5BUIS019W Business Analytics

Lecture 2

Introduction to Linear Programming

Outline

- Linear programming: definition
- Problem formulation
  - Objective function; Constraints; Decision variables
- Graphical solution procedure
  - Profit line; Optimal solution; Feasible region; Extreme points
  - Slope and y-intercept

PART 1

Linear Programming: Definition

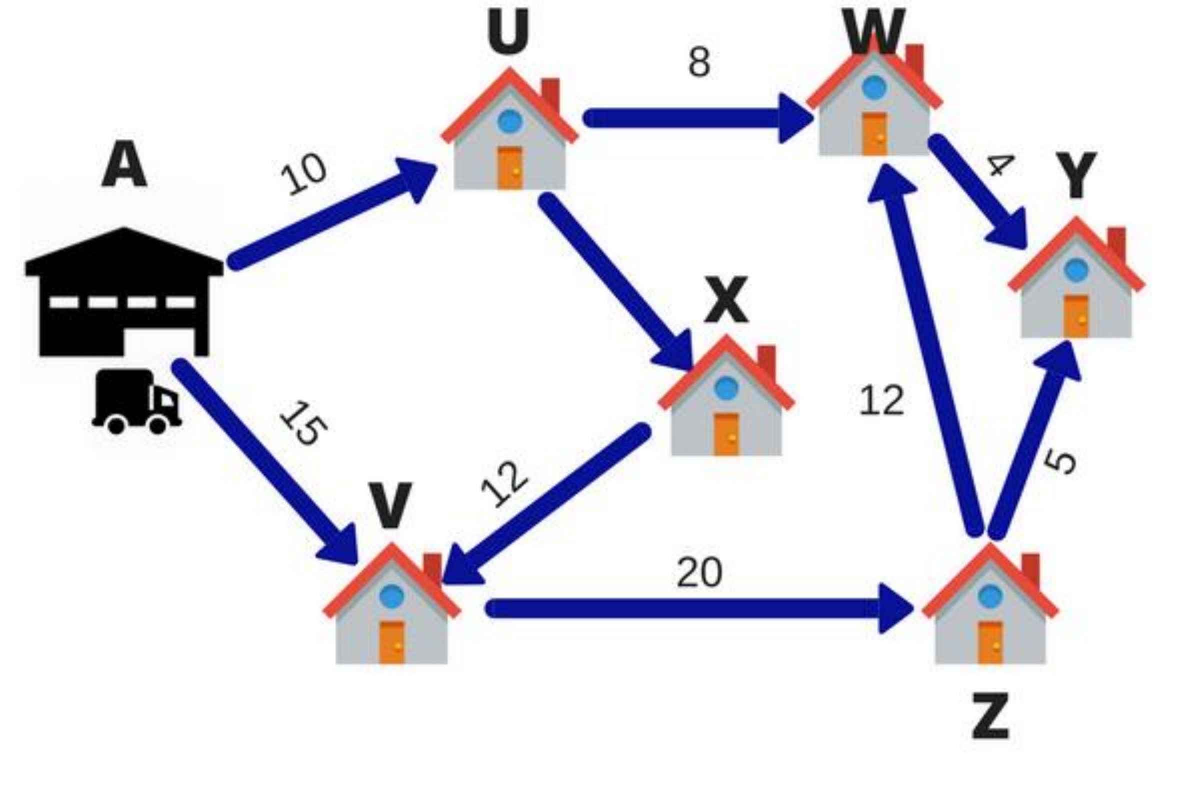
Linear Programming (LP)

- LP is a specific case of **mathematical optimisation**.

- It consists in finding the "best" value obtainable under specific conditions.
  - The **objective** is the maximization or minimization of some quantity;
  - Subject to a number of **constraints** that limit the degree to which the objective can be pursued.
  - Using **decision variables** which are under the control of the decision maker(s).

Let’s say a DHL delivery man has 6 packages to deliver in a day.

The warehouse is located at point A. The delivery destainations are U,V,W,X,Y and Z. The numbers on the line indicate the distance between the cities. To save time and fuel,he wants to take the shortest route.



The technique of choosing the shortest route is called linear programming.

Linear Programming is used for obtaining the most optimal solution for a problem given constraints. We formulate our real-life problem. Into a mathematical model. It involves an objective function ,inequalities subject to constraints.

Reflection

- Is the linear representation of the 6 points representative of the real world ?

No

It is a simplification as the real route may consist of multiple turns , U turns , signals and traffic jams .

Example: the designer problem

- A young designer decides to launch her own line of clothes and creates two types of shirts: ***A*** and ***B***.
- Both are made out of the same fabric. Each *shirt A* requires **2 square-meter** while each *shirt B* requires **3 square-meter**. She can buy only **19 square-meter** of the same fabric per week.
- Based on her past experience, she knows that she can sell a **maximum of 6** *shirts A* per week.
- Both shirts require **1 hour** of labour to produce. She has only **8 hours** per week to produce these shirts.
- *Shirt A* generates a unit profit of **£5** while *shirt B* generates a unit profit of **£7**.

$\Rightarrow$ *How many of the two types of shirt should the designer produce in order to maximise profit contribution?*

Example : the designer problem

- Understand the problem thoroughly
- Describe the **objective**
  - The <u>objective</u> is to maximise the total contribution to profit
- Describe each **constraint**:
  - <u>Constraint 1:</u> the amount of fabric used must be less than or equal to the available amount of fabric.
  - <u>Constraint 2:</u> the number of shirts A produced has to be less than or equal to a predefined quantity.
  - <u>Constraint 3:</u> the number of hours used to produce shirts must be less than or equal to the number of hours the designer is available.

Example : the designer problem

- Define the **decision variables**
  - Decision variable 1: number of shirt A to be produced ($\mathbf{x_1}$)
  - Decision variable 2: number of shirt B to be produced ($\mathbf{x_2}$)

PART 2

Problem formulation

Problem formulation

- Problem formulation or modelling is the process of translating a **verbal statement** of a problem **into a mathematical statement**.

Problem formulation

- Linear programming studies the cases where the function to optimise (i.e., **objective function**) and the **constraints** are expressed linearly.
- **Linear functions** are functions in which each variable appears in a separate term raised to the first power and is multiplied by a constant (which could be **0**).
- **Linear constraints** are linear functions that are restricted to be "*less than or equal to*", "*equal to*", or "*greater than or equal to*" a constant.

Example: the designer problem

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$\Rightarrow$ *How many of the two types of shirt should the designer produce in order to maximise profit contribution?*

Problem formulation: Example

- Write the **objective** in terms of decision variables:
  - The profit contribution made by producing shirt A:

    £5 x_1
  - The profit contribution made by producing shirt B:

    £7 x_2
  - Total profit contribution:

    $5x_1 + 7x_2$

  $\Rightarrow$ Objective function:

    $\max 5x_1 + 7x_2$

Problem formulation: example

- Write the **constraints** in terms of decision variables:

Constraint 1:

– Each *shirt A* requires 2 square-meter

$\rightarrow$ amount of fabric used for producing shirt A:

$$\mathbf{2\ x_1}$$

– Each *shirt B* requires 3 square-meter

$\rightarrow$ amount of fabric used for producing shirt B:

$$\mathbf{3\ x_2}$$

– Available fabric per week is 19 square-meter

$\Rightarrow$ $\mathbf{2\ x_1 + 3\ x_2 \leq 19}$

Problem formulation: example

- Write the **constraints** in terms of decision variables:

  - Constraint 2:

    $$\mathbf{x_1 \leq 6}$$

  - Constraint 3:

    $$\mathbf{x_1 + x_2 \leq 8}$$

  - Non-negativity constraints:

    $$\mathbf{x_1 \geq 0}$$

    $$\mathbf{x_2 \geq 0}$$

Problem formulation: Example

- The **complete mathematical model** for the designer problem can be summarised as follows:

$$\boxed{\text{Max} \quad 5x_1 + 7x_2}$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + 3x_2 &\leq 19 \\ x_1 &\leq 6 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\Rightarrow$ This is a maximisation problem

In brief…

- A linear programming problem may be defined as the problem of **maximizing or minimizing a linear function** subject to **linear constraints**.
- The constraints may be **equalities** or **inequalities**.

- Guidelines for model formulation:
  - Understand the problem thoroughly.
  - Describe the objective.
  - Describe each constraint.
  - Define the decision variables.
  - Write the objective in terms of the decision variables.
  - Write the constraints in terms of the decision variables.

PART 3

Graphical Solution Procedure

Graphical solution

- A linear programming problem involving **only two decision variables** can be solved using a graphical solution procedure.

Graphical solution

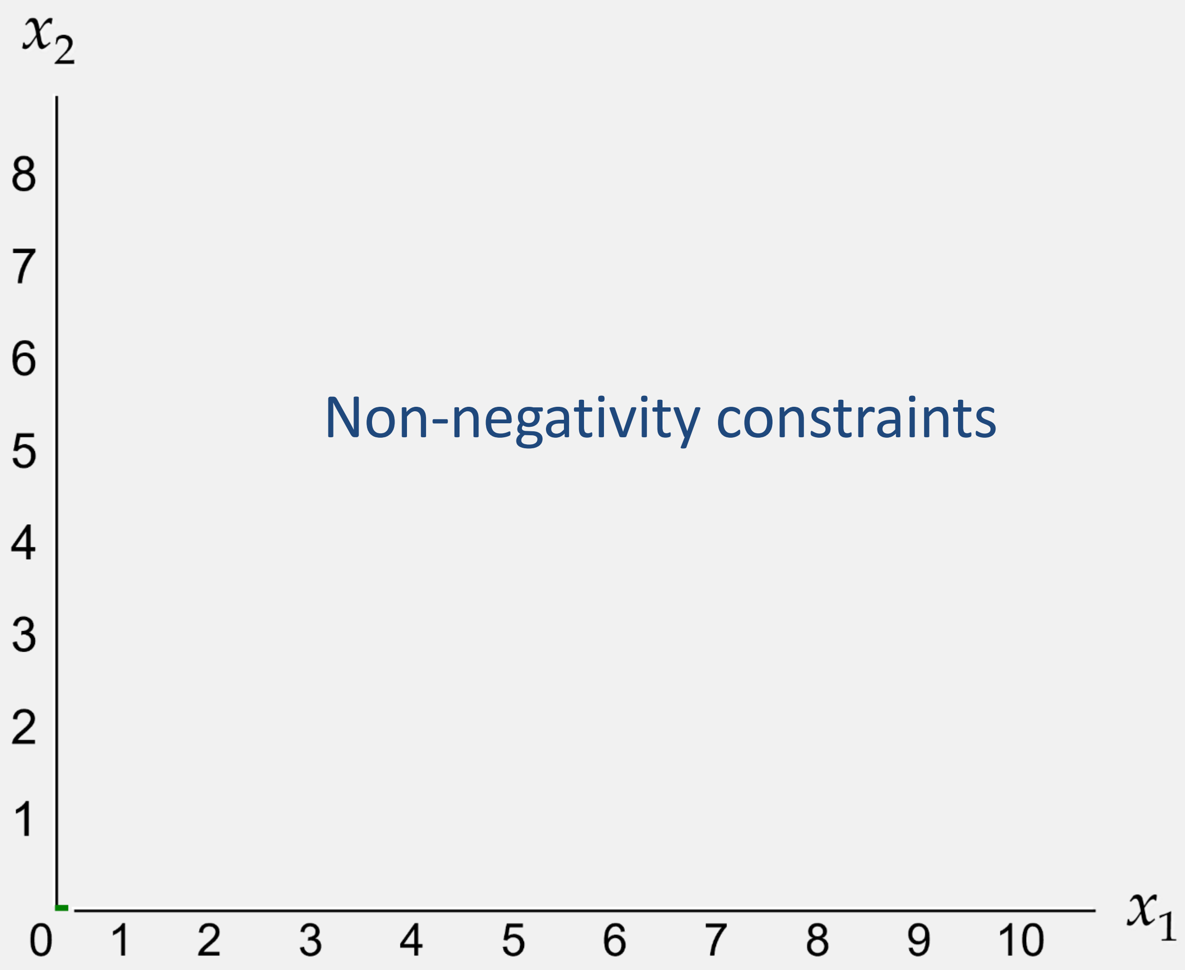
- Steps (for maximisation problems):
  1. Prepare a graph of the **feasible solutions for each of the constraints**, i.e., each inequality constraint is satisfied by a half-plane of points
  2. Determine the **feasible region** that satisfies all the constraints simultaneously, i.e., intersection of all the half-planes.
  3. Draw an **objective function line**.
  4. Move parallel objective function lines toward larger objective function values without entirely leaving the feasible region.
  5. Any **feasible solution** on the objective function line with the largest value is an **optimal solution**.

Example: Problem formulation

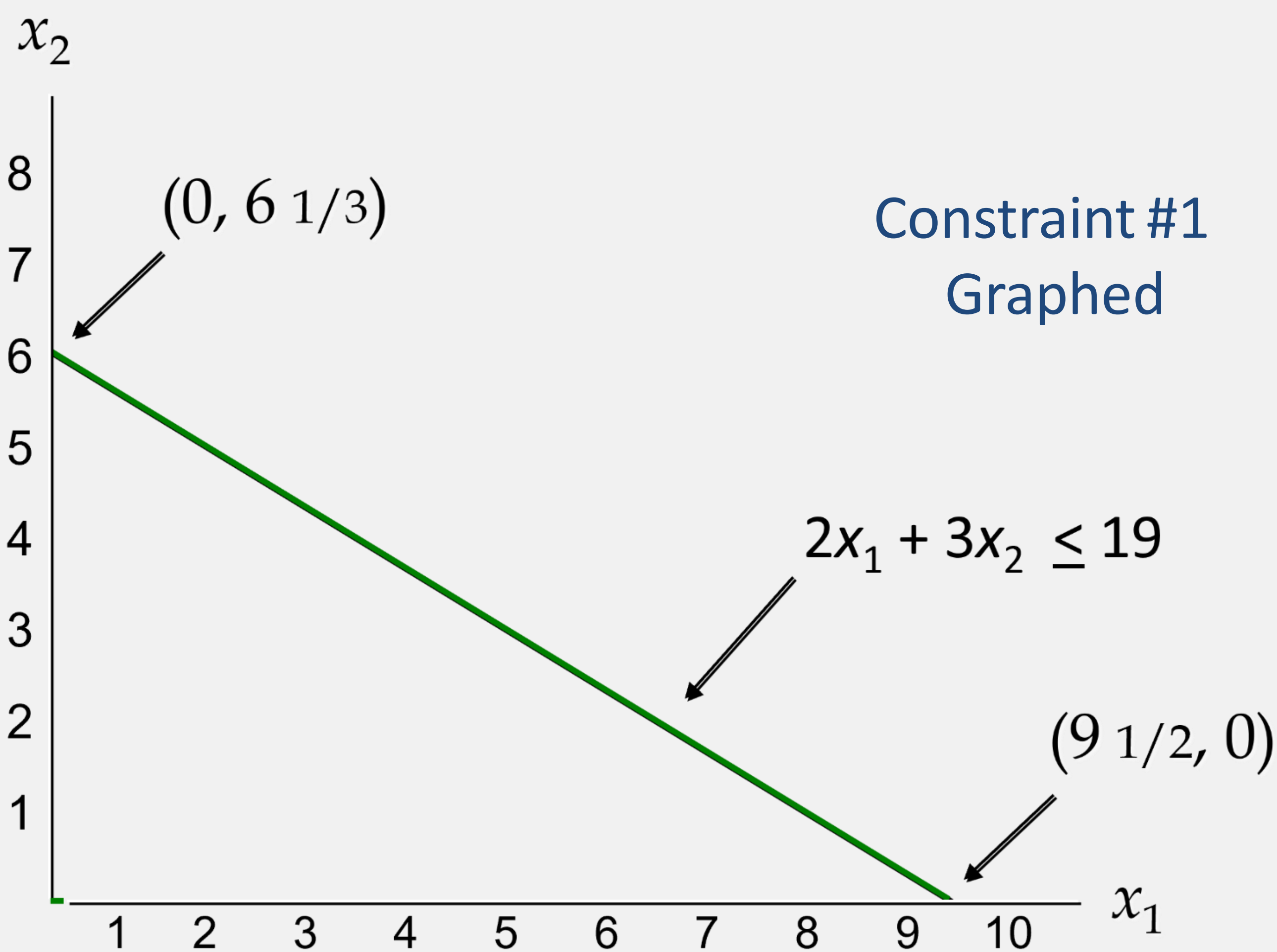
- The complete mathematical model for the designer problem can be summarised as follows:

$$
\begin{array}{ll}
\text{Max} & 5x_1 + 7x_2 \\
\\
\text{s.t.} & 2x_1 + 3x_2 \le 19 \\
 & x_1 \le 6 \\
 & x_1 + x_2 \le 8 \\
 & x_1, x_2 \ge 0
\end{array}
$$

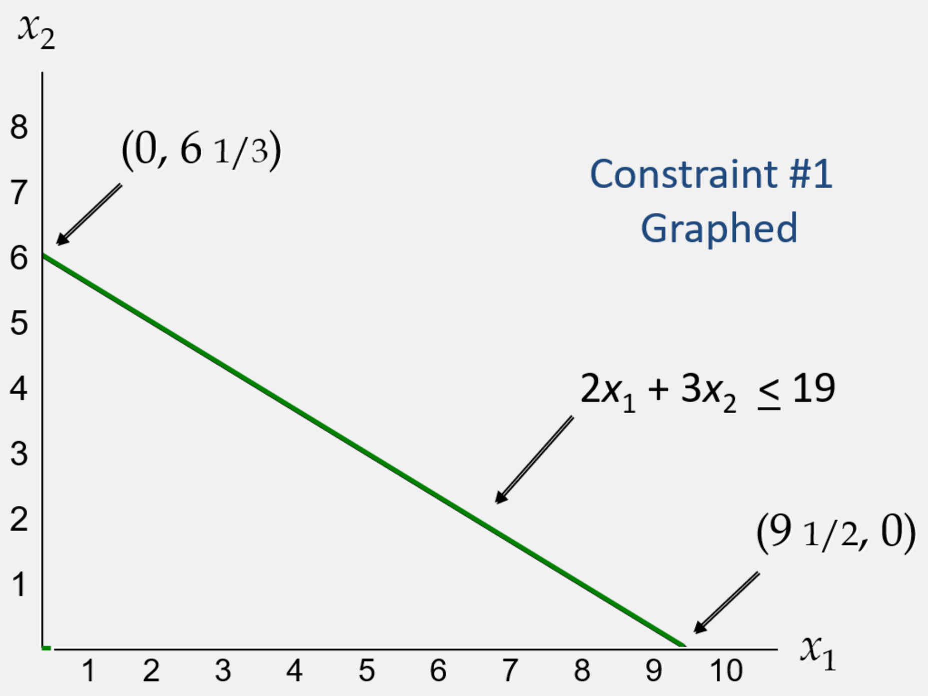
Example: Graphical Solution



Example: Graphical Solution



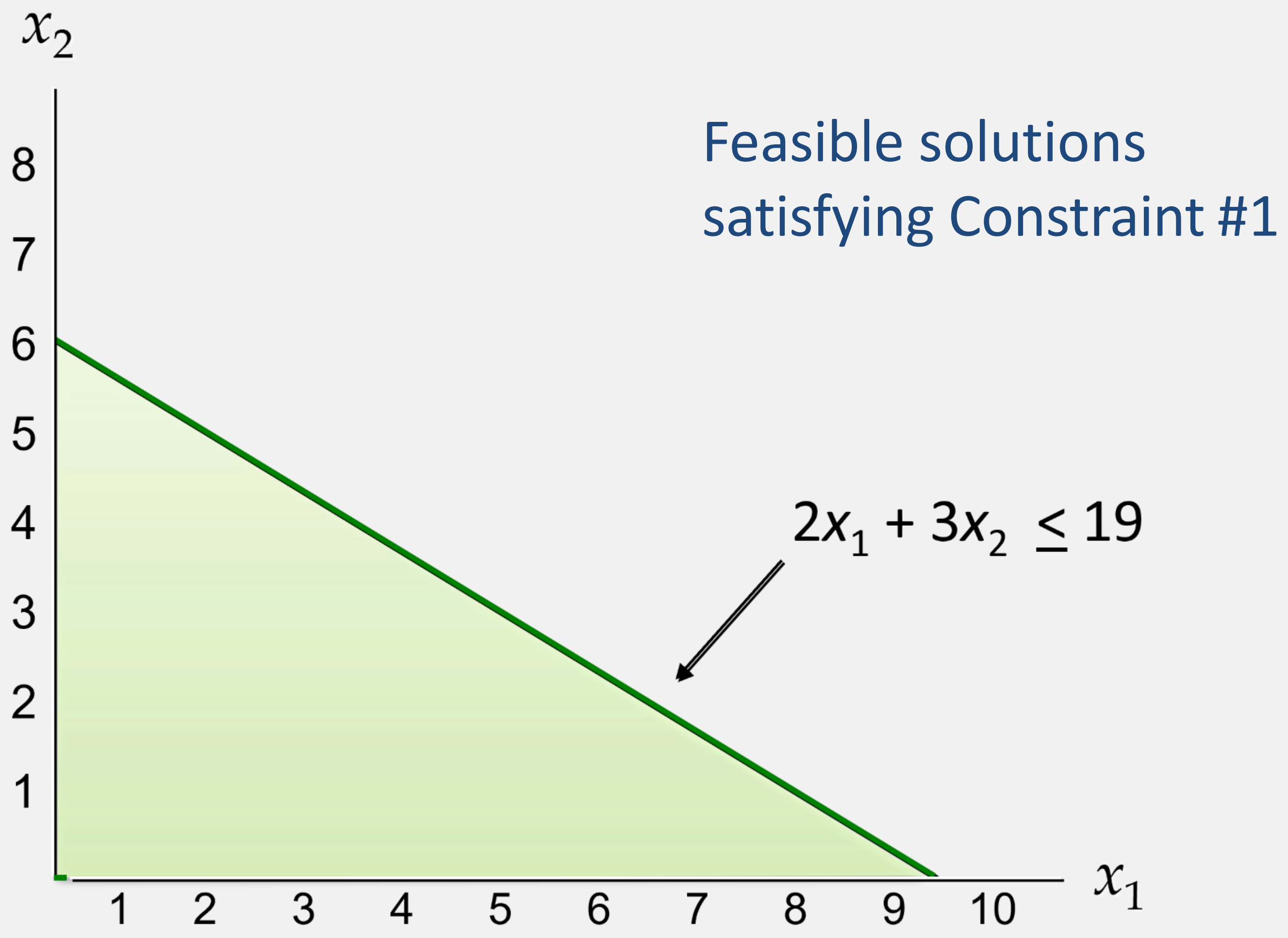
Example: Graphical Solution



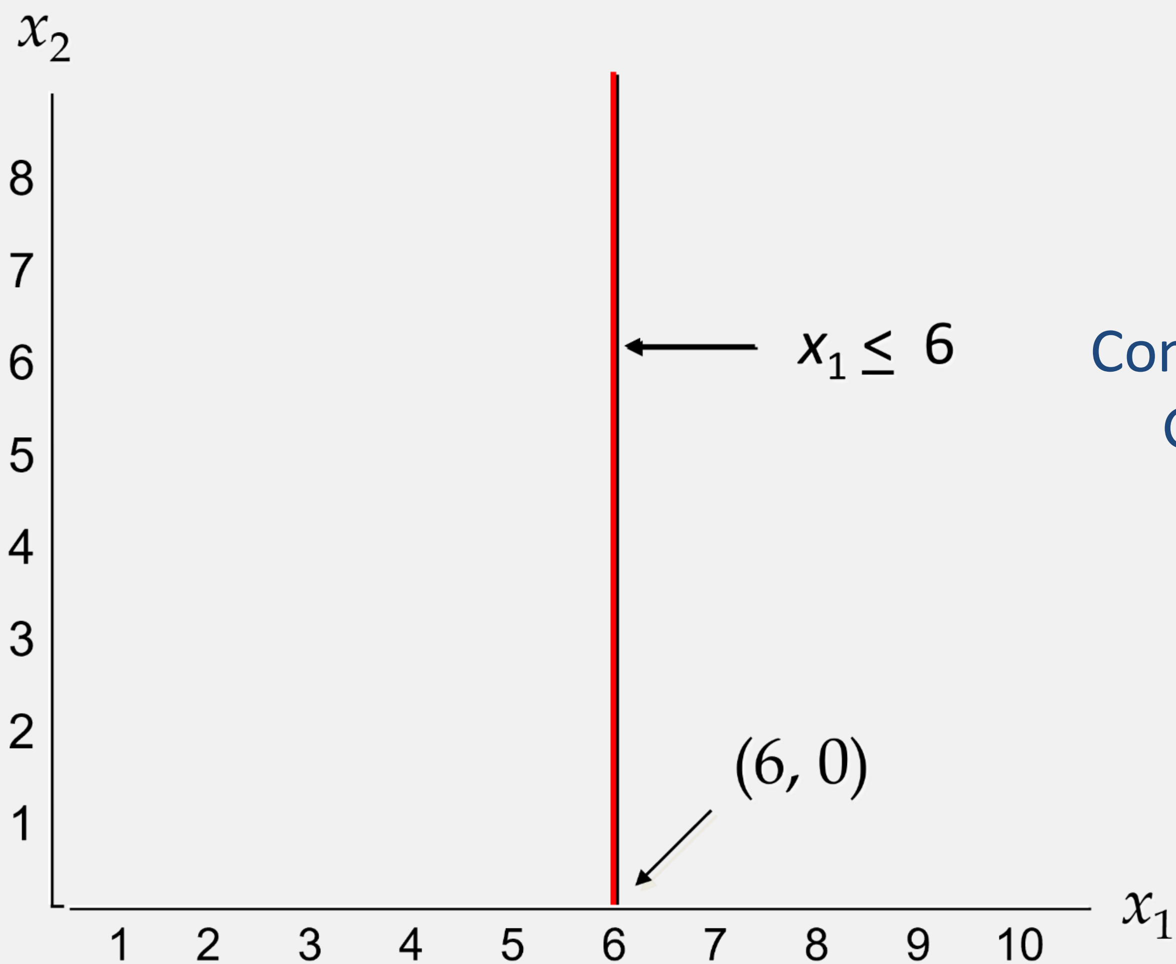
$$\boxed{2x_1 + 3x_2 = 19}$$

- When $x_1 = 0$, then $x_2 = 6.33$
- When $x_2 = 0$, then $x_1 = 9.5$
- Connect **(0, 6.33)** and **(9.5, 0)**

Example: Graphical Solution

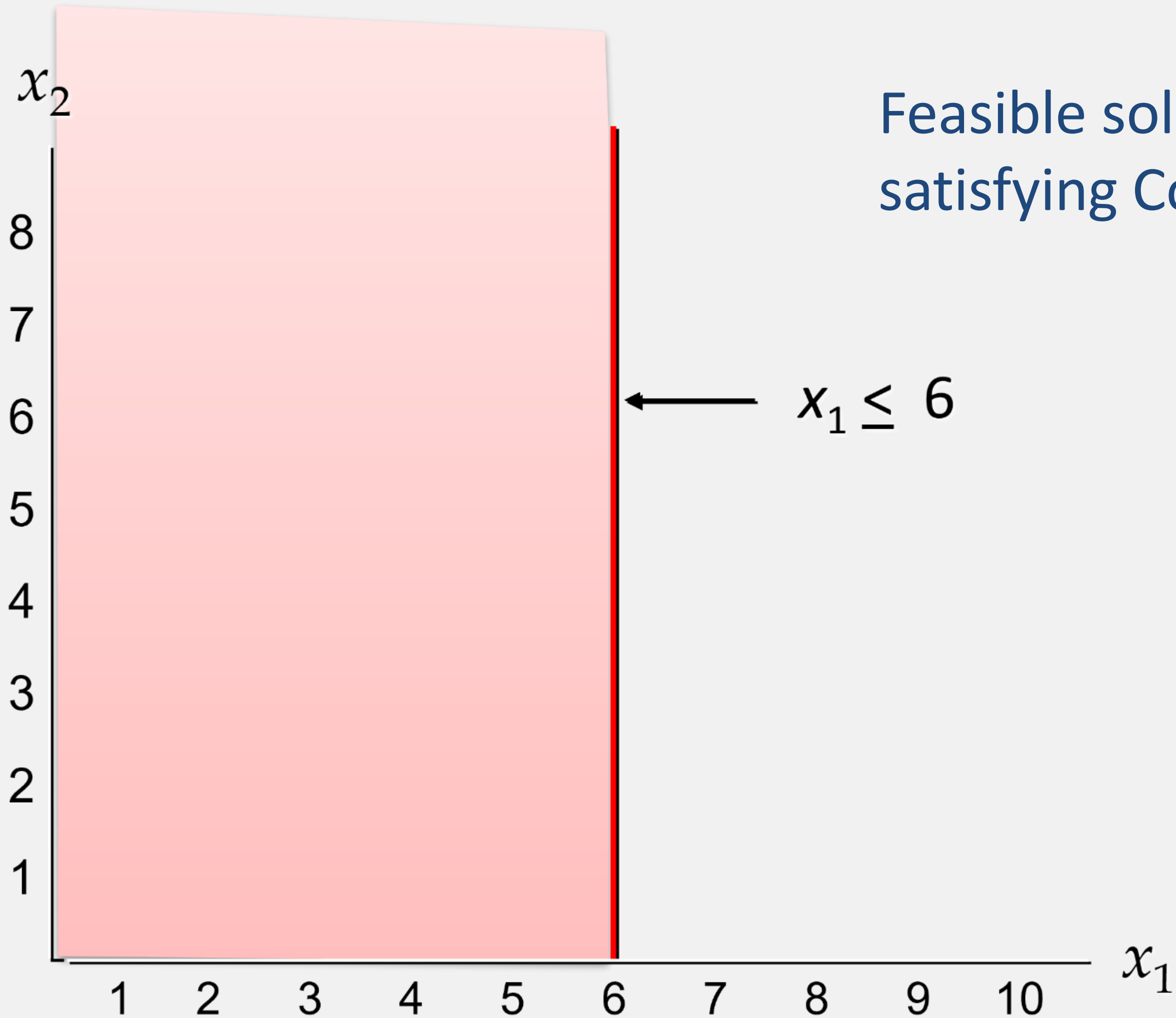


Example: Graphical Solution



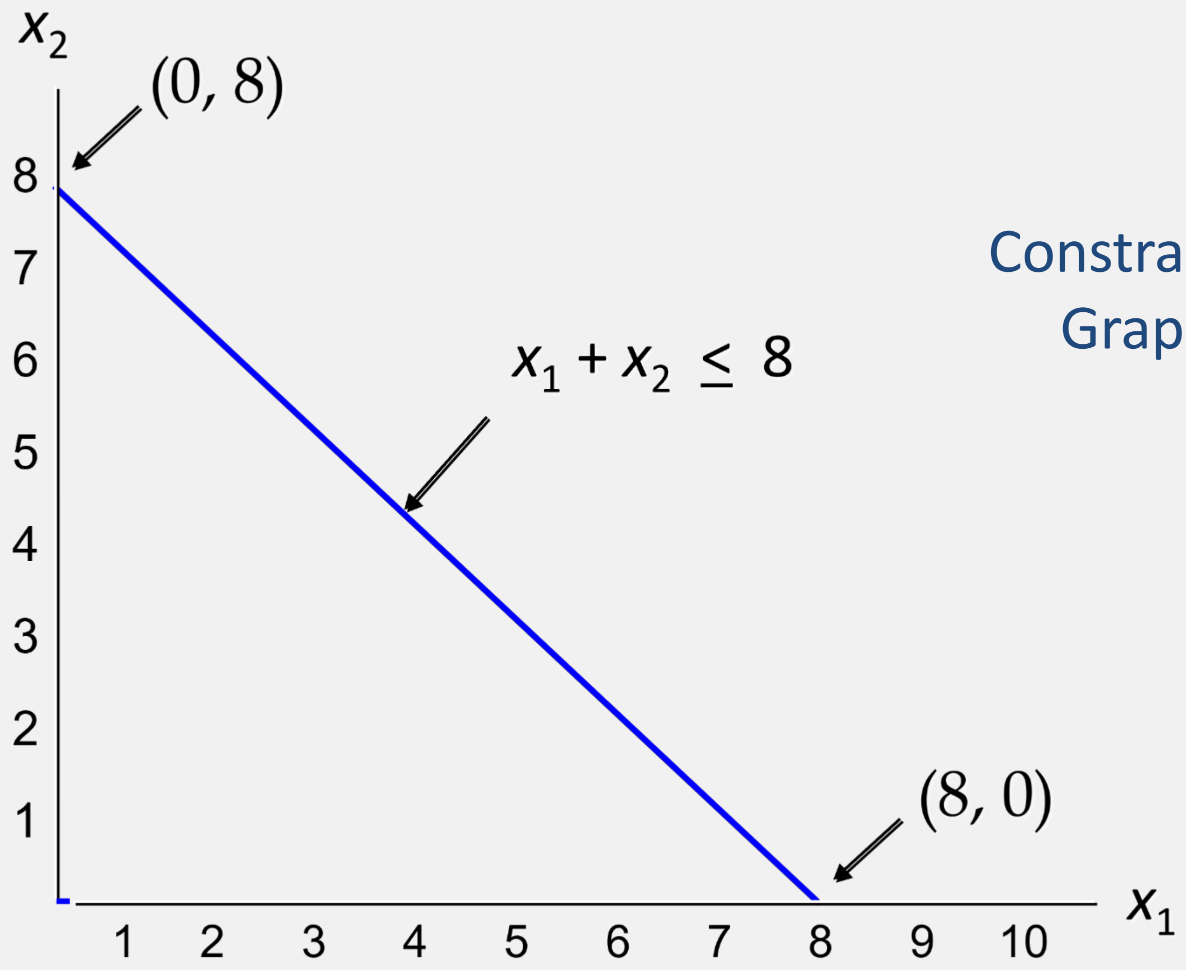
Constraint #2 Graphed

Example: Graphical Solution



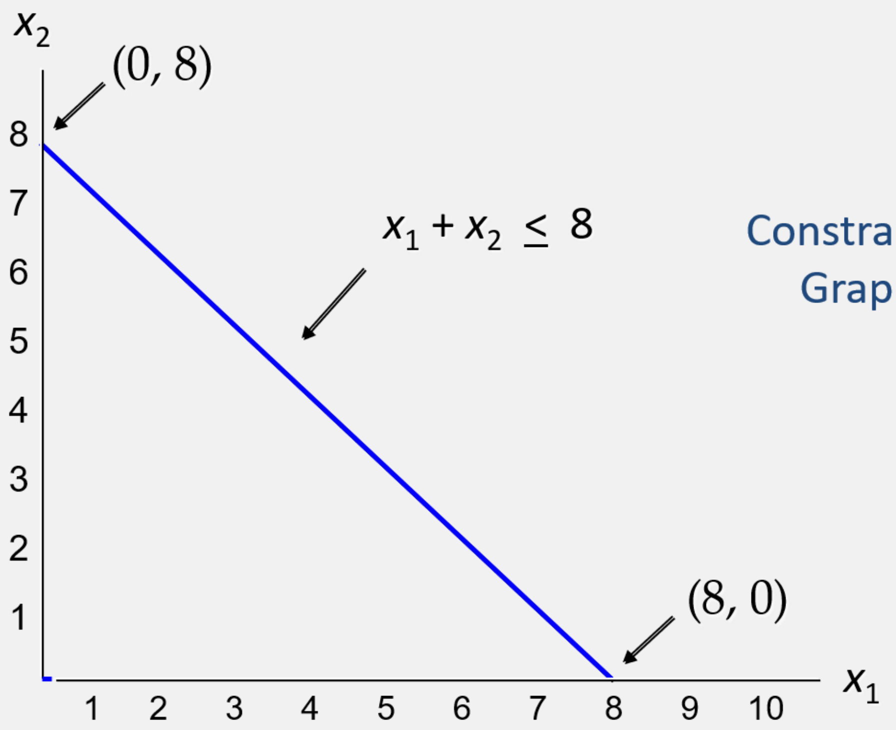
Feasible solutions satisfying Constraint #2

Example: Graphical Solution



Constraint #3 Graphed

Example: Graphical Solution

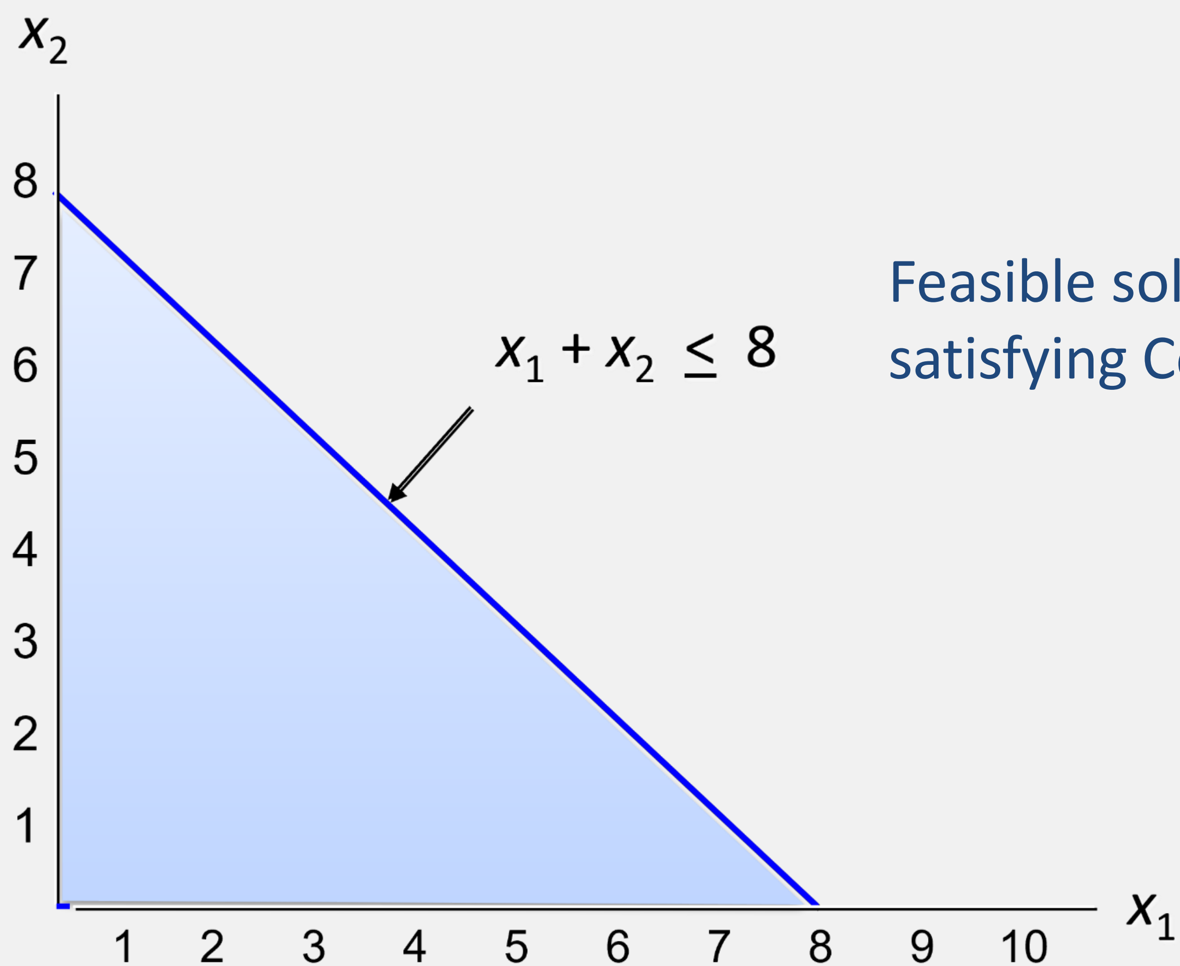


$$\boxed{x_1 + x_2 = 8}$$

Constraint #3 Graphed

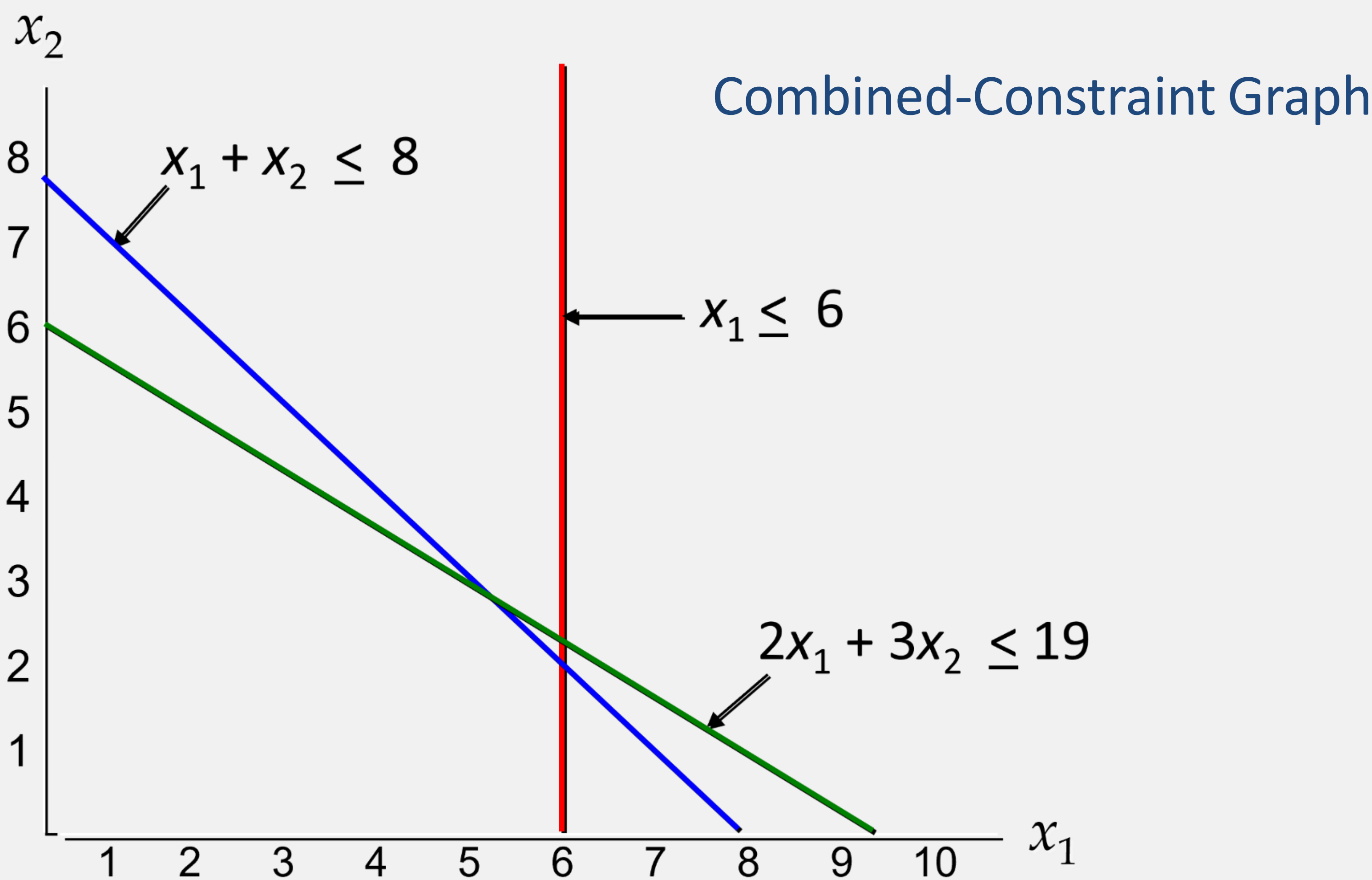
- When $x_1 = 0$, then $x_2 = 8$
- When $x_2 = 0$, then $x_1 = 8$
- Connect **(0, 8)** and **(8, 0)**

Example: Graphical Solution

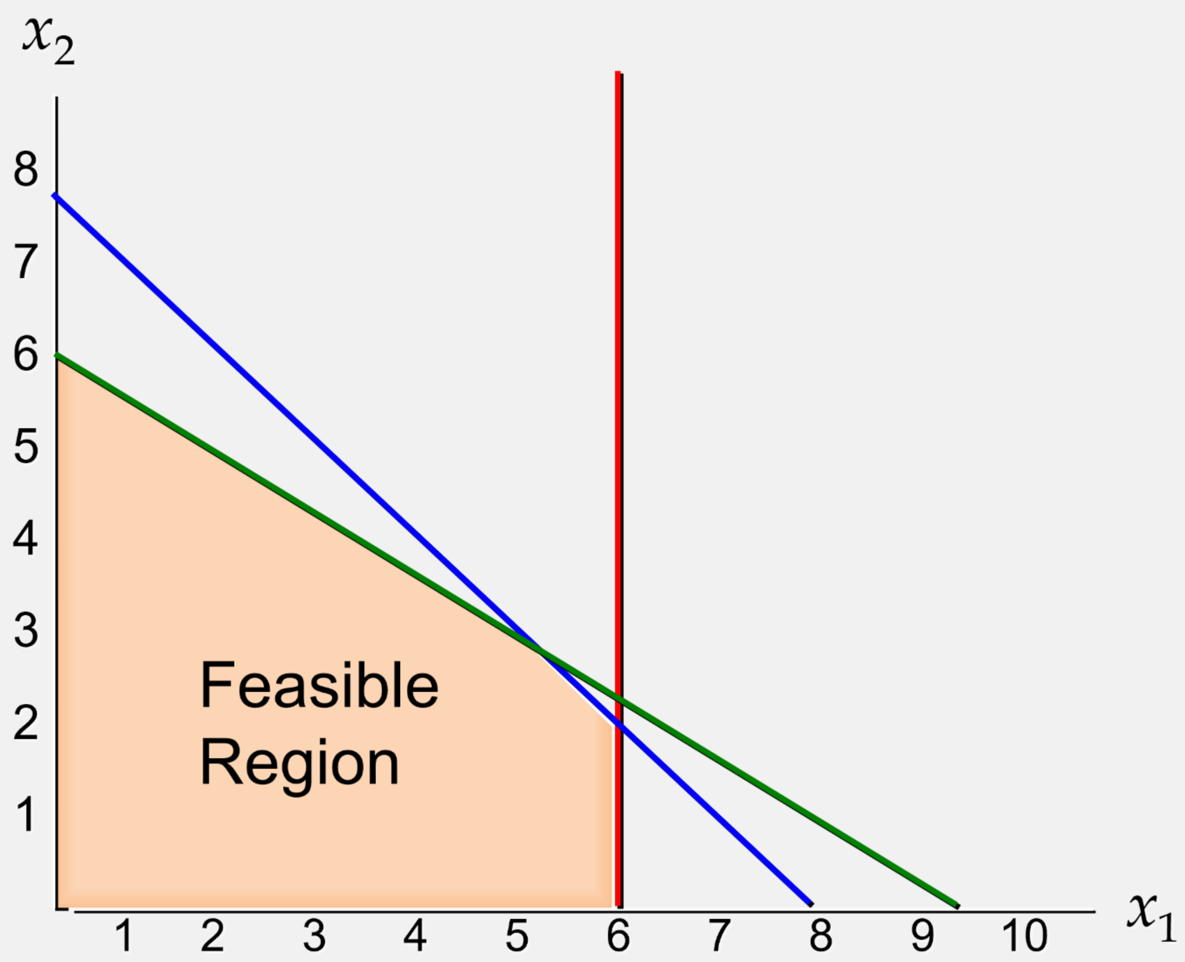


Feasible solutions satisfying Constraint #3

Example: Graphical Solution



Example: Graphical Solution



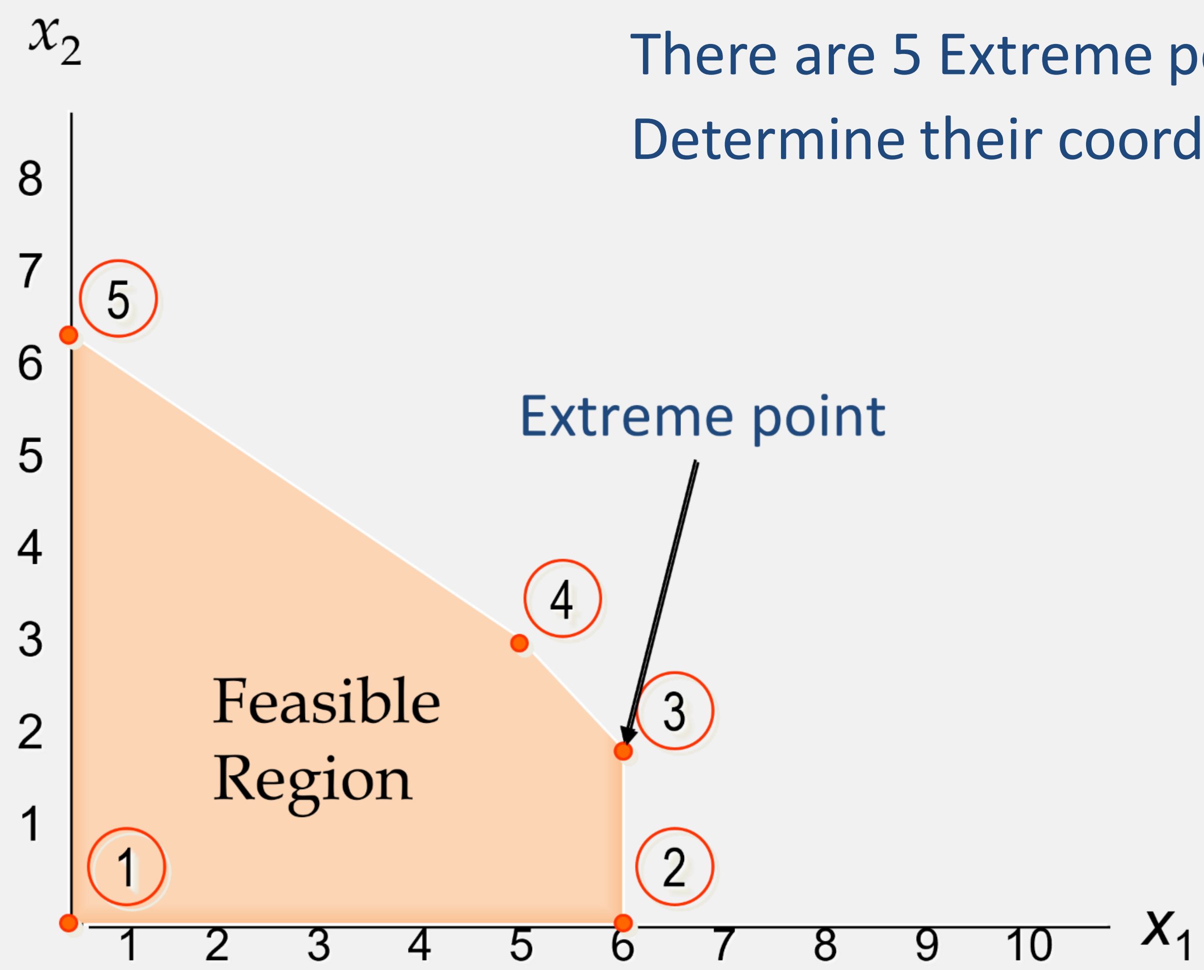
Extreme Points and the Optimal Solution

- The corners or vertices of the feasible region are referred to as the **extreme points**.
- An **optimal solution** to an LP problem can be found at an extreme point of the **feasible region**.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.

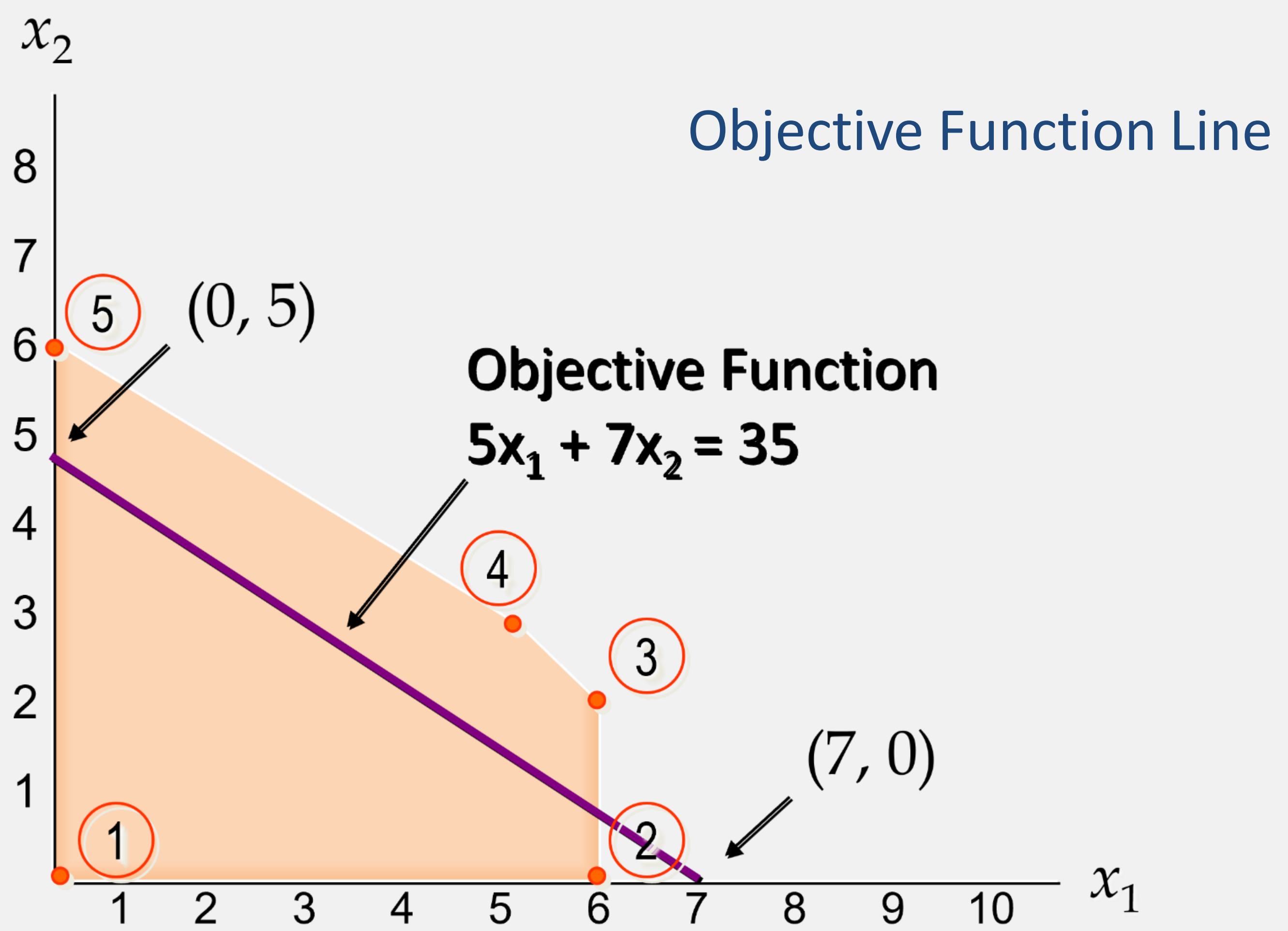
Example: Graphical Solution

There are 5 Extreme points.

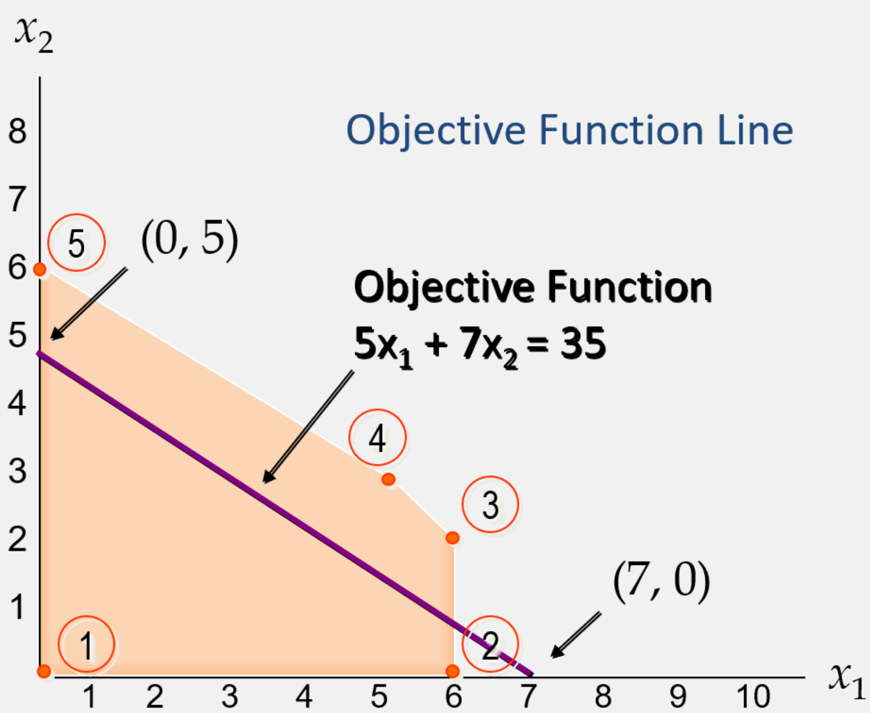
Determine their coordinates.



Example: Graphical Solution

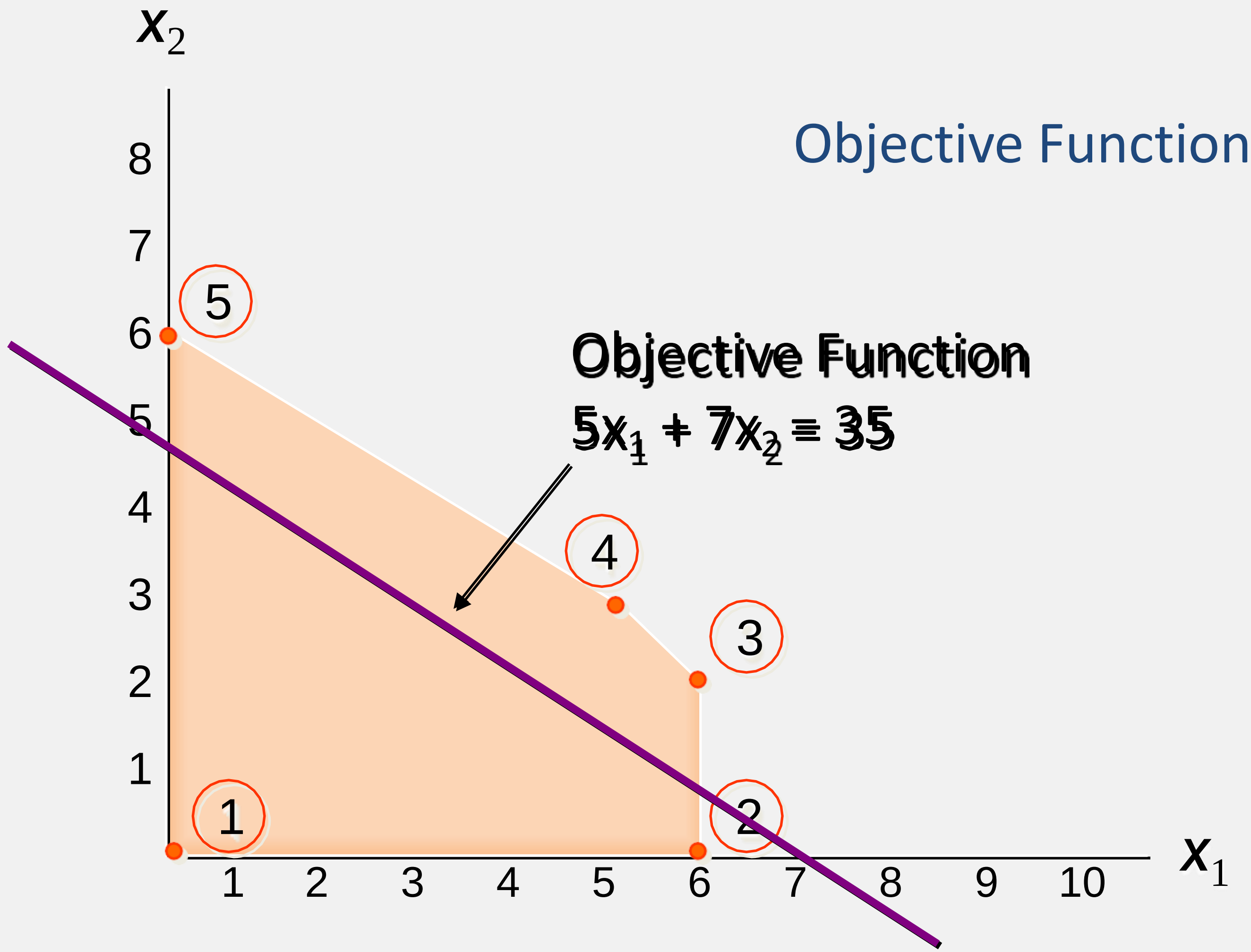


Example: Graphical Solution



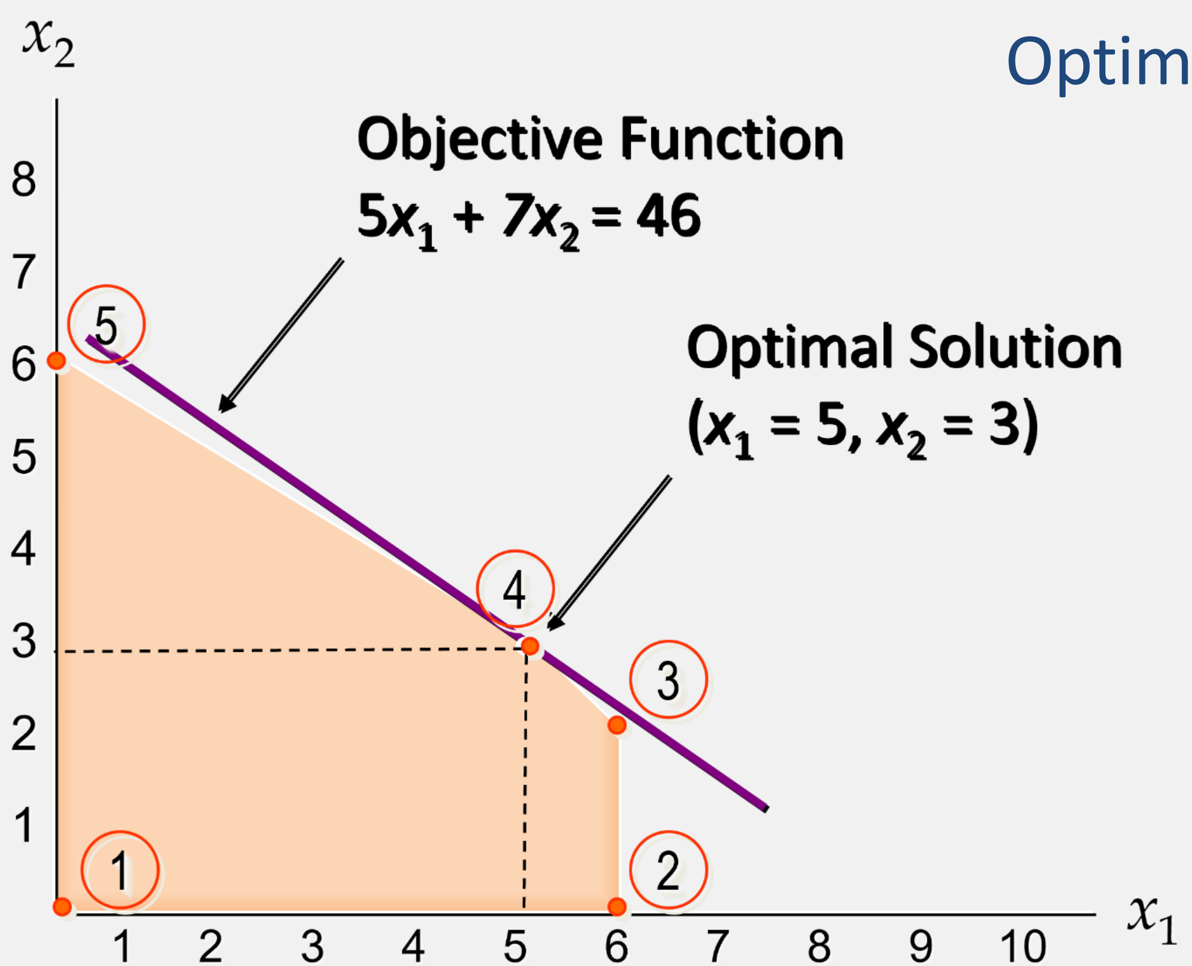
- Set the objective function equal to an **arbitrary constant** (e.g., 35) and graph it.
- For **$5x_1 + 7x_2 = 35$**, when **$x_1 = 0$**, then **$x_2 = 5$**; when **$x_2 = 0$**, then **$x_1 = 7$**.
- Connect **(7,0)** and **(0,5)**.

Example: Graphical Solution



Example: Graphical Solution

Optimal Solution



PART 4

Slope and y-intercept

Slope: formula

- Two points: $A(x_A, y_A)$ and $B(x_B, y_B)$

- Formula for the slope of the straight line going through A and B:

$$\frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}$$

Change in ***y*** going up or down

Change in ***x*** going from left to right

Slope and y-intercept

- Let us consider the following equation:

$$y = a x + b$$

Slope

The slope indicates how much ***y*** is changing for every unit of ***x***.

Gives the ***y*-intercept**

The equation line crosses the y-axis at ***b***

Example: equation

- Let us consider the following equation:

$$y = \frac{2}{3}x - 1$$

Slope

Two units up and three units over to the right

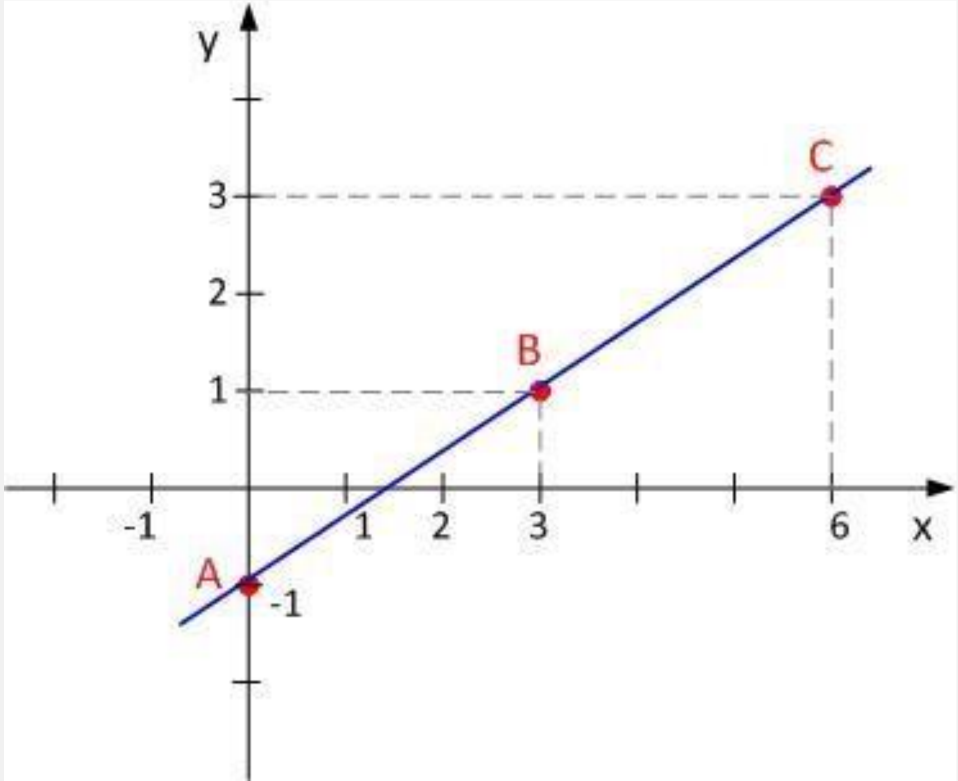
Gives the ***y*-intercept**

The equation line crosses the y-axis at ***-1***

Example: graphical solution

- Let us consider the following equation:

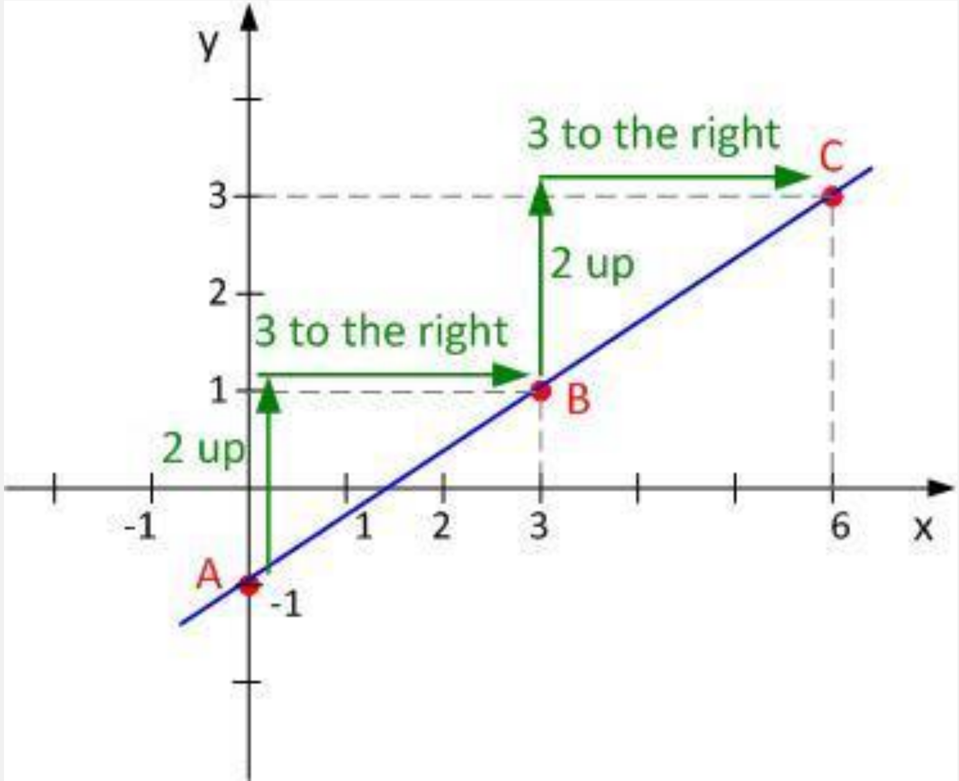
$$y = \frac{2}{3}x - 1$$



Example: graphical solution

- Let us consider the following equation:

$$y = \frac{2}{3}x - 1$$



Graphical solution: steps

- Draw the first point **(0, *b*)**;
- From this first point, use the slope ***a*** to draw the second point;
- From this second point, a third point could be obtained by using again the slope;
- With these points the line can be drawn.

Questions

Question 1

Let us consider the following equation: **$5x + 10y = 2$**

Write it in the following form: **$y = ax + b$**

Question 2

Consider the following inequality: **$-5 < -x/3 < -2$**

Multiplying through by (**-3**), what will you obtain?

Question 3

Consider the following inequality: **$2/3 < 1/x < 2$** where $x>0$

Find the range for **x**.