Fundamentals of Financial Mathematics

Design of a structured product

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Abstract

In this assignment we will discuss the design of two structured products, both with principle protection, of which the payoff depends on the performance of the stock of **Procter & Gamble** (PG). Both structured products are *Partially Principle Protected Notes* (PPPN). For each one we give a descriptive part in which we try to appeal to potential investors and a technical part in which we explain technicalities and profitabilities of selling the PPPN in question from the perspective of a bank.

Introduction

In general, the payoff for a PPPN at maturity is equal to 90% of the initial investment plus a premium that depends on the stock price at that maturity. In this homework it is assumed that an investor makes a single investment in one of the described PPPN at time $t = 0$. A redemption is only possible at the maturity $t = T$. The stock data of PG used throughout this assignment was taken from the New York Stock Exchange on Friday the 10th of December, 2021 at 16:58 and we found that $S_0 = \$154.51$. The maturity date for both PPPN will be Friday the 20th of January, 2023.

PPPN I

Description

Suppose an investor invests a single amount N at time $t = 0$ in PPPN I and assume the stock is worth S_0 at this moment. As was mentionned earlier, the client is assured a payment of $0.9N$ plus a premium at maturity $t = T$. This premium for PPPN I is determined as

$$\text{premium} = p\frac{N}{S_0} \times \begin{cases} 0 & \text{if } S_T \leq S_0 \\ S_T - S_0 & \text{if } S_T > S_0 \end{cases} \tag{1}$$

where p is called the participation rate of the customer.

Hence, if the stock price at maturity S_T exceeds the initial price S_0, for example with an increase of $a \times 100\%$, such that $S_T = (1 + a)S_0$, the investor receives an additional apN on top of the guaranteed $0.9N$. The total profit on the initial investment is thus $(ap - 0.1) \times 100\%$. Furthermore, the investor's losses are limited to just 10%.

We can ensure the investor a participation rate of 92.4%. The client would make a 5% profit on his or her investment if the stock were to increase by 16.2%. The average annual increase of PG over the past 3 years was 17.7%. If this trend persists, which is likely, the investor would have made profits while also protecting 90% of his or her investment from potential losses.

Technical construction

Let us once again assume an investment N at time $t = 0$. With this investment, we construct a portfolio of call options such that, at maturity, the payoff of this portfolio covers the entire amount the investor is due and the bank receives a fixed, market-independent margin.

First of all we make sure the guaranteed amount of $0.9N$ is covered by placing $0.9Ne^{-rT}$ on a risk free bank account with yield $r = 0.27\%$[1] and a time to maturity T.

Secondly, we construct a portfolio of call options to cover a potential payoff of the premium given in (1). In this case we just need to buy pN/S_0 number of call options of ask price X_1 with strike K close to (but less than) S_0.
Once this portfolio is set up, we have

$$\left(1 - 0.9e^{-rT} - p\frac{X_1}{S_0}\right) N$$

money left available from the initial investment. Placing this amount again on a similar risk free bank account as discussed above and equating this to a fixed margin of 1% for the bank, we find that

$$0.01 = \left(1 - 0.9e^{-rT} - p\frac{X_1}{S_0}\right) e^{rT}. \tag{2}$$

Note that by determining a combination of participation rate and profit margin in this manner, the profit margin for the bank is indeed independent of market performance.

Using the option data retrieved from [2] and equation (2), we can find that the maximal possible participation rate (which is most appealing for investors), while still providing a 1% profit margin for the bank, is equal to 92.4%. This means that the total payoff that the investor receives at maturity, assuming $S_T > S_0$, is equal to $0.9N + 0.924aN$, where a is once again the percentile increase of S_0. The client will receive his or her full initial investment N at maturity if the stock increased by 10.8%.

One could also set the profit margin for the bank equal to 2% (the left-hand-side of (2) then becomes 0.02). The most optimal participation rate found in this case is equal to 82.4%, which means that the total payoff for the client at maturity equals $0.9N + 0.824aN$, assuming $S_T > S_0$. Now, the stock price must increase 12.1% for the investor to receive the entire amount N.

We can plot the premium payoff in function of the stock price at maturity S_T for an initial investment of $N = \$1000$, as is done in figure 1. The stock price at $t = 0$ was found to be $S_0 = 154.51$. The margin for the bank is taken to be 1% and the participation rate 92.4%. In this case, we must buy 5.98 call options of ask price \$15.5 and strike \$150 to be able to cover the premium payoff in all different scenarios.

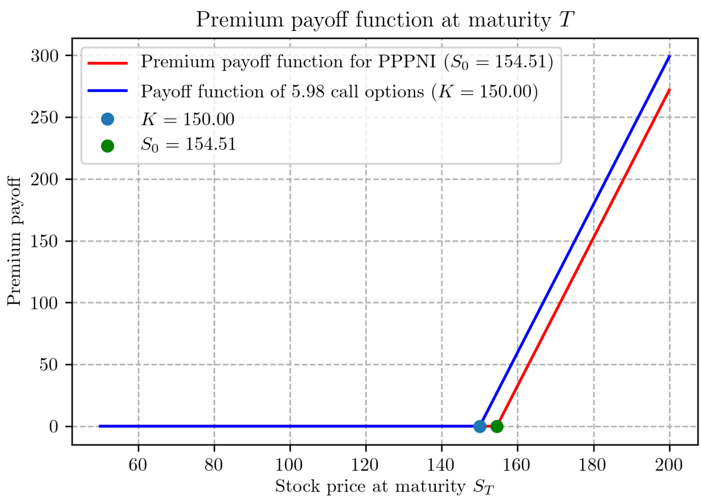

Figure 1: Plot of the premium payoff in function of S_T for PPPN I

We see that the payoff of the call option portfolio is always greater than or equal to the premium payoff that the bank is required to pay the customer.

PPPN II

Description

Suppose again an investor invests a single amount N at $t = 0$, but now in PPPN II. Assume the stock price at $t = 0$ was S_0. Here, the client is also assured a payment of $0.9N$ plus a premium at maturity $t = T$. The premium for PPPN II is determined as

$$\text{premium} = \begin{cases} 0 & \text{if } S_T \leq K \\ 0.1N & \text{if } K < S_T \leq 1.1S_0 \\ \dfrac{N}{S_0}(S_T - S_0) & \text{if } S_T > 1.1S_0, \end{cases} \tag{3}$$

where K is certain strike value. If the stock price exceeds this strike at maturity (but does not exceed $1.1S_0$), the investor will be due to full amount of his or her investment, $0.9N + 0.1N$. If the stock performs even better and rises above $1.1S_0$, say $S_T = aS_0$ where $a \geq 1.1$, the client will receive the full amount of the investment plus an additional percentage equal to $(a - 1.1) \times 100\%$.

It is also here important for the client to note that the average annual increase of PG over the past 3 years was about 17.7%. Such an increase would yield the client an extra 7.7% of the initial investment N, a total payoff at maturity of $1.077N$.

The total losses of the client are once again limited to 10%.

Technical construction

Assume again that a client makes an investment N at time $t = 0$ in the PPPN II with a premium as described in (3). With this money we will construct a portfolio such that the payoff that the client is due is covered in every possible scenario.

We start by placing $0.9e^{-rT}$ on a risk free bank account with interest rate r, to cover the guaranteed amount of $0.9N$.
Secondly, we will discuss the premium in two parts: a "digital" part, which entails $S_T \leq 1.1S_0$ and a "linear" part that entails $S_T > 1.1S_0$.

Digital part

To cover the digital part we buy and sell a specific amount of call options with well-determined ask and bid prices and strikes. Suppose we buy n call options of ask price X_1 and strike K_1, while also selling n call options of bid price X_2 and strike K_2. This call option portfolio has a cost of $n(X_2 - X_1)$ and a profit of $n(K_2 - K_1)$.
By means of buying and selling call options, we can approximate the digital nature of the premium in (3). When, for example, buying 1 call option of strike K and selling 1 call option of strike $K + h$, $h > 0$, we acquire a payoff that resembles a step-function as shown in figure 2.

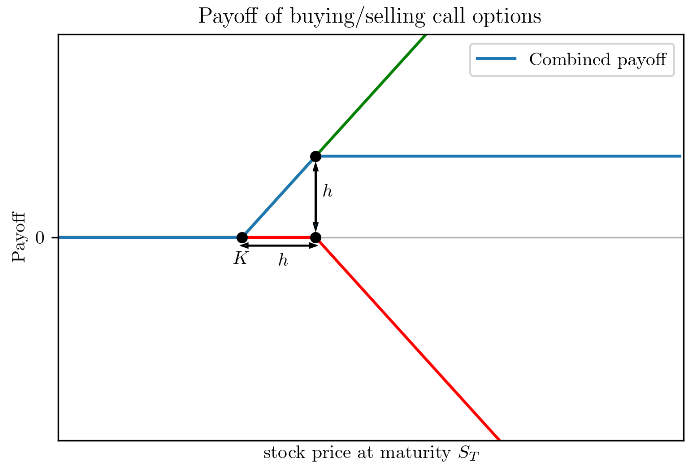

Figure 2: Illustration on how to approximate a step-function payoff by selling/buying call options

After buying/selling the mentionned call options we have an amount

$$\left[1 - 0.9e^{-rT} + n(X_2 - X_1)\right] N$$

left available to cover the second part of the premium. Since we want the profit margin for the bank to be market-independent, we can already set the profit of the call option portfolio equal to the payoff of the "digital" part of the premium. From this we can determine the required amount of call options to buy/sell and we find

$$n = \frac{0.1N}{K_2 - K_1},$$

and so the amount of money left available can be written as

$$\left[1 - 0.9e^{-rT} + \frac{0.1N}{K_2 - K_1}(X_2 - X_1)\right] N.$$

Linear part

We now tend to the linear part of the premium for which we only need to buy N/S_0 call options of ask price X_3 and strike K_3 such that

- $\left[1 - 0.9e^{-rT} + \frac{0.1N}{K_2 - K_1}(X_2 - X_1)\right] N \geq \frac{N}{S_0} X_3$
- $K_3 \leq S_0$ and $|K_3 - S_0|$ is small.

This way, the PPPN II is still interesting for potential investors, while the profit margin for the bank stays acceptable. The percentile profit margin for the bank, q, can then be found by

$$q = \left[1 - 0.9e^{-rT} + \frac{0.1N}{K_2 - K_1}(X_2 - X_1) - \frac{X_3}{S_0}\right] e^{rT}, \tag{4}$$

where we placed the total amount of money left after constructing the entire portfolio on the same risk free bank account.

Now, a potential investor also wants the strike value K in (3) to be as low as possible. The minimal K the bank can provide, whilst maintaining a decent enough profit margin, is $K = K_2$.

If we set the percentile profit margin for the bank equal to 1%, we can let our program run, using real market option data retriever from [2], to find the most optimal possible combination of call options to buy/sell. We did this for an initial investment of $N = \$1000$, a stock price $S_0 = \$154.51$, interest rate of the risk free bank account $r = 0.0027$ and time to maturity $T : 1.112$ years and found that the optimal amount of call options the buy and sell to cover the digital part is 20, with ask and bid prices equal to \$84.5 and \$83.05 and strikes \$65 and \$70 respectively. Additionally we buy 6.47 call options of ask price \$8.75 and strike \$165. For these values, the bank can maintain a profit margin of 1.1%. Finally we can plot the premium payoff in function of the stock price at maturity, S_T, as done below in figure 3. We see here aswell that the premium payoff is always less than or equal to the payoff of the call option portfolio.

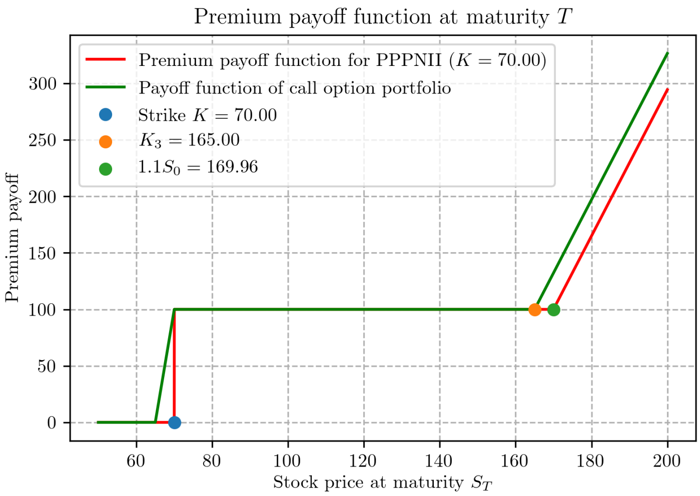

Figure 3: Plot of the premium payoff in function of S_T for PPPN II

Appendix

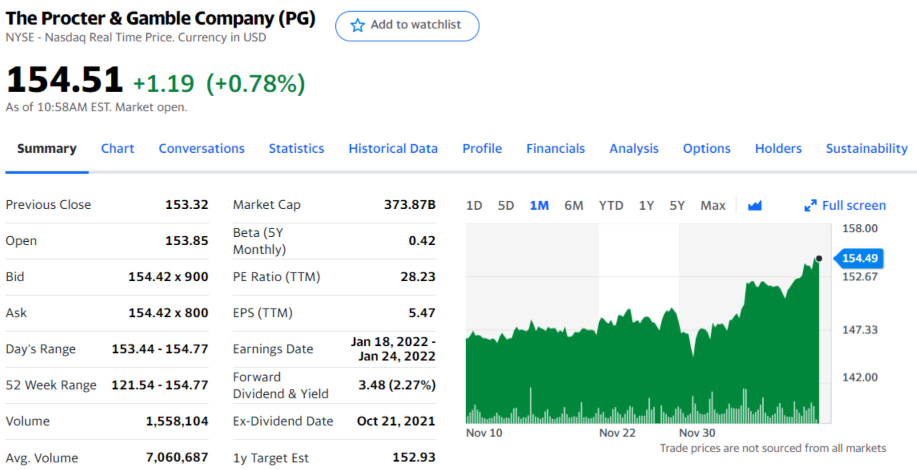

Figure 4: Screenshot of the PG stock price on 10 December 2021 at 16:58

Calls for January 20, 2023

Contract Name	Last Trade Date	Strike	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
PG230120C00065000	2021-10-20 2:49PM EST	65.00	76.42	79.50	84.50	0.00	-	37	0	0.00%
PG230120C00070000	2021-12-06 9:30AM EST	70.00	80.75	83.05	86.05	0.00	-	2	2	49.51%
PG230120C00075000	2021-11-10 6:47AM EST	75.00	60.00	78.10	80.75	0.00	-	1	0	43.53%
PG230120C00080000	2021-10-20 2:49PM EST	80.00	61.41	64.50	69.35	0.00	-	37	7	0.00%
PG230120C00085000	2021-11-05 8:40AM EST	85.00	61.05	63.30	67.50	0.00	-	2	7	0.00%
PG230120C00090000	2021-11-29 9:35AM EST	90.00	57.60	63.25	66.00	0.00	-	1	23	35.46%
PG230120C00095000	2021-11-02 9:48AM EST	95.00	49.11	53.75	57.95	0.00	-	1	112	0.00%
PG230120C00100000	2021-11-30 10:26AM EST	100.00	48.54	54.10	55.65	0.00	-	2	155	27.78%
PG230120C00105000	2021-11-29 3:52PM EST	105.00	45.05	48.85	50.80	0.00	-	2	147	25.99%
PG230120C00110000	2021-11-09 1:59PM EST	110.00	37.30	44.15	46.80	0.00	-	1	54	27.54%
PG230120C00115000	2021-12-07 3:39PM EST	115.00	38.80	40.65	41.35	0.00	-	1	536	23.19%
PG230120C00120000	2021-12-09 9:30AM EST	120.00	34.47	36.30	37.25	0.00	-	1	528	23.50%
PG230120C00125000	2021-12-09 1:00PM EST	125.00	31.43	32.00	33.00	0.00	-	10	542	22.76%
PG230120C00130000	2021-12-10 10:27AM EST	130.00	28.49	27.90	29.00	-0.09	-0.31%	1	543	22.23%
PG230120C00135000	2021-12-09 12:59PM EST	135.00	23.75	24.45	25.35	0.00	-	1	613	21.99%

Figure 5: Screenshot of the call option data for the PG stock on 10 December 2021 at 17:03 (the call options marked in blue are in-the-money)

PG230120C00135000	2021-12-09 12:59PM EST	135.00	23.75	24.45	25.35	0.00	-	1	613	21.99%
PG230120C00140000	2021-12-09 11:37AM EST	140.00	21.20	20.70	22.50	0.00	-	1	1,396	22.70%
PG230120C00145000	2021-12-09 3:03PM EST	145.00	17.83	17.65	18.50	0.00	-	3	4,752	20.93%
PG230120C00150000	2021-12-09 9:49AM EST	150.00	13.80	14.60	15.50	0.00	-	1	17,184	20.46%
PG230120C00155000	2021-12-10 9:48AM EST	155.00	12.45	12.20	13.00	+0.12	+0.97%	2	539	20.32%
PG230120C00160000	2021-12-10 9:52AM EST	160.00	10.45	9.95	10.60	+0.27	+2.65%	5	1,731	19.88%
PG230120C00165000	2021-12-09 2:04PM EST	165.00	8.30	8.35	8.75	0.00	-	2	5,417	19.84%
PG230120C00170000	2021-12-09 3:19PM EST	170.00	6.58	6.75	7.10	0.00	-	5	1,399	19.70%
PG230120C00175000	2021-12-10 9:32AM EST	175.00	4.70	5.10	5.65	-0.30	-6.00%	6	1,023	19.47%
PG230120C00180000	2021-12-09 1:04PM EST	180.00	4.10	4.00	4.60	0.00	-	10	292	19.56%
PG230120C00185000	2021-12-09 3:25PM EST	185.00	3.39	1.43	3.75	0.00	-	1	125	19.68%
PG230120C00190000	2021-12-10 9:49AM EST	190.00	2.81	2.44	2.91	+0.80	+39.80%	4	17	19.49%
PG230120C00195000	2021-11-26 9:38AM EST	195.00	1.50	1.24	2.92	0.00	-	10	157	21.02%
PG230120C00200000	2021-12-08 12:50PM EST	200.00	1.45	1.14	2.10	0.00	-	50	328	20.31%
PG230120C00210000	2021-12-08 2:10PM EST	210.00	1.10	0.92	1.22	+0.10	+10.00%	1	871	19.93%
PG230120C00220000	2021-12-09 2:41PM EST	220.00	0.70	0.64	0.92	0.00	-	5	32	20.80%

Figure 6: Screenshot of the call option data for the PG stock on 10 December 2021 at 17:04 (the call options marked in blue are in-the-money)

References

[1] U.S. Government. Daily Treasury Par Yield Curve Rates. https://www.treasury.gov/resource-center/data-chart-center/interest-rates/pages/TextView.aspx?data=yield, 12 2021.

[2] Yahoo! Finance. PG Option Data. https://finance.yahoo.com/quote/PG/options?p=PG, 12 2021.