Financial Engineering

Design & hedging of a structured product

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Abstract

In this assignment we will discuss the design and Δ-hedge of a structured product of which the payoff depends on the performance of the stock of **The Trade Desk Inc.** (TTD). The structured product in question is a *Twin-Win Certificate* (TWC). We first give a descriptive part in which we try to appeal to potential investors and a technical part afterwards where we explain technicalities (model calibration, product construction and pricing) and profitabilities of selling the TWC in question from the perspective of a bank. To conclude, a Δ-hedge for the product is discussed.

Introduction

A TWC is a structured product that offers the holder positive participation on the upside as well as the downside of the underlying asset (in our particular case the TTD stock) as long as the asset's price did not hit a lower barrier during the product's lifetime, i.e. the time between acquiring the TWC ($t = 0$) and maturity ($t = T$). No early redemption of the product is allowed. The stock market data used in this report was taken from the Nasdaq Stock Exchange on Wednesday 4th of May at 16:44 CEST. The initial price of the TTD stock found was $S_0 = \$56.88$ (see Fig.3 in the Appendix). The strike of the TWC is taken at $K = \$57$, the barrier $H = \$35 \approx 0.6S_0$ and the maturity date on January 19, 2024, so $T = 625/253$[1]. Furthermore, The Trade Desk Inc. states on their investors website they do not anticipate paying dividends in the foreseeable future [4], hence we set the dividend yield $q = 0$. To obtain a decent guess for the risk-free interest rate r, we interpolated the Daily Treasure Par Yield Curve Rates found on [5]. This gave the result of $r = 0.0275$.

Descriptive Part

Suppose an investor buys one TWC at time $t = 0$. The payoff to the investor at maturity $t = T$ consists of three different scenarios and depends on the performance of the underlying stock, a strike $K = \$57$ and lower barrier $H = \$35$:

Scenario 1. The stock price crossed the lower barrier H at least once during the lifetime of the certificate. The payout to the investor is the price of the underlying at maturity S_T.

Scenario 2. The stock price never crossed the lower barrier and ends up below the strike level K. The product pays out an amount equal to the strike plus the absolute difference between the strike and the stock price at maturity[2], i.e. $K + |S_T - K|$.

Scenario 3. The stock price never crossed the lower barrier and ends up above the strike level. The payoff is again the price of the underlying at maturity S_T.

It should be noted that the investor receives no principal protection. On the other hand, he or she enjoys partial protection due to the lower barrier as well as unlimited upside potential. When the price of the underlying asset drops below the barrier H or ends up above the strike K (or both), the

[1] Here we divide the number of days in the TWC's lifetime, i.e. 625, by 253, the average NASDAQ trading days.[6]

[2] Please note that the absolute value in this scenario is unnecessary, since $S_T \leq K$ is implied. We could have very well written $2K - S_T$.

TWC behaves as a stock that will be sold at a fixed maturity. While if the stock price does not drop below the barrier but stays beneath the strike K, the TWC still provides positive participation when a normal stock would not. This is because the payoff would then be that of scenario 2 and we note that

$$K + |S_T - K| \geq S_T.$$

Hence, the TWC issues an improved payoff to that of a regular stock, which immediately makes it a more interesting product. Combined with the fact that for the past (almost) two years the stock of TTD, although fluctuating heavily, has never dropped below \$35. Also, the stock would have to drop $\pm 40\%$ for it to break the barrier, which is unlikely.

To shortly summerize, a Twin-Win Certificate enables the holder to enjoy unlimited upside potential as well as providing partial protection and limited positive downside participation. However, the client should note that, when the stock price breaches the lower barrier, the TWC will behave as a regular stock.

Technical Part

In this section we will provide the issuer with some of the more technical notes such as the construction of the TWC from less exotic options, the calibration of the stochastic volatility- or Heston model, the pricing method using Monte Carlo simulations and to conclude we discuss the Δ-hedge required to mitigate the potential losses. We start by explaining how a TWC can be constructed from different, less exotic options.

Construction of a TWC

Since a TWC has a strike and a lower barrier, a well estimated guess would be to use a barrier call or put when constructing the financial product. Now, when the price of the underlying drops below the barrier or ends up above the strike, the product behaves as a normal stock. This would imply we are dealing with a *Down-And-Out Barrier Put option* (DOBP). However, a DOBP is worthless when the price breaches the barrier or when the price at maturity is above the strike, while the TWC is not. Hence, we require an extra option or derivative. A good pick would be a call with same maturity as the DOBP and strike equal to zero. This way we have a payout at maturity equal to the stock price at that time, if we were to find ourselves in scenario's 1 or 3. The only thing to fix now is the payout in scenario 2. The payoff of one DOBP with strike K, barrier H and time to maturity T is equal to

$$\text{DOBP} = (K - S_T)^+ \mathbb{I}\left(\min_{0 \leq t \leq T} S_t > H\right), \tag{1}$$

where $\mathbb{I}$ is equal to 1 if its argument is correct and 0 otherwise. In scenario 2, this becomes $K - S_T$, or, together with the zero strike call, $S_T + K - S_T$, which is not quite what we are looking for. Instead, use two DOBP, for then the payoff would be $S_T + 2(K - S_T) = 2K - S_T$, which is indeed exactly the payout of the TWC in scenario 2. Hence, the TWC is constructed out of one European call option with strike equal to zero and time to maturity T plus two DOBP with strike K, barrier H and again a time to maturity T. This payout is visualized in Fig.1.

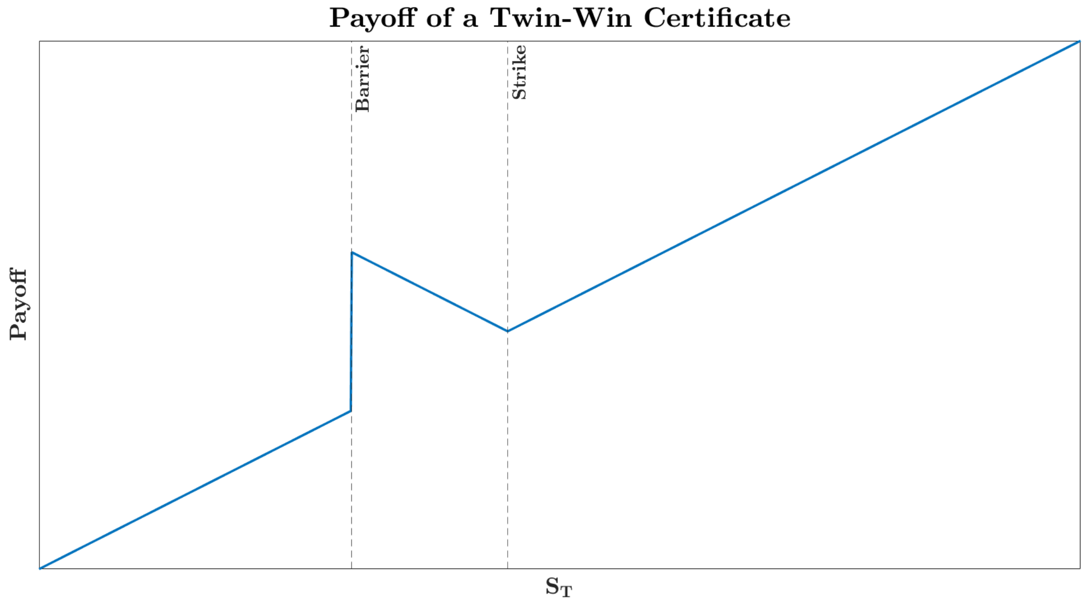


Figure 1: Visualization of the payoff of a TWC in function of the price of the underlying at maturity, T, assuming the price never dropped below the barrier H

Calibration of the Heston Model

The model used when pricing the TWC is the Heston model. The Heston model describes the evolution of the price of an underlying asset as well as the evolution of the volatility of said asset. It assumes that the volatility is neither constant nor deterministic but that it follows a stochastic process as is described below

$$\mathrm{d}S_t = (r-q)S_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t \qquad \text{with } S_0 \geq 0, \tag{2}$$

$$\mathrm{d}v_t = \kappa(\eta - v_t)\,\mathrm{d}t + \theta\sqrt{v_t}\,\mathrm{d}\tilde{W}_t \qquad \text{with } v_0 = \sigma_0^2 \geq 0. \tag{3}$$

The parameters in this model represent the following:

- S_t is the price of the underlying at time t
- r is the risk-free interest rate
- q is the dividend yield of the underlying
- v_t is the volatility at time t
- θ is the volatility of the volatility (or *vol-of-vol*)
- κ is the rate at which v reverts to θ
- η is level of mean reversion, as $t \to \infty$ the expected value of v_t tends to η
- W_t and $\tilde{W}_t$ are Wiener processes with correlation ρ.

To calibrate this model we used real market option data taken from Yahoo Finance [7] on Wednesday 4th of May. The data was imported directly from the website using Microsoft Excel and afterwards read out in the MATLAB-file *Data_cleaning.m*. In this file we extracted all option strikes and mid-prices for the call and put options with different maturities. We cleaned out any irregularities associated with very high or very low strikes and some of the "arbitrage opportunities" (e.g. prices of options with longer time to maturity but higher prices). The results of the cleaned data can be seen in Fig.4 in the Appendix.

The calibration process can be continued using the Fast-Fourier Transform (FFT) method for option valuation proposed by Peter Carr and Dilip Madan [1]. They derived a formula with which vanilla options can be valuated if a closed form for the characteristic function of the "log-price" process under the specified model is known. This formula is called the Carr-Madan formula and is given as follows in (4). Here, the formula is given for the price $C(K,T)$ of a European call option with strike K and time to maturity T [1]

$$C(K,T) = \frac{\exp[-\alpha \log(K)]}{\pi} \int_0^\infty \exp[-iv\log(K)]\varrho(v)\,\mathrm{d}v\,, \tag{4}$$

where

$$\varrho(v) = \frac{e^{-rT}\phi(v-(\alpha+1)i,T)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}, \tag{5}$$

with $\alpha > 0$ a constant such that the $(1+\alpha)$th moment of the price of the underlying exists and ϕ is the characteristic function of the "log-price" process at maturity time T. A similar equation can be obtained for a European put option using the Put-Call Parity.

Using the charateristic function for the Heston model, which was first derived by Heston [2], and the FFT of (4) as discussed in [1] we can obtain a range of various option prices. These prices can then be used in a Root-Mean-Square-Error (RMSE) with respect to the obtained market prices to optimize the parameters $\kappa, \eta, \theta, \rho$ and σ_0 of the Heston model. In the attached MATLAB-file *heston_characteristic.m* the characteristic function of the Heston model is defined and in *heston_FFT.m* the FFT method is applied, where we used Simpson's integration rule to approximate the integral. In the latter file we used the same values as proposed in the Carr-Madan paper [1]:

$$\begin{aligned} N &= 4096 \\ \alpha &= 1.5 \\ \eta &= 0.25, \end{aligned}$$

where this η is not the parameter from the Heston model but rather the spacing used in the FFT. The output of *heston_FFT.m* is a range of call or put option prices as predicted by the model. It is this MATLAB function we used in the RMSE:

$$\text{RMSE} = \sum_{i=1}^{n} \frac{[P_{\text{market},i} - P_{\text{model},i}(\kappa,\eta,\theta,\rho,\sigma_0)]^2}{n}, \tag{6}$$

where $P_{\text{market},i}$ is the option mid-price in the market for maturity i, $P_{\text{model,i}}$ is the option price as calculated by the model with *heston_FFT.m* and n is the total number of different option prices used to calculate the RMSE. Note that we have written $P_{\text{model},i}$ as a function of $\kappa, \eta, \theta, \rho$ and σ_0. Since we will vary these parameters when minimizing the RMSE and keep the strike K, risk-free interest rate r, dividends q and time to maturity T constant, we have not included these latter variables, even though $P_{\text{model},i}$ depends on them. A function for the RMSE is written in the MATLAB-file *RMSE_calibration.m*. We have only used data from option with a long time to maturity, since these are closer to the maturity of the TWC considered in this report and will hopefuly give a more accurate calibration.

Before minimizing this function, we require some constraints. For example, the correlation between the two Wiener processes, ρ, must lie within $(-1,1)$. Furthermore should $\kappa, \eta, \theta, \sigma_0 > 0$. To implement these constraints, the build-in MATLAB function *fmincon* is used. The resulting optimal values are obtained in the MATLAB-file *calibration.m* and can be seen in Table1.

Table 1: Table with calibrated parameters of the Heston model

Parameter	Optimal value
κ	2.0285
η	0.4155
θ	2.1082
ρ	-0.3234
σ_0	0.8177

Note that these parameter values do not comply with the so called "Feller" condition, i.e. $2\kappa\eta > \eta^2$. This extra condition would ensure the volatility process to be purely positive, however it is a sufficient condition and not a necessary one. This means we could still generate acceptable option prices using the parameter values in Table1.

These calibrated parameters yield a value for the RMSE equal to 0.443117 and can now be used to make a scatterplot that fits the market prices and their correspond model predicted prices. This is done in Fig.5 in the Appendix. Here, were can see that the prices for options with longer maturity seem to have a more accurate calibration.

Valuation

With the parameters of the Heston model calibrated accordingly, we can determine the price of the TWC by performing a Monte Carlo simulation. This is done by running $100,000$ different stock paths in MATLAB, each discretized to have daily steps. This discretization is achieved using the Milstein scheme [3]:

$$S_{i+1} = S_i[1 + (r-q)\Delta t + \sqrt{v_i \Delta t}\varepsilon_i^1] \tag{7}$$

$$v_{i+1} = v_i + \left[\kappa(\eta - v_i) - \frac{\theta^2}{4}\right]\Delta t + \theta\sqrt{v_i\Delta t}\varepsilon_i^2 + \frac{\theta^2}{4}\Delta t(\varepsilon_i^2)^2, \tag{8}$$

where ε^1 and ε^2 are correlated random numbers with correlation ρ. By taking standard normal random numbers $\varepsilon, \varepsilon^*$ and defining $\varepsilon^1, \varepsilon^2$ as

$$\varepsilon^1 = \varepsilon$$
$$\varepsilon^2 = \rho\varepsilon + \sqrt{1-\rho^2}\varepsilon^*,$$

they are constructed to have correlation ρ. Note that this is in fact the same parameter ρ that describes the correlation between the price- and volatility-process as in the Heston model. To ensure (8) is positive for every $i = 0, 1, \ldots, 99999$ we employ the reflective principle by taking the absolute value of the right-hand side. The first five simulated paths of the stock price with their respective TWC payoffs can be seen in Fig.2.

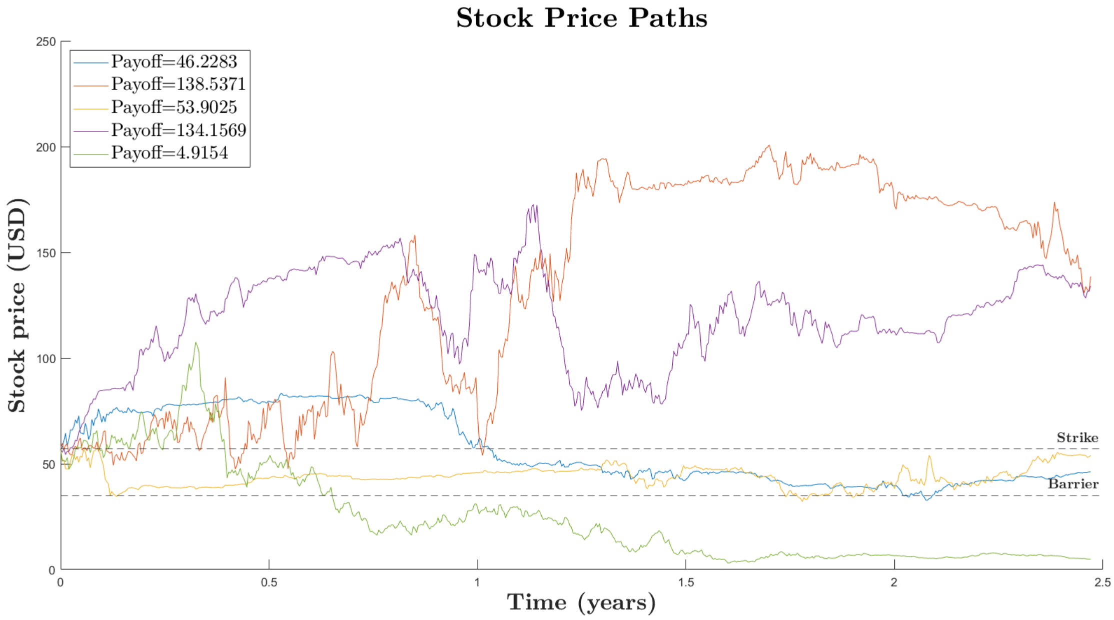


Figure 2: Different simulated paths of the stock price through time.

Every simulated path gives us a value of the stock price at maturity, from which we can derive the payout of TWC. Using the general derivative pricing formula

$$\text{Price(TWC)} = e^{-rT} \times \mathbb{E}_Q\left[\text{payout(TWC)}\right] \equiv \pi(S_0)$$

we can ultimately determine a price for the TWC with $K = \$57, H = \$35$ and $T = 625/253$. This is all done in the MATLAB-file *Monte_Carlo_Pricer.m* and to implement the payoff of a DOBP we have rewritten (1) as

$$\text{DOBP} = \max\{K - S_T, 0\} \times \max\left\{0, \frac{\min_{0\leq t\leq T} S_t - H}{|\min_{0\leq t\leq T} S_t - H|}\right\}.$$

Finally, we arrived at a price of $\pi(S_0) = \$57.4893$ for one TWC.

Hedging

We now explain how this short position in a TWC can be hedged using a Δ-hedge in the Heston model. This is done by calculating the Δ of the intrinsic value of the TWC, $\pi(S)$, defined as

$$\Delta \equiv \frac{\partial \pi}{\partial S}. \tag{9}$$

Unfortunately, this derivative can not be calculated analytically. However, we can still approximate Δ by using the stock paths generated in the Monte-Carlo simulation. First, we have to remember that the value, Π, of a Δ-hedge portfolio is given by

$$\Pi = -\pi + \int \frac{\partial \pi}{\partial S}\, \mathrm{d}S$$

and the change in value of such a portfolio over a time interval $[t, t + \mathrm{d}t]$[3] hence is

$$\mathrm{d}\Pi = -\,\mathrm{d}\pi + \Delta\,\mathrm{d}S\,.$$

[3] Please not that this time interval need not necessarily be small. We use the notation straight "d" to make a distinction between the $\Delta = \frac{\partial \pi}{\partial S}$.

The aim of a Δ-hedge is to buy (or sell) a Δ number of shares such that the change in value of the portfolio, $d\Pi$, is (as close as possible to) zero. In MATLAB, we can make Π a function of Δ and obtain the root of this function using the build-in function *fzero*. This is done in the MATLAB-file *Monte_Carlo_Pricer.m* for every generated path and afterwards an average over all Δ's is taken.

The average Δ obtained over the 100,000 paths is equal to 1.0287. It must be stated that this Δ will vary slightly every time the Monte-Carlo simulation if performed, however the value will always be greater than zero (and also be around 1.01) which implies that we must buy stocks.

If we now assume an investor buys $N = \$1,000,000$ worth of the TWC, which is equal to a total amount of $N/\pi(S_0) = 17,395$ certificates, the actual average amount of stocks the bank is required to buy to Δ-hedge itself is given by

$$\Delta \times \frac{N}{\pi(S_0)} = 17,893.$$

To conclude, we can also calculate the amount of money the bank can expect (on average) as a margin. The reasoning behind this is as follows:

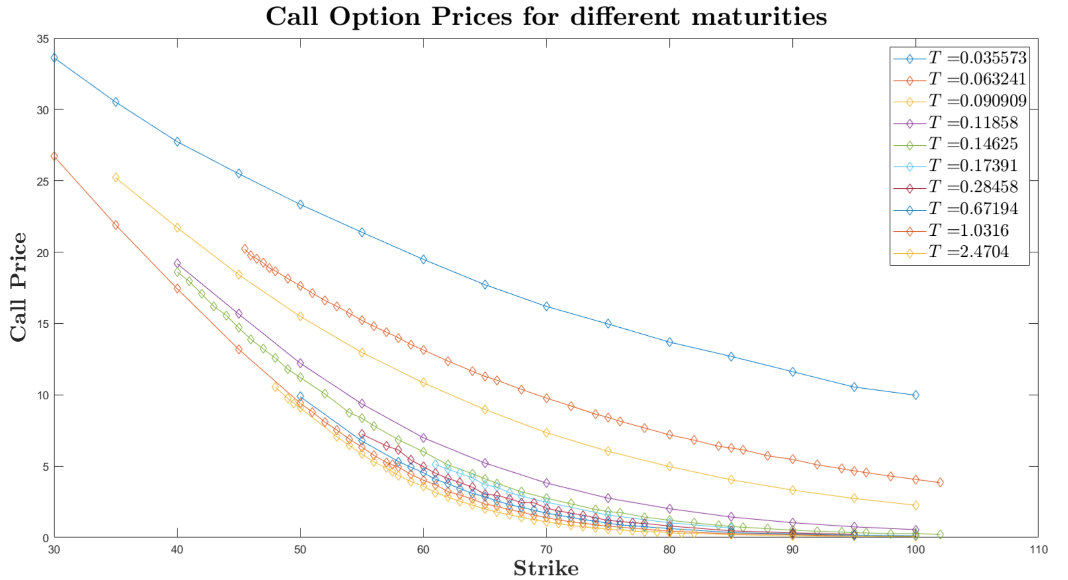
1. An investor buys N worth of the TWC, equal to total amount of $n = N/\pi(S0)$.
2. The bank buys an amount $n\Delta$ stocks and places $(N - n\Delta S_0)$ USD on a risk-free bank account (we assume the risk-free interest rate is always $r = 0.0275$ as stated in the Introduction).
3. At maturity, T, the investor is owed an amount $n\pi(S_T)$ USD while the bank receives $(N - n\Delta S_T)e^{rT} + n\Delta S_T$ USD. Here we have denoted the payoff of the TWC as $\pi(S_T)$, its value at maturity.
4. The total margin for the bank is equal to $(N - n\Delta S_T)e^{rT} + n\Delta S_T - n\pi(S_T)$ USD.

Once again, this is calculated for every path of the Monte-Carlo simulation and afterwards an average is taken. The simulation we ran to acquire a value for Δ equal to 1.0287 yielded an expected margin of \$2,895.7, which is approximately 0.29% of the initial investment. This amount also varies slightly for different simulations.

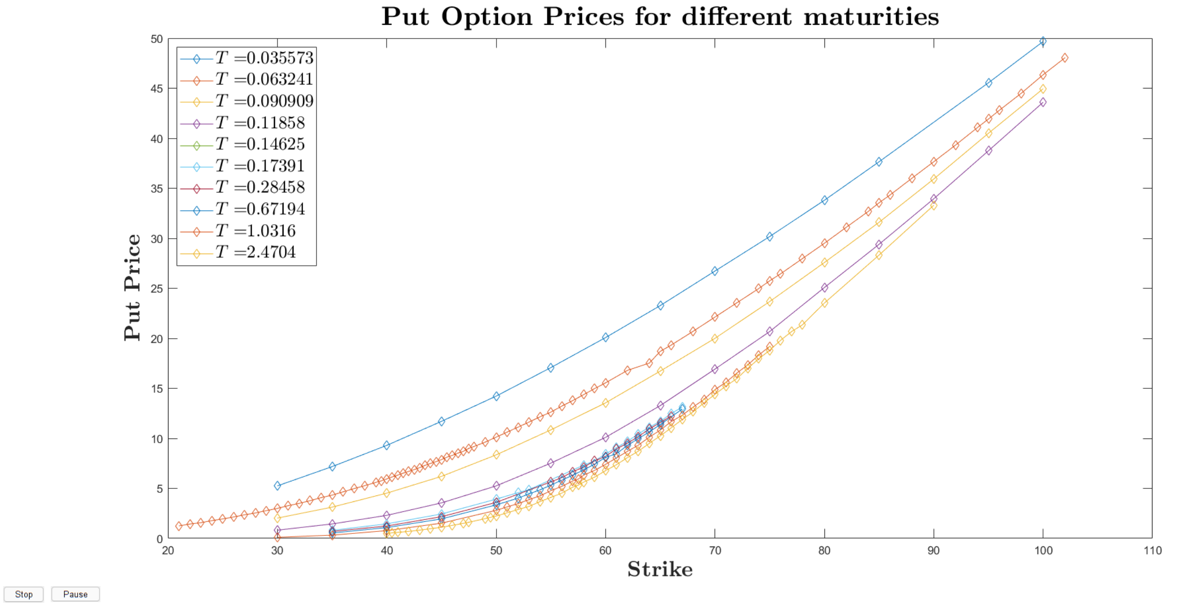
Note that these values for the Δ and the margin will stagnate as the number of generated stock price paths increases.

Appendix

Figure 3: Initial price of the TTD stock obtained from Yahoo! Finance on 4 May 2022 at 16:44:01 CEST

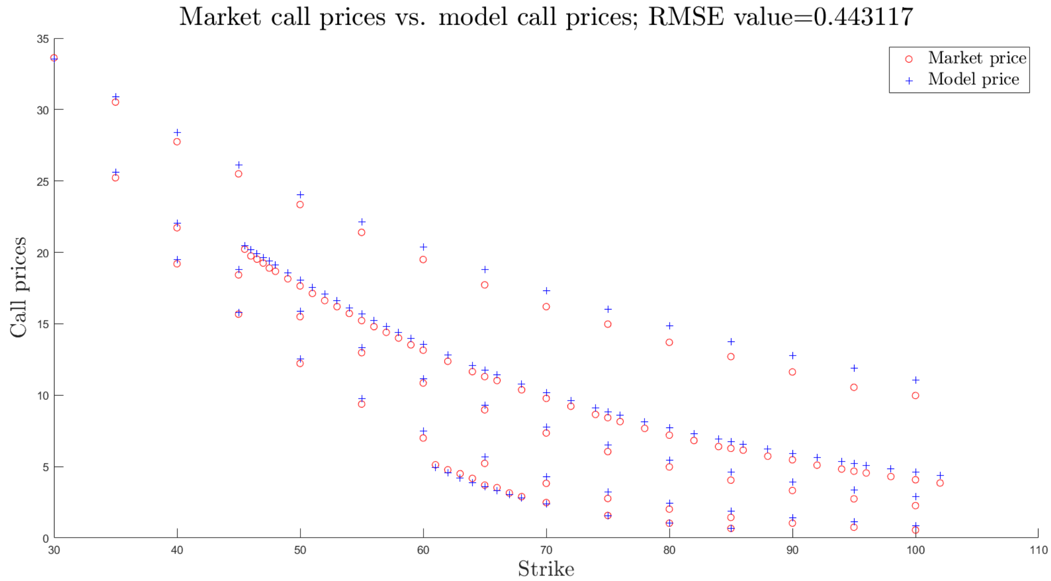


(a) Plot of the cleaned call option price data with respect to the strike for different maturities

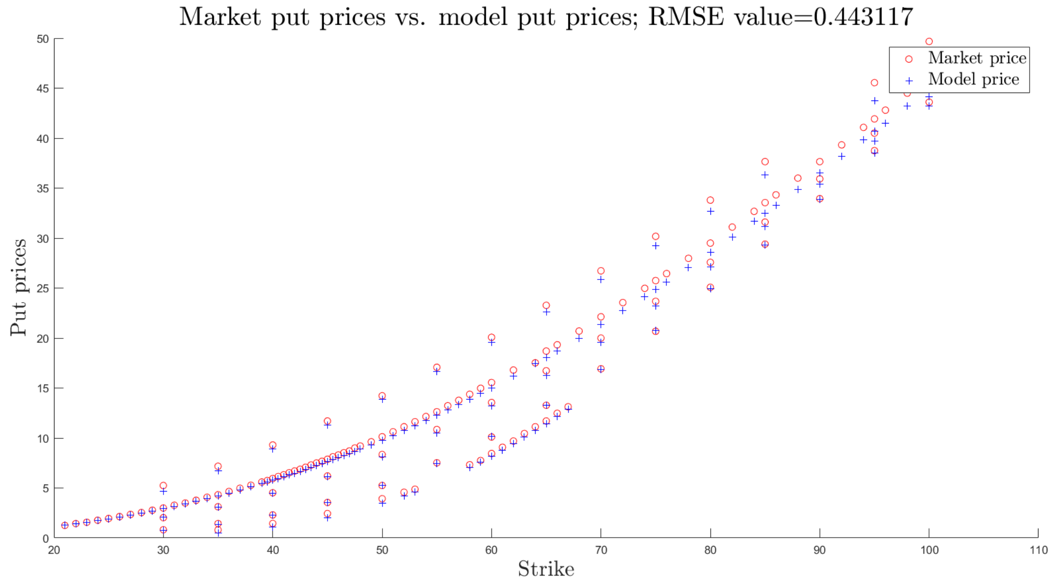


(b) Plot of the cleaned put option price data with respect to the strike for different maturities

Figure 4: Plots of cleaned option price data for different maturities



(a)



(b)

Figure 5: Scatterplot of the market prices and their corresponding model predicted prices for the calls (5a) and puts (5b)

References

[1] Peter Carr and Dilip Madan. Option valuation using the fast fourier transform. *Journal of computational finance*, 2(4):61–73, 1999.

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[3] GN Milstein. Approximate integration of stochastic differential equations. *Theory of Probability & Its Applications*, 19(3):557–562, 1975.

[4] The Trade Desk, Inc. Faq. https://investors.thetradedesk.com/shareholder-services/investor-faqs/#:~:text=Does%20The%20Trade%20Desk%20pay,and%20expansion%20of%20our%20business., 2022.

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[7] Yahoo! Finance. TTD Option Data. https://finance.yahoo.com/quote/TTD/options?p=TTD, 12 2022.