Sample Regression Function: $\hat{Y}_i = \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i}$

1、

$$\hat{u}_i = Y_i - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}$$

$$SSR = \sum_{i=1}^{n} \hat{u}_i^2 = \sum_{i=1}^{n} (Y_i - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2$$

2、

$$\frac{\partial SSR}{\partial \hat{\beta}_1} = \sum_{i=1}^{n} 2(Y_i - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})(-X_{1i})$$

$$\frac{\partial SSR}{\partial \hat{\beta}_2} = \sum_{i=1}^{n} 2(Y_i - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})(-X_{2i})$$

3、

$$\sum_{i=1}^{n} X_{1i} Y_i - \sum_{i=1}^{n} \hat{\beta}_1 X_{1i}^2 - \sum_{i=1}^{n} \hat{\beta}_2 X_{2i} X_{1i} = 0$$

$$\because \sum_{i=1}^{n} X_{1i} X_{2i} = 0$$

$$\therefore \hat{\beta}_1 = \frac{\sum_{i=1}^{n} X_{1i} Y_i}{\sum_{i=1}^{n} X_{1i}^2}$$

4、

$$\sum_{i=1}^{n} X_{2i} Y_i - \sum_{i=1}^{n} \hat{\beta}_1 X_{1i} X_{2i} - \sum_{i=1}^{n} \hat{\beta}_2 X_{2i}^2 = 0$$

$$\Rightarrow \hat{\beta}_2 = \frac{\sum_{i=1}^{n} X_{2i} Y_i}{\sum_{i=1}^{n} X_{2i}^2} - \hat{\beta}_1 \frac{\sum_{i=1}^{n} X_{1i} X_{2i}}{\sum_{i=1}^{n} X_{2i}^2}$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} X_{1i} Y_i - \hat{\beta}_2 \sum_{i=1}^{n} X_{2i} X_{1i}}{\sum_{i=1}^{n} X_{1i}^2} = \frac{\sum_{i=1}^{n} X_{1i} Y_i - \sum_{i=1}^{n} X_{2i} X_{1i} \left(\frac{\sum_{i=1}^{n} X_{2i} Y_i}{\sum_{i=1}^{n} X_{2i}^2} - \hat{\beta}_1 \frac{\sum_{i=1}^{n} X_{1i} X_{2i}}{\sum_{i=1}^{n} X_{2i}^2} \right)}{\sum_{i=1}^{n} X_{1i}^2}$$

$$\Rightarrow \hat{\beta}_1 \left[1 - \frac{\left(\sum_{i=1}^{n} X_{2i} X_{1i} \right)^2}{\sum_{i=1}^{n} X_{1i}^2 \cdot \sum_{i=1}^{n} X_{2i}^2} \right] = \frac{\sum_{i=1}^{n} X_{1i} Y_i \cdot \sum_{i=1}^{n} X_{2i}^2 - \sum_{i=1}^{n} X_{2i} X_{1i} \cdot \sum_{i=1}^{n} X_{2i} Y_i}{\sum_{i=1}^{n} X_{1i}^2 \cdot \sum_{i=1}^{n} X_{2i}^2}$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} X_{1i} Y_i \cdot \sum_{i=1}^{n} X_{2i}^2 - \sum_{i=1}^{n} X_{2i} X_{1i} \cdot \sum_{i=1}^{n} X_{2i} Y_i}{\sum_{i=1}^{n} X_{1i}^2 \cdot \sum_{i=1}^{n} X_{2i}^2 - \left(\sum_{i=1}^{n} X_{2i} X_{1i} \right)^2}$$

5.

Sample Regression Function:

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i}$$

For $\hat{\beta}_0$:

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) = 0$$

$$\frac{1}{n}\sum_{i=1}^{n} Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot \frac{1}{n}\sum_{i=1}^{n} X_{1i} - \hat{\beta}_2 \frac{1}{n}\sum_{i=1}^{n} X_{2i} = 0$$

$$\bar{Y} - \hat{\beta}_0 - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2 = 0$$

$$\Rightarrow \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2$$

6.

① For $\hat{\beta}_1$:

$$\sum_{i=1}^{n} X_{1i}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) = 0$$

$$\sum_{i=1}^{n} X_{1i}(Y_i - \bar{Y} + \hat{\beta}_1 \bar{X}_1 + \beta_2 \bar{X}_2 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) = 0$$

$$\sum_{i=1}^{n} X_{1i}(Y_i - \bar{Y}) + \hat{\beta}_1 \sum_{i=1}^{n} X_{1i}(\bar{X}_1 - X_{1i}) + \hat{\beta}_2 \sum_{i=1}^{n} X_{1i}(\bar{X}_2 - X_{2i}) = 0$$

$$\because \sum_{i=1}^{n}(\bar{X}_2 - X_{2i}) = 0 \Rightarrow \bar{X}_1 \sum_{i=1}^{n}(\bar{X}_2 - X_{2i}) = 0 \Rightarrow \sum_{i=1}^{n} X_{1i}(\bar{X}_2 - X_{2i}) = \sum_{i=1}^{n}(X_{1i} - \bar{X}_1)(\bar{X}_2 - X_{2i}) = 0$$

$$\therefore \sum_{i=1}^{n} X_{1i}(Y_i - \bar{Y}) + \hat{\beta}_1 \sum_{i=1}^{n} X_{1i}(\bar{X}_1 - X_{1i}) = 0$$

$$\sum_{i=1}^{n} X_{1i}(Y_i - \bar{Y}) - \hat{\beta}_1 \sum_{i=1}^{n} X_{1i}(X_{1i} - \bar{X}_1) = 0$$

Similarly, $\sum_{i=1}^{n} X_{1i}(X_{1i} - \bar{X}_1) = \sum_{i=1}^{n}(X_{1i} - \bar{X}_1)^2$, $\sum_{i=1}^{n} X_{1i}(Y_i - \bar{Y}) = \sum_{i=1}^{n}(X_{1i} - \bar{X}_1)(Y_i - \bar{Y})$

$$\therefore \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_{1i} - \bar{X}_1)(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_{1i} - \bar{X}_1)^2}$$

② They are the same under the condition that $\sum_{i=1}^{n}(X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2) = 0$