Stochastic Networks HW 3

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Note that for problem 3, $\theta(x)$ is the Heaviside step function.

3(a)

$k_{n,i}^{in} = \sum_{i=1}^{n} a_i b_j$, and $k_{n,i}^{in} = \sum_{i=1}^{n} a_j b_i$.

3(b)

$\langle k \rangle = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j$

3(c)

$\frac{\langle k_n^{in} k_n^{out} \rangle}{\langle k \rangle} = N \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j b_i b_j}{\sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j}$

3(d)

First, choose $N-1$ linearly independent vectors that are orthogonal to $\boldsymbol{b}$; they clearly have eigenvalue 0. Now, the remaining nonzero eigenvalue can be found by considering the action of the matrix,

$$(\boldsymbol{a}\boldsymbol{b}^T)\boldsymbol{u} = (\boldsymbol{b}^T\boldsymbol{u})\boldsymbol{a},$$

and the eigenvalue equation

$$(\boldsymbol{a}\boldsymbol{b}^T)\boldsymbol{u} = \lambda\boldsymbol{u},$$

from which we can see that the eigenvalue we seek is $\lambda = \boldsymbol{a}^T\boldsymbol{b}$, the corresponding right eigenvector is $\boldsymbol{u} = \boldsymbol{a}$, and the left eigenvector is $\boldsymbol{v}^T = \boldsymbol{b}^T$.

4(a)

The Jacobian is $J = K\boldsymbol{A} - (1 + 2\boldsymbol{x})\boldsymbol{I}$, so its eigenvalues are the sums of the eigenvalues of $K\boldsymbol{A}$ and of $-(1 + 2\mathbf{0})\boldsymbol{I}$, which are all -1, so the Jacobian has all negative (real parts of its) eigenvalues (and thus the fixed point is stable) iff the largest eigenvalue of $K\boldsymbol{A}$, $\lambda_{\max} \approx \frac{\langle \hat{k_{in}} \hat{k_{out}} \rangle}{\langle k \rangle} + \frac{1}{K}$, is less than 1.