.00, $0.13e^{-0.06\times 1}$, $0.33e^{-0.06\times 1}$, $0.26e^{-0.06\times 1}$, and 0.00, respectively. The least-squares relationship is

$$V = 2.038 - 3.335S + 1.356S^2$$

This gives the value of continuing at the 1-year point for paths 1, 4, 6, 7, 8 as 0.0139, 0.1092, 0.2866, 0.1175, and 0.1533, respectively. From Table 26.3 the value of exercising is 0.01, 0.17, 0.34, 0.18, and 0.22, respectively. This means that we should exercise at the 1-year point for paths 4, 6, 7, and 8. Table 26.6 summarizes the cash flows assuming that early exercise is possible at all three times. The value of the option is determined by discounting each cash flow back to time zero at the risk-free rate and calculating the

Table 26.6 Cash flows from option.

Path	$t = 1$	$t = 2$	$t = 3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00

mean of the results. It is

$$\tfrac{1}{8}(0.07e^{-0.06\times 3} + 0.17e^{-0.06\times 1} + 0.34e^{-0.06\times 1} + 0.18e^{-0.06\times 1} + 0.22e^{-0.06\times 1}) = 0.1144$$

Because this is greater than 0.10, it is not optimal to exercise the option immediately.

This method can be extended in a number of ways. If the option can be exercised at any time we can approximate its value by considering a large number of exercise points (just as a binomial tree does). The relationship between V and S can be assumed to be more complicated. For example we could assume that V is a cubic rather than a quadratic function of S. The method can be used where the early exercise decision depends on several state variables. A functional form for the relationship between V and the variables is assumed and the parameters are estimated using the least-squares approach, as in the example just considered.

The Exercise Boundary Parameterization Approach

A number of researchers, such as Andersen, have proposed an alternative approach where the early exercise boundary is parameterized and the optimal values of the parameters are determined iteratively by starting at the end of the life of the option and working backward.[28] To illustrate the approach, we continue with the put option example and assume that the eight paths shown in Table 26.3 have been sampled. In this case, the early exercise boundary at time t can be parameterized by a critical value of S, $S^*(t)$. If the asset price at time t is below $S^*(t)$ we exercise at time t; if it is above $S^*(t)$ we do not exercise at time t. The value of $S^*(3)$ is 1.10. If the stock price is above 1.10 when $t = 3$ (the end of the option's life) we do not exercise; if it is below 1.10 we exercise. We now consider the determination of $S^*(2)$.

Suppose that we choose a value of $S^*(2)$ less than 0.77. The option is not exercised at the 2-year point for any of the paths. The value of the option at the 2-year point for the eight paths is then 0.00, 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, 0.00, $0.20e^{-0.06\times 1}$, $0.09e^{-0.06\times 1}$, and 0.00, respectively. The average of these is 0.0636. Suppose next that $S^*(2) = 0.77$. The value of the option at the 2-year point for the eight paths is then 0.00, 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, 0.00, 0.33, $0.09e^{-0.06\times 1}$, and 0.00, respectively. The average of these is 0.0813. Similarly when $S^*(2)$ equals 0.84, 0.97, 1.07, and 1.08, the average value of the option at the 2-year point is 0.1032, 0.0982, 0.0938, and 0.0963, respectively. This analysis shows that the optimal value of $S^*(2)$ (i.e., the one that maximizes the average value of the option) is 0.84. (More precisely, it is optimal to choose $0.84 \leqslant S^*(2) < 0.97$.) When we choose this optimal value for $S^*(2)$, the value of the option at the 2-year point for the eight paths is 0.00, 0.00, 0.0659, 0.1695, 0.00, 0.33, 0.26, and 0.00, respectively. The average value is 0.1032.

We now move on to calculate $S^*(1)$. If $S^*(1) < 0.76$ the option is not exercised at the 1-year point for any of the paths and the value at the option at the 1-year point is $0.1032e^{-0.06\times 1} = 0.0972$. If $S^*(1) = 0.76$, the value of the option for each of the eight paths at the 1-year point is 0.00, 0.00, $0.0659e^{-0.06\times 1}$, $0.1695e^{-0.06\times 1}$, 0.0, 0.34, $0.26e^{-0.06\times 1}$, and 0.00, respectively. The average value of the option is 0.1008. Similarly when $S^*(1)$ equals 0.88, 0.92, 0.93, and 1.09 the average value of the option is 0.1283, 0.1202, 0.1215, and 0.1228, respectively. The analysis therefore shows that the optimal

[28] See L. Andersen, "A Simple Approach to the Pricing of Bermudan Swaptions in the Multifactor LIBOR Market Model," *Journal of Computational Finance*, 3, 2 (Winter 2000): 1–32.

value of $S^*(1)$ is 0.88. (More precisely, it is optimal to choose $0.88 \leqslant S^*(1) < 0.92$.) The value of the option at time zero with no early exercise is $0.1283e^{-0.06\times 1} = 0.1208$. This is greater than the value of 0.10 obtained by exercising at time zero.

In practice, tens of thousands of simulations are carried out to determine the early exercise boundary in the way we have described. Once the early exercise boundary has been obtained, the paths for the variables are discarded and a new Monte Carlo simulation using the early exercise boundary is carried out to value the option. Our American put option example is simple in that we know that the early exercise boundary at a time can be defined entirely in terms of the value of the stock price at that time. In more complicated situations it is necessary to make assumptions about how the early exercise boundary should be parameterized.

Upper Bounds

The two approaches we have outlined tend to underprice American-style options because they assume a suboptimal early exercise boundary. This has led Andersen and Broadie to propose a procedure that provides an upper bound to the price.[29] This procedure can be used in conjunction with any algorithm that generates a lower bound and pinpoints the true value of an American-style option more precisely than the algorithm does by itself.

SUMMARY

A number of models have been developed to fit the volatility smiles that are observed in practice. The constant elasticity of variance model leads to a volatility smile similar to that observed for equity options. The jump–diffusion model leads to a volatility smile similar to that observed for currency options. Variance-gamma and stochastic volatility models are more flexible in that they can lead to either the type of volatility smile observed for equity options or the type of volatility smile observed for currency options. The implied volatility function model provides even more flexibility than this. It is designed to provide an exact fit to any pattern of European option prices observed in the market.

The natural technique to use for valuing path-dependent options is Monte Carlo simulation. This has the disadvantage that it is fairly slow and unable to handle American-style derivatives easily. Luckily, trees can be used to value many types of path-dependent derivatives. The approach is to choose representative values for the underlying path function at each node of the tree and calculate the value of the derivative for each of these values as we roll back through the tree.

The binomial tree methodology can be extended to value convertible bonds. Extra branches corresponding to a default by the company are added to the tree. The roll-back calculations then reflect the holder's option to convert and the issuer's option to call.

Trees can be used to value many types of barrier options, but the convergence of the option value to the correct value as the number of time steps is increased tends to be slow. One approach for improving convergence is to arrange the geometry of the tree so

[29] See L. Andersen and M. Broadie, "A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options," *Management Science*, 50, 9 (2004), 1222–34.

that nodes always lie on the barriers. Another is to use an interpolation scheme to adjust for the fact that the barrier being assumed by the tree is different from the true barrier. A third is to design the tree so that it provides a finer representation of movements in the underlying asset price near the barrier.

One way of valuing options dependent on the prices of two correlated assets is to apply a transformation to the asset price to create two new uncorrelated variables. These two variables are each modeled with trees and the trees are then combined to form a single three-dimensional tree. At each node of the tree, the inverse of the transformation gives the asset prices. A second approach is to arrange the positions of nodes on the three-dimensional tree to reflect the correlation. A third approach is to start with a tree that assumes no correlation between the variables and then adjust the probabilities on the tree to reflect the correlation.

Monte Carlo simulation is not naturally suited to valuing American-style options, but there are two ways it can be adapted to handle them. The first involves using a least-squares analysis to relate the value of continuing (i.e, not exercising) to the values of relevant variables. The second involves parameterizing the early exercise boundary and determining it iteratively by working back from the end of the life of the option to the beginning.

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