Intersections

4/12

Implicit surface is a set of points that satisfy a constraint

$p \in \mathbb{R}^3 \quad f(p) = 0$

eg line

$$2x + 3y + 4 = 0$$

$$f(x, y) = 2x + 3y + 4$$

is $(3,6)$ on the line

$$f(3,6) = 2(3) + 3(6) + 4 = 28 \neq 0 \Rightarrow \text{not on line}$$

$$f(-11,6) = 2(-11) + 3(6) + 4 = 0 \Rightarrow \text{is on line}$$

Sphere:

$C = (c_x, c_y, c_z)$ center

r is radius $r > 0$

$$f(x,y,z) = (x-c_x)^2 + (y-c_y)^2 + (z-c_z)^2 - r^2$$

$\Updownarrow$ math step

Let $p = (x,y,z)$

$$f(p) = (p-c)\cdot(p-c) - r^2 \quad (\text{eqn } **)$$

any point $p \in \mathbb{R}^3$

s.t. $f(p) = 0 \Rightarrow p$ is on the surface of the sphere

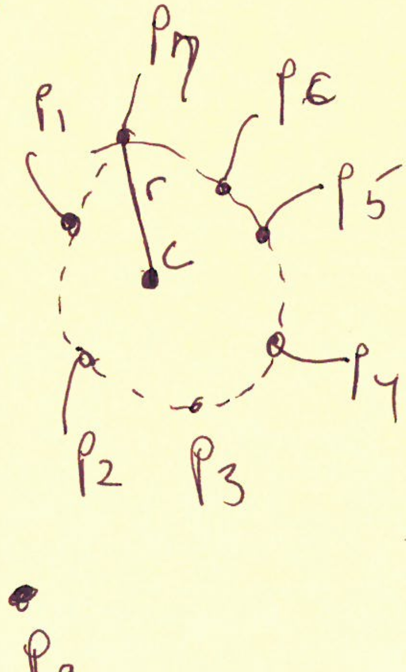
math step

$$f(\vec{p}) = (\vec{p}-\vec{c}) \cdot (\vec{p}-\vec{c}) - r^2$$

$$= \begin{bmatrix} p_x - c_x \\ p_y - c_y \\ p_z - c_z \end{bmatrix} \cdot \begin{bmatrix} p_x - c_x \\ p_y - c_y \\ p_z - c_z \end{bmatrix} - r^2$$

$$= (p_x - c_x)^2 + (p_y - c_y)^2 + (p_z - c_z)^2 - r^2$$

a picture



$f(p_1) = 0$

$f(p_2) = 0$

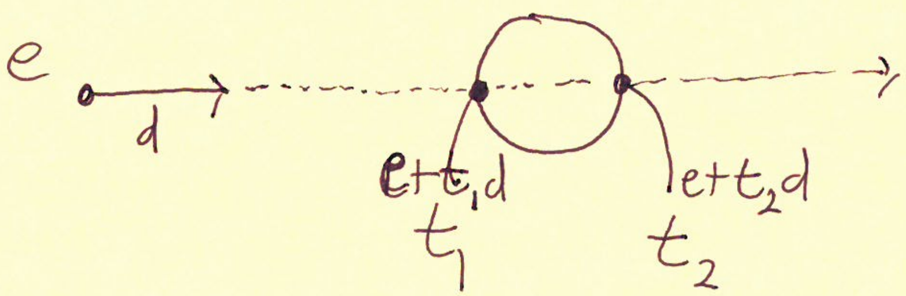
$\vdots$

$f(p_9) \neq 0$

Ray

$$p(t) = \vec{e} + t\vec{d} \qquad t \in \mathbb{R} \quad e, d \in \mathbb{R}^3$$

to solve ray sphere intersection



$t_1, t_2 \in \mathbb{R}$

$p(t_1)$ is on the ray

$p(t_2)$ is on the ray

$f(p(t_1))$ is on sphere (and ray)

$f(p(t_2))$ is on sphere (and ray)

Goal: Find t_1 and t_2 such that

$$f(p(t_i)) = 0$$

$$f(p(t)) = f(\vec{e} + t\vec{d})$$ by def of our ray

$$= ((\vec{e}+t\vec{d}) - \vec{c}) \cdot ((\vec{e}+t\vec{d}) - c) - r^2$$ (by eqn **)

⋮ some alg)

$$= \underbrace{(\vec{d}\cdot\vec{d})}_{A} t^2 + \underbrace{2\vec{d}\cdot(\vec{e}-\vec{c})}_{B} t + \underbrace{(\vec{e}-\vec{c})\cdot(\vec{e}-\vec{c}) - r^2}_{C}$$

$$= At^2 + Bt + C$$

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$D = B^2 - 4AC$ is discriminant

$D < 0 \Rightarrow$ no real solution (no intersection)

$D = 0 \Rightarrow$ a double root (tangental intersecton

$D > 0 \Rightarrow 2$ real solutions (2 intersections)

$f(q(t)) = f(z + td)$ by def of $q(t)$

$= ((z+td)^T - c)\cdot((z+td) - c) - r^2$

for ball

$= \underbrace{(d^T d)}_{A} t^2 + \underbrace{2d^T \cdot (z-c)}_{B} t + \underbrace{(z-c)^T (z-c) - r^2}_{C}$

$= At^2 + Bt + C$

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$D = B^2 - 4AC$ is discriminant

$D < 0 \Rightarrow$ no real solution (no intersection)

$D = 0 \Rightarrow$ 1 double root (tangential intersection)

$D > 0 \Rightarrow$ 2 real solutions (2 intersections)

Catastrophic cancelation

Subtract 2 numbers w/ floating point arith

relative error is greater than abs error

eg.

$.1234567890123456789 - .1234567890000000$

Consider

$-B \pm \sqrt{B^2 - 4AC}$

Case 1: A is very small

$-B \pm \sqrt{B^2 - 4(\approx 0)(C)} \simeq -B \pm \sqrt{B^2}$

use this eqn

rewrite

$$\boxed{q = -\left[B + \operatorname{sgn}(B)\sqrt{B^2 - 4AC}\right] \qquad t_1 = \frac{q}{2A} \qquad t_2 = \frac{2C}{q}}$$

$$\operatorname{sgn}(x) = \begin{cases} x<0 & -1 \\ x=0 & 0 \\ x>0 & 1 \end{cases}$$

Case 1 A is close to 0:

Case 1a: $B<0 \Rightarrow q = -\left[B + (-1)\sqrt{B^2 - 4AC}\right]$

$\simeq -\left[B + -|B|\right]$

Additional Ref: Wikipedia "Loss of sig"

Catastrophic cancellation

Cancellation of floating point with

relative error is greater than the error

eg.

.1234567890123456781 — .1234567891000000000

Consider

$-B \pm \sqrt{B^2 - 4AC}$

Case 1: A is very small

$-B \pm \sqrt{B^2 - 4(\approx 0)(C)} \approx -B \pm \sqrt{B^2}$

Case 1c

$q = -\left[B + \operatorname{sgn}(B)\sqrt{B^2 - 4AC}\right]$

$\operatorname{sgn}(x) = \begin{cases} x > 0 \\ x = 0 \\ x < 0 \end{cases}$

$x_1 = \frac{q}{2A} \qquad x_2 = \frac{2C}{q}$

Case 2: A is close to 0

Case 2a: $B < 0 \Rightarrow q = -\left[B + (-1)\sqrt{B^2 - 4AC}\right]$

$= -\left[B + -|B|\right]$

Worst case: W

Ray Δ intersection

Consider a parametric surface $f(u,v)$

and ray $\vec{e} + t\vec{d}$

expand our surface rep

eq *A*

$$\left.\begin{aligned} f_x(u,v) &= e_x + t d_x \\ f_y(u,v) &= e_y + t d_y \\ f_z(u,v) &= e_z + t d_z \end{aligned}\right\}$$

3 eqns and 3 unkrows u, v, t

Goal is rewrite to solve for u, v, t

write the eqn for the plane that Δabc is on as

$$f(u,v) = \vec{a} + u(\vec{b} - \vec{a}) + v(\vec{c} - \vec{a})$$

in its components

eq *B*

$$\begin{aligned} f_x(u,v) &= a_x + u(b_x - a_x) + v(c_x - a_x) \\ f_y(u,v) &= a_y + u(b_y - a_y) + v(c_y - a_y) \\ f_z(u,v) &= a_z + u(b_z - a_z) + v(c_z - a_z) \end{aligned}$$

combine eqn *A* & *B*

$$e_x + t d_x = a_x + u(b_x - a_x) + v(c_x - a_x)$$

$$e_y + t d_y = a_y + u(b_y - a_y) + v(c_y - a_y)$$

$$e_z + t d_z = a_z + u(b_z - a_z) + v(c_z - a_z)$$

move td $\longrightarrow$

move a $\longleftarrow$

write as a matrix

$$\begin{bmatrix} b_x - a_x & c_x - a_x & -d_x \\ b_y - a_y & c_y - a_y & -d_y \\ b_z - a_z & c_z - a_z & -d_z \end{bmatrix} \begin{bmatrix} u \\ v \\ t \end{bmatrix} = \begin{bmatrix} e_x - a_x \\ e_y - a_y \\ e_z - a_z \end{bmatrix}$$

Solve using Cramers rule

```
ray_tri_interset(ray r, vec3 a, vec3 b, vec3 c)
    compute u, v, t
    if (v<0 || 1<v) no intersect
elseif (u<0 || 1-v<u) no intersect
elseif (t<0) no intersect
    else intersect @ ray parm t
```

we know line through the ray intersects the Δ