PHENOMENOLOGICAL MODELS

An attempt is made here to represent the engineering behavior of a viscoelastic material through the use of simple mechanical elements, i.e., combinations of linear springs and dashpots. It is important to keep in mind that these models will describe the phenomenological behavior of the material, and in no way will they characterize the complex microstructural behavior that gives rise to the macroscopic response.

As engineers we wish to describe the component level response of the material, and leave the microstructural details to the material scientist. Conceptually, the engineer will adjust design parameters in order to optimize a structural component. Since we are not trained to implement adjustments in parameters associated with the design of the material (e.g., alloy percentages, seeding the material with dislocation sinks, etc.), then the expectation of designing a ***component*** from the microstructural level is a naive endeavor. It also ignores the fact that a number of phenomenological models have performed in a more than adequate fashion in component design.

Yet, the fact that these engineering models have worked well does not absolve the engineer from the responsibility of understanding what happens at the next level down. Here ignorance is not bliss, it is a dangerous state of mind.

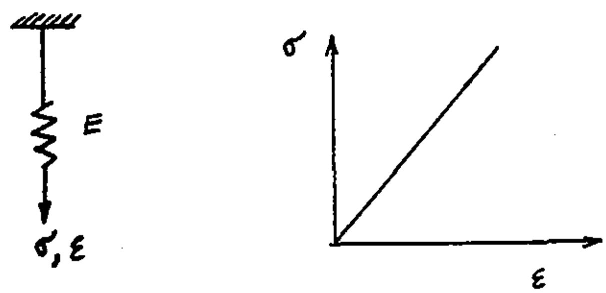
The models presented in the next sections include:

- Elastic model - a single spring
- Newtonian model - a single dashpot
- Maxwell model - a spring and dashpot connected in series
- Voigt (or Kelvin) model - a spring and dashpot connected in parallel
- Three element model
- Four element model
- Wiechert model - a generalized form of the Maxwell model

ELASTIC MODEL

This model, which simulates the uniaxial material response with a linear spring (see the figure below), represents the elastic behavior of a material. If the spring constant is characterized by Young's modulus (E), the applied force by stress (σ), and the spring extension is characterized by strain (ϵ), then the model can be used to capture the elastic behavior in either uniaxial tension or simple shear, i.e.,

$$\sigma = \epsilon E \qquad or \qquad \tau = \gamma G$$



Earlier in the course the multiaxial form this model was expressed as

$$\sigma_{ij} = C_{ijkl}\, \epsilon_{kl}$$

Although uniaxial forms of the models are developed through the combination of springs and dashpots, it is important to realize that as engineers we are looking to develop multiaxial formulations of these relationships.

Recall that differential operators were introduced earlier in the discussion relating to the mathematics associated with viscoelasticity. Specifically

$$P() = \sum_{k=0}^{N} p_k \frac{d^k()}{dt^k}$$

and

$$Q() = \sum_{k=0}^{M} q_k \frac{d^k()}{dt^k}$$

were introduced and these functions operated on stress and strain in the following manner

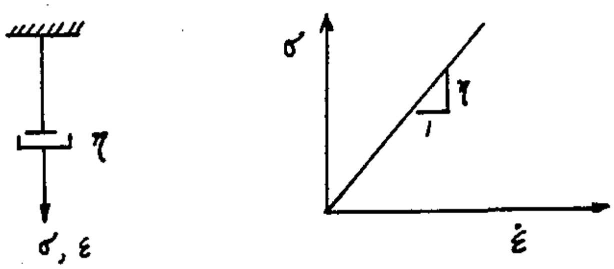
$$P[\sigma(t)] = Q[\epsilon(t)]$$

If this last expression is specialized to the elastic model given on the previous page, then N=0, M=0, $p_0=1$, and $q_0=E$.

NEWTONIAN MODEL

Another simple model is represented by the dashpot element in the figure below. In this particular model stress is proportionally related to the ***strain rate***, i.e.,

$$\sigma = \eta \frac{d\varepsilon}{dt}$$



This model is characterized by the fact that a constant applied stress produces a constant strain rate. Knowing that most materials eventually exhibit steady-state creep behavior under constant applied stress, it seems reasonable that this element will play an important role in capturing the time dependent behavior of a viscoelastic material.

Again, making use of the differential operators

$$P() = \sum_{k=0}^{N} p_k \frac{d^k()}{dt^k}$$

and

$$Q() = \sum_{k=0}^{M} q_k \frac{d^k()}{dt^k}$$

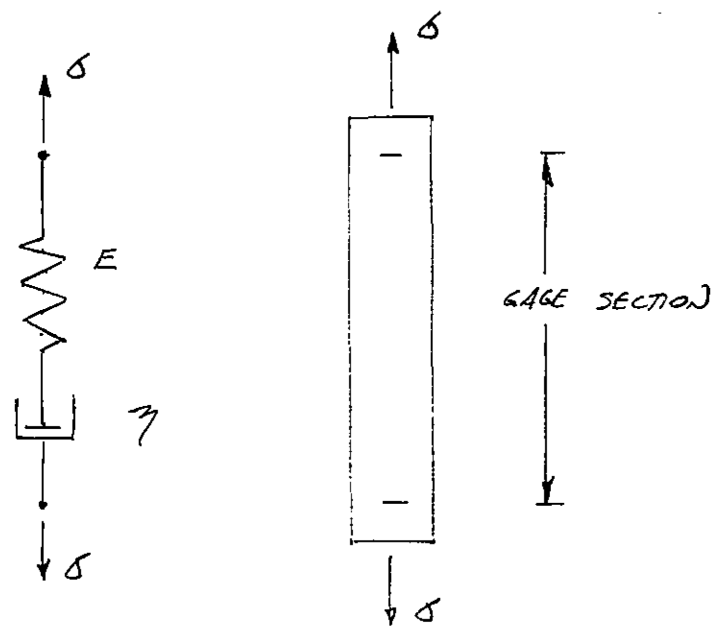
with

$$P[\sigma(t)] = Q[\epsilon(t)]$$

then for the Newtonian model $N=0$, $M=1$, $p_0=1$, $q_0=0$, $q_1=\eta$.

MAXWELL MODEL

This model is the first combined element model discussed here. The model consists of a spring and a dashpot combined in series, and is depicted in the following figure



The operator equation is obtained by noting two features of the model, i.e.,

- The stress in the dashpot element will be the same as the stress in the spring element. This is consistent with the equilibrium requirements of the tensile specimen depicted above.

- The total strain (ϵ) is equal to the summation of strains in each element, i.e.,

$$\epsilon = \epsilon_d + \epsilon_s$$

where ϵ_d is the strain in the dashpot, and ϵ_s is the strain in the spring. Again, this consistent with the kinematic requirements of the tensile specimen depicted above.

Expressing this last equation in a rate form yields

$$\dot{\epsilon} = \dot{\epsilon}_d + \dot{\epsilon}_s$$

where

$$\dot{\epsilon}_s = \frac{\dot{\sigma}_s}{E}$$

$$= \frac{\dot{\sigma}}{E}$$

and

$$\dot{\epsilon}_d = \frac{\sigma_d}{\eta}$$

$$= \frac{\sigma}{\eta}$$

Thus

$$\dot{\epsilon} = \frac{\sigma}{\eta} + \frac{\dot{\sigma}}{E}$$

$$\left(\frac{d}{dt}\right)\epsilon = \left[\frac{1}{\eta} + \left(\frac{1}{E}\right)\frac{d}{dt}\right]\sigma$$

Again, making use of the differential operators

$$P() = \sum_{k=0}^{N} p_k \frac{d^k()}{dt^k}$$

and

$$Q() = \sum_{k=0}^{M} q_k \frac{d^k()}{dt^k}$$

with

$$P[\sigma(t)] = Q[\epsilon(t)]$$

then for the Maxwell model N=1, M=1, $p_0=(1/\eta)$, $p_1=(1/E)$, $q_0=0$, $q_1=1$.

Next we will solve the governing differential equation for the Maxwell model under two applied boundary conditions, i.e.,

- Creep (constant applied stress)
- Stress Relaxation (constant applied strain)

assuming uniaxial load/displacement conditions.

SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

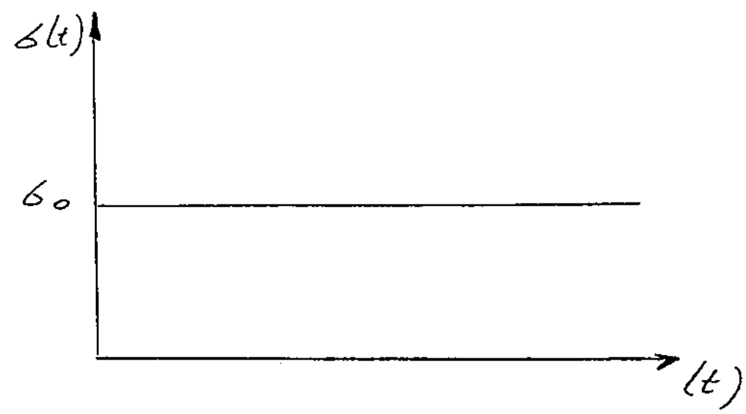
$$\frac{d\varepsilon}{dt} = \frac{\sigma}{\eta} + \left(\frac{1}{E}\right)\frac{d\sigma}{dt}$$

SUBJECT TO THE INITIAL AND BOUNDARY CONDITIONS

$$\varepsilon = 0 \quad @ \; t = 0$$

$$\sigma(t) = [H(t)]\,\sigma_0$$

GRAPHICALLY



THUS WE ARE ATTEMPTING TO SOLVE THE CREEP BOUNDARY VALUE PROBLEM.

SOLUTION:

SUBSTITUTE FOR σ IN THE DIFFERENTIAL EQUATION

$$\frac{d\varepsilon}{dt} = H(t)\left(\frac{\sigma_0}{\eta}\right) + \left(\frac{\sigma_0}{E}\right)\left(\frac{dH(t)}{dt}\right)$$

$$= \left(\frac{\sigma_0}{\eta}\right) H(t) + \left(\frac{\sigma_0}{E}\right)\delta(t)$$

NOW TAKE THE LAPLACE TRANSFORM OF BOTH SIDES OF THIS EXPRESSION

$$\mathcal{L}\left\{\frac{d\varepsilon}{dt}\right\} = \left(\frac{\sigma_0}{\eta}\right)\mathcal{L}\{H(t)\} + \left(\frac{\sigma_0}{E}\right)\mathcal{L}\{\delta(t)\}$$

$$sy - Y(0) = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right) + \left(\frac{\sigma_0}{E}\right)(1)$$

$$sy - 0 = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right) + \left(\frac{\sigma_0}{E}\right)(1)$$

$$y = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s^2}\right) + \left(\frac{\sigma_0}{E}\right)\left(\frac{1}{s}\right)$$

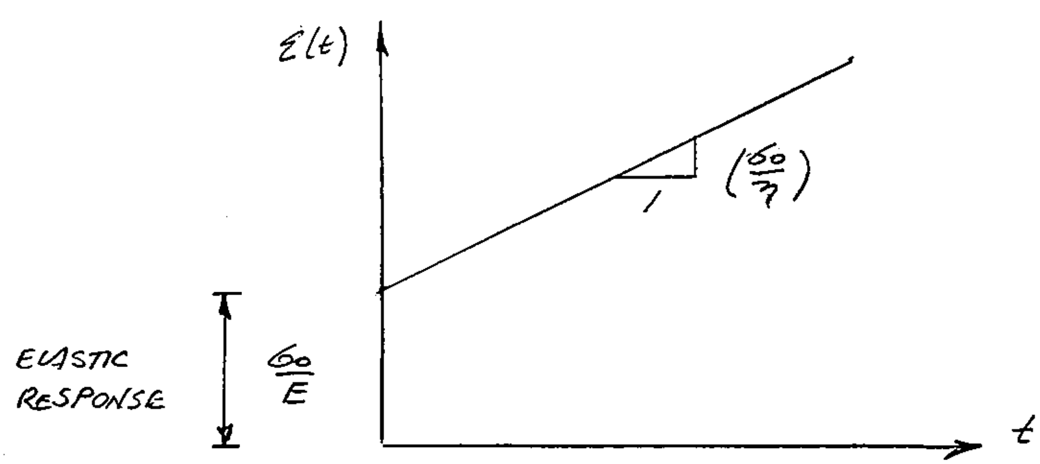
THUS

$$\varepsilon(t) = \mathcal{L}^{-1}\left\{\left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s^2}\right) + \left(\frac{\sigma_0}{E}\right)\left(\frac{1}{s}\right)\right\}$$

$$= \left(\frac{\sigma_0}{\eta}\right)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \left(\frac{\sigma_0}{E}\right)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= \left(\frac{\sigma_0}{\eta}\right)t + \left(\frac{\sigma_0}{E}\right)\delta(t)$$

GRAPHICALLY, THE STRAIN RESPONSE IS AS FOLLOWS



SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

$$\frac{d\varepsilon}{dt} = \frac{\sigma}{\eta} + \left(\frac{1}{E}\right)\frac{d\sigma}{dt}$$

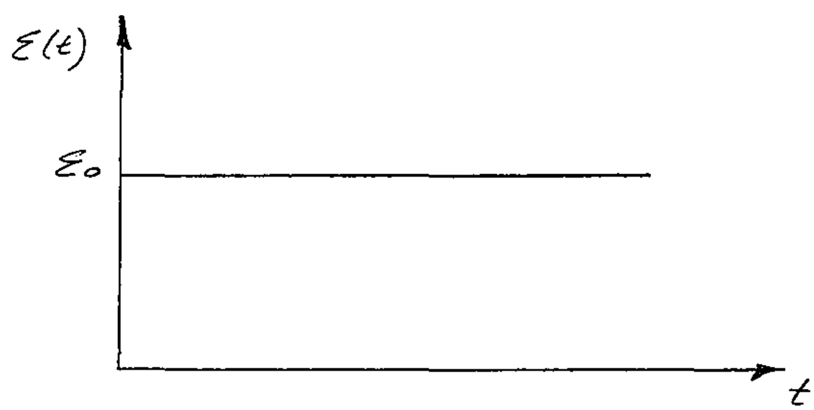
SUBJECT TO THE INITIAL CONDITION

$$\sigma = 0 \quad @ \quad t = 0$$

AND BOUNDARY CONDITION

$$\varepsilon(t) = [H(t)]\,\varepsilon_0$$

GRAPHICALLY



THUS WE ARE ATTEMPTING TO SOLVE THE STRESS RELAXATION BOUNDARY VALUE PROBLEM.

<u>SOLUTION</u>:

SUBSTITUTE FOR ε IN THE DIFFERENTIAL EQUATION

$$\frac{d}{dt}\left\{[H(t)]\,\varepsilon_0\right\} = \frac{\sigma}{\eta} + \left(\frac{1}{E}\right)\frac{d\sigma}{dt}$$

$$\varepsilon_o \left[\frac{d\,H(t)}{dt} \right] = \frac{\sigma}{\eta} + \left(\frac{1}{E}\right) \frac{d\sigma}{dt}$$

$$\varepsilon_o [\delta(t)] = \frac{\sigma}{\eta} + \left(\frac{1}{E}\right) \frac{d\sigma}{dt}$$

TAKING THE LAPLACE TRANSFORM OF BOTH SIDES OF THIS EXPRESSION YIELDS

$$\varepsilon_o \mathcal{L}\{\delta(t)\} = \left(\frac{1}{\eta}\right) \mathcal{L}\{\sigma\} + \left(\frac{1}{E}\right) \mathcal{L}\left\{\frac{d\sigma}{dt}\right\}$$

$$\varepsilon_o (1) = \left(\frac{1}{\eta}\right) y + \left(\frac{1}{E}\right) [s y - y(0)]$$

$$\varepsilon_o = \left[\left(\frac{1}{\eta}\right) + \left(\frac{s}{E}\right)\right] y - \left(\frac{1}{E}\right)(0)$$

$$\varepsilon_o = \left[\frac{E + \eta s}{\eta E}\right] y$$

$$y = \frac{\eta E \varepsilon_o}{E + \eta s}$$

$$= \frac{E \varepsilon_o}{\frac{E}{\eta} + s}$$

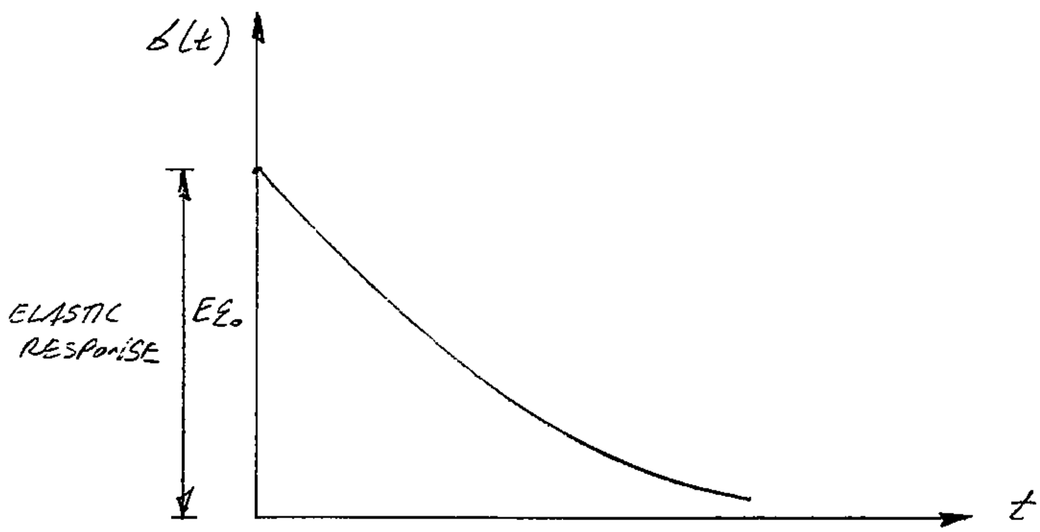
$$= E \varepsilon_o \left\{ \frac{1}{\frac{E}{\eta} + s} \right\}$$

TAKING THE INVERSE TRANSFORM OF THIS EXPRESSION YIELDS

$$\sigma(t) = E\varepsilon_0 \, \mathcal{L}^{-1}\left\{\frac{1}{s + \frac{E}{\eta}}\right\}$$

$$= (E\varepsilon_0)\, \exp\left[-\left(\frac{E}{\eta}\right)t\right]$$

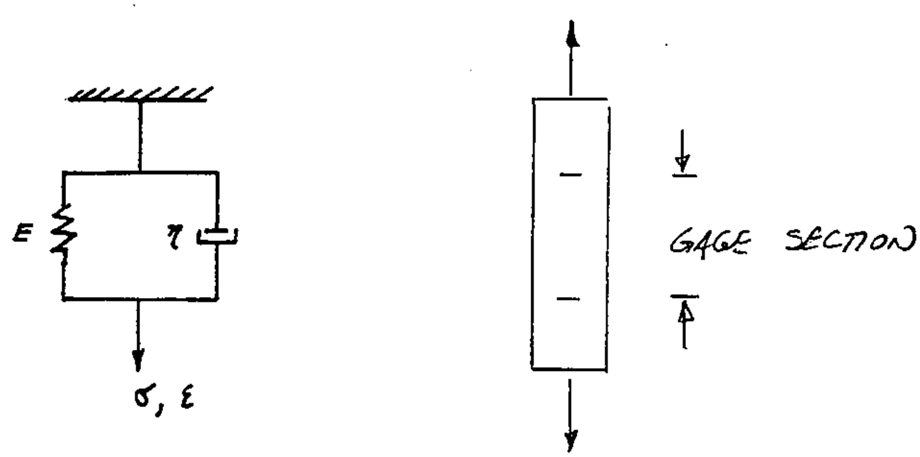
GRAPHICALLY



NOTE THAT IN THE LIMIT AS t APPROACHES INFINITY, σ APPROACHES ZERO

VOIGT (KELVIN) MODEL

The second combined model consists of a spring and a dashpot connected in parallel. This model is depicted in the following figure



The operator equation is obtained by noting two features of the model, i.e.,

- The strain in the dashpot element will be the same as the strain in the spring element. This is consistent with the kinematic requirements of the tensile specimen depicted above.

- The total stress (σ) is equal to the summation of the stress in each element, i.e.,

$$\sigma = \sigma_d + \sigma_s$$

  where σ_d is the stress in the dashpot, and σ_s is the stress in the spring. Again, this consistent with the equilibrium requirements of the tensile specimen depicted above.

With

$$
\begin{aligned}
\epsilon &= \epsilon_s \\
&= \frac{\sigma_s}{E}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\epsilon} &= \dot{\epsilon}_d \\
&= \frac{\sigma_d}{\eta}
\end{aligned}
$$

then

$$
\begin{aligned}
\sigma &= \sigma_s + \sigma_d \\
&= E\epsilon + \eta\,\dot{\epsilon} \\
&= \left[E + \eta\left(\frac{d}{dt}\right)\right]\epsilon
\end{aligned}
$$

Again, making use of the differential operators

$$
P() = \sum_{k=0}^{N} p_k \frac{d^k()}{dt^k}
$$

and

$$Q() = \sum_{k=0}^{M} q_k \frac{d^k()}{dt^k}$$

with

$$P[\sigma(t)] = Q[\epsilon(t)]$$

then for the Voigt model N=0, M=1, $p_0=1$, $q_0=E$, $q_1=\eta$.

Next we will solve the governing differential equation for the Voigt model under two applied boundary conditions, i.e.,

- Creep (constant applied stress)
- Stress Relaxation (constant applied strain)

assuming uniaxial load/displacement conditions.

SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

$$\sigma = E\varepsilon + \eta\left(\frac{d\varepsilon}{dt}\right)$$

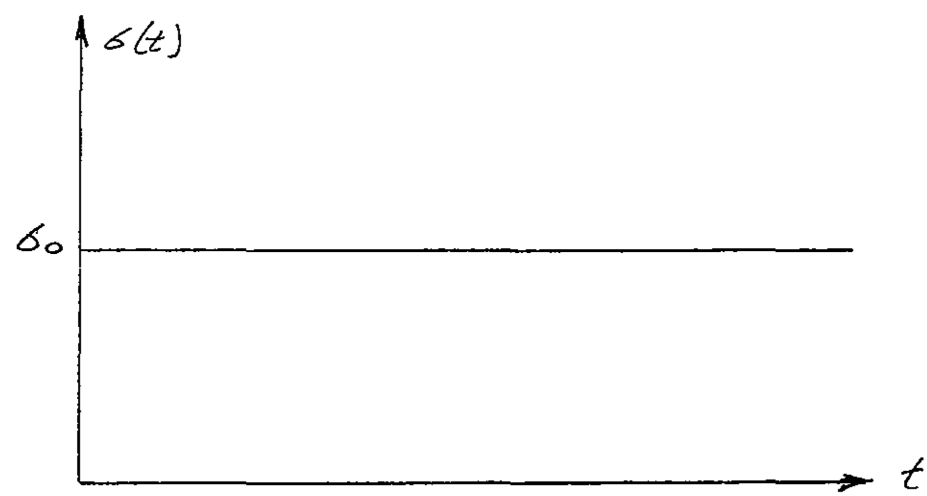
SUBJECT TO THE INITIAL CONDITION

$$\varepsilon = 0 \quad @ \quad t = 0$$

AND BOUNDARY CONDITIONS

$$\sigma(t) = [H(t)]\,\sigma_o$$

GRAPHICALLY



ONCE AGAIN WE ARE ATTEMPTING TO SOLVE THE CREEP BOUNDARY VALUE PROBLEM

<u>SOLUTION:</u>

SUBSTITUTE FOR σ IN THE DIFFERENTIAL EQUATION

$$[H(t)]\,\sigma_o = E\varepsilon + \eta\left(\frac{d\varepsilon}{dt}\right)$$

$$\frac{d\varepsilon}{dt} + \left(\frac{E}{\eta}\right)\varepsilon = H(t)\left(\frac{\sigma_0}{\eta}\right)$$

NOW TAKE THE LAPLACE TRANSFORM OF BOTH SIDES OF THIS EXPRESSION

$$\mathcal{L}\left\{\frac{d\varepsilon}{dt}\right\} + \left(\frac{E}{\eta}\right)\mathcal{L}\{\varepsilon\} = \left(\frac{\sigma_0}{\eta}\right)\mathcal{L}\{H(t)\}$$

$$sy - \varepsilon(0) + \left(\frac{E}{\eta}\right)y = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right)$$

$$\left[s + \left(\frac{E}{\eta}\right)\right]y - 0 = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right)$$

$$y = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right)\left[\frac{1}{s + (E/\eta)}\right]$$

UTILIZING THE METHOD OF PARTIAL FRACTIONS LET

$$\left(\frac{1}{s}\right)\left[\frac{1}{s + (E/\eta)}\right] = \frac{A}{s} + \frac{B}{s + (E/\eta)}$$

NOW MULTIPLY BOTH SIDES OF THIS EXPRESSION BY THE DENOMINATOR ON THE LEFT HAND SIDE OF THE EQUAL SIGN. THIS YIELDS

$$1 = A\left[s + (E/\eta)\right] + Bs$$

$$= (A+B)s + A(E/\eta)$$

WE NOW HAVE TWO EQUATIONS IN TWO UNKNOWNS, I.E.,

$$A + B = 0$$

$$A(E/\eta) = 1$$

THUS

$$A = \eta / E$$

$$B = -\eta / E$$

AND

$$\left(\frac{1}{s}\right)\left[\frac{1}{s + (E/\eta)}\right] = \left(\frac{\eta}{E}\right)\left(\frac{1}{s}\right) - \left(\frac{\eta}{E}\right)\left[\frac{1}{s + (E/\eta)}\right]$$

NOW

$$y = \left(\frac{\sigma_0}{\eta}\right)\left\{\left(\frac{\eta}{E}\right)\left(\frac{1}{s}\right) - \left(\frac{\eta}{E}\right)\left[\frac{1}{s + (E/\eta)}\right]\right\}$$

$$= \left(\frac{\sigma_0}{E}\right)\left(\frac{1}{s}\right) - \left(\frac{\sigma_0}{E}\right)\left[\frac{1}{s + (E/\eta)}\right]$$

THUS

$$\varepsilon(t) = \mathcal{L}^{-1}\left\{\left(\frac{\sigma_0}{E}\right)\left(\frac{1}{s}\right) - \left(\frac{\sigma_0}{E}\right)\left[\frac{1}{s + (E/\eta)}\right]\right\}$$

$$= \left(\frac{\sigma_0}{E}\right)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \left(\frac{\sigma_0}{E}\right)\mathcal{L}^{-1}\left\{\frac{1}{s + (E/\eta)}\right\}$$

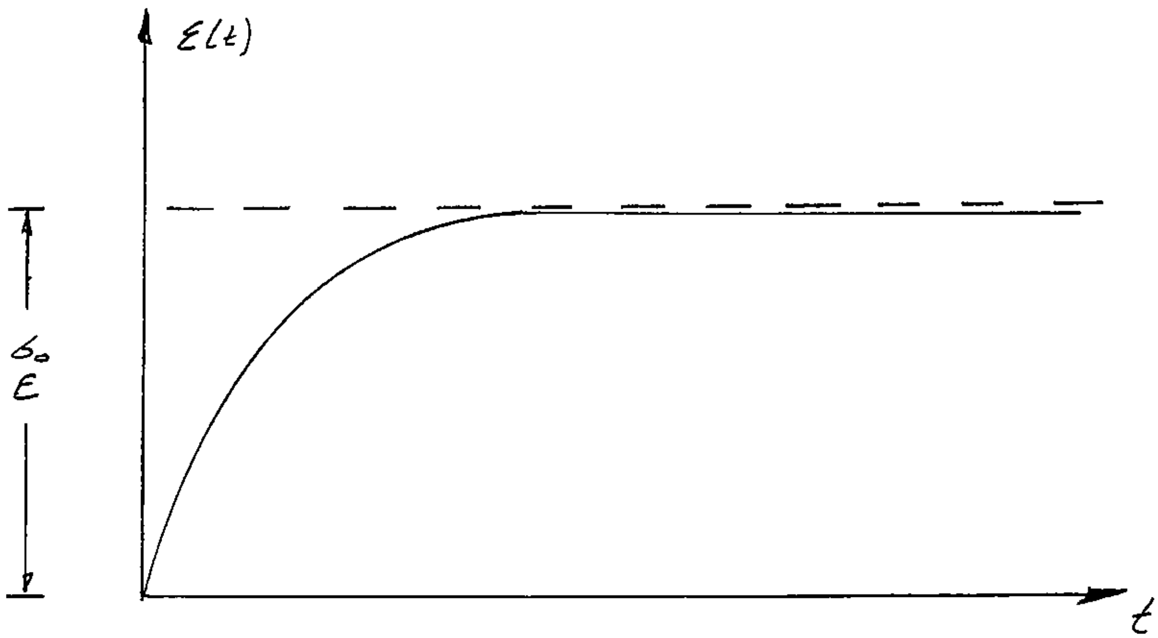
$$= \left(\frac{\sigma_0}{E}\right) H(t) - \left(\frac{\sigma_0}{E}\right) \exp\left[-\left(\frac{E}{\eta}\right) t\right]$$

NOTE THAT THE RATIO

$$\frac{\eta}{E}$$

IS SOMETIMES REFERRED TO AS THE "RETARDATION TIME" IN THE LITERATURE.

GRAPHICALLY, THE STRAIN RESPONSE IS AS FOLLOWS



NOTE THAT THIS MODEL COULD EASILY CAPTURE PRIMARY CREEP, HOWEVER FOR STEADY STATE CONDITIONS WHERE

$$\dot{\varepsilon}(t) = \text{CONSTANT}$$

THIS MODEL IS ONLY CAPABLE OF PREDICTING

$$\dot{\varepsilon}(t) = 0$$

SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

$$\sigma = E\varepsilon + \eta\left(\frac{d\varepsilon}{dt}\right)$$

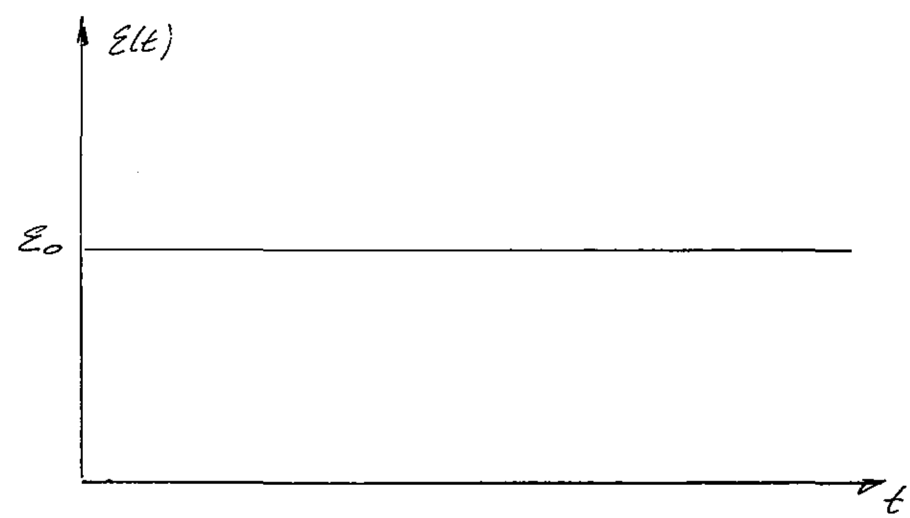
SUBJECT TO THE INITIAL CONDITION

$$\sigma = 0 \quad @ \quad t = 0$$

AND BOUNDARY CONDITION

$$\varepsilon(t) = [H(t)]\,\varepsilon_o$$

GRAPHICALLY



THUS WE ARE ATTEMPTING TO SOLVE THE STRESS RELAXATION BOUNDARY VALUE PROBLEM.

SOLUTION:

SUBSTITUTE FOR $\varepsilon(t)$ IN THE DIFFERENTIAL EQUATION

$$\sigma = E[H(t)]\,\varepsilon_o + \eta\frac{d}{dt}\left\{[H(t)]\,\varepsilon_o\right\}$$

$$\sigma = E\,[H(t)]\,\varepsilon_0 + \eta\left\{\left(\frac{dH}{dt}\right)\varepsilon_0 + H(t)\,\frac{d\varepsilon_0}{dt}\right\}$$

HOWEVER

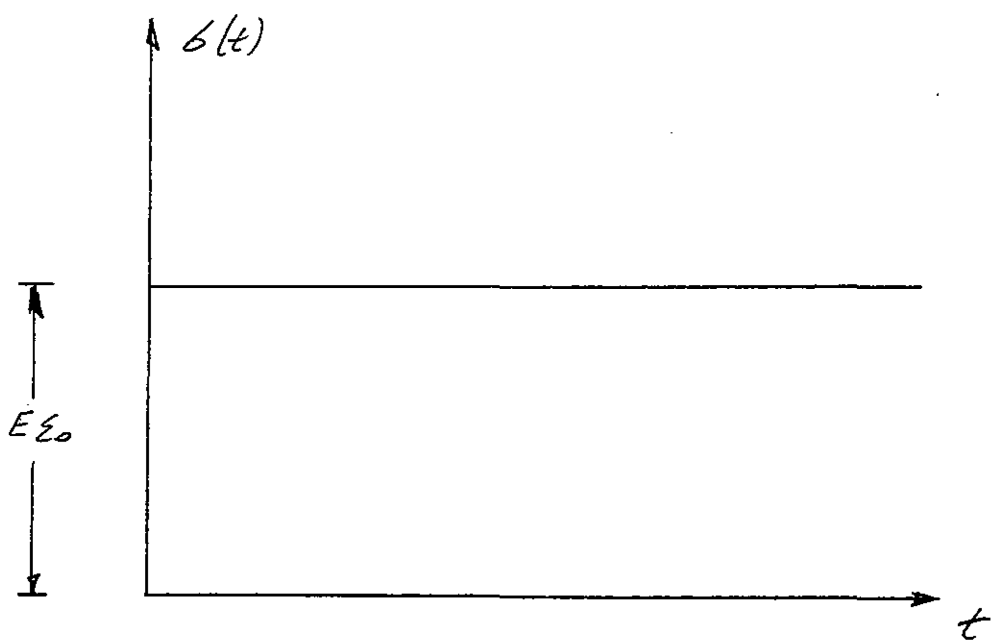
$$\frac{dH(t)}{dt} = 0$$

$$\frac{d\varepsilon_0}{dt} = 0$$

THUS

$$\sigma(t) = E\,\varepsilon_0\,[H(t)]$$

GRAPHICALLY



THREE ELEMENT MODEL

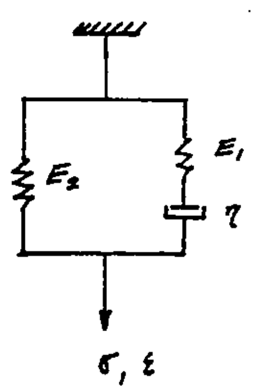
Recall that the Kelvin model

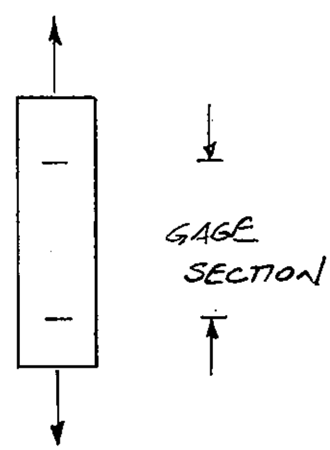
- captured the strain response for steady state creep
- captured the stress response adequately for relaxation tests
- could not model primary creep

Also recall that the Voigt model

- captured the primary creep response
- could not model steady state creep
- erroneously predicts constant stress for a relaxation test

It is reasonable to expect that if we combine the Kelvin and Voigt models with the simple spring or dashpot models (or with each other) that certain beneficial aspects of each model will be retained, and hopefully erroneous features will be eliminated. The first combination of the models places a spring in parallel with the Maxwell model. This model is depicted in the following figure





The operator equation is obtained by noting two features of this model, i.e.,

- The strain in the spring element will be the same as the strain in the Maxwell element. This is consistent with the kinematic requirements of the tensile specimen depicted above.

- The total stress (σ) is equal to the summation of the stress in each element, i.e.,

$$\sigma = \sigma_m + \sigma_s$$

where σ_m is the stress in the Maxwell element, and σ_s is the stress in the spring. Again, this consistent with the equilibrium requirements of the tensile specimen depicted above.

From the derivation of the Maxwell model

$$\left(\frac{d}{dt}\right)\epsilon = \left[\frac{1}{\eta} + \left(\frac{1}{E}\right)\frac{d}{dt}\right]\sigma_m$$

$$\sigma_m = \left[\frac{1}{\eta} + \left(\frac{1}{E}\right)\frac{d}{dt}\right]^{-1}\left(\frac{d}{dt}\right)\epsilon$$

The stress in the spring is given by the simple expression

$$\sigma_s = E_2\,\epsilon$$

thus

$$\sigma = \sigma_m + \sigma_s$$

$$= \left[\frac{1}{\eta} + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]^{-1}\left(\frac{d}{dt}\right)\epsilon + E_2\,\epsilon$$

$$\left[\frac{1}{\eta} + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma = \frac{d\epsilon}{dt} + \left(\frac{E_2}{\eta}\right)\epsilon + \left(\frac{E_2}{E_1}\right)\left(\frac{d\epsilon}{dt}\right)$$

$$\left[\frac{1}{\eta} + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma = \left[\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right)\left(\frac{d}{dt}\right)\right]\epsilon$$

Again, making use of the differential operators

$$P() = \sum_{k=0}^{N} p_k \frac{d^k()}{dt^k}$$

and

$$Q() = \sum_{k=0}^{M} q_k \frac{d^k()}{dt^k}$$

with

$$P[\sigma(t)] = Q[\epsilon(t)]$$

then for the three element model N=1, M=1, $p_0=1/\eta$, $p_1=1/E_1$, $q_0=E_2/\eta$, and q_1 has the following form

$$q_1 = \frac{E_1 + E_2}{E_1}$$

Again we will solve the governing differential equation for the Voigt model under two applied boundary conditions, i.e.,

- Creep (constant applied stress)

- Stress Relaxation (constant applied strain)

assuming uniaxial load/displacement conditions.

SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

$$\left[\left(\frac{1}{\eta}\right) + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma = \left[\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right)\frac{d}{dt}\right]\varepsilon$$

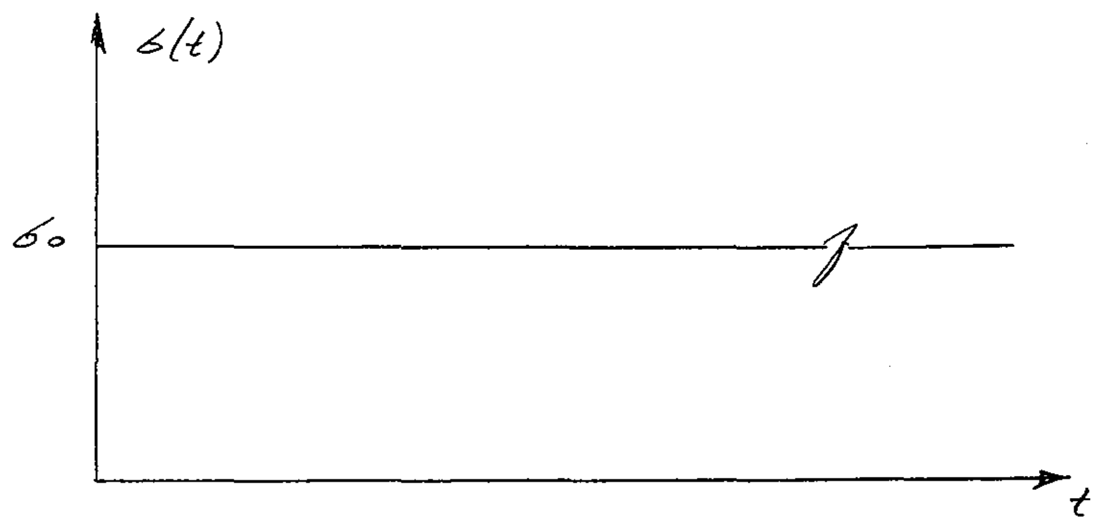
SUBJECT TO THE INITIAL CONDITION

$$\varepsilon = 0 \quad @ \quad t = 0$$

AND BOUNDARY CONDITIONS

$$\sigma(t) = [H(t)]\,\sigma_0$$

GRAPHICALLY



AGAIN WE ARE ATTEMPTING TO SOLVE THE CREEP BOUNDARY VALUE PROBLEM.

SOLUTION:

SUBSTITUTE FOR σ IN THE DIFFERENTIAL EQUATION. THUS

$$\left[\left(\frac{1}{\eta}\right) + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right][H(t)]\sigma_0 = \left[\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right)\frac{d}{dt}\right]\varepsilon$$

$$\left(\frac{\sigma_0}{\eta}\right) H(t) + \left(\frac{\sigma_0}{E_1}\right) \frac{dH(t)}{dt} = \left(\frac{E_2}{\eta}\right) \varepsilon + \left(\frac{E_1 + E_2}{E_1}\right) \frac{d\varepsilon}{dt}$$

$$\left(\frac{\sigma_0}{\eta}\right) H(t) + \left(\frac{\sigma_0}{E_1}\right) \delta(t) = \left(\frac{E_2}{\eta}\right) \varepsilon + \left(\frac{E_1 + E_2}{E_1}\right) \frac{d\varepsilon}{dt}$$

NOW TAKE THE LAPLACE TRANSFORM OF BOTH SIDES OF THIS EXPRESSION

$$\left(\frac{\sigma_0}{\eta}\right) \mathcal{L}\{H(t)\} + \left(\frac{\sigma_0}{E_1}\right) \mathcal{L}\{\delta(t)\}$$

$$= \left(\frac{E_2}{\eta}\right) \mathcal{L}\{\varepsilon\} + \left(\frac{E_1 + E_2}{E_1}\right) \mathcal{L}\left\{\frac{d\varepsilon}{dt}\right\}$$

$$\left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right) + \left(\frac{\sigma_0}{E_1}\right) = \left(\frac{E_2}{\eta}\right) y + \left(\frac{E_1 + E_2}{E_1}\right)\left\{ s y - \varepsilon(0)\right\}$$

$$\left[\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right) s\right] y = \left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right) + \left(\frac{\sigma_0}{E_1}\right)$$

$$y = \frac{\left(\frac{\sigma_0}{\eta}\right)\left(\frac{1}{s}\right) + \frac{\sigma_0}{E_1}}{\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right) s}$$

MULTIPLYING NUMERATOR AND DENOMINATOR BY S YIELDS

$$y = \frac{\left(\frac{\sigma_0}{\eta}\right) + \left(\frac{\sigma_0}{E_1}\right) s}{\left[\left(\frac{E_1 + E_2}{E_1}\right) s + \left(\frac{E_2}{\eta}\right)\right] s}$$

UTILIZING THE METHOD OF PARTIAL FRACTIONS, LET

$$\frac{\left(\frac{\sigma_0}{\eta}\right) + \left(\frac{\sigma_0}{E_1}\right)s}{\left[\left(\frac{E_1+E_2}{E_1}\right)s + \left(\frac{E_2}{\eta}\right)\right]s} = \frac{A}{\left(\frac{E_1+E_2}{E_1}\right)s + \left(\frac{E_2}{\eta}\right)} + \frac{B}{s}$$

NOW MULTIPLY BOTH SIDES OF THIS EXPRESSION BY THE DENOMINATOR ON THE LEFT HAND SIDE OF THE EQUAL SIGN. THIS YIELDS

$$\left(\frac{\sigma_0}{\eta}\right) + \left(\frac{\sigma_0}{E_1}\right)s = As + B\left[\left(\frac{E_1+E_2}{E_1}\right)s + \left(\frac{E_2}{\eta}\right)\right]$$

WE NOW HAVE TWO EQUATIONS IN TWO UNKNOWNS, I.E.,

$$\frac{\sigma_0}{\eta} = B\left(\frac{E_2}{\eta}\right)$$

AND

$$\frac{\sigma_0}{E_1} = A + B\left(\frac{E_1+E_2}{E_1}\right)$$

THUS

$$B = \frac{\sigma_0}{E_2}$$

AND

$$A = \frac{\sigma_0}{E_1} - \frac{\sigma_0}{E_2}\left(\frac{E_1+E_2}{E_1}\right)$$

$$= \left[1 - \left(\frac{E_1+E_2}{E_2}\right)\right]\frac{\sigma_0}{E_1}$$

$$= -\left(\frac{E_1}{E_2}\right)\frac{\sigma_0}{E_1}$$

$$= -\frac{\sigma_0}{E_1}$$

NOW

$$y = -\frac{\frac{\sigma_0}{E_1}}{\left(\frac{E_1+E_2}{E_1}\right)s + \left(\frac{E_2}{\eta}\right)} + \left(\frac{\sigma_0}{E_2}\right)\left(\frac{1}{s}\right)$$

TAKING THE INVERSE OF THIS EXPRESSION YIELDS

$$\varepsilon(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s + \frac{E_1 E_2}{\eta(E_1+E_2)}}\right\}\left(\frac{\sigma_0}{E_1+E_2}\right) + \left(\frac{\sigma_0}{E_2}\right)\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$= \left(\frac{\sigma_0}{E_2}\right)H(t) - \left(\frac{\sigma_0}{E_1+E_2}\right)\exp\left\{-\left[\frac{E_1 E_2}{\eta(E_1+E_2)}\right]t\right\}$$

NOTE THAT WHEN

$$t = 0$$

THEN

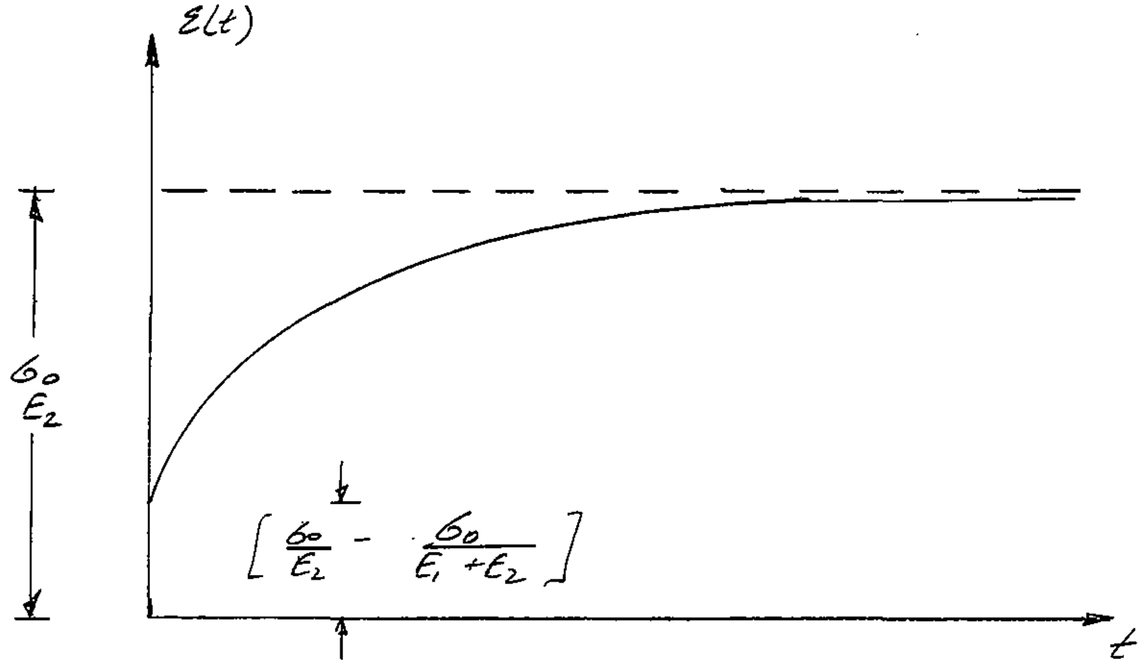
$$\varepsilon(0) = \frac{\sigma_0}{E_2} - \frac{\sigma_0}{E_1+E_2}$$

$$= \left[\frac{1}{E_2} - \frac{1}{E_1+E_2}\right]\sigma_0$$

AND IN THE LIMIT AS t APPROACHES INFINITY

$$\varepsilon(t\to\infty) = \frac{\sigma_0}{E_2}$$

GRAPHICALLY/



SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

$$\left[\left(\frac{1}{\eta}\right) + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma = \left[\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right)\frac{d}{dt}\right]\varepsilon$$

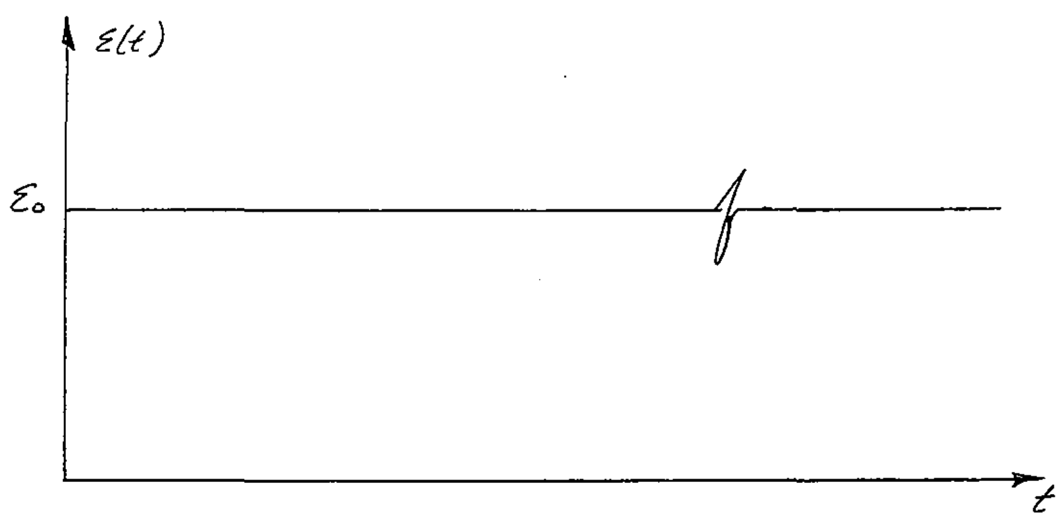
SUBJECT TO THE INITIAL CONDITION

$$\sigma = 0 \quad @\ t = 0$$

AND BOUNDARY CONDITIONS

$$\varepsilon(t) = [H(t)]\varepsilon_0$$

GRAPHICALLY



AGAIN, WE ARE ATTEMPTING TO SOLVE THE CLASSIC STRESS RELAXATION PROBLEM.

SOLUTION:

SUBSTITUTE FOR ε IN THE DIFFERENTIAL EQUATION. THUS

$$\left[\left(\frac{1}{\eta}\right) + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma = \left[\left(\frac{E_2}{\eta}\right) + \left(\frac{E_1 + E_2}{E_1}\right)\frac{d}{dt}\right][H(t)]\varepsilon_0$$

$$\left(\frac{1}{\eta}\right)\sigma + \left(\frac{1}{E_1}\right)\frac{d\sigma}{dt} = \left(\frac{\varepsilon_0 E_2}{\eta}\right)H(t) + \left[\frac{\varepsilon_0(E_1+E_2)}{E_1}\right]\frac{dH(t)}{dt}$$

$$\left(\frac{1}{\eta}\right)\sigma + \left(\frac{1}{E_1}\right)\frac{d\sigma}{dt} = \left(\frac{\varepsilon_0 E_2}{\eta}\right)H(t) + \left[\frac{\varepsilon_0(E_1+E_2)}{E_1}\right]\delta(t)$$

NOW TAKE THE LAPLACE TRANSFORM OF BOTH SIDES OF THIS EXPRESSION

$$\left(\frac{1}{\eta}\right)\mathcal{L}\{\sigma\} + \left(\frac{1}{E_1}\right)\mathcal{L}\left\{\frac{d\sigma}{dt}\right\}$$
$$= \left(\frac{\varepsilon_0 E_2}{\eta}\right)\mathcal{L}\{H(t)\} + \left[\frac{\varepsilon_0(E_1+E_2)}{E_1}\right]\mathcal{L}\{\delta(t)\}$$

$$\left(\frac{1}{\eta}\right)y + \left(\frac{1}{E_1}\right)\{sy - \sigma(0)\} = \left(\frac{\varepsilon_0 E_2}{\eta}\right)\left(\frac{1}{s}\right) + \frac{\varepsilon_0(E_1+E_2)}{E_1}$$

$$y\left\{\frac{1}{\eta} + \frac{s}{E_1}\right\} = \left(\frac{\varepsilon_0 E_2}{\eta}\right)\left(\frac{1}{s}\right) + \frac{\varepsilon_0(E_1+E_2)}{E_1}$$

$$y = \frac{\varepsilon_0\left\{\left(\frac{E_2}{\eta}\right)\left(\frac{1}{s}\right) + \left(\frac{E_1+E_2}{E_1}\right)\right\}}{\left[\frac{s}{E_1} + \frac{1}{\eta}\right]}$$

MULTIPLYING NUMERATOR AND DENOMINATOR BY S YIELDS

$$y = \frac{\varepsilon_0\left[\left(\frac{E_1+E_2}{E_1}\right)s + \left(\frac{E_2}{\eta}\right)\right]}{\left[\frac{s}{E_1} + \frac{1}{\eta}\right]s}$$

UTILIZING THE METHOD OF PARTIAL FRACTIONS LET

$$\frac{\varepsilon_0\left[\left(\frac{E_1+E_2}{E_1}\right)s + \frac{E_2}{\eta}\right]}{\left[\frac{s}{E_1} + \frac{1}{\eta}\right]s} = \frac{A}{\frac{s}{E_1} + \frac{1}{\eta}} + \frac{B}{s}$$

NOW MULTIPLY BOTH SIDES OF THIS EXPRESSION BY THE DENOMINATOR ON THE LEFT HAND SIDE OF THE EQUAL SIGN. THIS YIELDS

$$\mathcal{E}_0\left(\frac{E_1+E_2}{E_1}\right)s + \mathcal{E}_0\left(\frac{E_2}{\eta}\right) = As + B\left[\frac{s}{E_1}+\frac{1}{\eta}\right]$$

WE NOW HAVE TWO EQUATIONS IN TWO UNKNOWNS, I.E.

$$\mathcal{E}_0\left(\frac{E_2}{\eta}\right) = B\left(\frac{1}{\eta}\right)$$

AND

$$\mathcal{E}_0\left(\frac{E_1+E_2}{E_1}\right) = A + B\left(\frac{1}{E_1}\right)$$

THUS

$$B = E_2\mathcal{E}_0$$

AND

$$\mathcal{E}_0\left(\frac{E_1+E_2}{E_1}\right) = A + \mathcal{E}_0\left(\frac{E_2}{E_1}\right)$$

$$A = \mathcal{E}_0$$

NOW

$$y = \frac{\mathcal{E}_0}{\frac{s}{E_1}+\frac{1}{\eta}} + \frac{E_2\mathcal{E}_0}{s}$$

TAKING THE INVERSE TRANSFORM OF THIS EXPRESSION YIELDS

$$\sigma(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{E_1}{\eta}}\right\}(\mathcal{E}_0E_1) + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(\mathcal{E}_0E_2)$$

$$= (\mathcal{E}_0E_2)[H(t)] + (\mathcal{E}_0E_1)\exp\left[-\left(\frac{E_1}{\eta}\right)t\right]$$

NOTE THAT WHEN

$$t = 0$$

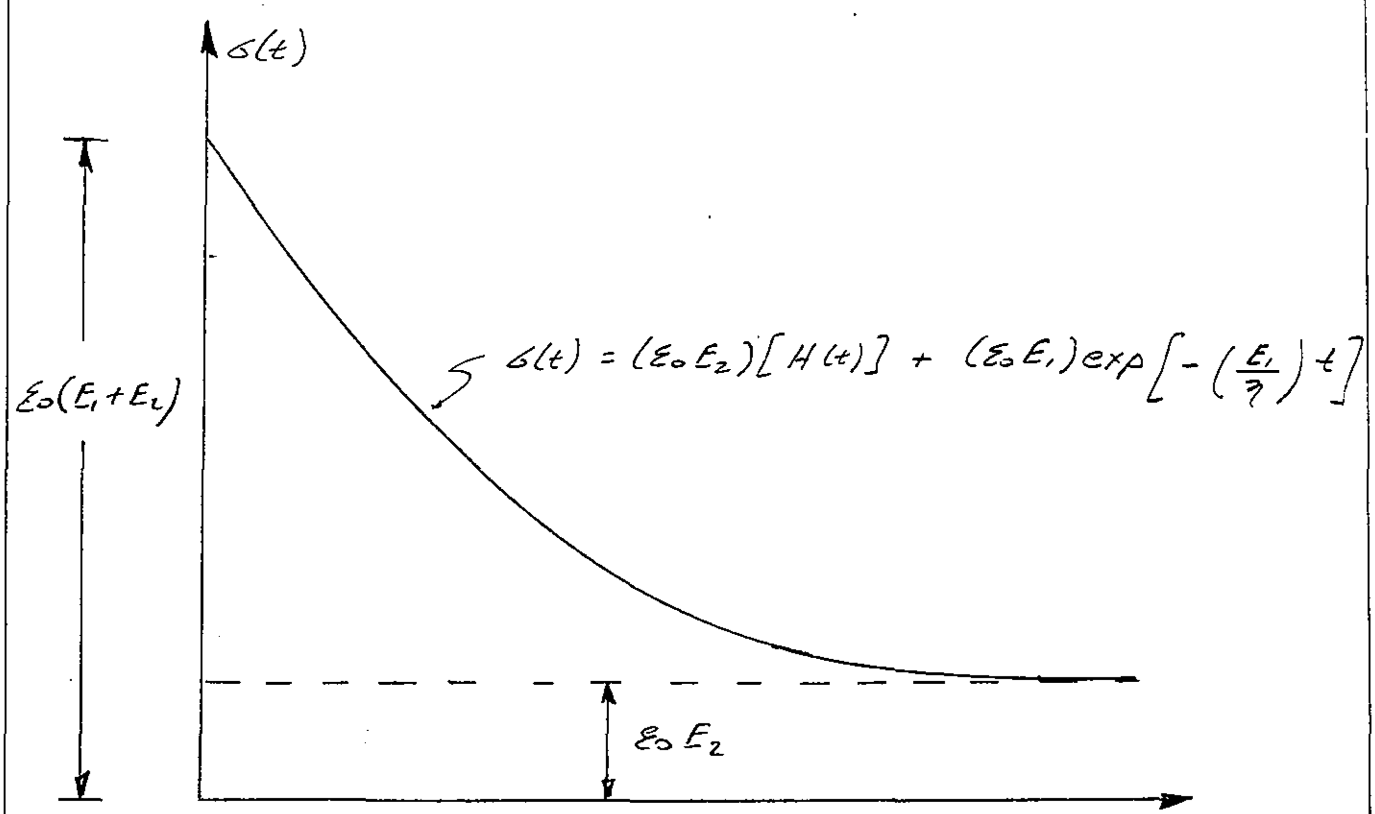
THEN

$$\sigma(0) = \mathcal{E}_0 E_2 + \mathcal{E}_0 E_1$$
$$= \mathcal{E}_0 (E_1 + E_2)$$

AND IN THE LIMIT AS t APPROACHES INFINITY

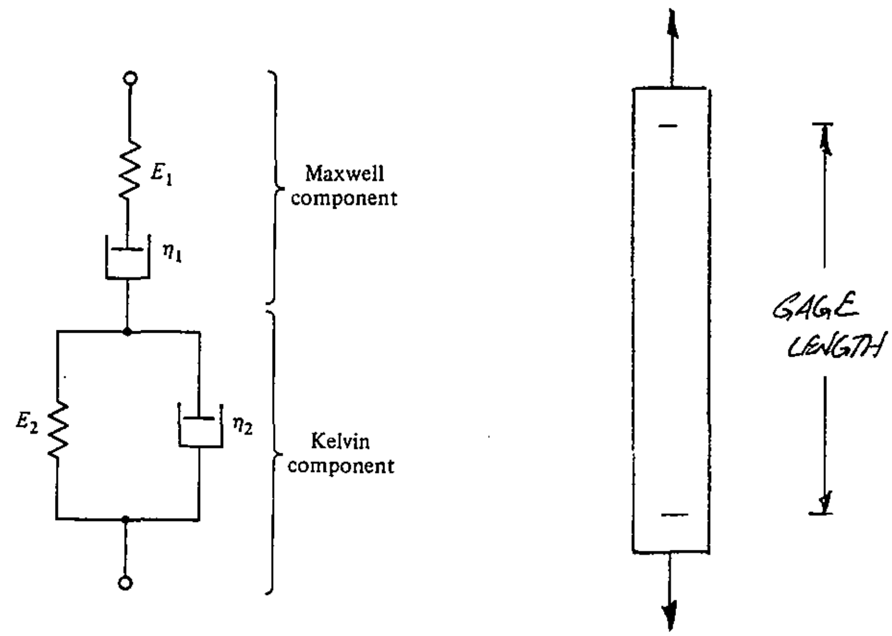
$$\sigma(t \to \infty) = \mathcal{E}_0 E_2$$

GRAPHICALLY



FOUR ELEMENT MODEL - BURGERS FLUID

Next, the theoretical structure is developed for a model that best represents observed material behavior, yet also minimizes the number spring and dashpot elements. This minimization also limits the number of material parameters needed to characterize the model. The model combines a Kelvin and a Maxwell element in series. This is depicted in the following figure



The operator equation is obtained by noting two features of this model, i.e.,

- The stress in the Kelvin element is the same as the stress in the Maxwell element. This is consistent with the equilibrium requirements of the tensile specimen depicted above. The equilibrium requirement is expressed mathematically as

$$\sigma = \sigma_k$$
$$= \sigma_m$$

where σ_m is the stress in the Maxwell element, and σ_k is the stress in the Kelvin model.

- The total strain (ϵ) is equal to the summation of the strain in each element, i.e.,

$$\epsilon = \epsilon_m + \epsilon_k$$

  Here ϵ_m is the strain in the Maxwell model, and ϵ_k is the strain in the Kelvin model. Expressing this last equation in rate form yields

$$\dot{\epsilon} = \dot{\epsilon}_m + \dot{\epsilon}_k$$

Recall that

$$\dot{\epsilon}_m = \frac{\sigma_m}{\eta_1} + \frac{\dot{\sigma}_m}{E_1}$$

$$\left(\frac{d}{dt}\right)\epsilon_m = \left[\frac{1}{\eta_1} + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma_m$$

was derived in the development of the Maxwell model, and

$$\sigma_k = \left[E_2 + \eta_2\left(\frac{d}{dt}\right)\right]\epsilon_k$$

was derived in the development of the Kelvin model. Taking the inverse of this last expression yields

$$\epsilon_k = \left[E_2 + \eta_2\left(\frac{d}{dt}\right)\right]^{-1}\sigma_k$$

The time-derivative of this relationship yields

$$\left(\frac{d}{dt}\right)\epsilon_k = \left(\frac{d}{dt}\right)\left[E_2 + \eta_2\left(\frac{d}{dt}\right)\right]^{-1}\sigma_k$$

Insertion of the derivatives of ϵ_m and ϵ_k into the rate equation for ϵ produces the following relationship

$$\left(\frac{d}{dt}\right)\epsilon = \left(\frac{d}{dt}\right)\epsilon_k + \left(\frac{d}{dt}\right)\epsilon_m$$

$$= \left(\frac{d}{dt}\right)\left[E_2 + \eta_2\left(\frac{d}{dt}\right)\right]^{-1}\sigma_k + \left[\frac{1}{\eta_1} + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma_m$$

Eliminating the inverse term leads to

$$\left[E_2 + \eta_2\left(\frac{d}{dt}\right)\right]\left(\frac{d}{dt}\right)\epsilon = \left(\frac{d}{dt}\right)\sigma + \left[E_2 + \eta_2\left(\frac{d}{dt}\right)\right]\left[\frac{1}{\eta_1} + \left(\frac{1}{E_1}\right)\frac{d}{dt}\right]\sigma$$

Note that the subscripts have been dropped on the stress terms due to equilibrium restrictions. Rearranging terms leads to

$$\left[E_2\left(\frac{d}{dt}\right) + \eta_2\left(\frac{d^2}{dt^2}\right)\right]\epsilon = \left[\left(\frac{d}{dt}\right) + \frac{E_2}{\eta_1} + \left(\frac{\eta_2}{\eta_1}\right)\left(\frac{d}{dt}\right) + \left(\frac{E_2}{E_1}\right)\left(\frac{d}{dt}\right) + \left(\frac{\eta_2}{E_1}\right)\left(\frac{d^2}{dt^2}\right)\right]\sigma$$

or

$$\left[E_2\left(\frac{d}{dt}\right) + \eta_2\left(\frac{d^2}{dt^2}\right)\right]\epsilon = \left[\frac{E_2}{\eta_1} + \left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)\left(\frac{d}{dt}\right) + \left(\frac{\eta_2}{E_1}\right)\left(\frac{d^2}{dt^2}\right)\right]\sigma$$

Again, making use of the differential operators

$$P() = \sum_{k=0}^{N} p_k \frac{d^k()}{dt^k}$$

and

$$Q() = \sum_{k=0}^{M} q_k \frac{d^k()}{dt^k}$$

with

$$P[\sigma(t)] = Q[\epsilon(t)]$$

then for the four element model N=2, M=2 and

$$p_0 = \frac{E_2}{\eta_1}$$

$$p_1 = 1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}$$

$$p_2 = \frac{\eta_2}{E_1}$$

$$q_0 = 0$$

$$q_1 = E_2$$

and

$$q_2 = \eta_2$$

Again we will solve the governing differential equation for the four-element model under two applied boundary conditions, i.e.,

- Creep (constant applied stress)

- Stress Relaxation (constant applied strain)

assuming uniaxial load/displacement conditions.

FIND THE LAPLACE TRANSFORM OF

$$\frac{d\,[H(t)]}{dt}$$

SOLUTION #1

BY DEFINITION

$$\delta(t-t_0) = \frac{d\,[H(t-t_0)]}{dt}$$

AND THE LAPLACE TRANSFORM OF THIS EXPRESSION

$$\mathcal{L}\{\delta(t-t_0)\} = \exp(-st_0)$$

WITH

$$t_0 = 0$$

THEN

$$\mathcal{L}\{\delta(t)\} = \exp(0)$$
$$= 1$$

THUS

$$\mathcal{L}\left\{\frac{d\,[H(t)]}{dt}\right\} = \mathcal{L}\{\delta(t)\}$$
$$= 1$$

SOLUTION #2

RECALL FROM THE DEVELOPMENT ON THE LAPLACE TRANSFORM OF DERIVATIVES THAT

$$\mathcal{L}\left\{\frac{d\,[H(t)]}{dt}\right\} = s\,\mathcal{L}\{H(t)\} - H(0)$$
$$= s\left(\frac{1}{s}\right) - H(0)$$

OR

$$\mathcal{L}\left\{\frac{d[H(t)]}{dt}\right\} = 1 - H(0)$$

HOWEVER, TO RECONCILE THIS WITH SOLUTION #1

$$H(0) = H(0^-)$$
$$= 0$$

THUS IF WE USE THE LAPLACE TRANSFORM OF DERIVATIVES ON THE HEAVISIDE STEP FUNCTION WE MUST EVALUATE THE FUNCTION AT $\underline{\underline{0^-}}$.

SOLVE THE FOLLOWING DIFFERENTIAL EQUATION

$$\left[E_2\left(\frac{d}{dt}\right) + \eta_2\left(\frac{d^2}{dt^2}\right)\right]\varepsilon = \left[\frac{E_2}{\eta_1} + \left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)\frac{d}{dt} + \left(\frac{\eta_2}{E_1}\right)\left(\frac{d^2}{dt^2}\right)\right]\sigma$$

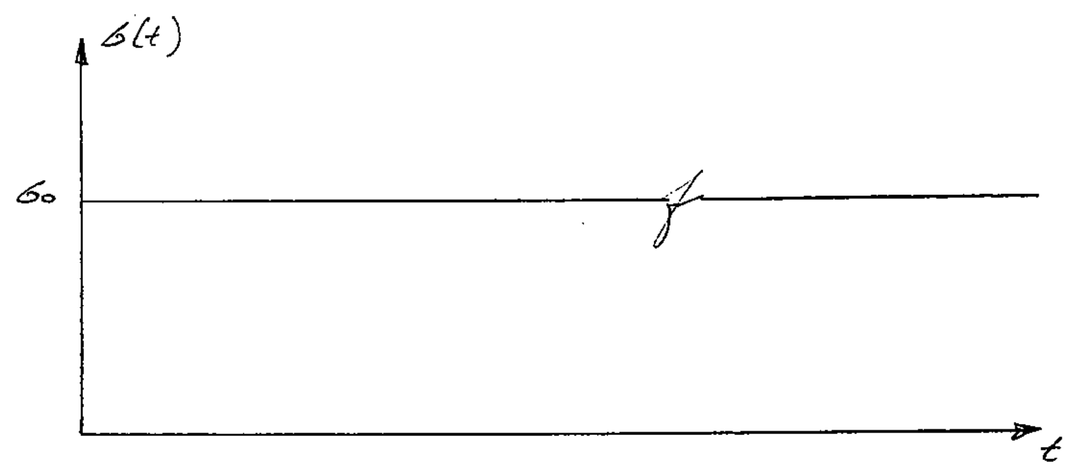
SUBJECT TO THE INITIAL CONDITION

$$\varepsilon = 0 \quad @ \quad t = 0$$

AND BOUNDARY CONDITIONS

$$\sigma(t) = [H(t)]\,\sigma_0$$

GRAPHICALLY



AGAIN, WE ARE ATTEMPTING TO SOLVE THE CLASSIC CREEP BOUNDARY VALUE PROBLEM.

SOLUTION:

SUBSTITUTE FOR σ IN THE DIFFERENTIAL EQUATION. THUS

$$\left[E_2\left(\frac{d}{dt}\right) + \eta_2\left(\frac{d^2}{dt^2}\right)\right]\varepsilon = \left[\frac{E_2}{\eta_1} + \left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)\frac{d}{dt} + \left(\frac{\eta_2}{E_1}\right)\left(\frac{d^2}{dt^2}\right)\right][H(t)]\,\sigma_0$$

NOW TAKE THE LAPLACE TRANSFORM OF BOTH SIDES OF THIS EXPRESSION

$$E_2 \mathcal{L}\left\{\frac{d\varepsilon}{dt}\right\} + \eta_2 \mathcal{L}\left\{\frac{d^2\varepsilon}{dt^2}\right\} = \left(\frac{E_2 \sigma_0}{\eta_1}\right) \mathcal{L}\{H(t)\}$$

$$+ \sigma_0\left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right) \mathcal{L}\left\{\frac{d[H(t)]}{dt}\right\}$$

$$+ \sigma_0\left(\frac{\eta_2}{E_1}\right) \mathcal{L}\left\{\frac{d^2[H(t)]}{dt^2}\right\}$$

NOTE THAT

$$\mathcal{L}\left\{\frac{d[H(t)]}{dt}\right\} = \mathcal{L}\{\delta(t)\}$$

$$= 1$$

HOWEVER

$$\mathcal{L}\left\{\frac{d^2[H(t)]}{dt^2}\right\} = \mathcal{L}\left\{\frac{d^2[f(t)]}{dt^2}\right\}$$

$$= s^2 \mathcal{L}\{H(t)\} - s H(0) - \delta(0)$$

$$= s^2\left(\frac{1}{s}\right) - s[H(0)] - \delta(0)$$

$$= s[1 - H(0)] - \delta(0)$$

NOW RECALL EARLIER THAT IF WE USE THE LAPLACE TRANSFORM OF DERIVATIVES WITH THE HEAVISIDE STEP FUNCTION THE INITIAL CONDITIONS MUST BE EVALUATED AT 0^-, THUS

$$\mathcal{L}\left\{\frac{d^2[H(t)]}{dt^2}\right\} = s[1 - H(0^-)] - \delta(0^-)$$

$$= s[1 - 0] - 0$$

$$= s$$

AND THE LAPLACE TRANSFORM OF THE DIFFERENTIAL EQUATION BECOMES

$$E_2\left[sy - \varepsilon(0)\right] + \eta_2\left[s^2 y - s\,\varepsilon(0) - \dot{\varepsilon}(0)\right] = \left(\frac{E_2\sigma_0}{\eta_1}\right)\frac{1}{s}$$

$$+\ \sigma_0\left[1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right](1) + \left(\frac{\sigma_0\eta_2}{E_1}\right)s$$

NOTE THAT

$$\dot{\varepsilon}(0^-) = 0 \qquad\qquad \dot{\varepsilon}(0^+) \neq 0$$

THUS

$$(E_2 s + \eta_2 s^2)y = \left(\frac{E_2\sigma_0}{\eta_1}\right)\frac{1}{s} + \left(\frac{\sigma_0\eta_2}{E_1}\right)s$$

$$+\ \sigma_0\left[1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right]$$

$$(E_2 s + \eta_2 s^2)y = \left[\sigma_0\left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)\right] + \left(\frac{\sigma_0 E_2}{\eta_1}\right)\frac{1}{s}$$

$$+\ \left(\frac{\sigma_0\eta_2}{E_1}\right)s$$

$$y = \frac{\dfrac{\sigma_0 E_2}{\eta_1} + \left[\sigma_0\left(1 + \dfrac{\eta_2}{\eta_1} + \dfrac{E_2}{E_1}\right)\right]s + \left(\dfrac{\sigma_0\eta_2}{E_1}\right)s^2}{(E_2 + \eta_2 s)s^2}$$

MAKING USE OF THE SOLUTION METHOD OF PARTIAL FRACTIONS LET

$$\frac{\dfrac{\sigma_0 E_2}{\eta_1} + \left[\sigma_0\left(1 + \dfrac{\eta_2}{\eta_1} + \dfrac{E_2}{E_1}\right)\right]s + \left(\dfrac{\sigma_0\eta_2}{E_1}\right)s^2}{(E_2 + \eta_2 s)s^2}$$

$$= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(E_2 + \eta_2 s)}$$

MULTIPLYING BOTH SIDES BY THE DENOMINATOR ON THE LEFT HAND SIDE OF THE EQUAL SIGN YIELDS

$$\frac{\sigma_0 E_2}{\eta_1} + \sigma_0\left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)s + \left(\frac{\sigma_0 \eta_2}{E_1}\right)s^2 = A(E_2 + \eta_2 s) + B(E_2 + \eta_2 s)s + Cs^2$$

WE NOW HAVE THREE EQUATIONS IN THREE UNKNOWNS

$$C + B\eta_2 = \frac{\sigma_0 \eta_2}{E_1}$$

ALSO

$$BE_2 + A\eta_2 = \sigma_0\left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)$$

AND

$$AE_2 = \frac{\sigma_0 E_2}{\eta_1}$$

THUS

$$A = \frac{\sigma_0}{\eta_1}$$

THEN

$$BE_2 + \left(\frac{\sigma_0}{\eta_1}\right)\eta_2 = \sigma_0\left(1 + \frac{\eta_2}{\eta_1} + \frac{E_2}{E_1}\right)$$

$$BE_2 = \sigma_0\left(1 + \frac{E_2}{E_1}\right)$$

$$B = \frac{\sigma_0}{E_2}\left(\frac{E_1 + E_2}{E_1}\right)$$

$$= \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right)$$

FINALLY

$$C + \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right)\eta_2 = \frac{\sigma_0 \eta_2}{E_1}$$

$$C = \frac{\sigma_0 \eta_2}{E_1}\left(1 - \frac{E_1 + E_2}{E_2}\right)$$

$$= \frac{\sigma_0 \eta_2}{E_1}\left(\frac{E_2 - E_1 - E_2}{E_2}\right)$$

$$= -\frac{\sigma_0 \eta_2}{E_1}\left(\frac{E_1}{E_2}\right)$$

$$= -\frac{\sigma_0 \eta_2}{E_2}$$

NOW

$$y = \left(\frac{\sigma_0}{\eta_1}\right)\left(\frac{1}{s^2}\right) + \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right)\left(\frac{1}{s}\right) - \left(\frac{\sigma_0 \eta_2}{E_2}\right)\frac{1}{(E_2 + \eta_2 s)}$$

TAKING THE INVERSE LAPLACE TRANSFORM OF THIS EXPRESSION YIELDS

$$\varepsilon(t) = \left(\frac{\sigma_0}{\eta_1}\right)\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right)\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$
$$- \left(\frac{\sigma_0 \eta_2}{E_2}\right)\mathcal{L}^{-1}\left\{\frac{1}{E_2 + \eta_2 s}\right\}$$

$$= \left(\frac{\sigma_0}{\eta_1}\right)t + \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right)H(t)$$
$$- \left(\frac{\sigma_0 \eta_2}{E_2}\right)\left(\frac{1}{\eta_2}\right)\mathcal{L}^{-1}\left\{\frac{1}{(E_2/\eta_2) + s}\right\}$$

$$= \left(\frac{\sigma_0}{\eta_1}\right)t + \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right)H(t)$$
$$- \left(\frac{\sigma_0}{E_2}\right)\exp\left[-\left(\frac{E_2}{\eta_2}\right)t\right]$$

AT

$$t = 0^+$$

THEN

$$\varepsilon(0^+) = 0 + \sigma_0\left(\frac{E_1 + E_2}{E_1 E_2}\right) - \frac{\sigma_0}{E_2}$$

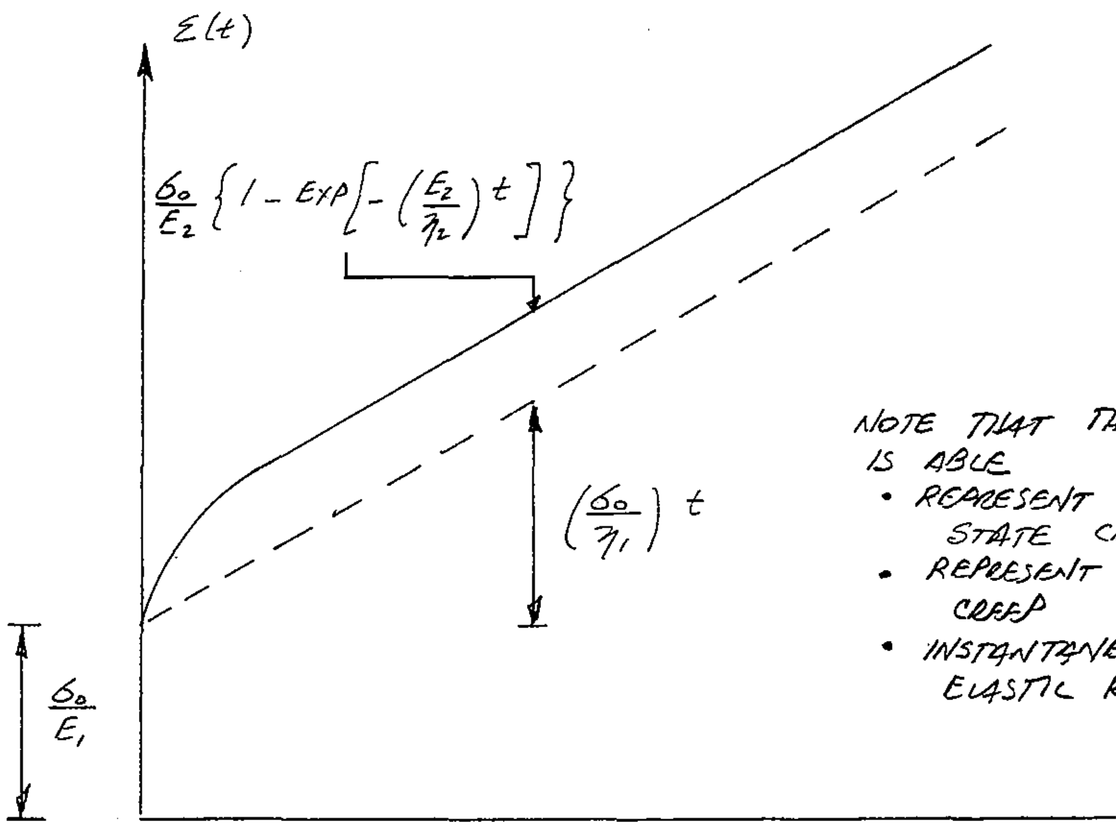
$$= \frac{\sigma_0}{E_2} + \frac{\sigma_0}{E_1} - \frac{\sigma_0}{E_2}$$

$$= \frac{\sigma_0}{E_1}$$

AS t APPROACHES INFINITY THE TERM

$$\left(\frac{\sigma_0}{\eta_1}\right) t$$

APPROACHES INFINITY IN A LINEAR FASHION. GRAPHICALLY THE STRAIN FUNCTION FOR THE FOUR ELEMENT MODEL CAN BE DEPICTED AS FOLLOWS



NOTE THAT THIS MODEL IS ABLE

- REPRESENT STEADY STATE CREEP
- REPRESENT PRIMARY CREEP
- INSTANTANEOUS ELASTIC RESPONSE

THE CREEP COMPLIANCE, RELAXATION MODULUS & HEREDITARY INTEGRALS

A number of viscoelastic models have been studied making use of two fundamental experiments, i.e.,

- Creep test
- Relaxation test

The functional relationships developed for the creep test can be generalized by the following expression

$$\epsilon(t) = \sigma_0 J(t)$$

where J(t) is known as the creep compliance. A similar expression can be developed for the relaxation test. This expression takes the following form

$$\sigma(t) = \epsilon_0 G(t)$$

MAXWELL

where G(t) is the relaxation modulus. Note that for the ~~Kelvin~~ model

$$J(t) = \frac{t}{\eta} + \frac{\cancel{H(t)}}{E} = \left(\frac{1}{E} + \frac{t}{\eta}\right)$$

$$G(t) = E\left\{\exp\left[-\left(\frac{E}{\eta}\right)t\right]\right\}$$

Similar relationships can be developed for all the models presented. Note that linearity is assumed for these relationships, i.e., if

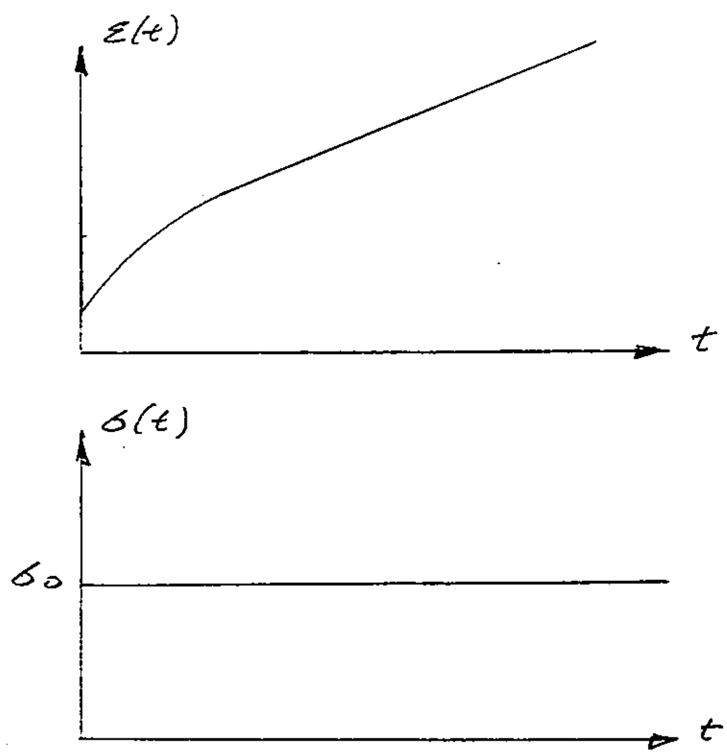
$$\varepsilon(t) = \sigma_0 J(t)$$

then

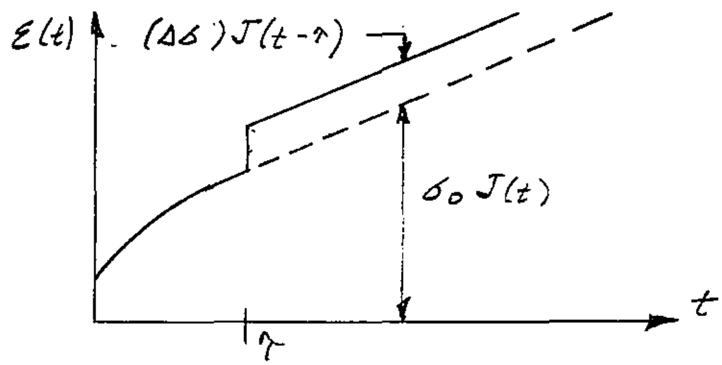
$$2\,\varepsilon(t) = 2\,\sigma_0 J(t)$$

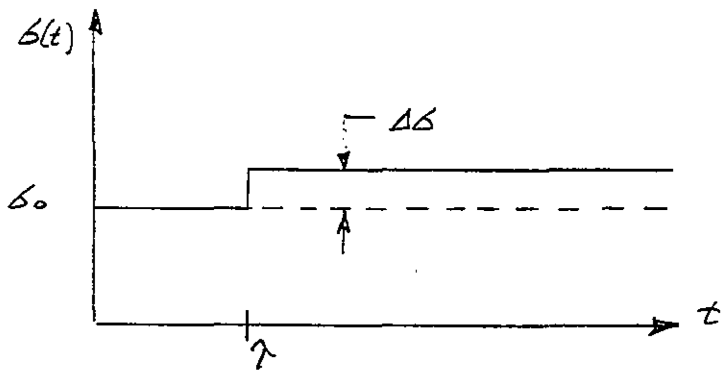
Thus if we double the load ($2\sigma_0$) the strain response is doubled. In engineering mechanics a material that exhibits this kind of behavior is referred to as a "linear material."

Recall that during a creep test the basic response of a uniaxial test specimen to a step load can be represented graphically as follows:

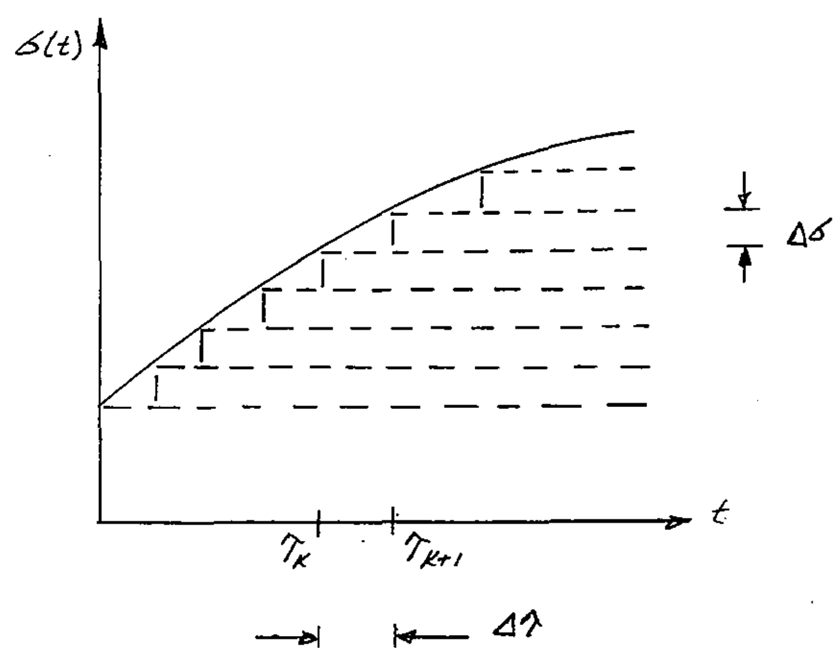


Now if at some time τ the load in a creep test is increased from σ_0 to $(\sigma_0 + \Delta\sigma)$ then because the material is linear the response to the increased load would appear as follows





Now consider the situation where the load is increased as follows



As was indicated in the figure, the smooth increase in load can be approximated as a series of small step increases where

$$\Delta\sigma = \left(\frac{\Delta\sigma}{\Delta\tau}\right)\Delta\tau$$

In the limit as $\Delta\tau$ approaches zero this expression becomes

$$d\sigma = \left(\frac{d\sigma}{d\tau}\right)d\tau$$

If the material is linear the strain response can be characterized as

$$\epsilon(t) = \sigma_0 J(t) + \lim_{N\to\infty}\left\{\sum_{i=1}^{N} J(t - \tau_i)\,\Delta\sigma_i\right\}$$

The second term represents a Riemann sum, thus

$$\epsilon(t) = \sigma_0 J(t) + \int_{0^+}^{t} J(t - \tau)\,d\sigma$$

Substituting for $d\sigma$ yields

$$\varepsilon(t) = \sigma_0 J(t) + \int_{0^+}^{t} J(t - \tau)\left[\frac{d\sigma(\tau)}{d\tau}\right] d\tau$$

Note that if the stress history can be represented functionally, i.e., if $\sigma(t)$ is known, then the integral above can be evaluated. In addition, if the stress function includes the instaneous load application then the first term can be tucked inside the integral by extending the lower limit of integration. Thus

$$\varepsilon(t) = \int_{0}^{t} J(t - \tau)\left[\frac{d\sigma(\tau)}{d\tau}\right] d\tau$$

A similar integral can be developed for the stress relaxation test where

$$\sigma(t) = \int_{0}^{t} G(t - \tau)\left[\frac{d\varepsilon(\tau)}{d\tau}\right] d\tau$$

These two integrals are referred to as ***HERIDITARY INTEGRALS*** in viscoelasticity. They are hereditary in the sense that to evaluate either the entire stress history, or the entire strain history must be known a priori.

To close this discussion, consider the standard creep test where

$$\sigma(t) = \sigma_0 H(t)$$

The derivative of this expression with respect to time yields

$$\frac{d\sigma}{dt} = \sigma_0 \, [\delta \, (t)]$$

Thus

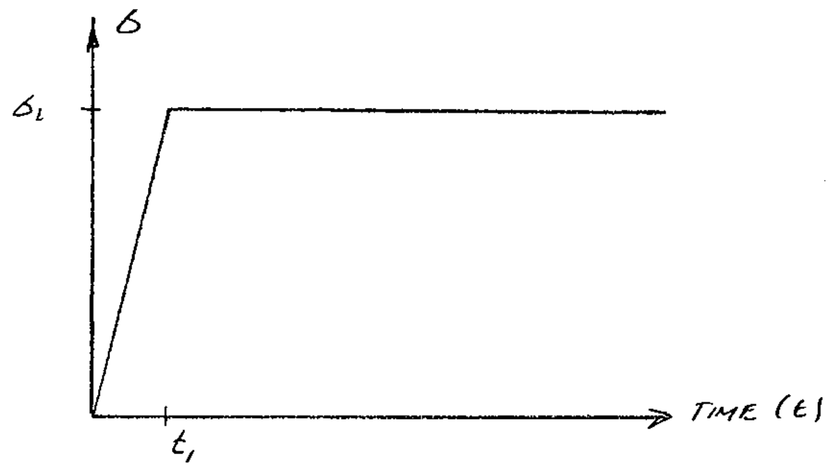
$$\varepsilon \, (t) = \int_0^t J(t - \tau) \left\{ \sigma_0 \, [\delta \, (\tau)] \right\} d\tau$$

which due to the sifting property of the delta function yields

$$\varepsilon \, (t) = \sigma_0 \, J(t)$$

The conclusion here is that the hereditary integrals will simplify to correct expressions.

FOR THE FOLLOWING LOAD HISTORY



DEVELOP AN EXPRESSION FOR $\varepsilon(t)$ USING THE HERIDITARY INTEGRAL APPROACH. ASSUME A MAXWELL MODEL.

SOLUTION

THE LOAD HISTORY CAN BE CHARACTERIZED MATHEMATICALLY AS

$$\sigma(t) = \left(\frac{t}{t_1}\right)\sigma_1 \qquad 0 \le t \le t_1$$

$$\sigma(t) = \sigma_1 \qquad t \ge t_1$$

RECALL THAT FOR A MAXWELL MODEL

$$\varepsilon(t) = \sigma_0 J(t) + \int_0^t J(t-\tau)\left[\frac{d\sigma(t)}{d\tau}\right] d\tau$$

WITH

$$\sigma_0 = 0$$

THEN

$$\varepsilon(t) = (0) J(t) + \int_0^t J(t-\tau)\left[\frac{d\sigma(\tau)}{d\tau}\right] d\tau$$

$$= \int_0^t J(t-\tau)\left[\frac{d\sigma(\tau)}{d\tau}\right] d\tau$$

GIVEN THE EXPRESSIONS FOR $\sigma(t)$, THEN

$$\frac{d\sigma}{dt} = \begin{cases} \dfrac{\sigma_1}{t_1} & 0 \le t \le t_1 \\ 0 & t \ge t_1 \end{cases}$$

THUS

$$\varepsilon(t) = \int_0^{t_1} J(t-\tau)\left(\frac{\sigma_1}{t_1}\right) d\tau - \int_{t_1}^{t} J(t-\tau)(0)\, d\tau$$

$$= \int_0^{t_1} J(t-\tau)\left(\frac{\sigma_1}{t_1}\right) d\tau$$

FOR A MAXWELL MODEL, BY DEFINITION

$$J(t-\tau) = \frac{1}{E} + \frac{t-\tau}{\eta}$$

THUS

$$\varepsilon(t) = \int_0^{t_1} \left[\frac{1}{E} + \frac{t}{\eta} - \frac{\tau}{\eta}\right]\left(\frac{\sigma}{t_1}\right) d\tau$$

$$= \left(\frac{1}{E} + \frac{t}{\eta}\right)\left(\frac{\sigma_1}{t_1}\right)\int_0^{t_1} d\tau - \left(\frac{1}{\eta}\right)\left(\frac{\sigma_1}{t_1}\right)\int_0^{t_1} \tau\, d\tau$$

$$= \left(\frac{\sigma_1}{t_1}\right)\left[\left(\frac{1}{E} + \frac{t}{\eta}\right)\right]\tau\Big|_0^{t_1} - \left(\frac{1}{\eta}\right)\left(\frac{\tau^2}{2}\right)\Big|_0^{t_1}\Big\}$$

$$= \left(\frac{\sigma_1}{t_1}\right)\left[\left(\frac{1}{E} + \frac{t}{\eta}\right)t_1 - \left(\frac{1}{\eta}\right)\left(\frac{t_1^2}{2}\right)\right\}$$

$$= \sigma_1\left[\frac{1}{E} + \frac{t}{\eta} - \frac{t_1}{2\eta}\right]$$

$$= \sigma_1\left[\frac{1}{E} + \frac{2t - t_1}{2\eta}\right] \qquad t \ge t_1$$

LAPLACE TRANSFORMS OF THE OPERATORS

RECALL THAT THE LAPLACE TRANSFORM OF $f(t)$ IS DEFINED AS

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

ALSO RECALL THAT

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = -f(0) + s\bar{f}(s)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = -\frac{df(0)}{dt} - s f(0) + s^2 \bar{f}(s)$$

OR IN GENERAL

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = -\frac{d^{n-1}f(0)}{dt^{n-1}} - \dots - s^{n-1} f(0) + s^n \bar{f}(s)$$

NOW RECALL THAT

$$P[\sigma(t)] = \sum_{k=0} p_k \left[\frac{d^k \sigma(t)}{dt^k}\right]$$

THE LAPLACE TRANSFORM OF THIS EXPRESSION IS

$$\mathcal{L}\{P[\sigma(t)]\} = \bar{P}(s)\,\bar{\sigma}(s) - \sum_{k=1} p_k \left[s^{k-1}\sigma(0) + s^{k-2}\frac{d\sigma(0)}{dt} + \dots + \frac{d^{k-1}\sigma(0)}{dt^{k-1}} \right]$$

SIMILARLY RECALL THAT

$$Q[\varepsilon(t)] = \sum_{k=0} q_k \left[\frac{d^k \varepsilon(t)}{dt^k}\right]$$

THE LAPLACE TRANSFORM OF THIS EXPRESSION LEADS TO

$$\mathcal{L}\{Q[\varepsilon(t)]\} = \bar{Q}(s)\,\bar{\varepsilon}(s) - \sum_{k=1} q_k \left[s^{k-1}\varepsilon(0) + s^{k-2}\frac{d\varepsilon(0)}{dt} + \dots + \frac{d^{k-1}\varepsilon(0)}{dt^{k-1}} \right]$$

THUS THE LAPLACE TRANSFORM OF

$$P[\sigma(t)] = Q[\varepsilon(t)]$$

LEADS TO

$$\bar{P}(s)\,\bar{\sigma}(s) - \sum_{k=1} p_k \left[s^{k-1}\sigma(0) + s^{k-2}\frac{d\sigma(0)}{dt} + \cdots + \frac{d^{k-1}\sigma(0)}{dt^{k-1}} \right]$$

$$= \bar{Q}(s)\,\bar{\varepsilon}(s) - \sum_{k=1} q_k \left[s^{k-1}\varepsilon(0) + s^{k-2}\frac{d\varepsilon(0)}{dt} + \cdots + \frac{d^{k-1}\varepsilon(0)}{dt^{k-1}} \right]$$

WHERE

$$\bar{P}(s) = \sum_{k=0} p_k s^k$$

$$\bar{Q}(s) = \sum_{k=0} q_k s^k$$

IF WE ASSUME THAT

$$\sigma(0) = \frac{d\sigma(0)}{dt} = \cdots = \frac{d^{k-1}\sigma(0)}{dt^{k-1}} = 0$$

$$\varepsilon(0) = \frac{d\varepsilon(0)}{dt} = \cdots = \frac{d^{k-1}\varepsilon(0)}{dt^{k-1}} = 0$$

WHICH MUST BE ENFORCED AT 0^-, THEN

$$\bar{P}(s)\,\bar{\sigma}(s) = \bar{Q}(s)\,\bar{\varepsilon}(s)$$

IF WE NOW LOOK AT A CREEP TEST WITH

$$\sigma_0 = 1$$

THEN

$$\sigma = (1)\,H(t)$$

$$\varepsilon = J(t)$$

THE TRANSFORMS OF THESE ARE

$$\bar{\sigma}(s) = \frac{1}{s}$$

$$\bar{\varepsilon}(s) = \bar{J}(s)$$

THUS

$$\bar{P}(s)\,\bar{\sigma}(s) = \bar{Q}(s)\,\bar{\varepsilon}(s)$$

$$\bar{P}(s)\left(\frac{1}{s}\right) = \bar{Q}(s)\,\bar{J}(s)$$

WHICH LEADS TO

$$\bar{J}(s) = \frac{\bar{P}(s)}{s\,\bar{Q}(s)}$$

IF WE LOOK AT A RELAXATION TEST WITH

$$\varepsilon_0 = 1$$

THEN

$$\varepsilon(t) = H(t)$$

$$\sigma(t) = G(t)$$

THE TRANSFORMS OF THESE ARE

$$\bar{\varepsilon}(s) = \frac{1}{s}$$

$$\bar{\sigma}(s) = \bar{G}(s)$$

WHICH LEADS TO

$$\bar{P}(s)\,\bar{\sigma}(s) = \bar{Q}(s)\,\bar{\varepsilon}(s)$$

$$\bar{P}(s)\,\bar{G}(s) = \bar{Q}(s)\left(\frac{1}{s}\right)$$

$$\bar{G}(s) = \frac{\bar{Q}(s)}{s\,\bar{P}(s)}$$

SINCE

$$s\,\bar{G}(s) = \frac{\bar{Q}(s)}{\bar{P}(s)}$$

AND

$$s\,\bar{J}(s) = \frac{\bar{P}(s)}{\bar{Q}(s)}$$

THEN

$$s^2 \bar{G}(s) \; \bar{J}(s) = \left[\frac{\bar{Q}(s)}{\bar{P}(s)}\right]\left[\frac{\bar{P}(s)}{\bar{Q}(s)}\right]$$

$$= 1.0$$

OR

$$\bar{G}(s) \; \bar{J}(s) = \frac{1}{s^2}$$

CYCLIC LOADING

THE REAL EXPONENTIAL FUNCTION HAS THE FOLLOWING PROPERTY

$$e^{x+y} = \exp(x+y)$$
$$= \exp(x)\exp(y)$$

THE COMPLEX FUNCTION

$$z = x + iy$$

CAN HELP DEFINE THE FOLLOWING COMPLEX EXPONENTIAL FUNCTION

$$\exp(z) = \exp(x+iy)$$
$$= \exp(x)\exp(iy)$$
$$= \exp(x)\left[\cos(y) + i\sin(y)\right]$$

FROM WHICH WE CAN EXTRACT THE EULER FORMULA

$$\exp(iy) = \cos(y) + i\sin(y)$$

SUPPOSE WE SUBJECT A UNIAXIAL TEST SPECIMEN TO A SINUSOIDAL LOAD HISTORY

$$\sigma(t) = (\sigma_0)\exp(i\omega t)$$
$$= (\sigma_0)\cos(\omega t) + (i\sigma_0)\sin(\omega t)$$

WE NOW CONSIDER THE STEADY STATE RESPONSE OF THE TEST SPECIMEN TO THIS FORCE VIBRATION, WITH CONSTANT FREQUENCY ω. FOR A LINEAR SYSTEM, BOTH STRESS AND STRAIN WILL VARY SINUSOIDALLY WITH THE SAME FORCING FREQUENCY, ω, THUS

$$\varepsilon(t) = (\varepsilon_0)\exp(i\omega t)$$

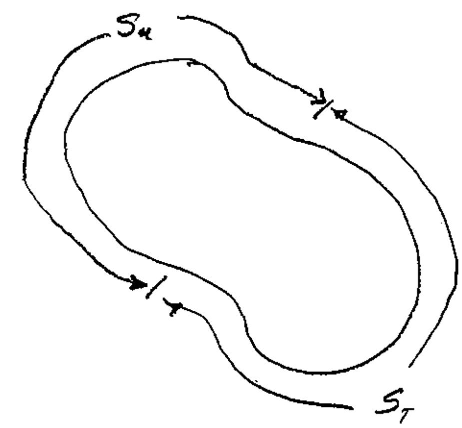
BECAUSE OF THE INFLUENCE OF DELAYED ELASTICITY AND VISCOUS FLOW, THE STRESS AND STRAIN WILL, IN GENERAL NOT BE IN PHASE. THUS IN GENERAL

$$\varepsilon(t) = (\varepsilon_0)\exp\left[i\omega t + i\delta t\right]$$
$$= (\varepsilon_0)\exp(i\delta t)\exp(i\omega t)$$

WHERE δ IS A PHASE ANGLE.

CORRESPONDANCE PRINCIPLE

REVISITING POTATO MECHANICS ONLY THIS TIME WITH A VISCO-ELASTIC MATERIAL



ASSUMPTIONS:

1. INITIALLY THE BODY IS UNDISTURBED. AT t_0 THE BODY IS AT REST. THUS WE HAVE ZERO INITIAL CONDITIONS

2. ALSO, S_u AND S_T ARE TIME INDEPENDENT, THUS WE CAN ONLY CONSIDER CREEP OR RELAXATION

WITH THESE TWO ASSUMPTIONS WE NEED TO FIND A SOLUTION OF THE FORM

$$\sigma_{ij}(x_k, t) \qquad \varepsilon_{ij}(x_k, t)$$

$$B_i(x_k, t) \qquad u_i(x_k, t)$$

WHERE THE FOLLOWING EQUATIONS MUST BE SATISFIED FOR ALL TIMES

$$\frac{\partial \sigma_{ij}}{\partial x_j} + B_i = 0 \qquad \varepsilon_{ij} = \left(\frac{1}{2}\right)\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

$$T_i = \sigma_{ij} n_j \rightarrow T_i(x_k, t) = \hat{T}_i \quad \text{ON } S_T$$

$$u_i = \hat{u}_i \quad \text{ON } S_u$$

IF WE TAKE LAPLACE TRANSFORMS OF THE ABOVE

$$\bar{\sigma}_{ij}(x_k, s) \qquad \bar{\varepsilon}_{ij}(x_k, s)$$

$$\bar{u}_i(x_k, s) \qquad \bar{B}_i(x_k, s)$$

THUS THE FOLLOWING EQUATIONS MUST BE SATISFIED

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \bar{B}_i = 0 \qquad \bar{\varepsilon}_{ij} = \frac{1}{2}\left[\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right]$$

$$T_i = \bar{\sigma}_{ij} \bar{n}_j \longrightarrow \bar{T}_i(x_k, s) = \hat{\bar{T}} \quad \text{ON } S_T$$