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Tempered fractionally integrated process with stable noise as a transient anomalous diffusion model

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Abstract

We present here the autoregressive tempered fractionally integrated moving average (ARTFIMA) process obtained by taking the tempered fractional difference operator of the non-Gaussian stable noise. The tempering parameter makes the ARTFIMA process stationary for a wider range of the memory parameter values than for the classical autoregressive fractionally integrated moving average, and leads to semi-long range dependence and transient anomalous behavior. We investigate ARTFIMA dependence structure with stable noise and construct Whittle estimators. We also introduce the stable Yaglom noise as a continuous version of the ARTFIMA model with stable noise. Finally, we illustrate the usefulness of the ARTFIMA process on a trajectory from the Golding and Cox experiment.

Keywords: stable distribution, tempered fractional calculus, codifference, Whittle method, Yaglom noise

(Some figures may appear in colour only in the online journal)

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1. Introduction

Revolutionary insights provided by novel experimental techniques such as single-particle tracking (SPT) using superresolution microscopy and supercomputer simulations have prompted a flurry of new developments. The classical diffusion is described by the Brownian motion (BM) which mean-squared displacement (MSD) is linear. However, in many experiments, the linearity of MSD is violated and non-Gaussian distributions are observed. In particular, the presence of memory in the system, induced, e.g., by traps or long-ranged correlations, is one of the possible origins leading to the emergence of anomalous diffusion that is typically characterised by the MSD of the power-law form $E[X^2(t)] \sim t^a$, with the anomalous diffusion exponent $0 < a \neq 1$ ($a < 1$ corresponds to subdiffusion, whereas $a > 1$ leads to superdiffusion) [1, 2]. A second modern class of diffusive dynamics violating Brownian assumptions is the one in which the Gaussian property is not present, the non-Gaussian diffusion. In many of these observations, the MSD is still linear, in others, the diffusion is both anomalous and non-Gaussian [3–7].

Classically, there are three anomalous diffusion models. One is the continuous-time random walk (CTRW) that combines variable lengths and waiting times for each jump. The second is the fractional Brownian motion (FBM) which is the unique Gaussian self-similar process with stationary increments. Its increments are called fractional Gaussian noise and they are zero-mean power-law correlated [8]. For subdiffusion, which corresponds to the Hurst exponent $H < 1/2$, the noise is anticorrelated with a negative sign of the displacement correlations. This process is often considered an approximation to the more physically viable generalised Langevin equation (GLE) [9, 10]. The third model is a diffusion of a particle in a (statistically) fractal environment such as a percolation cluster close to criticality [11]. This process has similar features as subdiffusive FBM and GLE but only has approximate analytical solutions [11].

However, they do not exhaust all possible sources of anomalous diffusion. Let us mention here a generalization of FBM that was proved to be useful in modeling of anomalous diffusion data, namely fractional Lévy stable motion (FLSM), see [3, 12]. Although that process for the stability index less than 2 (non-Gaussian stable case) does not have finite variance so the ensemble average MSD diverges, the time average MSD may exhibit either subdiffusion, normal diffusion, or superdiffusion [3].

While typically anomalous diffusion refers to the power-law behaviour with a fixed α, an increasing number of systems are reported in which the local scaling exponent of the MSD is an explicit function of time, $a(t)$. Such transient behaviour has, for instance, been observed for green fluorescent proteins in cells or for the motion of lipid molecules in protein-crowded bilayer membranes [13–15].

All those models are continuous-time processes. However, the experimental data are always discrete. This, and the need for a wider class of possible distributions underlying the data has led to the idea of applying classical time series models [16], which were introduced in the econometric context, to modeling of SPT data [17–22]. Time series models are stationary for an appropriate choice of their parameters, so they are mainly fitted to the increments of the observed SPT trajectories. The main model in this context is the autoregressive fractionally integrated moving average (ARFIMA). This process for the noise/innovations belonging to the domain of attraction of Lévy stable law, when aggregated and normalized, converges to FLSM (as a special case, to FBM when the noise belongs to the domain of attraction of Gaussian law) [23]. As a consequence, there is a strong and natural relation between discrete-time stationary time series models and their continuous-time self-similar counterparts, e.g. the self-similarity parameter H of FLSM can be written as $d + 1/\alpha$, where d is the memory parameter of the time series model [3].

Recently, tempered anomalous diffusion processes have been introduced. In the context of continuous-time processes, different tempered fractional Brownian motions (TFBM's) and fractional Lévy stable motion (TFLSM) were first proposed and studied in [24–27]. Such processes account for transient dynamics, for example, TFBM introduced in [27] can describe the subdiffusion–diffusion and subdiffusion–subdiffusion crossovers observed in lipid bilayer systems [28]. Transience is closely related to the property of semi-long range dependence (semiLRD). The increments of a stochastic process are said to display semiLRD when their autocovariance function decays hyperbolically over small lags and exponentially fast over large lags [29, 30].

The discrete-time process corresponding in the limit sense to continuous-time tempered fractional processes is the autoregressive tempered fractionally integrated moving average (ARTFIMA) [31–33]. It is an important model to consider when describing data that were traditionally modeled by the ARFIMA. In the finite variance case, it has a spectral density converging to a constant at zero. Moreover, its covariance function is absolutely summable. The ARTFIMA model has been already applied to various datasets to model climate, stock returns, water turbulence and hydrology [33, 34]. In [35] two tempered linear and non-linear time series models were introduced, namely ARTFIMA with α-stable noise and ARTFIMA with generalized autoregressive conditional heteroskedasticity (GARCH) noise (ARTFIMA-GARCH). Those processes were found to be very successful in modeling of solar flares.

The α-stable ARTFIMA process is obtained by taking tempered fractional integral (or derivative) of an ARMA model with α-stable noise, see [35]. The tempering parameter makes the α-stable ARTFIMA process stationary and for any $d \in \mathcal{R}\backslash\mathcal{Z}^-$. The tempering parameter also manifests its role in the codifference: for small values of the tempering parameter, the codifference of the α-stable ARTFIMA model follows the power law, but eventually decays exponentially fast. We call this property of the ARTFIMA model semiLRD.

The remainder of this paper is organized as follows. In Section 2, we recall the α-stable ARTFIMA model and prove its causality and invertibility. Moreover, we compute the dependence structure of the α-stable ARTFIMA model. In section 3, we employ the Whittle method to study parameter estimation of the model, including the estimators' consistency and asymptotic distributions. In section 4, we introduce tempered fractional Langevin equation based on tempered fractional calculus. We call the stochastic process driven based on this equation as α-stable Yaglom noise. We also show that the connection between an α-stale ARTFIMA model and α-stable Yaglom noise. Sections 5 and 6 contain the simulation results of the α-stable ARTFIMA model and an application of this model to the Golding and Cox dataset [36], respectively. All proofs can be found in the appendix A.

2. ARTFIMA model with non-Gaussian stable noise

This section defines the ARTFIMA model with non-Gaussian stable noise/innovations and studies its essential properties, including causality, invertibility, and dependence structure. First, we provide some basic definitions and assumptions.

Tempered fractional integrated operator is defined by

$$\Delta^{-d,\lambda} f(x) = (I - \mathrm{e}^{-\lambda} B)^{-d} f(x) = \sum_{j=0}^{\infty} \omega_{-d,\lambda}(j) f(x - j), \tag{1}$$

where $d \in \mathbb{R} - \mathbb{Z}_{-}$, the tempering parameter $\lambda > 0$, $BX(t) = X(t-1)$ is the backward shift operator, and

$$\omega_{-d,\lambda}(j) := \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \mathrm{e}^{-\lambda j} \tag{2}$$

with the gamma function $\Gamma(d) = \int_0^\infty \mathrm{e}^{-x} x^{d-1} \, \mathrm{d}x$. If $d < 0$, we view $\Delta^{-d,\lambda}$ as a tempered fractional difference operator.

A non-Gaussian symmetric α-stable ($S\alpha S$) random variable $Z_\alpha = \{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ has the characteristic function

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta Z_\alpha(t)}] = \mathrm{e}^{-\sigma^\alpha |\theta|^\alpha |t|}, \quad \theta \in \mathbb{R},\ 0 < \alpha < 2, \tag{3}$$

where the parameter σ is called the scale parameter of Z_α. It can be shown that

$$\mathbb{E}[|Z|^p] = \begin{cases} \mathbb{E}[|\sigma^{-1} Z|^p]\sigma^p = c(p,a)\sigma^p & \text{if } 0 < p < \alpha \\ \infty & \text{if } p \geqslant \alpha \end{cases}, \tag{4}$$

where the constant $c(p,\alpha)$ does not depend on the scale parameter σ, see [37, 38]. We will use the following two assumptions for the rest of the paper.

Assumption 1. We assume $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ is a sequence of i.i.d. $S\alpha S$ random variables with the index stability $1 < \alpha < 2$ and scale parameter $\sigma = 1$.

Assumption 2. Let $\Phi_p(z)$ and $\Theta_q(z)$ be polynomials with real coefficients defined by $\Phi_p(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$, and $\Theta_q(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ respectively. We assume that Φ_p and Θ_q have no common roots and that Φ_p has no roots in the closed unit disk $\{z : |z| \leqslant 1\}$.

Definition 2.1. The stochastic process $\{X(t)\}_{t\in\mathbb{Z}}$ is said to be a non-Gaussian $S\alpha S$ autoregressive tempered fractional integrated moving average, denoted by ARTFIMA(p,d,λ,q), model if $\{X(t)\}_{t\in Z}$ satisfies the tempered fractional difference equations

$$\Phi_p(B)X(t) = \Theta(B)\Delta^{-d,\lambda} Z_\alpha(t), \tag{5}$$

where $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$, Φ_p, and Θ_q satisfy under assumptions 1 and 2, $d \in \mathbb{R} - \mathbb{Z}_{-}$, and $\lambda > 0$.

The next proposition shows the ARTFIMA(p,d,λ,q) with non-Gaussian stable innovations is causal and invertible.

Proposition 2.1. *Let $\{X(t)\}_{t\in\mathbb{Z}}$ be the non-Gaussian $S\alpha S$ ARTFIMA(p,d,λ,q) given by equation (5). Then,*

(a) $X(t)$ has the moving average representation

$$X(t) = X_{p,d,\lambda,q}(t) = \sum_{j=0}^{\infty} a_{-d,\lambda}(j) Z_\alpha(t-j), \tag{6}$$

where

$$a_{-d,\lambda}(j) = \sum_{s=0}^{j} \omega_{-d,\lambda}(s) b(j-s) \tag{7}$$

with $\Theta_q(z)\Phi_p(z)^{-1} = \sum_{j=0}^{\infty} b(j) z^j$ for $|z| \leqslant 1$, and $\omega_{-d,\lambda}(j)$ is given by equation (2). Moreover, the series in equation (6) converges a.s. and in L^ν for any $\nu < \alpha$.

(*b*) *X*(*t*) *is invertible. That is*

$$Z_\alpha(t) = \sum_{j=0}^{\infty} c_{d,\lambda}(j)X(t-j), \tag{8}$$

where

$$c_{d,\lambda}(j) = \sum_{s=0}^{j} \omega_{d,\lambda}(j)c(j-s) \tag{9}$$

with $\frac{\Phi_p(z)}{\Theta_q(z)} = \sum_{j=0}^{\infty} c(j)z^j$ *for* $|z| \leqslant 1$ *with* $\omega_{d,\lambda}(j) = \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)}\mathrm{e}^{-\lambda j}$. *Moreover, the series in equation* (8) *converges a.s. and in* L^ν *for any* $\nu < \alpha$.

Remark 2.2. Sabzikar and Surgailis [32] introduced another tempered fractionally integrated time series in the following sense: a stochastic process $X^{\mathit{II}}_{p,d,\lambda,q}$ is called non-Gaussian SαS ARTFIMA(p, d, λ, q) if

$$X^{\mathit{II}}_{p,d,\lambda,q}(t) = \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda j} a_{-d}(j)Z_\alpha(t-k), \quad t \in \mathbb{Z}, \tag{10}$$

where

$$a_{-d}(j) = \sum_{s=0}^{j} \omega_{-d}\,(s)b(j-s), \quad k \geqslant 0,$$

and $\omega_{-d}(j) = \omega_{-d,0}(j) = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}$. We note that $\{X_{p,d,\lambda,q}\}$ in equation (6) and $\{X^{\mathit{II}}_{p,d,\lambda,q}\}$ in equation (10) are two different stochastic processes since $a_{-d,\lambda}(j) \neq \mathrm{e}^{-\lambda j} a_{-d}(j)$ in general. However, these two processes are the same if $\Phi_p(z) = \Theta_q(z) = 1$ which means $X_{0,d,\lambda,0} = X^{\mathit{II}}_{0,d,\lambda,0}$. In this paper, we focus on the $X_{p,d,\lambda,q}$.

We close this section by investigating the asymptotic dependence structure of the non-Gaussian $S\alpha S$ ARTFIMA sequences. Since the covariance function stops working in the presence of non-Gaussian stable distributions, we use an alternative distance measure called codifference. Let $Y(n) = \sum_{j=0}^{\infty} c(j)Z_\alpha(n-j)$ be a casual moving average where $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ satisfies under assumption 1. A codifference $\tau(n)$ of $X(0)$ and $X(n)$ is defined by

$$\tau(n) := \tau(X(0), X(n)) = \sum_{j=0}^{\infty} \left[|c(j)|^\alpha + |c(j+n)|^\alpha - |c(j) - c(j+n)|^\alpha\right]. \tag{11}$$

We refer the reader to [37] for more properties of the codifference.

The next theorem provides the asymptotic behavior of the codifference of the non-Gaussian $S\alpha S$ ARTFIMA process given by equation (6). Without loss of generality, we may consider $p = q = 0$.

Theorem 2.3. *Let* $X_{d,\lambda}(t) = \sum_{j=0}^{\infty} \omega_{-d,\lambda}(j)Z_{t-j}$ *be a non-Gaussian* $S\alpha S$ *ARTFIMA*$(0, d, \lambda, 0)$ *with the codifference* $\tau_{d,\lambda}(n) = \tau_{d,\lambda}(X(0), X(n))$. *Then*

$$\lim_{n\to\infty} \frac{\tau_{d,\lambda}(n)}{\mathrm{e}^{-\lambda n} n^{d-1}} = \frac{\alpha}{\Gamma(d)} \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda j}(\omega_{-d,\lambda}(j))^{\alpha-1}$$

for $\alpha \in (1, 2)$, $d \in \mathbb{R} - \mathbb{Z}_-$,*and* $\lambda > 0$.

A stationary casual moving average representation $Y(n)$ with finite second moments innovations, i.e. $\mathbb{E}(Y^2) < \infty$ is called to have long memory if $\sum_{n=0}^{\infty} \gamma(n) = \infty$ where $\gamma(n) = \mathrm{Cov}(Y(0), Y(n))$. For a stationary non-Gaussian $S\alpha S$ $Y(n)$, i.e., $\mathbb{E}(Y^2(n)) = \infty$, $Y(n)$ is called to have long memory if

$$\sum_{n=0}^{\infty} |\tau(n)| = \infty, \tag{12}$$

where $\tau(n)$ is the codifference defined in equation (11).

Corollary 2.4. *The non-Gaussian $S\alpha S$ ARTFIMA$(0, d, \lambda, 0)$ does not have long memory in the sense of equation* (12).

Remark 2.5. According to corollary 2.4, the non-Gaussian $S\alpha S$ ARTFIMA$(0, d, \lambda, 0)$ is not long-range dependent. But, it does exhibit semiLRD under theorem 2.3. That is, the sum $\sum_{n=0}^{\infty} \tau_{d,\lambda}(n)$ tends to infinity as λ approaches to zero.

Define the periodogram for $X(t)$ as

$$I_X(\omega) = n^{-2/\alpha} \left| \sum_{t=1}^{n} X(t) \mathrm{e}^{-\mathrm{i}\omega t} \right|, -\pi < \omega \leqslant \pi$$

and let $I_{Z_\alpha}(\omega)$ be the periodogram of the innovations $Z_\alpha(t)$. For any $h \geqslant 0$ define

$$\rho(h) = \frac{\sum_{j=0}^{\infty} a_{-d,\lambda}(j) a_{-d,\lambda}(j+h)}{\sum_{j=0}^{\infty} a_{-d,\lambda}^2(j)}$$

and the sample correlation function

$$\hat{\rho}_n(h) = \frac{C(h)}{C(0)}, \qquad C(h) = \sum_{t=1}^{n} X(t) X(t+h).$$

Theorem 2.6. *Let $\{X(t)\}_{t \in \mathbb{Z}}$ be a non-Gaussian stable ARTFIMA(p, d, λ, q) given by equation* (6) *with innovations $\{Z_\alpha(t)\}_{t \in \mathbb{Z}}$ satisfying under assumptions* 3 *and* 4*. Then for any positive integer l,*

$$\left(\frac{n}{\ln n} \right)^{1/\alpha} (\hat{\rho}_n(h) - \rho(h), 1 \leqslant h \leqslant l) \xrightarrow{d} (S(1), \ldots, S(l))',$$

where

$$S(h) = \sum_{j=1}^{\infty} (\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h)) \frac{Y(j)}{Y(0)},$$

and $Y(0), Y(1), \ldots, Y(k)$ are independent $S\alpha S$ random variables. Moreover, $Y(0) \stackrel{d}{=} S_{\alpha/2}(C_{\alpha/2}^{-2}, 1, 0)$ is a positive $\alpha/2$-stable, $\{Y(k)\}_{k \geqslant 1}$ are i.i.d. $S\alpha S$ with scale parameter $\sigma = C_\alpha^{-1/\alpha}$, where $C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}$ for $\alpha \in (1, 2)$.

3. Parameter estimation

In this section, we prove the consistency and asymptotic distribution of a variant Whittle estimator for ARTFIMA(p, d, λ, q). The Whittle's method for non-Gaussian stable ARMA(p, q) model was studied in [39]. In [40] the authors studied statistical inference for the non-Gaussian stable ARFIMA(p, d, q) when $d \in (0, 1 - \frac{1}{\alpha})$ and $\alpha \in (1, 2)$. In [41] the authors investigated the parameter estimation for stable ARFIMA(p, d, q) when $d \in (-1/2, 0)$ and $\alpha \in (2/3, 2]$. The authors in [33] considered statistical inference, including the parameter estimation and asymptotic normality for ARTFIMA time series but with finite second moments innovations. Here, we deal with the ARTFIMA model with non-Gaussian α-stable innovations, which means the ARTFIMA model does not have a finite second moment. Although the Whittle method was also applied in [33], we will show that the limit distributions of the estimators in the non-Gaussian α-stable case are non-Gaussian, which contrasts with the asymptotic normality in [33]. In all these references, there is a function which is called *power transfer function* and it plays an important role to show the consistency of the Whittle estimator. We define the *tempered power transform function*

$$g(\omega, \beta) := \left| \frac{\Theta_q(\mathrm{e}^{-\mathrm{i}\omega}, \beta)}{\Phi_p(\mathrm{e}^{-\mathrm{i}\omega}, \beta)(1 - \mathrm{e}^{-(\lambda+\mathrm{i}\omega)})^d} \right|^2, \tag{13}$$

where $\boldsymbol{\beta} = (\phi_1, \ldots, \phi_p, d, \lambda, \theta_1, \ldots, \theta_q)$ is the $(p + q + 2)$ dimensional vector belongs to the parameter space

$$E := \{\beta : \phi_p, \theta_q \neq 0, \Phi_p(z), \Theta_q(z) \neq 0 \quad \text{for } |z| \leqslant 1,\ d \in \mathbb{R} - \mathbb{Z}_-,\ \lambda > 0\}. \tag{14}$$

To establish consistency and asymptotic distribution of the Whittle estimators, see equation (17), we set up some assumptions on the innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ of the non-Gaussian $S\alpha S$ ARTFIMA(p, d, λ, q) given by equation (6).

Assumption 3. The innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ satisfy under the following three assumptions:

(a) $E|Z_\alpha(1)|^\nu < \infty$, for some $\nu > 0$,
(b) $n^{1-2\delta/\alpha} \to 0$, as $n \to \infty$, for $\delta = \min(1, \nu) > \alpha/2$,
(c) $\lim_{x\to 0} \limsup_{n\to\infty} \mathbb{P}\left(n^{-2/\alpha} \sum_{t=1}^n Z_\alpha^2(t) \leqslant x\right) = 0$.

Assumption 4. The innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ belong to the domain of normal attraction (DNA) of a symmetric α-stable, random variable $Z(t) \in \mathrm{DNA}(\alpha)$ for some $0 < \alpha < 2$. That is $\frac{1}{n^{1/\alpha}} \sum_{t=1}^n Z(t) \Rightarrow^d Y$, where Y is $S\alpha S$ random variable.

Let $\mathbf{X} = (X(1), \ldots, X(n))$ be a realization of the non-Gaussian stable ARTFIMA(p, d, λ, q) with the sample size n. Define the self-normalised periodogram

$$\tilde{I}_n(\omega) := \frac{\left|\sum_{t=1}^n X(t)\mathrm{e}^{-\mathrm{i}t\omega}\right|^2}{\sum_{t=1}^n X^2(t)}, \quad -\pi \leqslant \omega \leqslant \pi. \tag{15}$$

Let β a vector in the parameter space E in equation (14). Define

$$\sigma_n^2(\beta) = \int_{-\pi}^{\pi} \frac{\tilde{I}_n(\omega)}{g(\omega, \beta)}\, \mathrm{d}\omega, \tag{16}$$

where $g(\omega, \beta)$ is given by equation (15).

Definition 3.1. Let β_0 denote the true parameter values of β. The estimators of β_0 based on $\mathbf{X} = (X(1), \ldots, X(n))$ are defined by

$$\beta_n := \arg\min\{\sigma_n^2(\beta) : \beta \in E\}. \tag{17}$$

Theorem 3.2. *Let β_0 be the true parameter, β_n be the value of β minimizing $\sigma_n^2(\beta)$, and the innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ satisfying assumption 3. Then*

$$\beta_n \xrightarrow{p} \beta_0 \quad \text{and} \quad \sigma_n^2(\beta_n) \xrightarrow{p} \frac{2\pi}{\gamma(0)},$$

as $n \to \infty$, where $\gamma(0) ={}_2F_1(d;\ d;\ 1;\ e^{-2\lambda})$.

Theorem 3.3. *Let β_0 and β_n be the same as in theorem 3.2, and the innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ satisfying assumption 4. Then*

$$\left(\frac{n}{\ln n}\right)^{1/\alpha} (\beta_n - \beta_0) \xrightarrow{p} 4\pi \mathbf{W}^{-1}(\beta_0) \frac{1}{Y(0)} \sum_{k=1}^{\infty} Y(k) b(k), \tag{18}$$

where $Y(0) \stackrel{d}{=} S_{\alpha/2}(C_{\alpha/2}^{-2}, 1, 0)$ is a positive $\alpha/2$-stable, $\{Y(t)\}_{t\in\mathbb{N}}$ are i.i.d. SαS with scale parameter $\sigma = C_\alpha^{-1/\alpha}$, $\mathbf{W}^{-1}(\beta_0)$ is the inverse of the matrix

$$\mathbf{W}(\beta_0) = \int_{-\pi}^{\pi} \left\{\frac{\partial \log g(\omega, \beta_0)}{\partial \beta}\right\} \left\{\frac{\partial \log g(\omega, \beta_0)}{\partial \beta}\right\}' \mathrm{d}\omega$$

and

$$b(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\omega} g_{p,d,\lambda,q}(\omega, \beta_0) \left(\frac{\partial g^{-1}(\omega, \beta_0)}{\partial \beta}\right) \mathrm{d}\omega,$$

where $g^{-1}(\omega, \beta) = \frac{1}{g(\omega,\beta)}$.

Corollary 3.4. *Let $\{X(t)\}_{t\in\mathbb{Z}}$ is a stable ARTFIMA$(0, d, \lambda, 0)$ with the innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ satisfying assumption 4. Let $\widehat{d}$ and $\widehat{\lambda}$ are the Whittle estimators of the parameters d and λ respectively. Then equation (18) holds with*

$$\mathbf{W} = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix},$$

where

$$w_{1,1} = \int_{-\pi}^{\pi} \left(\log(1 - 2\,\mathrm{e}^{-\lambda} \cos \nu + \mathrm{e}^{-2\lambda})\right)^2 \mathrm{d}\nu, \tag{19}$$

$$w_{2,2} = \frac{d^2\,\mathrm{e}^{-2\lambda}}{1 - \mathrm{e}^{-2\lambda}}, \tag{20}$$

$$w_{1,2} = v_{2,1} = d \ln(1 - \mathrm{e}^{-2\lambda}) \tag{21}$$

and

$$b_k = \begin{pmatrix} -\dfrac{1}{k}\,\mathrm{e}^{-\lambda k} \\ d\,\mathrm{e}^{-\lambda k} \end{pmatrix}.$$

4. Tempered fractional Langevin equation of the second kind

In this section, we apply tempered fractional calculus to introduce tempered fractional Langevin equation. First, let us recall the definitions of tempered fractional integrals and derivatives as some of their essential properties. We begin with the definition of a tempered fractional integral.

Definition 4.1. For any $f \in L^p(\mathbb{R})$ (where $1 \leqslant p < \infty$), the positive and negative tempered fractional integrals are defined by

$$\mathbb{I}_{+}^{\eta,\lambda} f(t) = \frac{1}{\Gamma(\eta)} \int_{-\infty}^{+\infty} f(u)(t-u)_{+}^{\eta-1} \mathrm{e}^{-\lambda(t-u)_{+}} \mathrm{d}u \tag{22}$$

and

$$\mathbb{I}_{-}^{\eta,\lambda} f(t) = \frac{1}{\Gamma(\eta)} \int_{-\infty}^{\infty} f(u)(u-t)_{+}^{\eta-1} \mathrm{e}^{-\lambda(u-t)_{+}} \mathrm{d}u \tag{23}$$

respectively, for any $\eta > 0$ and $\lambda > 0$, where $\Gamma(\eta) = \int_0^{+\infty} \mathrm{e}^{-x} x^{\eta-1}\, \mathrm{d}x$ is the Euler gamma function, and $(x)_+ = xI(x > 0)$.

When $\lambda = 0$ these definitions reduce to the (positive and negative) Riemann–Liouville fractional integral [42, 43], which extends the usual operation of iterated integration to a fractional order. When $\lambda = 1$, the operator equation (22) is called the Bessel fractional integral [43, section 18.4].

Tempered fractional integrals are bounded linear operators on $L^p(\mathbb{R})$ such that

$$\|\mathbb{I}_{\pm}^{\eta,\lambda} f\|_p \leqslant \lambda^{-\eta} \|f\|_p \tag{24}$$

for all $f \in L^p(\mathbb{R})$. They also have semi-group property in the sense that

$$\mathbb{I}_{\pm}^{\eta,\lambda} \mathbb{I}_{\pm}^{\beta,\lambda} f = \mathbb{I}_{\pm}^{\eta+\beta,\lambda} f \tag{25}$$

for all $\eta, \beta > 0$ and all $f \in L^p(\mathbb{R})$. TFI has a relationship with the Fourier transform. Recall that the Fourier transform

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kx} f(x) \mathrm{d}x$$

for functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be extended to an isometry (a linear onto map that preserves the inner product) on $L^2(\mathbb{R})$ such that

$$\widehat{f}(k) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \mathrm{e}^{-\mathrm{i}kx} f(x)\, \mathrm{d}x \tag{26}$$

for any $f \in L^2(\mathbb{R})$, see for example [44, theorem 6.6.4]. Therefore, for any $\eta > 0$ and $\lambda > 0$, we have

$$\mathcal{F}[\mathbb{I}_{\pm}^{\eta,\lambda} f](k) = \hat{f}(k)(\lambda \pm \mathrm{i}k)^{-\eta} \tag{27}$$

for all $f \in L^1(\mathbb{R})$ and all $f \in L^2(\mathbb{R})$.

The inverse operator of TFI is called tempered fractional derivative. We define the positive and negative tempered fractional derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$\mathbb{D}_{+}^{\eta,\lambda} f(t) = \lambda^{\eta} f(t) + \frac{\eta}{\Gamma(1-\eta)} \int_{-\infty}^{t} \frac{f(t) - f(u)}{(t-u)^{\eta+1}} \, \mathrm{e}^{-\lambda(t-u)} \, \mathrm{d}u \tag{28}$$

and

$$\mathbb{D}_{-}^{\eta,\lambda} f(t) = \lambda^{\eta} f(t) + \frac{\eta}{\Gamma(1-\eta)} \int_{t}^{+\infty} \frac{f(t) - f(u)}{(u-t)^{\eta+1}} \, \mathrm{e}^{-\lambda(u-t)} \, \mathrm{d}u \tag{29}$$

respectively, for any $0 < \eta < 1$ and any $\lambda > 0$. Assuming f and f' are in $L^1(\mathbb{R})$. Then the tempered fractional derivative $\mathbb{D}_{+}^{\eta,\lambda} f(t)$ exists and

$$\mathcal{F}[\mathbb{D}_{\pm}^{\eta,\lambda} f](k) = \widehat{f}(k)(\lambda \pm \mathrm{i}k)^{\eta} \tag{30}$$

for any $0 < \eta < 1$ and any $\lambda > 0$. We can extend the definition of tempered fractional derivatives to a suitable class of functions in $L^2(\mathbb{R})$. For any $\eta > 0$ and $\lambda > 0$ we may define the fractional Sobolev space

$$W^{\eta,2}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + k^2)^{\eta} |\widehat{f}(k)|^2 \, \mathrm{d}k < \infty \right\}, \tag{31}$$

which is a Banach space with norm $\|f\|_{\eta,\lambda} = \|(\lambda^2 + k^2)^{\eta/2} \, \widehat{f}(k)\|_2$. The space $W^{\eta,2}(\mathbb{R})$ is the same for any $\lambda > 0$ (typically we take $\lambda = 1$) and all the norms $\|f\|_{\eta,\lambda}$ are equivalent, since $1 + k^2 \leqslant \lambda^2 + k^2 \leqslant \lambda^2(1 + k^2)$ for all $\lambda \geqslant 1$, and $\lambda^2 + k^2 \leqslant 1 + k^2 \leqslant \lambda^{-2}(1 + k^2)$ for all $0 < \lambda < 1$. Then we define the positive (resp., negative) tempered fractional derivative $\mathbb{D}_{\pm}^{\eta,\lambda} f(t)$ of a function $f \in W^{\eta,2}(\mathbb{R})$ is defined as the unique element of $L^2(\mathbb{R})$ with Fourier transform $\widehat{f}(k)(\lambda \pm \mathrm{i}k)^{\eta}$ for any $\eta > 0$ and any $\lambda > 0$. TFI and TFD have the following relationships: for any $\eta > 0$ and $\lambda > 0$, we have

$$\mathbb{D}_{\pm}^{\eta,\lambda} \mathbb{I}_{\pm}^{\eta,\lambda} f(t) = f(t) \tag{32}$$

for any function $f \in L^2(\mathbb{R})$, and

$$\mathbb{I}_{\pm}^{\eta,\lambda} \mathbb{D}_{\pm}^{\eta,\lambda} f(t) = f(t) \tag{33}$$

for any $f \in W^{\eta,2}(\mathbb{R})$.

Now, we introduce the *tempered fractional Langevin equation* by the following differential equation:

$$(\mathbb{D}_{+}^{d,\lambda} Y_{d,\alpha,\lambda})(t) = l_{\alpha}(t), \tag{34}$$

where $\lambda > 0$, $d > 0$, and $l_{\alpha}(t)$ is the α-stable noise. By taking the TFI, $\mathbb{I}_{+}^{d,\lambda}$, of both sides of equation (34), we get

$$Y_{d,\alpha,\lambda}(t) = \mathbb{I}_{+}^{d,\lambda} l_{\alpha}(t) \tag{35}$$

$$= \frac{1}{\Gamma(d)} \int_{-\infty}^{t} (t-x)^{d-1} \, \mathrm{e}^{-\lambda(t-x)} \, l_{\alpha}(x) \mathrm{d}x \tag{36}$$

$$= \frac{1}{\Gamma(d)} \int_{-\infty}^{t} (t-s)^{d-1} \mathrm{e}^{-\lambda(t-s)} L_{\alpha}(\mathrm{d}s), \tag{37}$$

where $L_\alpha(s) = \int_0^s l_\alpha(x)\mathrm{d}x$ is the α-stable Lévy motion. The α-stable noise $l_\alpha(x) = L'(x)$ can be viewed as the (weak) derivative of an α-stable Lévy stable motion on $x \in \mathbb{R}$. Therefore $Y_{d,\alpha,\lambda}(t)$ has a stable distribution with the scale parameter

$$\sigma^\alpha = \int_0^\infty x^{\alpha(d-1)}\mathrm{e}^{-\lambda\alpha x}\,\mathrm{d}x = \frac{\Gamma(\alpha(d-1)+1)}{(\lambda\alpha)^{\alpha(d-1)}(\Gamma(d))^\alpha}$$

provided $d > 1 - \frac{1}{\alpha}$ and $1 < \alpha < 2$. We call $Y_{d,\alpha,\lambda}(t)$ as *α-stable Yaglom noise*.

Remark 4.1. The Yaglom noise [45–47] is a stochastic model for turbulence in continuous time. The Yaglom noise $Y(t)$ can be considered as the tempered fractional integral of a white noise. Using the definition $\mathbb{I}_+^{\eta,\lambda}$ equation (4.1), we have

$$Y_{d,2,\lambda}(t) = \mathbb{I}_+^{d,\lambda}W(t) = \frac{1}{\Gamma(d)}\int_{-\infty}^t (t-x)^{d-1}\,\mathrm{e}^{-\lambda(t-x)}\;W(x)\mathrm{d}x \tag{38}$$

for any $d > \frac{1}{2}$ and $\lambda > 0$, where the white noise $W(x) = B'(x)$ is the (weak) derivative of a BM on $x \in \mathbb{R}$ such that $\mathbb{E}[B(x)^2] = \sigma^2|x|$.

The authors in [48] computed the codifference of α-stable Yaglom noise and showed that

$$\tau(t) \sim c_\alpha(\lambda; d)d_\alpha(\lambda; d)C_\alpha(\lambda, d)t^{d-1}\,\mathrm{e}^{-\lambda t} \tag{39}$$

as $t \to \infty$, where

$$d_\alpha(\lambda; d) = -\alpha\frac{\Gamma((d-1)(\alpha-1)+1)}{(\lambda\alpha)^{(d-1)(\alpha-1)+1}}, \tag{40}$$

$$c_\alpha(\lambda; d) = \left(\frac{\sqrt{2\lambda}\lambda^{d-1}}{\Gamma(d)}\right)^\alpha, \tag{41}$$

$$C_\alpha(\lambda, d) = \exp\left\{-2c_\alpha(\lambda; d)\frac{\Gamma(\alpha(d-1)+1)}{(\lambda\alpha)^{1+\alpha(d-1)}}\right\}. \tag{42}$$

By comparing the asymptotic behavior of the codifference of $Y_{d,\alpha,\lambda}$ in equation (39) with the asymptotic behavior of the codifference of α-stable ARTFIMA$(0, d, \lambda, 0)$ in theorem 2.3 we arrive at the conclusion that these two processes have the same power-law exponent and exponential tempering factor. This observation motivates us to prove that the α-stable ARTFIMA$(0, d, \lambda, 0)$ process $X(t) = \Delta^{-d,\lambda}Z_\alpha(t)$ converges to an α-stable Yaglom noise.

Theorem 4.2. *Let $d > 1 - \frac{1}{\alpha}$ and $\lambda > 0$, and suppose that $\{Z_\alpha(t)\}$ is an i.i.d. sequence of $S\alpha S$ random variables with $1 < \alpha < 2$. Then*

$$N^{1-\frac{1}{\alpha}-d}\Delta^{-d,\frac{\lambda}{N}}Z_\alpha(t) \Rightarrow Y_{d,\alpha,\lambda}(t)$$

in distribution as $N \to \infty$, where $Y_{d,\alpha,\lambda}(t)$ is the α-stable Yaglom noise equation (35).

The next proposition provides a scaling property of $Y_{d,\alpha,\lambda}(t)$.

Proposition 4.3. *The process $Y_{d,\alpha,\lambda}$ in equation (35) satisfies*

$$\{Y_{d,\alpha,\lambda}(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{c^{d-1+1/\alpha}Y_{d,\alpha,c\lambda}(t)\right\}_{t\in\mathbb{R}}, \quad \forall\; c > 0. \tag{43}$$

While theorem 4.2 shows that the α-stable ARTFIMA$(0, d, \lambda, 0)$ is a discrete time version of an α-stable Yaglom noise (by assuming λ depends on N), the following theorem shows that

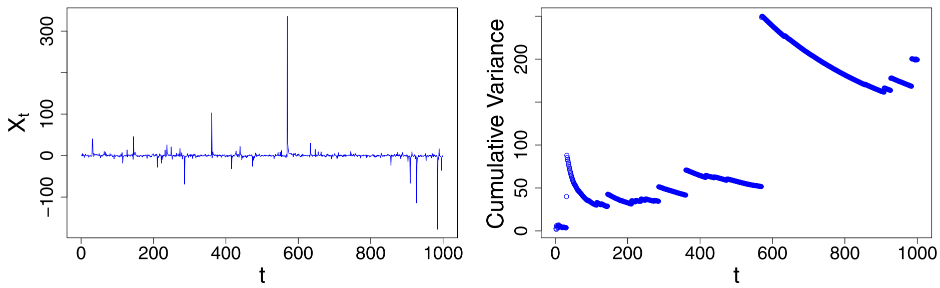

Figure 1. Plot of 10 000 samples of ARTFIMA (left) with non-Gaussian stable innovations $\alpha = 1.3, d = 0.1, \lambda = 0.045$ along with a cumulative variance plot (right) showing diverging variance due to stable distribution.

the normalized partial sums of ARTFIMA$(0, d, \frac{\lambda}{N}, 0)$ converges to tempered fractional stable motion of the second kind (TFSM II) with the moving average representation

$$Z^{II}_{H,\alpha,\lambda}(t) := \int_{\mathbb{R}} h_{H,\alpha,\lambda}(t;y) M_\alpha(\mathrm{d}y), \qquad t \in \mathbb{R} \tag{44}$$

with respect to α-stable Lévy process M_α with integrand

$$h_{H,\alpha,\lambda}(t;y) := (t-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(t-y)_+} - (-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(-y)_+} + \lambda \int_0^t (s-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(s-y)_+}\, \mathrm{d}s, \qquad y \in \mathbb{R}. \tag{45}$$

Theorem 4.4. *Let $S_N^{d,\frac{\lambda}{N}}(t) = \sum_{k=1}^{[Nt]} \Delta^{-d,\frac{\lambda}{N}} Z_\alpha(t)$. Then*

$$N^{-H} S_N^{d,\lambda_N}(t) \Rightarrow \Gamma(d+1)^{-1} Z^{II}_{H,\alpha,\lambda_*}(t), \tag{46}$$

where $Z^{II}_{H,\alpha,\lambda_*}$ is a TFSM II as defined in equation (44) and $H = d + \frac{1}{\alpha} > 0$.

Remark 4.5. TFSM II and its Gaussian counterpart tempered fractional Brownian motion of second kind (TFBM II) were recently introduced by Sabzikar and Surgailis [26], the above processes being closely related to the tempered fractional stable motion (TFSM) and the TFBM defined in Meerschaert and Sabzikar [25] and Meerschaert and Sabzikar [24], respectively. As shown in [26], TFSM and TFSM II are different processes, especially striking are their differences as $t \to \infty$.

5. Simulation results

In this section, we simulate the non-Gaussian stable ARTFIMA model with the index stability $\alpha \in (1, 2)$. We use the Durbin–Levinson algorithm [49] in which the innovations are non-Gaussian stable random variables. We modify the `artsim` method in the `artfima` R package to generate simulations of a stable ARTFIMA time series. We consider ARTFIMA$(0, 0.1, 0.045, 0)$ of length 10 000 when $\alpha \in \{1.3, 1.8\}$. The time series plot and the corresponding cumulative variance plot that illustrates infinite variance [50] are shown in figures 1 and 2.

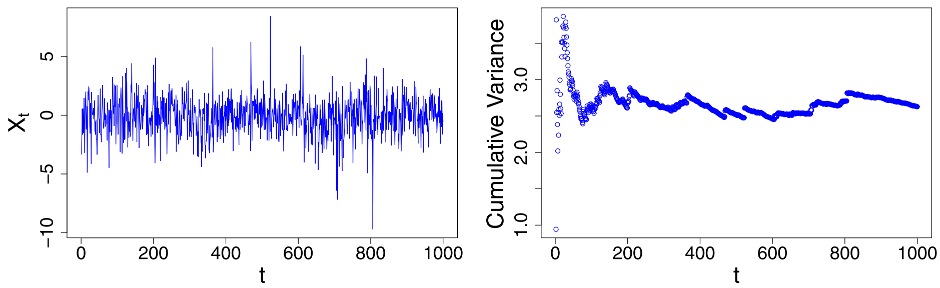

Figure 2. Plot of 10 000 samples of ARTFIMA (left) with non-Gaussian stable innovations $\alpha = 1.8, d = 0.1, \lambda = 0.045$ along with a cumulative variance plot (right) showing diverging variance due to stable distribution.

Table 1. Whittle estimators of d, λ for different stable time series along with average, bias, mean square error (MSE) and 95% percentile bootstrap confidence intervals of the estimates.

	$d = 0.1$				$\lambda = 0.045$			
α	Mean	Bias	MSE	CI	Mean	Bias	MSE	CI
1.3	0.104	0.004	5.0×10^{-6}	[0.088, 0.121]	0.059	0.014	1.30×10^{-5}	[0.003, 0.151]
1.8	0.116	0.016	1.5×10^{-4}	[0.085, 0.136]	0.082	0.037	2.2×10^{-4}	[5.9×10^{4}, 0.208]

For the time series shown in figures 1 and 2, we compute the Whittle estimates of parameters d and λ for 1000 Monte Carlo simulations of the time series. We calculate the bias, MSE, and 95% percentile bootstrap confidence interval for the time series as shown in table 1. We can see that the parameter estimates exhibit very low bias and MSE. The 95% bootstrap confidence interval and the bias for parameter λ is larger than that of d and the quality of the estimators is better for smaller α.

6. Application to mRNA dataset

For the first time, a method for tracking single RNA molecules in *Escherichia coli* cells was presented in the work of Golding and Cox [36]. The method was sensitive to single copies of mRNA, and, using the method, individual molecules were followed for many hours in living cells. The authors found that this motion resembles subdiffusion. In [3] it was discovered that FLSM can describe the motion of individual fluorescently labeled mRNA molecules. That work extended the results of [51] where an efficient test based on p-variation for distinguishing between CTRW and FBM was introduced.

A statistical test based on FLSM to classify trajectories of the mRNA dataset was proposed in [12]. The results of analysis of both mRNA trajectory coordinates showed that 21 out of the overall 27 trajectories can be modeled by the α-stable distribution and among them 12 are non-Gaussian. Here, we analyze only the trajectory no. 4 which has the longest continuous part, namely consists of 970 observations. We refer it as $\times$**4** in figure 3. It is apparent from the ACF plot that the data are not stationary.

To make the dataset stationary, we difference this dataset by lag 1 so that $X_t = X_t - X_{t-1}$ where $t = 2, 3, \ldots, 970$, see the top left panel of figure 4. We confirm both unit root and trend

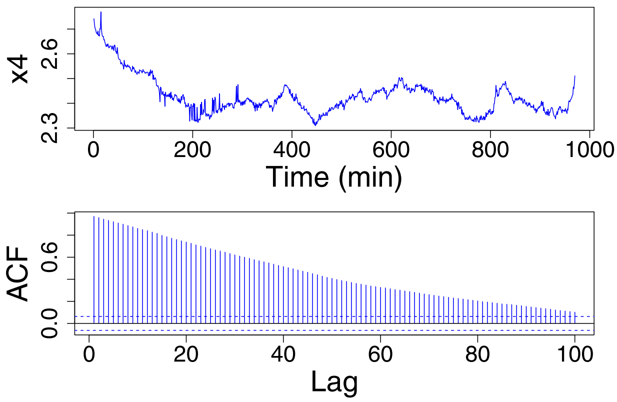

Figure 3. Time series (top) and ACF (bottom) of time series of $\times 4$.

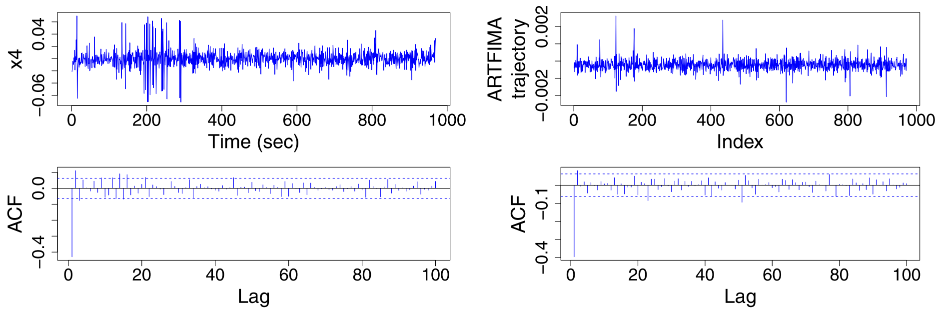

Figure 4. Time series and ACF of $\times 4$ differenced by lag 1 (top left and bottom left), a simulated ARTFIMA trajectory (top right) and ACF of ARTFIMA with parameters from table 2 (bottom right).

stationary of the differenced data using statistical tests. The augmented Dickey–Fuller test [52] fails to reject the alternate hypothesis of unit-root stationarity with a p-value of 0.01 and fails to reject null hypothesis of trend stationarity using the KPSS test [53] with a p-value of 0.1. The ACF plot in the bottom left panel of figure 4 for the differenced data indicates evidence of long memory. To check if the data follow heavy tails, we estimate the stability parameter α using the quantile based McCulloch estimation [54]. The value of the α estimate for the data is 1.8 which indicates a stable time series with infinite second moment. We use the Ljung–Box test [55] to confirm the lack of independence of data. We fit the differenced dataset with ARTFIMA(p, d, λ, q) model. The order of the models $p = 2$ and $q = 0$ was chosen to minimize the AIC. We use the `artfima` R package to fit the differenced data to the ARTFIMA model. The ARTFIMA parameter estimates, AIC, and BIC of this model are shown in table 2. In the top right panel of figure 4 we can see a sample trajectory of the fitted ARTFIMA model and in the bottom right panel ACF of the model. By comparing it with the ACF for the data we can see that the model replicates the data well.

The residual plot along with ACF plot of residuals are shown in figure 5. We can see the ACF shows no significant correlations which suggests the goodness of fit of the ARTFIMA

Table 2. ARTFIMA model Whittle estimates of d, λ, ϕ_1, ϕ_2 for differenced $\times 4$ trajectory of mRNA data.

d	λ	ϕ_1	ϕ_2	AIC	BIC
-0.149	0.216	-0.365	-0.029	-5593.54	-5564.28

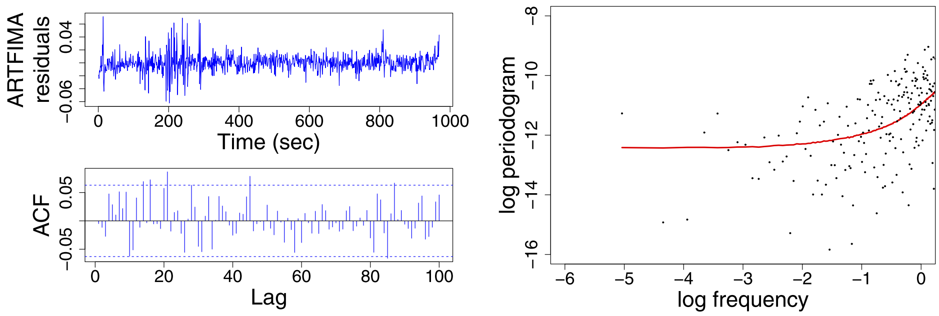

Figure 5. Plot of residuals, ACF of residuals from fitted ARTFIMA model (left) and data periodogram with the averaged periodogram of 10 000 simulations of the fitted ARTFIMA model (right).

model. We verify that the residuals are i.i.d. by using the Ljung–Box test [55]. The obtained p-value is 0.88 which also indicates independence of residuals.

Finally, we perform the McLeod–Li test [56], which is essentially the Ljung–Box test for squared residuals, to check for conditional heteroscedascity. The test returned p-value less than 0.001, so it rejects the null hypothesis of independence of the square residuals which indicates the presence of the heteroscedastic effect in the series (which is also visible in the top left panel of figure 4). This suggests that a GARCH component could be added to the model to account for this effect, so resulting in the ARTFIMA-GARCH model (for an application of the classical ARFIMA-GARCH model to the biological data see [21]). This can be a subject of future research.

To gain insight into the goodness of fit, we now investigate the periodogram statistic, which approximates the spectral density in the finite variance case, and follow the procedure presented in [35]. The log–log plots of the averaged periodogram of the fitted ARTFIMA model along with periodogram of the data are shown in figure 5. The averaged periodogram for the fitted model was generated from Monte Carlo simulations by averaging the periodograms calculated for each of the simulated 10 000 trajectories of the model. We can observe that the averaged periodogram of ARTFIMA model fits the data periodogram very well, especially for high frequencies. This suggests that the tempered model can describe the dependence structure of the data.

7. Conclusions

One of the approaches allowing to explain the emergence of non-Gaussianity in the SPT experiments is that the measured particle motion does not correspond to samples of one distribution, but to a mixture of individual subdistributions stemming from local dynamics characterised by time- and space-dependent parameters. We concentrated here on α-stable processes which are natural extensions of the Gaussian models via the generalized central limit theorem [37, 38].

In this paper, we presented the ARTFIMA process with stable noise/innovations and its main properties along with estimation techniques. Such process is a tempered ARFIMA which is considered classical in the context of discrete-time anomalous dynamic models [17]. Tempering leads to semiLRD which means that the autocodifference function is always summable (contrary to the behaviour of the non-tempered ARFIMA) unless the parameter λ goes to zero. In the Gaussian case, the second moment of the partial sum process switches from the power-law behavior (corresponding to anomalous diffusion) to linear (standard diffusion), hence exhibiting a transient dynamic [29].

We also showed that the α-stable ARTFIMA$(0, d, \lambda, 0)$ is a discrete time version of the α-stable Yaglom noise. The Yaglom noise was found to be a solution of the introduced tempered fractional Langevin equation.

Finally, we illustrated the usefulness of the ARFTIMA process in the context of SPT experiments by analyzing the longest trajectory from the Golding and Cox data. We showed that the fitted ARTFIMA model fits well the observed trajectory, which was justified by residual and periodogram analysis.

We strongly believe that the process can serve as the first choice for a discrete-time transient anomalous diffusion model not only in the context of biological data but also other complex systems.

Acknowledgments

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Proofs

Proof of proposition 2.1. Define $A_\lambda(B) := \Theta_q(B)\Phi_p(B)^{-1}(1 - \mathrm{e}^{-\lambda}B)^{-d}$ and write $\Theta_q(z)\Phi_p(z)^{-1} = \sum_{j=0}^{\infty} b(j)z^j$ for $|z| \leqslant 1$. Then

$$A_\lambda(z) = \Theta_q(z)\Phi_p(z)^{-1}(1 - \mathrm{e}^{-\lambda}z)^{-d} \tag{A.1}$$

$$= \left(\sum_{s=0}^{\infty} b(s)z^s\right)\left(\sum_{i=0}^{\infty} \omega_{-d,\lambda}(i)z^i\right) = \sum_{j=0}^{\infty} a_{-d,\lambda}(j)z^j, \tag{A.2}$$

where

$$a_{-d,\lambda}(j) = \sum_{s=0}^{j} \omega_{-d,\lambda}(s)b(j-s) \tag{A.3}$$

for $j \geqslant 0$. Since $X(t)$ satisfies $\Phi_p(B)X(t) = \Theta_q(B)\Delta^{-d,\lambda}Z(t)$, we can write

$$X(t) = \frac{\Theta_q(B)}{\Phi_p(B)}\Delta^{-d,\lambda}Z(t) \tag{A.4}$$

$$= \left(\frac{\Theta_q(B)}{\Phi_p(B)}(1-\mathrm{e}^{-\lambda}B)^{-d}\right)Z(t) \tag{A.5}$$

$$= \left(\sum_{j=0}^{\infty} a_{-d,\lambda}(j)B^j\right)Z(t) = \sum_{j=0}^{\infty} a_{-d,\lambda}(j)Z(t-j) =: X_{p,d,\lambda,q}, \tag{A.6}$$

where $a_{-d,\lambda}(j)$ is given by equation (A.3) and hence this proves equation (6). Next, in order to show the series in equation (6) converges a.s. and in L^ν for any $\nu < \alpha$, we need to show that $\sum_{j=0}^{\infty} |a_{-d,\lambda}(j)|^\nu < \infty$ for some $\nu < \alpha$. Consider the case $1 \leqslant \nu < \alpha < 2$ and write

$$\begin{aligned}\sum_{j=0}^{\infty} |a_{-d,\lambda}(j)| &= \sum_{j=0}^{\infty}\left|\sum_{s=0}^{j}\omega_{-d,\lambda}(s)b(j-s)\right| \\ &= \sum_{j=0}^{\infty}|(\omega_{-d,\lambda}*b)(j)|^\nu \\ &= \|\omega_{-d,\lambda}*b\|_\nu^\nu,\end{aligned} \tag{A.7}$$

where $(\omega_{-d,\lambda}*b)(j) = \sum_{s=0}^{j}\omega_{-d,\lambda}(s)b_{j-s}$. Under assumption 2, $|\Theta_q(z)/\Phi_p(z)| < \infty$, for $|z| \leqslant 1+\varepsilon$, and the convergence of the series $\Theta_q(z)/\Phi_p(z)$ implies that $|b_j| \leqslant C(1+\varepsilon)^{-j}$ for $j \geqslant 0$, see [57, theorem 7.2.3]. Therefore $b(j) \in L^1$. On the other side,

$$\sum_{j=0}^{\infty}|\omega_{-d,\lambda}(j)|^\nu = \sum_{j=0}^{M}|\omega_{-d,\lambda}(j)|^\nu + \Gamma(\nu)^{-d}\sum_{j=M+1}^{\infty}\left(j^{\,d-1}\,\mathrm{e}^{-\lambda j}\right)^\nu < \infty \tag{A.8}$$

and hence $\omega_{-d,\lambda}(j) \in L^\nu$. Now, by applying Young inequality we have

$$\|\omega_{-d,\lambda}*b\|_\nu \leqslant \|b\|_1\|\omega_{-d,\lambda}\|_\nu < \infty \tag{A.9}$$

and this shows $\sum_{j=0}^{\infty}|a_{-d,\lambda}(j)|^\nu < \infty$ for some $1 \leqslant \nu < \alpha < 2$. Consequently, the series in equation (6) converges a.s. and in L^ν for any $1 \leqslant \nu < \alpha < 2$. Now, Minkowski's inequality implies

$$\mathbb{E}|X_{p,d,\lambda,q}(t)|^\nu \leqslant 2\mathbb{E}|Z(0)|^\nu\sum_{j=0}^{\infty}|a_{-d,\lambda}(j)|^\nu < \infty \tag{A.10}$$

for $1 < \nu < \alpha < 2$. The proof of part (a) is completed now.

To prove part (b), let $C_{d,\lambda}(B) := \Phi_p(B)/\Theta_q(B)(1-\mathrm{e}^{-\lambda}B)^d$. Write $\Phi(z)/\Theta(z) = \sum_{j=0}^{\infty}c(j)z^j$ for $|z| \leqslant 1$ so that

$$\begin{aligned} C_{d,\lambda}(z) &= \frac{\Phi_p(z)}{\Theta_q(z)}(1-\mathrm{e}^{-\lambda}z)^d \\ &= \left(\sum_{s=0}^{\infty} c(s)z^s\right)\left(\sum_{i=0}^{\infty}\omega_{d,\lambda}(i)z^i\right) \\ &= \sum_{j=0}^{\infty} c_{d,\lambda}(j)z^j, \end{aligned} \tag{A.11}$$

where

$$c_{d,\lambda}(j) = \sum_{s=0}^{j}\omega_{d,\lambda}(s)c(j-s) \tag{A.12}$$

for $j \geqslant 0$. Since $Z(t)$ satisfies $\Phi_p(B)X(t) = \Theta_q(B)\Delta^{-d,\lambda}Z(t)$, we can write

$$Z(t) = \frac{\Phi_p(B)}{\Theta_q(B)}\Delta^{d,\lambda}X(t) = \left(\sum_{j=0}^{\infty} c_{d,\lambda}(j)B^j\right)X(t) = \sum_{j=0}^{\infty} c_{d,\lambda}(j)X(t-j), \tag{A.13}$$

where $c_{d,\lambda}(j)$ is given by equation (A.12) and hence this proves equation (8). Next, in order to show the series in equation (8) converges a.s. and in L^ν for any $\nu < \alpha$, one need to verify that $\sum_{j=0}^{\infty} |c_{d,\lambda}(j)|^\nu < \infty$ for some $\nu < \alpha$. But, the proof is similar to part (a) and hence we omit the details. The proofs of part (b) and proposition 2.1 is completed now. □

Proof of theorem 2.3. From equation (11) we have

$$\tau_{d,\lambda}(n) := \tau(X_{d,\lambda}(0), X_{d,\lambda}(n)) \tag{A.14}$$

$$= \sum_{j=0}^{\infty}\left[|\omega_{-d,\lambda}(j)|^\alpha + |\omega_{-d,\lambda}(j+n)|^\alpha - |\omega_{-d,\lambda}(j) - \omega_{-d,\lambda}(j+n)|^\alpha\right] \tag{A.15}$$

$$:= I_1 + I_2, \tag{A.16}$$

where $I_1 = \sum_{j=0}^{\infty} |\omega_{-d,\lambda}(j+n)|^\alpha$ and $I_2 = \sum_{j=0}^{\infty} |\omega_{-d,\lambda}(j)|^\alpha - |\omega_{-d,\lambda}(j) - \omega_{-d,\lambda}(j+n)|^\alpha$. Since $\omega_{-d,\lambda}(j) \sim \frac{1}{\Gamma(d)}\,\mathrm{e}^{-\lambda j} j^{d-1}$ as $j \to \infty$, we may work with the asymptotic form of $\omega_{-d,\lambda}(j)$. For any $j > 0$,

$$\mathrm{e}^{\lambda n} n^{-(d-1)}|\omega_{-d,\lambda}(j)|^\alpha = C\,\mathrm{e}^{\lambda n} n^{-(d-1)}|\mathrm{e}^{-\lambda\alpha(n+j)}(n+j)^{d-1}| \tag{A.17}$$

$$= \mathrm{e}^{-\lambda\alpha j}\,\mathrm{e}^{-\lambda n(\alpha-1)}\left(\frac{j+n}{n^{1/\alpha}}\right)^{\alpha(d-1)} \to 0 \quad \text{as } n \to \infty. \tag{A.18}$$

since $1 < \alpha \leqslant 2$. We note that

$$\sup_{n>1}\left(\left|\mathrm{e}^{\lambda n} n^{-(d-1)}\omega_{-d,\lambda}(n+j)\right|\right) = C\sup_{n>1}\left(\left|\mathrm{e}^{-\lambda\alpha j}\,\mathrm{e}^{-\lambda n(\alpha-1)}\left(\frac{j+n}{n^{1/\alpha}}\right)^{\alpha(d-1)}\right|\right) \tag{A.19}$$

$$\leqslant \begin{cases} \mathrm{e}^{-\lambda\alpha j} & \text{for } d-1 \leqslant 0, \\ \mathrm{e}^{-\lambda\alpha j}(1+j)^{\alpha(d-1)} & \text{for } d-1 > 0, \end{cases} \tag{A.20}$$

which belongs to $L^1(0,\infty)$. Now, using equations (A.17), and (A.19), the dominated convergence theorem implies that

$$\mathrm{e}^{\lambda n} n^{(d-1)} I_1 = C\,\mathrm{e}^{\lambda n} n^{(d-1)} \sum_{j=0}^{\infty} (j+n)^{\alpha(d-1)} \mathrm{e}^{-\lambda\alpha(j+n)} \to 0, \quad \text{as } n \to \infty. \tag{A.21}$$

Next, we show $\mathrm{e}^{\lambda n} n^{-(d-1)} I_2 \to \alpha\,\mathrm{e}^{-\lambda\alpha j} j^{(d-1)(\alpha-1)}$ as $n \to \infty$. For each $j > 0$,

$$\mathrm{e}^{\lambda n} n^{-(d-1)} \left[|\omega_{-d,\lambda}(j)|^{\alpha} - |\omega_{-d,\lambda}(j) - \omega_{-d,\lambda}(j+n)|^{\alpha}\right] =: |a_n|^{\alpha} - |a_n - b_n|^{\alpha}, \tag{A.22}$$

where $a_n = \left(\frac{j}{n^{1/\alpha}}\right)^{(d-1)} \mathrm{e}^{-\lambda(j-n/\alpha)}$ and $b_n = \left(\frac{n+j}{n^{1/\alpha}}\right)^{(d-1)} \mathrm{e}^{-\lambda j}\,\mathrm{e}^{-\lambda n(1-1/\alpha)}$. It is obvious that $a_n \to \infty$ and $b_n \to 0$ as $n \to \infty$. Then using $|a_n|^{\alpha} - |a_n - b_n|^{\alpha} \sim \alpha |b_n||a_n|^{\alpha-1}$, as $n \to \infty$, we get

$$\begin{aligned} &\mathrm{e}^{\lambda n} n^{-(d-1)} \left[|\omega_{-d,\lambda}(j)|^{\alpha} - |\omega_{-d,\lambda}(j) - \omega_{-d,\lambda}(j+n)|^{\alpha}\right] \\ &\qquad \sim \alpha\,\mathrm{e}^{-\alpha\lambda j} \left(\frac{n+j}{nj}\right)^{d-1} j^{\,\alpha(d-1)} \end{aligned} \tag{A.23}$$

consequently,

$$\mathrm{e}^{\lambda n} n^{-(d-1)} \left[|\omega_{-d,\lambda}(j)|^{\alpha} - |\omega_{-d,\lambda}(j) - \omega_{-d,\lambda}(j+n)|^{\alpha}\right] \to \alpha\,\mathrm{e}^{-\alpha\lambda j} j^{(\alpha-1)(d-1)}. \tag{A.24}$$

Now, using the fact that $||a_n - b_n|^{\alpha} - |a_n|^{\alpha}| \leqslant b_n^{\alpha} + \alpha b_n {a_n}^{\alpha-1}$, for $a, b \geqslant 0$ and $1 < \alpha \leqslant 2$, we have

$$\begin{aligned} &\sup_{n>1} \left|\mathrm{e}^{\lambda n} n^{-(d-1)} \left[|\omega_{-d,\lambda}(j)|^{\alpha} - |\omega_{-d,\lambda}(j) - \omega_{-d,\lambda}(j+n)|^{\alpha}\right]\right| \\ &\qquad \leqslant \sup_{n>1} b_n^{\alpha} + \alpha \sup_{n\geqslant 1} a_n\, b_n^{\alpha-1} \end{aligned} \tag{A.25}$$

$$\leqslant \begin{cases} \mathrm{e}^{-\lambda\alpha j}\left[1 + \alpha(j+1)^{d-1} j^{(d-1)(\alpha-1)}\right] & \text{if } d-1 \leqslant 0 \\ \mathrm{e}^{-\lambda\alpha j}\left[\alpha(j+1)^{\alpha(d-1)} \alpha j^{(d-1)}\right] & \text{if } (d-1) > 0 \end{cases}, \tag{A.26}$$

which belongs to L^1. From equations (A.22)–(A.25), the dominated convergence theorem implies that

$$\mathrm{e}^{\lambda n} n^{-\alpha(d-1)} I_2 \to \alpha \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda\alpha j} j^{(d-1)(\alpha-1)} \quad \text{as } n \to \infty. \tag{A.27}$$

Finally, equations (A.21) and (A.27) together yield

$$\lim_{n\to\infty} \frac{\tau_{d,\lambda}(n)}{\mathrm{e}^{-\lambda n} n^{(d-1)}} = (\Gamma(d))^{-\alpha} \sum_{j=0}^{\infty} \alpha \mathrm{e}^{-\lambda\alpha j} \omega_{-d}^{\alpha-1}(j) \tag{A.28}$$

for $1 < \alpha < 2$ and $d \in \mathbb{R} - \mathbb{Z}_-$. The proof is completed now. □

Proof of corollary 2.4. Theorem 2.3 implies that $\sum_{j=0}^{\infty} |\tau_{d,\lambda}(n)| < \infty$ for any $d \in \mathbb{R} - \mathbb{Z}_-$ and $\alpha \in (1, 2)$ and hence the statement holds. □

Lemma A.1. *Let $\{X(t)\}_{t\in\mathbb{Z}}$ be the ARTFIMA(p, d, λ, q) given by equation (6) with innovations $\{Z_\alpha(t)\}_{t\in\mathbb{Z}}$ satisfies under assumption 3. Then*

$$\sum_{j=-\infty}^{\infty} |a_{-d,\lambda}(j)|^{\delta} |j| < \infty$$

for $\delta = \min(1, \nu)$.

Proof of lemma 1. We use the asymptotic behavior of $\omega_{-d,\lambda}(j)$ to conclude the statement of the lemma as follows:

$$\sum_{j=0}^{\infty} |\omega_{-d,\lambda}(j)|^{\delta} \; |j| = \sum_{j=0}^{M} |\omega_{-d,\lambda}(j)|^{\delta} \; |j| + C \sum_{j=M+1}^{\infty} |j^{\,d-1}\, \mathrm{e}^{-\lambda j}|^{\delta} \; |j| < \infty$$

for any $d \in \mathbb{R} - \mathbb{Z}_-$ and this completes the proof. □

Proof of theorem 2.6. The proof follows by [58, theorem 13.3.1] if we can verify the claim that $\sum_{j=0}^{\infty} |j||\omega_{-d,\lambda}(j)|^{\delta} < \infty$ for $\delta \in (0, \alpha) \cap [0, 1]$. But this claim proved in lemma 2 and this completes the proof. □

Lemma A.2. *Let $f(\omega, \beta)$ be any continuous function on $[-\pi, \pi] \times E$. Then as $n \to \infty$,*

$$\sup_{\beta\in\bar{E}} \left| \int_{-\pi}^{\pi} f(\omega, \beta)\tilde{I}_n(\omega)\mathrm{d}\omega - \frac{1}{\gamma(0)} \int_{-\pi}^{\pi} f(\omega, \beta)g(\omega, \beta)\mathrm{d}\omega \right| \xrightarrow{p} 0, \tag{A.29}$$

where $\bar{E}$ denotes the closure of E, and $\gamma(0) =_2 F_1(d;\; d;\; 1;\; \mathrm{e}^{-2\lambda})$.

Proof of lemma 2. The proof follows by the similar lines as in [40, lemma 2.2] with the only exception that we need to verify $\rho_n(h) \xrightarrow{d} \rho(h)$ as $n \to \infty$. But, we have shown this claim in theorem 2.6 and hence this completes the proof. □

Lemma A.3. *Suppose that $\beta_i = (\phi_1, \ldots, \phi_{p_i}, d_i, \lambda_i, \theta_1, \ldots, \theta_{q_i}) \in E$ for $i = 1, 2$. If $\beta_1 \neq \beta_2$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(\omega, \beta_1)}{g(\omega, \beta_2)} > 1.$$

Proof of lemma 3. Let $|z| < 1$ and $\beta \in E$. Define

$$A(z, \beta) = \frac{\Theta(z, \beta)}{\Phi(z, \beta)(1 - \mathrm{e}^{-\lambda(\beta)}z)^{d(\beta)}} = \sum_{j=0}^{\infty} a_{-d(\beta),\lambda(\beta)}(j)z^j,$$

where $a_{-d(\beta),\lambda(\beta)}(j)$ is given by equation (A.3). We used $-d(\beta)$ and $\lambda(\beta)$ to stress on the point that d and λ are the elements of the vector β.

We also define

$$C(z, \beta) = \frac{1}{A(z, \beta)} = \sum_{j=0}^{\infty} c_{d(\beta),\lambda(\beta)}(j)z^j,$$

where $c_{d,\lambda}(j)$ is given by equation (A.12). Let $\xi(n)$ be a sequence of i.i.d Gaussian random variables with zero mean and unit variance, and $X(n,\beta_1)=\sum_{j=0}^{\infty}a_{-d(\beta_1),\lambda(\beta_1)}(j)\xi(n-j)$ be a Gaussian ARTFIMA model with the parameter β_1. By mimicking a similar argument in [58, section 13.2], we can minimize Var $\left(X(n+1,\beta_1)-\sum_{j=0}^{\infty}\theta(j)X(n-j,\beta_1)\right)$ if and only if $\theta(j)=-c_{d(\beta_1),\lambda(\beta_1)}(j+1)$ and the smallest value of the variance is one. Since $\beta_1\neq\beta_2$, we have $C(\cdot,\beta_1)\neq C(\cdot,\beta_2)$, and hence Var $\left(X(n+1,\beta_1)+\sum_{j=0}^{\infty}c_{d(\beta_2),\lambda(\beta_2)}(j+1)X(n-j,\beta_1)\right)>1$. This leads to complete the proof because the variance equals

$$\begin{aligned}\sum_{j=0}^{\infty}&\left|\sum_{k=0}^{j}c_{d(\beta_2),\lambda(\beta_2)}(j)a_{-d(\beta_1),\lambda(\beta_1)}(j-k)\right|^2\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|C(\mathrm{e}^{-\mathrm{i}\omega},\beta_2)A(\mathrm{e}^{-\mathrm{i}\omega},\beta_1)\right|^2\,\mathrm{d}\omega\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{A(\mathrm{e}^{-\mathrm{i}\omega},\beta_1)}{A(\mathrm{e}^{-\mathrm{i}\omega},\beta_2)}\right|^2\,\mathrm{d}\omega\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{g(\omega,\beta_1)}{g(\omega,\beta_2)}\mathrm{d}\omega>1.\end{aligned}$$

□

Proof of theorem 3.2. We first note that $g(\omega,\beta)$ is a continuous function on $[-\pi,\pi]\times E$ and hence we can apply lemma 2 by taking $f(\omega,\beta)=\frac{1}{g(\omega,\beta)}$. Therefore,

$$\sigma_n^2(\beta_n)=\int_{-\pi}^{\pi}\frac{\tilde{I}_n(\omega)}{g(\omega,\beta_n)}\xrightarrow{p}\frac{2\pi}{\gamma(0)}$$

as logs as we show $\beta_n\xrightarrow{p}\beta_0$ as $n\to\infty$. This can be done by applying 3 and mimicking a similar arguments as in [40, theorem 1.1]. □

Proof of theorem 3.3. The proof follows a similar path of the proof of [40, theorem 1.2.], and hence we omit the details. □

Proof of corollary 3.4. The elements of the covariance matrix $\mathbf{W}$ can be obtained as in [33, theorem 3.3]. We only need to compute b_k as follows: recall that

$$b_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}k\omega}g(\omega,\beta_0)\frac{\partial g^{-1}(\omega,\beta_0)}{\partial\beta}\,\mathrm{d}\omega, \tag{A.30}$$

where $g(\omega,\beta_0)=\frac{|\Theta_q(\mathrm{e}^{-\mathrm{i}\omega})|^2}{\Phi_p(\mathrm{e}^{-\mathrm{i}\omega})|^2}(1-2\,\mathrm{e}^{-\lambda}\cos(\omega)+\mathrm{e}^{-2\lambda})^{-d}$. Let $p=q=0$ so that $\beta_0=(d,\lambda)$ and $g^{-1}(\omega,\beta_0)=(1-2\,\mathrm{e}^{-\lambda}\cos(\omega)+\mathrm{e}^{-2\lambda})^{d}$. By taking the derivative of $g^{-1}(\omega,\beta_0)$ with respect to d and λ, we get

$$\frac{\partial g^{-1}(\omega,\beta_0)}{\partial d}=\ln(1-2\,\mathrm{e}^{-\lambda}\cos(\omega)+\mathrm{e}^{-2\lambda})(1-2\,\mathrm{e}^{-\lambda}\cos(\omega)+\mathrm{e}^{-2\lambda})^{d} \tag{A.31}$$

and

$$\frac{\partial g^{-1}(\omega,\beta_0)}{\partial\lambda}=2d\,\mathrm{e}^{-\lambda}(\cos(\omega)-\mathrm{e}^{-\lambda})(1-2\,\mathrm{e}^{-\lambda}\cos(\omega)+\mathrm{e}^{-2\lambda})^{d-1}. \tag{A.32}$$

By putting equations (A.31) and (A.32) into equation (A.30), we get

$$b_k = \begin{pmatrix} b_{1k} \\ b_{2k} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\omega}\ \ln(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})\mathrm{d}\omega \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{\mathrm{e}^{\mathrm{i}k\omega}2d\,\mathrm{e}^{-\lambda}(\cos(\omega)-\mathrm{e}^{-\lambda})}{(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})}\mathrm{d}\omega \end{pmatrix}.$$

Next, we simplify b_{1k} and b_{2k}. For b_{1k}, using the standard formula [59, No 6, p 924]

$$\int_0^{\pi} \ln(1-2a\ \cos(\omega)+a^2)\cos(n\omega)\mathrm{d}\omega = -\frac{\pi}{n}a^n, \quad [a^2<1],$$

we have

$$\begin{aligned} b_{1k} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\omega}\ \ln(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\pi} \ln(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})\cos(k\omega)\mathrm{d}\omega = \frac{1}{\pi}\left(\frac{-\pi}{k}\right)\mathrm{e}^{-\lambda k} \\ &= \frac{-1}{k}\mathrm{e}^{-\lambda k}. \end{aligned}$$

For b_{2k}, we write

$$\begin{aligned} b_{2k} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}k\omega}2d\,\mathrm{e}^{-\lambda}(\cos(\omega)-\mathrm{e}^{-\lambda})}{(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})}\mathrm{d}\omega \\ &= \frac{d\,\mathrm{e}^{-\lambda}}{\pi}\left[\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}k\omega}\ \cos(\omega)}{(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})}\mathrm{d}\omega \quad - \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}k\omega}\ \mathrm{e}^{-\lambda}}{(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})}\mathrm{d}\omega\right] \\ &= \frac{d\,\mathrm{e}^{-\lambda}}{\pi}[I_1-I_2]. \end{aligned}$$

For I_2, we obtain

$$I_2 = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}k\omega}\ \mathrm{e}^{-\lambda}}{(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})}\mathrm{d}\omega = \frac{2\pi\mathrm{e}^{-\lambda(k+1)}}{1-\mathrm{e}^{-2\lambda}}, \tag{A.33}$$

where we use the standard formula [59, No 2, p 606]

$$\int_0^{\pi} \frac{\cos(n\omega)}{(1-2a\ \cos(\omega)+a^2)}\mathrm{d}\omega = \frac{\pi a^n}{1-a^2}$$

provided $a^2<1$. For I_1, we obtain

$$I_1 = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}k\omega}\ \cos(\omega)}{(1-2\,\mathrm{e}^{-\lambda}\ \cos(\omega)+\mathrm{e}^{-2\lambda})}\mathrm{d}\omega = \pi\left(\frac{1+\mathrm{e}^{-2\lambda}}{1-\mathrm{e}^{-2\lambda}}\right)\mathrm{e}^{-\lambda(k-1)}, \tag{A.34}$$

where we use the standard formula [59, No 4, p 606]

$$\int_0^{\pi} \frac{\cos(\omega)\cos(n\omega)}{(1-2a\ \cos(\omega)+a^2)}\mathrm{d}\omega = \frac{\pi}{2}\frac{1+a^2}{1-a^2}a^{n-1}$$

provided $a^2<1$. Now, using equations (A.33) and (A.34), we get

$$\begin{aligned} b_{2k} &= \frac{d\,\mathrm{e}^{-\lambda}}{\pi}[I_1 - I_2] \\ &= \frac{d\,\mathrm{e}^{-\lambda}}{\pi}\left[\pi\left(\frac{1+\mathrm{e}^{-2\lambda}}{1-\mathrm{e}^{-2\lambda}}\right)\mathrm{e}^{-\lambda(k-1)} - \frac{2\pi\mathrm{e}^{-\lambda(k+1)}}{1-\mathrm{e}^{-2\lambda}}\right] \\ &= d\,\mathrm{e}^{-\lambda k}. \end{aligned}$$

The proof of the corollary is complete now. □

The proof of theorem 4.2 requires the following two simple lemmas.

Lemma A.4. *Given $d > 1 - \frac{1}{\alpha}$ and $\lambda > 0$, define*

$$\tilde{\omega}_{d,\lambda}(j) = \frac{1}{\Gamma(d)} j^{\,d-1}\,\mathrm{e}^{-\lambda j} \tag{A.35}$$

for $j \geqslant 1$, and $\tilde{\omega}_{d,\lambda}(j) = 0$ for $j \leqslant 0$. Then as $N \to \infty$ as have

$$N^{\alpha-1-\alpha d}\sum_{j=0}^{\infty}\left|\tilde{\omega}_{d,\lambda/N}(j)\right|^{\alpha} \to \frac{1}{\Gamma(d)^{\alpha}}\int_{-\infty}^{t}\left|(t-x)^{d-1}\,\mathrm{e}^{-\lambda(t-x)}\right|^{\alpha}\mathrm{d}x. \tag{A.36}$$

Proof of lemma 4. For any t real, a change of variable $j = [Nt] - M$ in the sum yields

$$\begin{aligned} N^{\alpha-1-\alpha d}\sum_{j=1}^{\infty}|\tilde{\omega}_{d,\lambda}(j)|^{\alpha} &= N^{\alpha-1-\alpha d}\sum_{j=1}^{\infty}\left|\frac{1}{\Gamma(d)} j^{\,d-1}\,\mathrm{e}^{-\frac{\lambda}{N}j}\right|^{\alpha} \\ &= \frac{N^{\alpha-1-\alpha d}}{\Gamma(d)^{\alpha}}\sum_{m=-\infty}^{[Nt]-1}\left|([Nt]-m)^{d-1}\,\mathrm{e}^{-\frac{\lambda}{N}([Nt]-m)}\right|^{\alpha} \\ &= \frac{1}{\Gamma(d)^{\alpha}}\left\{\frac{1}{N}\sum_{m=-\infty}^{[Nt]-1}\left|\left(\frac{[Nt]}{N} - \frac{m}{N}\right)^{\alpha-1}\right.\right. \\ &\qquad \left.\left.\times\ \exp\left[-\lambda\left(\frac{[Nt]}{N} - \frac{m}{N}\right)\right]\right|^{\alpha}\right\} \\ &\to \frac{1}{\Gamma(d)^{\alpha}}\int_{-\infty}^{t}\left|(t-x)^{d-1}\,\mathrm{e}^{-\lambda(t-x)}\right|^{\alpha}\mathrm{d}x \end{aligned}$$

as $N \to \infty$ by the definition of the Riemann integral. □

Lemma A.5. *Let $d > 1 - \frac{1}{\alpha}$, $\lambda > 0$, and*

$$\omega_{-d,\lambda}(j) = (-1)^j\frac{d!}{j!(d-j)!}\,\mathrm{e}^{-\lambda j} \tag{A.37}$$

for $j \geqslant 1$, and $\omega_{-d,\lambda}(j) = 0$ for $j \leqslant 0$. Then as $N \to \infty$ as have

$$N^{\alpha-1-\alpha d}\sum_{j=1}^{\infty}\left|\omega_{-d,\lambda/N}(j)\right|^{\alpha} \to \frac{1}{\Gamma(d)^{\alpha}}\int_{-\infty}^{t}\left|(t-x)^{d-1}\,\mathrm{e}^{-\lambda(t-x)}\right|^{\alpha}\mathrm{d}x. \tag{A.38}$$

Proof of lemma 5. It follows from Stirling's approximation that

$$\omega_{-d,\lambda/N}(j) = (-1)^j \frac{d!}{j!(d-j)!} \mathrm{e}^{-\frac{\lambda}{N}} \sim \frac{d}{\Gamma(1+d)} j^{d-1} \mathrm{e}^{-\frac{\lambda}{N}} = \tilde{\omega}_{d,\lambda/N}(j)$$

as $j \to \infty$ where $\tilde{\omega}_{d,\lambda}(j)$ is from equation (A.35). Hence, for any $\epsilon > 0$ there exists some positive integer M such that

$$(1-\epsilon)\tilde{\omega}_{d,\lambda/N}(j) < \omega_{-d,\lambda/N}(j) < (1+\epsilon)\tilde{\omega}_{d,\lambda/N}(j) \tag{A.39}$$

for all $j > M$. It follows that

$$\lim_{N\to\infty} N^{\alpha-1-\alpha d} \sum_{j=0}^{\infty} |\omega_{-d,\lambda/N}(j)|^{\alpha} \tag{A.40}$$

$$\leqslant \lim_{N\to\infty} N^{\alpha-1-\alpha d} \left[\sum_{j=0}^{M} |\omega_{-d,\lambda/N}(j)|^{\alpha} + (1+\epsilon)^{\alpha} \sum_{j=M+1}^{\infty} |\tilde{\omega}_{d,\lambda/N}(j)|^{\alpha} \right] \tag{A.41}$$

$$\leqslant \lim_{N\to\infty} N^{\alpha-1-\alpha d} \left[\sum_{j=0}^{M} |\omega_{-d,\lambda/N}(j)|^{\alpha} + (1+\epsilon)^{\alpha} \sum_{j=0}^{\infty} |\tilde{\omega}_{d,\lambda/N}(j)|^{\alpha} \right] \tag{A.42}$$

$$\leqslant \frac{(1+\epsilon)^{\alpha}}{\Gamma(d)^{\alpha}} \int_{-\infty}^{t} \left|(t-x)^{d-1} \mathrm{e}^{-\lambda(t-x)}\right|^{\alpha} \mathrm{d}x, \tag{A.43}$$

since

$$\lim_{N\to\infty} N^{\alpha-1-\alpha d} \sum_{j=0}^{M} |\omega_{-d,\lambda/N}(j)|^{\alpha} = 0.$$

Similarly, $N^{\alpha-1-\alpha d} \sum_{j=0}^{M} |\tilde{\omega}_{d,\lambda/N}(j)|^{\alpha} \to 0$, so that

$$\lim_{N\to\infty} N^{\alpha-1-\alpha d} \sum_{j=0}^{\infty} |\omega_{-d,\lambda/N}(j)|^{\alpha}$$

$$\geqslant \lim_{N\to\infty} N^{\alpha-1-\alpha d} \left[\sum_{j=0}^{M} |\omega_{-d,\lambda/N}(j)|^{\alpha} + (1-\epsilon)^{\alpha} \sum_{j=M+1}^{\infty} |\tilde{\omega}_{d,\lambda/N}(j)|^{\alpha} \right]$$

$$= (1-\epsilon)^{\alpha} \lim_{N\to\infty} N^{\alpha-1-\alpha d} \sum_{j=0}^{\infty} |\omega_{-d,\lambda/N}(j)|^{\alpha}$$

$$= \frac{(1-\epsilon)^{\alpha}}{\Gamma(d)^{\alpha}} \int_{-\infty}^{t} \left|(t-x)^{d-1} \mathrm{e}^{-\lambda(t-x)}\right|^{\alpha} \mathrm{d}x$$

by lemma 4. Since $\epsilon > 0$ is arbitrary, equation (A.38) follows. □

Proof of theorem 4.2. Compute the characteristic function of $N^{1-\frac{1}{\alpha}-d}\Delta^{-d,\frac{\lambda}{N}} Z_{\alpha}(t)$ and take the limit as $N \to \infty$ using lemma 4 to see that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\exp\left\{\mathrm{i}\theta N^{\alpha-1-\alpha d}\Delta^{-d,\frac{\lambda}{N}} Z_\alpha(t)\right\}\right] \\
&\quad = \exp\left\{-\lim_{N\to\infty} N^{\alpha-1-\alpha d}\theta^\alpha \sum_{j=0}^{\infty}\left|\omega_{-d,\frac{\lambda}{N}}(j)\right|^\alpha\right\} \\
&\quad = \exp\left\{-\frac{\theta^\alpha}{\Gamma(d)^\alpha}\int_{-\infty}^{t}\left|(t-x)^{d-1}\,\mathrm{e}^{-\lambda(t-x)}\right|^\alpha \mathrm{d}x\right\} \\
&\quad = \mathbb{E}\left[\exp\{\mathrm{i}\theta Y_{d,\alpha,\lambda}(t)\}\right].
\end{aligned}$$

Since convergence of characteristic functions implies convergence in distribution, this completes the proof. □

Proof of proposition equation (43). The scaling property equation (43) follows from

$$\begin{aligned}
&\left(Y_{d,\alpha,\lambda}(ct_i) = \frac{1}{\Gamma(d)}\int_{-\infty}^{ct_i}(ct_i - s)^{d-1}\,\mathrm{e}^{-\lambda(ct_i - s)}L_\alpha(\mathrm{d}s);\quad i = 1,\ldots,n\right) \\
&\quad = \left(\frac{c^{d-1}}{\Gamma(d)}\int_{-\infty}^{y}(t_i - y)^{d-1}\,\mathrm{e}^{-\lambda c(t_i - y)}\,L_\alpha(c\mathrm{d}y);\quad i = 1,\ldots,n\right) \\
&\quad \overset{\mathrm{fdd}}{=} \left(\frac{c^{d-1+1/\alpha}}{\Gamma(d)}\int_{-\infty}^{y}(t_i - y)^{d-1}\,\mathrm{e}^{-\lambda c(t_i - y)}\,L_\alpha(\mathrm{d}y);\quad i = 1,\ldots,n\right) \\
&\quad = \left(c^{d-1+1/\alpha}Y_{d,\alpha,c\lambda}(t_i);\quad i = 1,\ldots,n\right),
\end{aligned}$$

where we used the fact $\{L_\alpha(ct)\} \overset{\mathrm{fdd}}{=} \{c^{\frac{1}{\alpha}}L_\alpha(t)\}$ to get the main result.

□

Proof of theorem 4.4. The proof follows as a special case of [32, theorem 4.3(c)] and hence we omit the proof. □

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