Calculation of the Average GW Response for TDI-X2.0

April 24, 2025

Contents

1 Compute the Specified Form of $< \left| X_{2.0}^{GW} \right|^2 >$

To compute the averaged response to GW using a semi-analytical approach, we evaluate $< \left| X_{2.0}^{GW} \right|^2 >$.We have the following notations:

$$\hat{k} = -[\cos\beta\cos\lambda, \cos\beta\sin\lambda, \sin\beta] \tag{1}$$

$$x_i = a\cos(\alpha) + ae(\sin\alpha\cos\alpha\sin\beta_i - (1+\sin^2\alpha)\cos\beta_i) \tag{2}$$

$$y_i = a\sin(\alpha) + ae(\sin\alpha\cos\alpha\cos\beta_i - (1+\cos^2\alpha)\sin\beta_i) \tag{3}$$

$$z_i = -\sqrt{3}ae\cos(\alpha - \beta_i) \tag{4}$$

$$\hat{n}_{23} = \frac{1}{L}[x_2 - x_3, y_2 - y_3, z_2 - z_3] \tag{5}$$

$$\vec{u} = [\sin\lambda, -\cos\lambda, 0] \tag{6}$$

$$\vec{v} = [-\sin\beta\cos\lambda, -\sin\beta\sin\lambda, \cos\beta] \tag{7}$$

$$F_{rs}^{+} = \hat{n}_{rs}^{i}\hat{n}_{rs}^{j}[\epsilon_{ij}^{+}\cos 2\psi + \epsilon_{ij}^{\times}\sin 2\psi] \tag{8}$$

$$F_{rs}^{\times} = \hat{n}_{rs}^{i}\hat{n}_{rs}^{j}[-\epsilon_{ij}^{+}\sin 2\psi + \epsilon_{ij}^{\times}\cos 2\psi] \tag{9}$$

$$X_{1.5}^{GW} = (\omega L)\sin(\omega L)e^{-i[\Phi(t-\hat{k}\vec{R}_1)-\omega L]}\{A_+[F_{13}^{+}\Upsilon_{13} - F_{12}^{+}\Upsilon_{12}] + A_\times[F_{13}^{\times}\Upsilon_{13} - F_{12}^{\times}]\} \tag{10}$$

$$X_{2.0}^{GW} = 2i\sin(2\omega L)e^{-2i\omega L}X_{1.5}^{GW} \tag{11}$$

$$F_{rs}^{+} = \hat{n}_{rs}^{i}\hat{n}_{rs}^{j}[\epsilon_{ij}^{+}\cos 2\psi + \epsilon_{ij}^{\times}\sin 2\psi] \tag{12}$$

$$F_{rs}^{\times} = \hat{n}_{rs}^{i}\hat{n}_{rs}^{j}[-\epsilon_{ij}^{+}\sin 2\psi + \epsilon_{ij}^{\times}\cos 2\psi] \tag{13}$$

$$\Upsilon_{rs} = Sinc[\frac{\omega L}{2}(1 - \hat{k}.\hat{n}_{rs})]e^{-i\frac{\omega L}{2}(1-\hat{k}.\hat{n}_{rs})} + Sinc[\frac{\omega L}{2}(1 - \hat{k}.\hat{n}_{sr})]e^{-i\frac{\omega L}{2}(3+\hat{k}.\hat{n}_{sr})} \tag{14}$$

Using another notations:

$$F_X^{+} \equiv \frac{1}{4}[F_{13}^{+}\Upsilon_{13} - F_{12}^{+}\Upsilon_{12}] \tag{15}$$

$$F_X^{+} \equiv \frac{1}{4}[F_{13}^{\times}\Upsilon_{13} - F_{12}^{\times}\Upsilon_{12}] \tag{16}$$

in order to get the compact form:

$$X_{1.5}^{GW} = (\omega L)\sin(\omega L)e^{-i[\Phi(t-\hat{k}\vec{R}_1)-\omega L]}(4A_+F_X^{+} + 4A_\times F_X^{\times}) \tag{17}$$

$$X_{2.0}^{GW} = 2i\sin(2\omega L)e^{-2i\omega L}(\omega L)\sin(\omega L)e^{-i[\Phi(t-\hat{k}\vec{R}_1)-\omega L]}(4A_+F_X^{+} + 4A_\times F_X^{\times}) \tag{18}$$

We need to compute $< \left|X_{2.0}^{GW}\right|^2 >$ where $< \ >$ is used for the polarization and sky averaging:

$$< ... >= \frac{1}{2\pi}\int_0^{2\pi} d\psi \frac{1}{4\pi}\int d^2\Omega \tag{19}$$

Then we get:

$$\begin{aligned} < \left|X_{2.0}^{GW}\right|^2 > =& < \left|-4\sin^2(2\omega L)e^{-4i\omega L}\left|X_{1.5}^{GW}\right|^2\right| > \\ &= 64(\omega L)^2\sin^2\omega L\sin^2 2\omega L\Big[A_+^2 < (F_X^+)^2 > + A_\times^2 < (F_X^\times)^2 > \\ &\quad + 2A_+A_\times(< F_X^+(F_X^\times)^* + (F_X^+)^*F_X^\times >)\Big] \end{aligned} \tag{20}$$

Next, we compute the specific form ofF_X^+ and $F_X^\times$:

Given $\beta_1 = \lambda, \beta_2 = \frac{2}{3}\pi + \lambda$ and $\beta_3 = \frac{4}{3}\pi + \lambda$, we compute the differences $x_2 - x_3$, $y_2 - y_3$, and $z_2 - z_3$:

$$x_2 - x_3 = ae\Big[\sin\alpha\cos\alpha(\sin\beta_2 - \sin\beta_3) - (1+\sin^2\alpha)(\cos\beta_2 - \cos\beta_3)\Big] \tag{21}$$

Applying trigonometric identities:

$$\sin\beta_2 - \sin\beta_3 = 2\cos\left(\frac{\beta_2+\beta_3}{2}\right)\sin\left(\frac{\beta_2-\beta_3}{2}\right) \tag{22}$$

$$= -\sqrt{3}\cos\lambda \tag{23}$$

$$\cos\beta_2 - \cos\beta_3 = -2\sin\left(\frac{\beta_2+\beta_3}{2}\right)\sin\left(\frac{\beta_2-\beta_3}{2}\right) \tag{24}$$

$$= \sqrt{3}\sin\lambda \tag{25}$$

Thus:

$$x_2 - x_3 = -\sqrt{3}ae\left[\sin\alpha\cos\alpha\cos\lambda + (1+\sin^2\alpha)\sin\lambda\right] \tag{26}$$

Similarly, it can be obtained that:

$$y_2 - y_3 = \sqrt{3}ae\left[\sin\alpha\cos\alpha\sin\lambda + (1+\cos^2\alpha)\cos\lambda\right] \tag{27}$$

$$z_2 - z_3 = -3ae\sin(\alpha - \lambda) \tag{28}$$

$$x_3 - x_1 = \sqrt{3}ae\left[-\sin\alpha\cos\alpha\cos\left(\lambda + \frac{2\pi}{3}\right) + (1+\sin^2\alpha)\sin\left(\lambda + \frac{2\pi}{3}\right)\right] \tag{29}$$

$$y_3 - y_1 = \sqrt{3}ae\left[-\sin\alpha\cos\alpha\sin\left(\lambda + \frac{2\pi}{3}\right) - (1+\cos^2\alpha)\cos\left(\lambda + \frac{2\pi}{3}\right)\right] \tag{30}$$

$$z_3 - z_1 = 3ae\sin\left(\alpha - \lambda - \frac{\pi}{3}\right) \tag{31}$$

$$x_1 - x_2 = \sqrt{3}ae\left[-\sin\alpha\cos\alpha\cos\left(\lambda + \frac{\pi}{3}\right) + (1+\sin^2\alpha)\sin\left(\lambda + \frac{\pi}{3}\right)\right] \tag{32}$$

$$y_1 - y_2 = \sqrt{3}ae\left[-\sin\alpha\cos\alpha\sin\left(\lambda + \frac{\pi}{3}\right) - (1+\cos^2\alpha)\cos\left(\lambda + \frac{\pi}{3}\right)\right] \tag{33}$$

$$z_1 - z_2 = 3ae\sin\left(\alpha - \lambda + \frac{\pi}{3}\right) \tag{34}$$

Therefore:

$$\hat{n}_{23} = \frac{1}{L}\begin{bmatrix} -\sqrt{3}ae\left[\sin\alpha\cos\alpha\cos\lambda + (1+\sin^2\alpha)\sin\lambda\right] \\ \sqrt{3}ae\left[\sin\alpha\cos\alpha\sin\lambda + (1+\cos^2\alpha)\cos\lambda\right] \\ -3ae\sin(\alpha - \lambda) \end{bmatrix} \tag{35}$$

$$\hat{n}_{31} = \frac{1}{L}\begin{bmatrix} \sqrt{3}ae\left[-\sin\alpha\cos\alpha\cos\left(\lambda + \frac{2\pi}{3}\right) + (1+\sin^2\alpha)\sin\left(\lambda + \frac{2\pi}{3}\right)\right] \\ \sqrt{3}ae\left[-\sin\alpha\cos\alpha\sin\left(\lambda + \frac{2\pi}{3}\right) - (1+\cos^2\alpha)\cos\left(\lambda + \frac{2\pi}{3}\right)\right] \\ 3ae\sin\left(\alpha - \lambda - \frac{\pi}{3}\right) \end{bmatrix} \tag{36}$$

$$\hat{n}_{12} = \frac{1}{L}\begin{bmatrix} \sqrt{3}ae\left[-\sin\alpha\cos\alpha\cos\left(\lambda + \frac{\pi}{3}\right) + (1+\sin^2\alpha)\sin\left(\lambda + \frac{\pi}{3}\right)\right] \\ \sqrt{3}ae\left[-\sin\alpha\cos\alpha\sin\left(\lambda + \frac{\pi}{3}\right) - (1+\cos^2\alpha)\cos\left(\lambda + \frac{\pi}{3}\right)\right] \\ 3ae\sin\left(\alpha - \lambda + \frac{\pi}{3}\right) \end{bmatrix} \tag{37}$$

Then we can compute that:

$$\hat{n}^i_{rs}\hat{n}^j_{rs}\epsilon^+_{ij} = [\hat{n}^1_{rs}, \hat{n}^2_{rs}, \hat{n}^3_{rs}] \begin{bmatrix} u_1u_1 - v_1v_1 & u_1u_2 - v_1v_2 & u_1u_3 - v_1v_3 \\ u_2u_1 - v_2v_1 & u_2u_2 - v_2v_2 & u_2u_3 - v_2v_3 \\ u_3u_1 - v_3v_1 & u_3u_2 - v_3v_2 & u_3u_3 - v_3v_3 \end{bmatrix} \begin{bmatrix} \hat{n}^1_{rs} \\ \hat{n}^2_{rs} \\ \hat{n}^3_{rs} \end{bmatrix} \tag{38}$$

$$= [\hat{n}^1_{rs}, \hat{n}^2_{rs}, \hat{n}^3_{rs}] \left[\begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix} - \begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix} \right] \begin{bmatrix} \hat{n}^1_{rs} \\ \hat{n}^2_{rs} \\ \hat{n}^3_{rs} \end{bmatrix} \tag{39}$$

$$= [\hat{n}^1_{rs}, \hat{n}^2_{rs}, \hat{n}^3_{rs}] \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix} \begin{bmatrix} \hat{n}^1_{rs} \\ \hat{n}^2_{rs} \\ \hat{n}^3_{rs} \end{bmatrix} - [\hat{n}^1_{rs}, \hat{n}^2_{rs}, \hat{n}^3_{rs}] \begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix} \begin{bmatrix} \hat{n}^1_{rs} \\ \hat{n}^2_{rs} \\ \hat{n}^3_{rs} \end{bmatrix} \tag{40}$$

$$= [\hat{n}^1_{rs}, \hat{n}^2_{rs}, \hat{n}^3_{rs}] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [u_1, u_2, u_3] \begin{bmatrix} \hat{n}^1_{rs} \\ \hat{n}^2_{rs} \\ \hat{n}^3_{rs} \end{bmatrix} - [\hat{n}^1_{rs}, \hat{n}^2_{rs}, \hat{n}^3_{rs}] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} [v_1, v_2, v_3] \begin{bmatrix} \hat{n}^1_{rs} \\ \hat{n}^2_{rs} \\ \hat{n}^3_{rs} \end{bmatrix} \tag{41}$$

$$\hat{n}^i_{rs}\hat{n}^j_{rs}\epsilon^+_{ij} = (\hat{u} \cdot \hat{n}_{rs})^2 - (\hat{v} \cdot \hat{n}_{rs})^2 \tag{42}$$

Similarly, it can be obtained that:

$$\hat{n}^i_{rs}\hat{n}^j_{rs}\epsilon^\times_{ij} = 2(\hat{u} \cdot \hat{n}_{rs})(\hat{v} \cdot \hat{n}_{rs}) \tag{43}$$

We have already calculated the expression for $\hat{n}_{rs}$. Using equations 8 and 9 ,we can compute the specific form of equation 15 and 16.

Using a Python-based implementation, we modeled the LISA constellation's motion over a 1000-second period, with time steps of 50 seconds, and evaluated the response functions at frequencies $f = 10^{-4}$, 10^{-2}, and 1.0 Hz. The program computes F^+_X and $F^\times_X$ by the equations we just calculated.

The numerical results confirm that $\langle |F^+_X|^2 \rangle = \langle |F^\times_X|^2 \rangle$ across all tested frequencies. This equivalence is visualized in Figure 1, where the curves for $\langle |F^+_X|^2 \rangle$ (blue solid) and $\langle |F^\times_X|^2 \rangle$ (red dashed) overlap perfectly.

Table 1: Averaged Antenna Response of Different Frequency

Frequency (Hz)	$\langle \lvert F_X^+ \rvert^2 \rangle$	$\langle \lvert F_X^\times \rvert^2 \rangle$
10^{-4}	2.39997791	2.39997791
10^{-2}	2.18736945	2.18736945
1.0	0.00184269935	0.00184269935

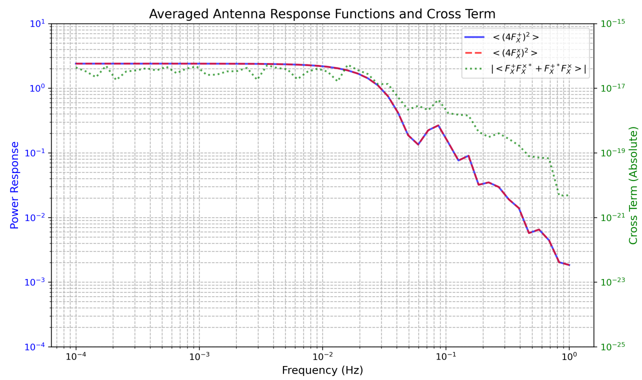

Figure 1:

Next, we use this program to calculate $< F_X^+(F_X^\times)^* + (F_X^+)^* F_X^\times >$. The results, listed in Table 2 showing near-zero values, indicate that $< F_X^+(F_X^\times)^* + (F_X^+)^* F_X^\times >$ is effectively 0 within the limits of numerical precision.

Table 2: The Cross-correlation Term

Frequency (Hz)	$\langle \left\lvert F_X^+(F_X^\times)^* + (F_X^+)^* F_X^\times \right\rvert \rangle$
10^{-4}	$2.0795233272504503 \times 10^{-15}$
10^{-2}	$2.1172286586545604 \times 10^{-15}$
1.0	$1.9465964365455394 \times 10^{-18}$

Substituting the above results into the equation 20, we get the average response TDI X2.0 to GW:

$$< \left| X_{2.0}^{GW} \right|^2 > = 64(\omega L)^2 \sin^2 \omega L \sin^2 2\omega L < (F_X^+)^2 > [A_+^2 + A_\times^2] \tag{44}$$

$$< R_{L,X_{2.0}}(f) > = 64(\omega L)^2 \sin^2 \omega L \sin^2 2\omega L < (F_X^+)^2 > \tag{45}$$

2 Approximation of $<(F_X^+)^2>$

Now we need to compute the approximation of $<(F_X^+)^2>$ at low frequencies ($\omega L << 1$).

For $X_{1.5}$TDI in the long wavelength limit(LISA-based frame) we obtain

$$\tilde{X}_{1.5} \approx (4\omega L)\sin\omega L\frac{\sqrt{3}}{2}(F_+\tilde{h}_+ + F_\times\tilde{h}_\times) \tag{46}$$

where

$$F_+ = -\frac{1}{2}(1+\cos^2\theta)\cos 2\phi\ cos2\psi - \cos\theta\ \sin 2\phi\ \sin 2\psi \tag{47}$$

$$F_\times = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi\ cos2\psi - \cos\theta\ \sin 2\phi\ \cos 2\psi \tag{48}$$

The value of $<(F_+)^2>$ and $<(F_\times)^2>$ is calculated to be 0.2. Using the definition:

$$SNR^2 = 4\mathrm{Re}\left(\int_0^{f_{\max}} df\frac{\tilde{X}\tilde{X}^*}{S_n(f)}\right) \tag{49}$$

then:

$$<SNR^2> = <(F_X^+)^2> \frac{3}{4}\ 4\mathrm{Re}\left(\int_0^{f_{\max}} df\ (4\omega L)^2\sin^2(\omega L)\frac{\tilde{h}_+^2+\tilde{h}_\times^2}{S_{n,X_{1.5}}(f)}\right) \tag{50}$$

$$= 4\mathrm{Re}\left(\int_0^{f_{\max}} df(4\omega L)^2\sin^2(\omega L)\frac{3}{20}\frac{\tilde{h}_+^2+\tilde{h}_\times^2}{S_{n,X_{1.5}}(f)}\right) \tag{51}$$

Using the definition:

$$<SNR^2> = 4\mathrm{Re}\left(\int_0^{f_{\max}} df\frac{\tilde{h}_+^2+\tilde{h}_\times^2}{S_n(f)/<|R_L|^2>}\right) \tag{52}$$

then:

$$<\left|R_{L,X_{1.5}}^{LW}\right|^2> = \frac{3}{20}(4\omega L)^2\sin^2(\omega L) \tag{53}$$

adding additional factor:

$$<|R_{L,X_{1.5}}|^2> = \frac{3}{20}(4\omega L)^2\sin^2(\omega L)\frac{1}{1+0.6(\omega L)^2} \tag{54}$$

That is to say:

$$<(4F_X^+)^2> \approx 16\ \frac{3}{20}\frac{1}{1+0.6(\omega L)^2} \tag{55}$$

Plotting to verify the accuracy of the approximation:

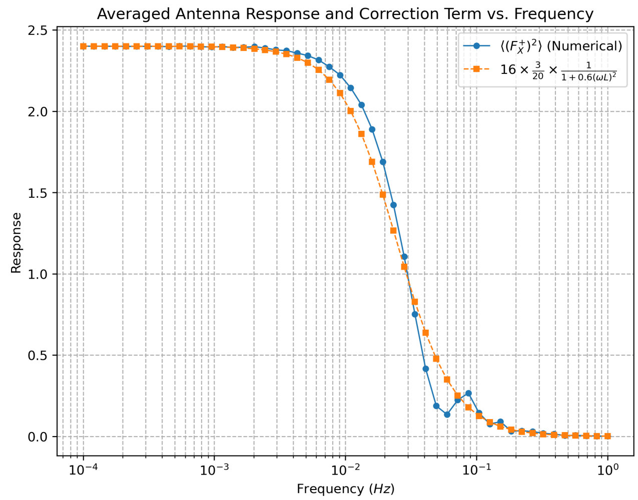

Figure 2:

3 Finally compute the average response

Using equation 11:

$$< |R_{L,X_{2.0}}|^2 >= 64(\omega L)^2 \sin^2 \omega L \sin^2 2\omega L \, \frac{3}{20} \frac{1}{1+0.6(\omega L)^2} \tag{56}$$

The Average GW Response for TDI-X2.0 can be plotted:

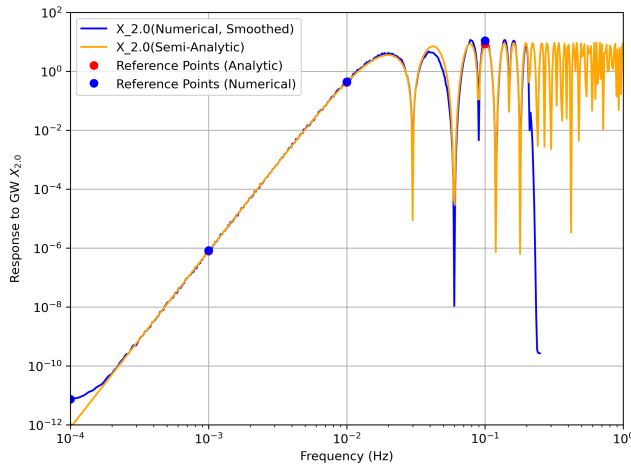

Figure 3: The Average GW Response for TDI-X2.0

Table 3: The numerical values of the response of TDI $X_{2.0}$ to GW for some fixed frequencies

Frequency (Hz)	Semi-Analytic	$R_{\text{LISACode},X_{2.0}}$
0.00010	7.945098e-13	7.372150e-12
0.00100	7.896301e-07	8.349357e-07
0.01000	4.251647e-01	4.427565e-01
0.10000	8.519737e+00	1.104438e+01

4 Sensitivity

Using the definition:

$$S_{h,X} = \frac{S_{OMS} + (3 + \cos(2\omega L))S_{acc}}{(\omega L)^2 < (F_x^+)^2 >} \tag{57}$$

$$\sqrt{S_{OMS}}(f) = 15 \left[\frac{\text{pm}}{\sqrt{\text{Hz}}}\right] \sqrt{1 + \left(\frac{2 \times 10^{-3}}{f}\right)^4} \tag{58}$$

$$\sqrt{S_{acc}}(f) = 3 \left[\frac{\text{fm}.s^{-2}}{\sqrt{\text{Hz}}}\right] \sqrt{1 + \left(\frac{0.4 \times 10^{-3}}{f}\right)^2} \sqrt{1 + \left(\frac{f}{8 \times 10^{-3}}\right)^4} \tag{59}$$

The strain sensitivity to only X, $\sqrt{S_{h,X}}$ can be plotted:

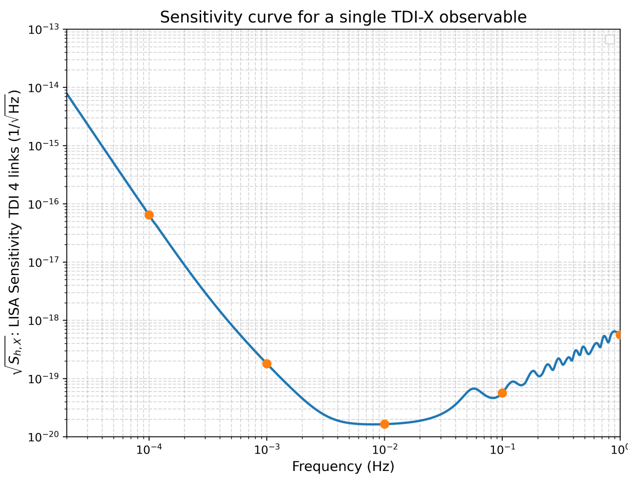

Figure 4:

Table 4: The numerical values of the strain sensitivity curve for some fixed frequencies

f (Hz)	$S_{h,X}$	S_h
0.00010	4.226903e-33	2.113451e-33
0.00100	3.265832e-38	1.632916e-38
0.01000	2.719033e-40	1.359516e-40
0.10000	3.218440e-39	1.609220e-39
1.00000	3.176069e-37	1.588034e-37