Journal of Physics A: Mathematical and Theoretical

PAPER • **OPEN ACCESS**

Limit properties of Lévy walks

To cite this article: Marcin Magdziarz and Tomasz Zorawik 2020 *J. Phys. A: Math. Theor.* **53** 504001

OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **53** (2020) 504001 (22pp) https://doi.org/10.1088/1751-8121/abc43c

Limit properties of Lévy walks

Marcin Magdziarz* **and Tomasz Zorawik**

Hugo Steinhaus Center, Faculty of Pure and Applied Mathematics Wroclaw University of Science and Technology, Wyspianskiego 27, 50-370 Wroclaw, Poland

E-mail: Marcin.Magdziarz@pwr.wroc.pl

Received 12 August 2020, revised 19 October 2020
Accepted for publication 23 October 2020
Published 19 November 2020

Abstract

In this paper we study properties of the diffusion limits of three different models of Lévy walks (LW). Exact asymptotic behavior of their trajectories is found using LePage series representation. We also prove an existing conjecture about total variation of LW sample paths. Based on this conjecture we verify martingale properties of the limit processes for LW. We also calculate their probability density functions and apply this result to determine the potential density of the associated non-symmetric α-stable processes. The obtained theoretical results for continuous LW can be used to recognize and verify this type of processes from anomalous diffusion experimental data. In particular they can be used to estimate parameters from experimental data using maximum likelihood methods.

Keywords: Lévy walk, asymptotic properties, martingale, potential measure, variation

(Some figures may appear in colour only in the online journal)

1. Introduction

Lévy walks (LW) were introduced in pioneering papers written by Shlesinger *et al* [29] in 1982 and Klafter *et al* [14] in 1987. Since then these stochastic processes have quickly become useful models in physics and biology describing superdiffusion [11, 23]. Among the most spectacular applications of LW one should mention: modeling of migration of swarming bacteria [1], light transport in a special optical material known as Lévy glass [2], foraging patterns of animals [5] or blinking nanocrystals [22]. There is also a recent article in Nature where authors show that LW can be used to describe transport of certain proteins (neuronal mRNP)—see [31]. For more applications we refer the curious reader to the review paper [34].

*Author to whom any correspondence should be addressed.

LW can be defined in the framework of continuous time random walks (CTRW) [6, 13, 30]. This approach is used in our manuscript. It should be added that LWs can also be defined using subordinated Langevin equations, see [7, 17] for the details. Let T_i be a sequence of independent, identically distributed (IID) waiting times such that $\mathbf{P}(T_i > x) \approx Cx^{-\alpha}$ when $x \to \infty$, $\alpha \in (0, 1)$, $C > 0$. Here by $f(x) \approx g(x)$ we mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. The jumps are defined as $J_i = V_i T_i$ where V_i govern the direction of the jump. It is assumed that V_i are IID with $\mathbf{P}(V_i = 1) = p$ and $\mathbf{P}(V_i = -1) = q$, $q = 1 - p$, $p \in [0, 1]$. Notice that waiting times T_i and jumps J_i are dependent. This coupling allows the CTRW process to have heavy-tailed jumps—a phenomenon which is often observed in experimental data—but still the process can have a finite mean square displacement (variance) which makes it a suitable physical model [8].

Note that the parameter α is restricted here to the case $\alpha \in (0, 1)$. This is due to the fact that for $\alpha > 1$ the diffusion limit of LW is the well-known and studied α-stable Lévy motion (Lévy flight) [16]. This is in sharp contrast with the case $\alpha \in (0, 1)$, for which the diffusion limit is a completely different, much more complicated process [16, 17].

The renewal process is given by $N(t) = \max\left\{n \geqslant 0 : \sum_{i=1}^{n} T_i \leqslant t\right\}$. In mathematical and physical literature there appear three different types of LW [8, 16]. Wait-first Lévy walk (also known as undershooting Lévy walk) is defined as:

$$R(t) = \sum_{i=1}^{N(t)} J_i. \tag{1}$$

The second model—jump-first Lévy walk (overshooting Lévy walk) differs from wait-first LW by changing the order of waiting and jumping:

$$\tilde{R}(t) = \sum_{i=1}^{N(t)+1} J_i. \tag{2}$$

Jump-first LW $\tilde{R}(t)$ seems to be similar to the wait-first model $R(t)$, but has completely different properties—for instance its second moment diverges. This makes it less suitable for physical applications but still is very interesting in the context of limit diffusion processes. Finally Lévy walk (or continuous LW) is given by

$$L(t) = R(t) + \left(t - \sum_{i=1}^{N(t)} T_i\right) V_{N(t)+1}. \tag{3}$$

$L(t)$ is the most useful process from a physical point of view—it has a finite second moment and continuous trajectories. Figure 1 shows sample trajectories of these three different LW.

We are going to analyze diffusion limit processes of LW. Waiting times T_i belong to the domain of attraction of α-stable processes [24]:

$$n^{-1/\alpha} \sum_{i=1}^{[nt]} T_i \xrightarrow[n\to\infty]{d} U_\alpha(t), \tag{4}$$

where $\xrightarrow[n\to\infty]{d}$ denotes convergence of all finite dimensional distributions and $(U_\alpha(t), t \geqslant 0)$ is an α-stable subordinator (strictly increasing Lévy process) with Laplace transform $\mathbf{E}\exp(-sU_\alpha(t)) = \exp(-C\Gamma(1-\alpha)ts^\alpha)$. For simplicity we set the scaling constant to

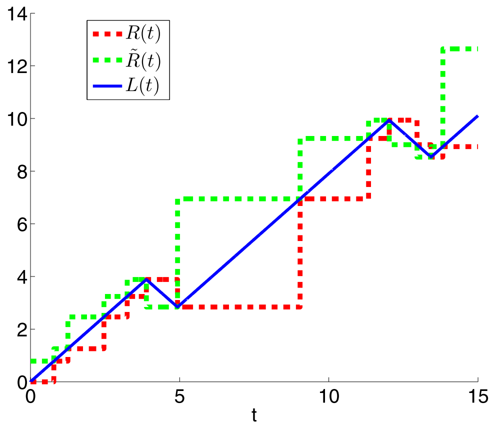

Figure 1. Trajectories of three different LW, $\alpha = 0.75$, $p = 0.5$.

$C = \frac{1}{\Gamma(1-\alpha)}$. For jumps J_i we have [24]

$$n^{-1/\alpha}\sum_{i=1}^{[nt]} J_i \underset{n\to\infty}{\overset{d}{\to}} M_\alpha(t), \tag{5}$$

where $(M_\alpha(t), t \geqslant 0)$ is an α-stable process with Fourier transform $\mathbf{E}\,\exp(-ikM_\alpha(t)) = \exp(-t\,(p(ik)^\alpha + q(-ik)^\alpha))$. We also have joint convergence for T_i and J_i [16, 24]:

$$n^{-1/\alpha}\sum_{i=1}^{[nt]} (J_i, T_i) \underset{n\to\infty}{\overset{d}{\to}} (M_\alpha(t), U_\alpha(t)), \tag{6}$$

where $((M_\alpha(t), U_\alpha(t)), t \geqslant 0)$ is the Lévy process with Lévy triplet $(0, 0, \mu_{(M_\alpha,U_\alpha)})$ and its Lévy measure $\mu_{(M_\alpha,U_\alpha)}$ equals

$$\mu_{(M_\alpha,U_\alpha)}(\mathrm{d}x_1, \mathrm{d}x_2) = p\delta_{x_2}(\mathrm{d}x_1)\nu_{U_\alpha}(\mathrm{d}x_2) + q\delta_{-x_2}(\mathrm{d}x_1)\nu_{U_\alpha}(\mathrm{d}x_2), \tag{7}$$

where $\nu_{U_\alpha}(\mathrm{d}s) = \frac{\alpha}{\Gamma(1-\alpha)}s^{-1-\alpha}\mathrm{d}s$ is the Lévy measure of $U_\alpha(t)$ and $\delta_x(\mathrm{d}y)$ is the Dirac delta measure. It follows that $M_\alpha(t)$ and $U_\alpha(t)$ have jumps of equal magnitude $|\Delta M_\alpha(t)| = \Delta U_\alpha(t)$ almost surely (a.s.) for all $t > 0$. Let $S_\alpha(t)$ be the α-stable inverse subordinator that is the first passage time process of the α-stable subordinator $U_\alpha(t)$: $S_\alpha(t) = \inf\{\tau : U_\alpha(\tau) > t\}$. Now we can define limit processes of LW. For jump models of LW we have [17, 18]

$$\frac{1}{n}R(nt) \underset{n\to\infty}{\overset{J_1}{\to}} (M_\alpha^-(S_\alpha(t)))^+ \overset{\text{def}}{:=} X(t) \tag{8}$$

and

$$\frac{1}{n}\tilde{R}(nt) \underset{n\to\infty}{\overset{J_1}{\to}} M_\alpha(S_\alpha(t)) \overset{\text{def}}{:=} Y(t). \tag{9}$$

For continuous Lévy walk we have [16]

$$\frac{1}{n}L(nt) \underset{n\to\infty}{\overset{J_1}{\to}} Z(t), \tag{10}$$

where

$$Z(t) = \begin{cases} X(t) & \text{when } t \in \mathcal{R}(U_\alpha) \\ X(t) + \dfrac{t - G(t)}{H(t) - G(t)}(Y(t) - X(t)) & \text{when } t \notin \mathcal{R}(U_\alpha) \end{cases}. \tag{11}$$

In the above equation $\mathcal{R}(U_\alpha) = \{U_\alpha(t): \ t \geqslant 0\} \subset [0, \infty)$ is the set of values of $U_\alpha(t)$, $G(t) = (U_\alpha^-(S_\alpha(t)))^+$ is the moment of last jump before t and $H(t) = (U_\alpha(S_\alpha(t)))^+$ is the moment of first jump after t. The symbol $\underset{n\to\infty}{\overset{J_1}{\to}}$ denotes weak convergence in J_1 Skorokhod topology [3, 10].

In this article we determine the exact asymptotic behavior of trajectories of the limit processes and their total variation (proving an already existing conjecture [16]) using LePage series representation for stable process $(M_\alpha(t), U_\alpha(t))$. Using the proven conjecture we also verify martingale properties of the limit processes. Moreover in section 3 we determine the distributions of all limit processes $X(t)$, $Y(t)$ and $Z(t)$ for all values of $p \in (0, 1)$ and $\alpha \in (0, 1)$. Although the limit processes are composition of α-stable processes their probability density functions (PDFs) are given by elementary functions. In the last section we take advantage of the calculated densities and obtain formulas for the potential densities of the associated non-symmetric α-stable processes.

The obtained theoretical results for continuous LW can be used to recognize and verify this type of processes from anomalous diffusion experimental data. In particular, the explicit density functions can be used to estimate parameters from experimental data using maximum likelihood methods.

2. Properties of the limit processes

2.1. Asymptotic properties of trajectories

We are going to analyze the limit processes $X(t)$, $Y(t)$ and $Z(t)$ for different values of the parameter $p \in [0, 1]$ which governs the probability of jumping up and down. To underline the dependence on p sometimes we are going to write it in the subscript: $X_p(t)$, $Y_p(t)$ and $Z_p(t)$. Let us begin with the non symmetric case $p = 1$. The lower bound asymptotic behavior for the process $X_1(t)$ is known:

Theorem 1 (Theorem 3.4 in [4]). *Let $f: (0, \infty) \to (0, \infty)$ be a continuous strictly increasing function with $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ and $\liminf_{x\to\infty} \frac{f(x)}{f(2x)} > 0$. Then for wait-first Lévy walk $X_1(t)$ we have almost surely*

$$\liminf_{t\to\infty} \frac{X_1(t)}{f(t)} = \begin{cases} 0 & \textit{if} \displaystyle\int_1^\infty f^\alpha(x) x^{-1-\alpha} \mathrm{d}x = \infty \\ \infty & \textit{if} \displaystyle\int_1^\infty f^\alpha(x) x^{-1-\alpha} \mathrm{d}x < \infty \end{cases}. \tag{12}$$

In particular it follows that

$$\liminf_{t\to\infty} \frac{X_1(t)}{t} \log^{1/\alpha}(t) = 0 \tag{13}$$

and

$$\liminf_{t\to\infty} \frac{X_1(t)}{t} \log^{1/\alpha+\varepsilon}(t) = \infty \tag{14}$$

almost surely for $\varepsilon > 0$. The following easy to prove fact concerns the upper bound asymptotics.

Proposition 1. *For the process $X_1(t)$ we have*

$$\limsup_{t\to\infty} \frac{X_1(t)}{t} = 1 \quad a.s. \tag{15}$$

Proof. The process $X_1(t)$ is bounded, that is $X_1(t) \leqslant t$ almost surely. When $p = 1$ we also have $X_1(t) = G(t) = (U_\alpha^-(S_\alpha(t)))^+$. Moreover there exists a sequence of jumping times t_n of $U_\alpha(t)$ such that $t_n \to \infty$. Now since $X_1(t_n) = G(t_n) = t_n$ the proof is completed. □

Notice that the case $p = 1$ is trivial for continuous Lévy walk since $Z_1(t) = t$. Due to the symmetry analogous results can be obtained immediately for $p = 0$. Let us move to the more complicated case $p \in (0, 1)$.

Theorem 2. *Wait-first Lévy walk $X_p(t)$ with $p \in (0, 1)$ has the following asymptotic behavior*

$$\limsup_{t\to\infty} \frac{X_p(t)}{t} = 1 \quad and \quad \liminf_{t\to\infty} \frac{X_p(t)}{t} = -1 \quad a.s \tag{16}$$

Proof. Fourier transform of the law of $M_\alpha(t)$ can be written as

$$\mathbf{E}\ \exp(-ikM_\alpha(t))\ \exp\left(-t\cos\left(\frac{\pi\alpha}{2}\right)|k|^\alpha\left(1 + i\beta\ \operatorname{sign}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right), \tag{17}$$

where $\beta = 2p - 1$. Therefor the limit process $M_\alpha(t)$ admits the following LePage series representation (see [15] and example 3.10.3 in [27])

$$(M_\alpha(t),\ 0 \leqslant t \leqslant 1) \stackrel{d}{=} \left(D_\alpha \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} \mathbf{1}(W_i \leqslant t),\ 0 \leqslant t \leqslant 1\right), \tag{18}$$

where $D_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)}$, $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $(\gamma_i)_{i\in\mathbb{N}}$ is a sequence of IID random variables satisfying

$$\mathbf{P}(\gamma_i = 1) = 1 - \mathbf{P}(\gamma_i = -1) = p, \tag{19}$$

$(\Gamma_i)_{i\in\mathbb{N}}$ is a sequence of arrival times of a Poisson process with unit arrival rate and $(W_i)_{i\in\mathbb{N}}$ is a sequence of IID random variables uniformly distributed on $[0, 1]$. These sequences are assumed to be independent. This representation holds for $t \in [0, 1]$, but since Lévy processes have independent and stationary increments we can extend it for all $t \in [0, \infty)$

$$(M_\alpha(t),\ 0 \leqslant t < \infty) \stackrel{d}{=} \left(D_\alpha \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \gamma_{i,j} \Gamma_{i,j}^{-1/\alpha} \mathbf{1}(W_{i,j} \leqslant t),\ 0 \leqslant t < \infty\right), \tag{20}$$

where $[\gamma_{i,j}]$ where $\{\gamma_{i,j}\}$ is a matrix of IID variables such that

$$\mathbf{P}(\gamma_{i,j} = 1) = 1 - \mathbf{P}(\gamma_{i,j} = -1) = p, \tag{21}$$

$(\Gamma_{1,j}, \Gamma_{2,j}, \ldots)$ is a sequence of arrival times of a Poisson process $N_j(t)$ with unit arrival rate for each $j \in \mathbb{N}$ (the processes $N_j(t)$ are independent) and $[W_{i,j}]$ is a matrix of independent random variables such that $W_{i,j}$ is uniformly distributed on $[j-1, j]$ for each $i \in \mathbb{N}$. We assume that the matrices and sequences are independent. Intuitively $D_\alpha \Gamma_{i,j}^{-1/\alpha}$ describe magnitude of the jumps, $\gamma_{i,j}$ govern their direction and $W_{i,j}$ correspond to moments of their occurrence. We can apply this representation also to the limit Lévy process $(M_\alpha(t), S_\alpha(t))$ obtaining

$$((M_\alpha(t), U_\alpha(t)),\ 0 \leqslant t < \infty)$$
$$\stackrel{d}{=} \left(D_\alpha \left(\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \gamma_{i,j} \Gamma_{i,j}^{-1/\alpha} \mathbf{1}(W_{i,j} \leqslant t), \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \Gamma_{i,j}^{-1/\alpha} \mathbf{1}(W_{i,j} \leqslant t) \right),\ 0 \leqslant t < \infty \right). \tag{22}$$

This is a consequence of the fact that $|\Delta M_\alpha(t)| = \Delta U_\alpha(t)$ for all t almost surely. Now if we remove all the direction components $\gamma_{i,j}$ from the series representation (we change the direction of all jumps to the upward one), we obtain the series representation of the process $(U_\alpha(t), U_\alpha(t))$. theorem 1 assures us that

$$\liminf_{t\to\infty} \frac{U_\alpha^-(S_\alpha(t))}{t} = 0 \quad \text{a.s.} \tag{23}$$

This in turn means that infinitely often (i.o.) as $t \to \infty$ we can observe 'large' jumps of the process $U_\alpha(t)$ compared to the value of this process itself at time point t. More precisely let $\varepsilon > 0$. Then

$$\Delta U_\alpha(t) > (1-\varepsilon) U_\alpha(t) \quad \text{i.o.} \quad \text{as} \quad t \to \infty \ \text{ almost surely.} \tag{24}$$

We are going to show that for $p \in (0, 1)$ we also observe both 'large' positive and 'large' negative jumps of the process $M_\alpha(t)$ compared to the value of the process $U_\alpha(t)$ i.o. as $t \to \infty$. To achieve this goal we number the elements of the matrix $[i, j]_{i,j\in\mathbb{N}}$ - let $P_n = (i_n, j_n)$. We choose the sequence P_n such that it satisfies the condition $j_n \to \infty$. This can be done for instance by numbering the elements of the matrix using diagonals.

Now we use the series representation and define two sequences of events—$(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$. Let $A_n = \{D_\alpha \Gamma_{i_n,j_n}^{-1/\alpha} > (1-\varepsilon) U_\alpha(W_{i_n,j_n})\}$ so A_n describes the event that at time W_{i_n,j_n} there was a 'large' jump. Since $W_{i_n,j_n} \in [j_n - 1, j_n]$, $j_n \to \infty$ from equation (24) we get that $\mathbf{P}(\limsup_{n\to\infty} A_n) = 1$ almost surely. Next let $B_n = \{\gamma_{(i_n,j_n)} = 1\}$. From Borel–Cantelli lemma we get that $\mathbf{P}(\limsup_{n\to\infty} B_n) = 1$ because $p \in (0, 1)$. Since $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are independent it follows that $\mathbf{P}(\limsup_{n\to\infty}(A_n \cap B_n)) = 1$.

This means that $\Delta M_\alpha(t) > (1-\varepsilon) U_\alpha(t)$ i.o. as $t \to \infty$ and consequently $\Delta X_p(t) > (1-\varepsilon)t$ i.o. as t goes to infinity. Observe that the bound $|X_p(t)| \leqslant t$ yields

$$\frac{X_p(t)}{t} = \frac{X_p(t - \Delta X_p(t)) + \Delta X_p(t)}{t} \geqslant \frac{-(t - \Delta X_p(t)) + \Delta X_p(t)}{t}$$
$$= \frac{2\Delta X_p(t) - t}{t}. \tag{25}$$

Therefore almost surely

$$\frac{X_p(t)}{t} \geqslant 2(1-\varepsilon) - 1 = 1 - 2\varepsilon \tag{26}$$

i.o. as $t \to \infty$. The choice of $\varepsilon > 0$ was arbitrary thus we conclude that

$$\limsup_{t\to\infty} \frac{X_p(t)}{t} = 1 \quad \text{a.s.} \tag{27}$$

The proof of the second part of theorem ($\liminf_{t\to\infty} \frac{X_p(t)}{t} = -1$ almost surely) is analogous.□

The next corollaries concern continuous Lévy walk $Z_p(t)$.

Corollary 1. *Let $p \in (0, 1)$. Continuous Lévy walk $Z_p(t)$ has the following asymptotic behavior*

$$\limsup_{t\to\infty} \frac{Z_p(t)}{t} = 1 \quad \textit{and} \quad \liminf_{t\to\infty} \frac{Z_p(t)}{t} = -1 \quad \textit{a.s.} \tag{28}$$

Proof. The values of $X_p(t)$ and $Z_p(t)$ coincide for $t \in [0, \infty] : \Delta X_p(t) \neq 0$. Notice that equation (26) holds also i.o. as $t \to \infty$ for t such that $X_p(t) \neq 0$. Thus we deduce that

$$\frac{Z_p(t)}{t} \geqslant -\varepsilon + (1 - \varepsilon) = 1 - 2\varepsilon \tag{29}$$

i.o. as $t \to \infty$ almost surely for arbitrary $\varepsilon > 0$. The proof of the lower bound asymptotics is analogous. □

Corollary 2. *Let $p \in (0, 1)$. Then $Z_p(t) = 0$ i.o. as $t \to \infty$ almost surely.*

Proof. This is an immediate consequence of the previous corollary combined with the fact that $Z_p(t)$ has continuous trajectories. □

It remains an open problem to find the asymptotic behavior of trajectories of jump-first Lévy walk $Y_p(t)$.

2.2. Total variation

In this section we calculate the total variation for the process $X(t)$, $Y(t)$ and $Z(t)$ on a time interval $[0, T]$, $T > 0$ using LePage series representation. The result for the limit process of continuous Lévy walk proves the hypothesis from [16]. In the next section we also use these findings to verify martingale properties of the limit processes. For $a < b$ let $v_a^b(W(t))$ denote the total variation of the process $W(t)$ on the time interval $[a, b]$ that is

$$v_a^b(W(t)) = \sup_P \sum_{j=1}^{n} |W(s_j) - W(s_{j-1})|, \tag{30}$$

where supremum is taken over all partitions $P = \{a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b\}$ of the interval $[a, b]$. Set $p \in [0, 1]$.

Proposition 2. *The total variation of the process $X_p(t)$ on a time interval $[0, T]$ equals*

$$v_0^T(X_p(t)) = G(T), \tag{31}$$

where $G(T) = (U_\alpha^-(S_\alpha(T)))^+$.

Proof. The process $X_p(t)$ is a pure-jump process so

$$v_0^T(X_p(t)) = \sup_P \sum_{j=1}^{n} |X_p(s_j) - X_p(s_{j-1})| = \sum_{t\leqslant T} |\Delta X_p(t)|. \tag{32}$$

Now we take advantage of the series representation given by equation (22) and use the fact that changing direction of all jumps of $X_p(t)$ to positive we obtain the jumps of the process $X_1(t)$

$$\sum_{t\leqslant T} |\Delta X_p(t)| = \sum_{t\leqslant T} \Delta X_1(t) = X_1(T) = G(T). \tag{33}$$

□

The proof of the following result for the process $Y_p(t)$ is analogous.

Proposition 3. *The total variation of the process $Y_p(t)$ on a time interval $[0, T]$ equals*

$$v_0^T(Y_p(t)) = H(T), \tag{34}$$

where $H(T) = (U_\alpha(S_\alpha(T)))^+$.

We can now move to the continuous case—the theorem below appears as hypothesis in [16].

Theorem 3. *The total variation of the process $Z_p(t)$ on a time interval $[0, T]$ equals*

$$v_0^T(Z_p(t)) = T. \tag{35}$$

Proof. Recall that

$$Z_p(t) = \begin{cases} X_p(t) & \text{when } t \in \mathcal{R}(U_\alpha) \\ X_p(t) + \dfrac{t - G(t)}{H(t) - G(t)}(Y_p(t) - X_p(t)) & \text{when } t \notin \mathcal{R}(U_\alpha) \end{cases}, \tag{36}$$

where $\mathcal{R}(U_\alpha)$ is a (countable) set of values of the process $U_\alpha(t)$, $G(t) = (U_\alpha^-(S_\alpha(t)))^+$ and $H(t) = (U_\alpha(S_\alpha(t)))^+$. The total variation is additive

$$v_0^T(Z_p(t)) = v_0^{G(T)}(Z_p(t)) + v_{G(T)}^T(Z_p(t)). \tag{37}$$

The trajectory of $Z_p(t)$ is continuous Moreover during calculation of $v_0^T(Z_p(t)) = \sup_P \sum_{j=1}^n |Z_p(s_j) - Z_p(s_{j-1})|$ instead of taking supremum over all partitions $P = \{0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = T\}$ of the interval $[0, G(T)]$ we can without changing the final result restrict ourselves to the partitions P' which contain points only from the set $\mathcal{R}(U_\alpha)$. This is because if we take a partition P_0 of the interval $[0, G(T)]$ which contains a point $t_0 \notin \mathcal{R}(U_\alpha)$ we can construct a corresponding partition P_0' in the following way: we remove from P_0 the point t_0 and add points (if they are not already there) $G(t_0)$ and $H(t_0)$ which are in the set $\mathcal{R}(U_\alpha) \cap [0, G(T)]$. Then since the trajectory on the interval $[G(t_0), H(t_0)]$ is a straight line the variation calculated based on P_0' is not smaller than the variation calculated based on P_0. The values of the processes $Z_p(t)$ and $X_p(t)$ coincide for $t \in \mathcal{R}(U_\alpha)$. Also notice that the values of the process $X_p(t)$ are constant on the interval $[G(t_0), H(t_0))$, so the total variation for the process $X_p(t)$ can also be calculated based only on partitions which contain only points from the set $\mathcal{R}(U_\alpha)$. Therefore from proposition 2 we get

$$v_0^{G(T)}(Z_p(t)) = v_0^{G(T)}(X_p(t)) = v_0^T(X_p(t)) = G(T). \tag{38}$$

On the interval $[G(T), T]$ the trajectory of $Z_p(t)$ is a straight line so

$$v_{G(T)}^T(Z_p(t)) = T - G(T). \tag{39}$$

Therefore $v_0^T(Z_p(t)) = v_0^{G(T)}(Z_p(t)) + v_{G(T)}^T(Z_p(t)) = T$. □

Remark 1. The above proof remains valid also in case of a multidimensional Lévy walk [16]. The only difference is in the series representation given by equation (22). Instead of random variables $\gamma_{i,j}$ we have unit random vectors.

2.3. Martingale properties

In this section we will verify if the limit processes are martingales with respect to the natural filtration. Recall that the natural filtration for the process $(W(t), t \geqslant 0)$ is defined as $\mathcal{F}_t^W = \sigma(W(s) : s \leqslant t)$ and the process $W(t)$ is a martingale with respect to the natural filtration if

(a) $\mathbf{E}\,|W(t)| < \infty$,

(b) $\mathbf{E}[W(t)|\mathcal{F}_s^W] = W(s)$ a.s. for $0 \leqslant s < t$.

Here $\sigma(W(s) : s \leqslant t)$ is the sigma algebra generated by W up to time t.

Proposition 4. *The limit process $X_p(t)$ for wait-first Lévy walk is a martingale if and only if $p = 1/2$ (symmetric case).*

Proof. First assume that $p \neq 1/2$. Then $\mathbf{E}X_p(t) = t\mathbf{E}X_p(1)$ since $X_p(t)$ is self-similar with an index of self-similarity equal to 1 and $\mathbf{E}X_p(1) \neq 0$ which follows from the lack of symmetry (the exact distribution of $X_p(t)$ is given in theorem 4). Thus the expected value is not constant over time and $X_p(t)$ is not a martingale in this case. Next assume $p = 1/2$. We will show that in this case $X_p(t)$ is a martingale. Since $|X_p(t)| \leqslant t$ we have $\mathbf{E}\,|X_p(t)| < \infty$. It remains to prove the second martingale condition. Let $0 \leqslant s < t$, then almost surely

$$\begin{aligned} \mathbf{E}[X_p(t)|\mathcal{F}_s^{X_p}] &= \mathbf{E}[X_p(t) + (X_p(t) - X_p(s))|\mathcal{F}_s^{X_p}] \\ &= X_p(s) + \mathbf{E}[X_p(t) - X_p(s)|\mathcal{F}_s^{X_p}] \\ &= X_p(s) + \mathbf{E}\left[\sum_{w\in(s,t]} \Delta X_p(w) \middle| \mathcal{F}_s^{X_p}\right]. \end{aligned} \tag{40}$$

Once again we use the series representation (equation (22)). Notice that the directions of all jumps of $X_p(t)$ on an interval $(s,t]$ are defined by the random variables $\gamma_{i,j}$ which are independent of the previous history of the process on a time interval $[0,s]$. The length of the next jump after time s may depend on this history but the direction does not. Since for all $i, j \in \mathbb{N}$ we have $\mathbf{E}\gamma_{i,j} = 0$ it follows that

$$\mathbf{E}\left[\sum_{w\in(s,t]} \Delta X_p(w) \middle| \mathcal{F}_s^{X_p}\right] = 0 \tag{41}$$

and therefore $\mathbf{E}[X_p(t)|\mathcal{F}_s^{X_p}] = X_p(t)$ almost surely for $p = 1/2$. □

The situation is different for the jump-first process.

Proposition 5. *The process $Y_p(t)$ is not a martingale for all $p \in [0,1]$.*

Proof. The first martingale condition is not fulfilled—$\mathbf{E}\,|Y_p(t)| = \infty$—this can be calculated directly using the distribution of $Y_p(t)$ which is given in theorem 5. □

Finally we look at the continuous process $Z_p(t)$. It is not a martingale similarly as $Y_p(t)$ but for different reasons (contrary to $Y_p(t)$, the process $Z_p(t)$ is bounded).

Proposition 6. *The process $Z_p(t)$ is not a martingale for all $p \in [0,1]$.*

Proof. The limit process $Z_p(t)$ has continuous trajectories. We also proved that it is of bounded variation ($v_0^T(Z_p(t) = t)$. However from martingale theory we now that the only continuous process of bounded variation is the constant process (theorem 30.4 in [26]). Thus $Z_p(t)$ is not a martingale. □

3. Probability distributions of the limit processes

In this section we are going to find explicit formulas for PDFs of the limit processes $X_p(t)$, $Y_p(t)$ and $Z_p(t)$ for all values of parameter $p \in (0,1)$ and $\alpha \in (0,1)$. It turns out that these densities are given by elementary functions. This extends results from [8] where PDFs were obtained only in the symmetric case $p = q = 1/2$. Also the results from this section simplify findings from [19] where PDFs of the jump processes $X_p(t)$ and $Y_p(t)$ were expressed using integrals of special functions.

Theorem 4. *Let $p \in (0,1)$. The PDF $p(x,t)$ of the process $X_p(t)$ equals*

$$p(x,t) = \frac{1}{t}\phi\left(\frac{x}{t}\right), \tag{42}$$

where

$$\phi(y) = \begin{cases} \dfrac{p\sin(\alpha\pi)}{\pi}\dfrac{(1-y)^{\alpha}y^{-1+\alpha}}{p^2(1-y)^{2\alpha}+q^2(1+y)^{2\alpha}+2pq(1-y^2)^{\alpha}\cos(\alpha\pi)} & \text{when } 1 > y > 0 \\ \dfrac{q\sin(\alpha\pi)}{\pi}\dfrac{(1+y)^{\alpha}(-y)^{-1+\alpha}}{q^2(1-y)^{2\alpha}+p^2(1+y)^{2\alpha}+2pq(1-y^2)^{\alpha}\cos(\alpha\pi)} & \text{when } -1 < y < 0 \\ 0 & \text{when } |y| > 1. \end{cases} \tag{43}$$

Proof. In the proof we will use techniques from [8, 9]. Fourier–Laplace transform of the density $p(x,t)$ of the process $X(t)$ equals [18]

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}kx-st}\, p(x,t)\mathrm{d}t\ \mathrm{d}x = \frac{s^{\alpha-1}}{p(s-ik)^{\alpha}+q(s+ik)^{\alpha}} = \frac{1}{s}g\left(\frac{ik}{s}\right), \tag{44}$$

where

$$g(\xi) = \frac{1}{p(1+\xi)^{\alpha}+q(1-\xi)^{\alpha}}. \tag{45}$$

Our process $X(t)$ is self-similar with an index of self-similarity equal to 1 so the PDF has the form $p(x,t) = \frac{1}{t}\phi\left(\frac{x}{t}\right)$ for a certain scaling function ϕ. In such scenario it is possible to invert Fourier–Laplace transform using Sokhotski–Weierstrass theorem. We have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}kx-st}\frac{1}{t}\phi\left(\frac{x}{t}\right)\mathrm{d}t\ \mathrm{d}x &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-(\mathrm{i}ky+s)t}\phi(y)\,\mathrm{d}t\ \mathrm{d}y \\ &= \int_{-\infty}^{\infty}\frac{1}{iky+s}\phi(y)\,\mathrm{d}t \\ &= \frac{1}{s}\mathbf{E}\frac{1}{\frac{ik}{s}X(1)+1}. \end{aligned} \tag{46}$$

Therefore

$$g(\xi) = \mathbf{E}\frac{1}{\xi X(1)+1}. \tag{47}$$

The density of $\phi(x)$ of the random variable $X(1)$ can be written as

$$\phi(x) = \mathbf{E}\delta(x - X_1(1)). \tag{48}$$

From Sokhotsky–Weierstrass theorem we get [32]

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon}=\text{P.V.}\frac{1}{x}\mp i\pi\delta(x), \tag{49}$$

where P.V. denotes the principal value. Thus

$$\delta(x)=-\,\text{Im}\,\lim_{\varepsilon\to 0}\frac{1}{x+i\varepsilon}. \tag{50}$$

After combining the equations (48) and (50) we obtain

$$\phi(x)=-\,\text{Im}\,\lim_{\varepsilon\to 0}\mathbf{E}\frac{1}{x-X(1)+i\varepsilon}=-\lim_{\varepsilon\to 0}\,\text{Im}\,\frac{1}{x+i\varepsilon}\mathbf{E}\frac{1}{1-\frac{X(1)}{x+i\varepsilon}}. \tag{51}$$

Therefore

$$\begin{aligned}\phi(x)&=-\frac{1}{\pi}\lim_{\varepsilon\to 0}\,\text{Im}\left(\frac{1}{x+i\varepsilon}\mathbf{E}\frac{1}{1-\frac{X_1(1)}{x+i\varepsilon}}\right)\\&=-\frac{1}{\pi}\lim_{\varepsilon\to 0}\,\text{Im}\left(\frac{1}{x+i\varepsilon}g\left(-\frac{1}{x+i\varepsilon}\right)\right).\end{aligned} \tag{52}$$

Now we apply the above formula for the function $g(\xi)$ given by equation (45)

$$\phi(y)=-\frac{1}{\pi}\lim_{\varepsilon\to 0}\,\text{Im}\left(\frac{1}{y+i\varepsilon}\frac{1}{p\Big(1-\frac{1}{y+i\varepsilon}\Big)^{\alpha}+q\Big(1+\frac{1}{y+i\varepsilon}\Big)^{\alpha}}\right). \tag{53}$$

For $y\in(0,1)$ we have

$$\begin{aligned}\phi(y)&=-\frac{1}{\pi}\,\text{Im}\left(\frac{1}{y}\frac{1}{p\Big(\frac{1}{y}-1\Big)^{\alpha}(-1)^{\alpha}+q\Big(1+\frac{1}{y}\Big)^{\alpha}}\right)\\&=-\frac{1}{y\pi}\,\text{Im}\left(\frac{1}{p\Big(\frac{1}{y}-1\Big)^{\alpha}(\cos(\pi\alpha)+i\sin(\pi\alpha))+q\Big(1+\frac{1}{y}\Big)^{\alpha}}\right)\\&=-\frac{1}{y\pi}\,\text{Im}\left(\frac{q\Big(1+\frac{1}{y}\Big)^{\alpha}+p\Big(\frac{1}{y}-1\Big)^{\alpha}(\cos(\pi\alpha)-i\sin(\pi\alpha))}{\Big(q\Big(1+\frac{1}{y}\Big)^{\alpha}+p\Big(\frac{1}{y}-1\Big)^{\alpha}\cos(\pi\alpha)\Big)^{2}+\Big(p\Big(\frac{1}{y}-1\Big)^{\alpha}\sin(\pi\alpha)\Big)^{2}}\right)\\&=\frac{1}{y\pi}\frac{p\Big(\frac{1}{y}-1\Big)^{\alpha}\sin(\pi\alpha)}{p^{2}\Big(\frac{1}{y}-1\Big)^{2\alpha}+q^{2}\Big(1+\frac{1}{y}\Big)^{2\alpha}+2pq\Big(\frac{1}{y^{2}}-1\Big)^{\alpha}\cos(\alpha\pi)}\\&=\frac{p\sin(\alpha\pi)}{\pi}\frac{(1-y)^{\alpha}y^{-1+\alpha}}{p^{2}(1-y)^{2\alpha}+q^{2}(1+y)^{2\alpha}+2pq\big(1-y^{2}\big)^{\alpha}\cos(\alpha\pi)}.\end{aligned} \tag{54}$$

When $y\in(-1,0)$ we perform analogous calculations or alternatively take advantage of the following symmetry $p_p(x,t)=p_q(-x,t)$ where $p_p(x,t)$ and $p_q(x,t)$ denote the PDFs of the processes $X_p(t)$ and $X_q(t)$ respectively . Next when $|y|>1$

$$\mathrm{Im}\left(\frac{1}{y}\frac{1}{p\left(1-\frac{1}{y}\right)^{\alpha}+q\left(1+\frac{1}{y}\right)^{\alpha}}\right)=0 \tag{55}$$

and $\phi(y)=0$. □

The PDF of the jump-first process $Y_p(t)$ is calculated in the theorem below.

Theorem 5. *Let $p\in(0,1)$. The PDF $p(x,t)$ of the process $Y_p(t)$ is given by*

$$p(x,t)=\frac{1}{t}\phi\left(\frac{x}{t}\right), \tag{56}$$

where

$$\phi(y)=\begin{cases}\dfrac{pq\sin(\alpha\pi)}{\pi|y|}\dfrac{|(1+y)^{\alpha}-(1-y)^{\alpha}|}{p^2(1-y)^{2\alpha}+q^2(1+y)^{2\alpha}+2pq\left(1-y^2\right)^{\alpha}\cos(\alpha\pi)} & \text{when } |y|<1\\ \dfrac{p\sin(\pi\alpha)}{\pi y}\dfrac{1}{p(y-1)^{\alpha}+q(y+1)^{\alpha}} & \text{when } y>1\\ \dfrac{q\sin(\pi\alpha)}{-\pi y}\dfrac{1}{q(-y-1)^{\alpha}+p(-y+1)^{\alpha}} & \text{when } y<-1.\end{cases} \tag{57}$$

Proof. We proceed similarly as in the proof of the previous theorem. Fourier–Laplace transform of the density $p(x,t)$ of the process $Y(t)$ equals [18]

$$\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{e}^{-\mathrm{i}kx-st}p(x,t)\,\mathrm{d}t\,\mathrm{d}x=g\left(\frac{ik}{s}\right), \tag{58}$$

where

$$g(\xi)=1-\frac{p\xi^{\alpha}+q(-\xi)^{\alpha}}{p(1+\xi)^{\alpha}+q(1-\xi)^{\alpha}}. \tag{59}$$

Therefore

$$\phi(y)=-\frac{1}{\pi}\lim_{\varepsilon\to0}\,\mathrm{Im}\left(\frac{1}{y+i\varepsilon}\frac{p\left(\frac{-1}{y+i\varepsilon}\right)^{\alpha}+q\left(\frac{1}{y+i\varepsilon}\right)^{\alpha}}{p\left(1-\frac{1}{y+i\varepsilon}\right)^{\alpha}+q\left(1+\frac{1}{y+i\varepsilon}\right)^{\alpha}}\right). \tag{60}$$

For $y\in(0,1)$ this yields

$$\begin{aligned}\phi(y)&=-\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{y}\frac{p\frac{1}{y^{\alpha}}(\cos(\pi\alpha)+i\sin(\pi\alpha))+q\frac{1}{y^{\alpha}}}{p\left(\frac{1}{y}-1\right)^{\alpha}(\cos(\pi\alpha)+i\sin(\pi\alpha))+q\left(1+\frac{1}{y}\right)^{\alpha}}\right)\\ &=-\frac{1}{y\pi}\,\mathrm{Im}\left(\frac{\left(p\frac{1}{y^{\alpha}}(\cos(\pi\alpha)+i\sin(\pi\alpha))+q\frac{1}{y^{\alpha}}\right)\left(q\left(1+\frac{1}{y}\right)^{\alpha}+p\left(\frac{1}{y}-1\right)^{\alpha}(\cos(\pi\alpha)-i\sin(\pi\alpha))\right)}{\left(q\left(1+\frac{1}{y}\right)^{\alpha}+p\left(\frac{1}{y}-1\right)^{\alpha}\cos(\pi\alpha)\right)^2+\left(p\left(\frac{1}{y}-1\right)^{\alpha}\sin(\pi\alpha)\right)^2}\right)\\ &=\frac{-1}{y\pi}\frac{-p\sin(\pi\alpha)\left(\frac{1}{y}-1\right)^{\alpha}(p\cos(\pi\alpha)+q)\frac{1}{y^{\alpha}}+p\frac{1}{y^{\alpha}}\sin(\pi\alpha)\left(q\left(1+\frac{1}{y}\right)^{\alpha}+p\left(\frac{1}{y}-1\right)^{\alpha}\cos(\pi\alpha)\right)}{p^2\left(\frac{1}{y}-1\right)^{2\alpha}+q^2\left(1+\frac{1}{y}\right)^{2\alpha}+2pq\left(\frac{1}{y^2}-1\right)^{\alpha}\cos(\alpha\pi)}\\ &=\frac{pq\sin(\alpha\pi)}{\pi y}\frac{(1+y)^{\alpha}-(1-y)^{\alpha}}{p^2(1-y)^{2\alpha}+q^2(1+y)^{2\alpha}+2pq\left(1-y^2\right)^{\alpha}\cos(\alpha\pi)}.\end{aligned} \tag{61}$$

For $y > 1$ the result is different

$$\phi(y) = -\frac{1}{\pi}\ \mathrm{Im}\left(\frac{1}{y}\frac{p\frac{1}{y^{\alpha}}\left(\cos(\pi\alpha) + i\sin(\pi\alpha)\right) + q\frac{1}{y^{\alpha}}}{p\left(1-\frac{1}{y}\right)^{\alpha} + q\left(1+\frac{1}{y}\right)^{\alpha}}\right)$$
$$= \frac{p\sin(\pi\alpha)}{\pi y}\frac{1}{p(y-1)^{\alpha} + q(y+1)^{\alpha}}. \tag{62}$$

□

Finally we move to the continuous process $Z(t)$.

Theorem 6. *Let $p \in (0, 1)$. The density $p(x, t)$ of the process $Z_p(t)$ equals*

$$p(x,t) = \frac{1}{t}\phi\left(\frac{x}{t}\right), \tag{63}$$

where

$$\phi(y) = \frac{pq\sin(\pi\alpha)}{\pi}$$
$$\times \frac{(1-y)^{\alpha-1}(1+y)^{\alpha} + (1+y)^{\alpha-1}(1-y)^{\alpha}}{p^2(1-y)^{2\alpha} + q^2(1+y)^{2\alpha} + 2pq(1-y)^{\alpha}(1+y)^{\alpha}\cos(\pi\alpha)}$$
$$\times \mathbf{1}_{(-1,1)}(y) \tag{64}$$

Proof. The proof is analogous as in the case of jump processes $X(t)$ and $Y(t)$. Fourier–Laplace transform $p(x, t)$ of $Z(t)$ has the form [16]

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}kx-st}p(x,t)\,\mathrm{d}t\ \mathrm{d}x = g\left(\frac{ik}{s}\right), \tag{65}$$

where

$$g(\xi) = \frac{p(1+\xi)^{\alpha-1} + q(1-\xi)^{\alpha-1}}{p(1+\xi)^{\alpha} + q(1-\xi)^{\alpha}}. \tag{66}$$

This implies that

$$\phi(y) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\ \mathrm{Im}\left(\frac{1}{y+i\varepsilon}\frac{p\left(1-\frac{1}{y+i\varepsilon}\right)^{\alpha-1} + q\left(1+\frac{1}{y+i\varepsilon}\right)^{\alpha-1}}{p\left(1-\frac{1}{y+i\varepsilon}\right)^{\alpha} + q\left(1+\frac{1}{y+i\varepsilon}\right)^{\alpha}}\right). \tag{67}$$

When $y \in (0, 1)$ we obtain

$$\phi(y) = -\frac{1}{\pi}\ \mathrm{Im}\left(\frac{1}{y}\frac{p\left(\frac{1}{y}-1\right)^{\alpha-1}\left(\cos(\pi\alpha-\pi) + i\sin(\pi\alpha-\pi)\right) + q\left(1+\frac{1}{y}\right)^{\alpha-1}}{p\left(\frac{1}{y}-1\right)^{\alpha}\left(\cos(\pi\alpha) + i\sin(\pi\alpha)\right) + q\left(1+\frac{1}{y}\right)^{\alpha}}\right)$$
$$= -\frac{1}{y\pi}\ \mathrm{Im}\left(\frac{p\left(\frac{1}{y}-1\right)^{\alpha-1}\left(-\cos(\pi\alpha) - i\sin(\pi\alpha)\right) + q\left(1+\frac{1}{y}\right)^{\alpha-1}}{\left(q\left(1+\frac{1}{y}\right)^{\alpha} + p\left(\frac{1}{y}-1\right)^{\alpha}\cos(\pi\alpha)\right)^2 + \left(p\left(\frac{1}{y}-1\right)^{\alpha}\sin(\pi\alpha)\right)^2}\right.$$

$$\times\left(q\left(1+\frac{1}{y}\right)^{\alpha}+p\left(\frac{1}{y}-1\right)^{\alpha}(\cos(\pi\alpha)-i\sin(\pi\alpha))\right)\Bigg)$$

$$=\frac{1}{y\pi}\frac{p\sin(\pi\alpha)\left(\frac{1}{y}-1\right)^{\alpha-1}\left(q\left(1+\frac{1}{y}\right)^{\alpha}+p\left(\frac{1}{y}-1\right)^{\alpha}\cos(\pi\alpha)\right)}{p^2\left(\frac{1}{y}-1\right)^{2\alpha}+q^2\left(1+\frac{1}{y}\right)^{2\alpha}+2pq\left(\frac{1}{y^2}-1\right)^{\alpha}\cos(\alpha\pi)}$$

$$+\frac{1}{y\pi}\frac{p\sin(\pi\alpha)\left(\frac{1}{y}-1\right)^{\alpha}\left(-p\cos(\pi\alpha)\left(\frac{1}{y}-1\right)^{\alpha-1}+q\left(1+\frac{1}{y}\right)^{\alpha-1}\right)}{p^2\left(\frac{1}{y}-1\right)^{2\alpha}+q^2\left(1+\frac{1}{y}\right)^{2\alpha}+2pq\left(\frac{1}{y^2}-1\right)^{\alpha}\cos(\alpha\pi)}$$

$$=\frac{pq\sin(\alpha\pi)}{\pi}\frac{(1-y)^{\alpha-1}(1+y)^{\alpha}+(1-y)^{\alpha}(1+y)^{\alpha-1}}{p^2(1-y)^{2\alpha}+q^2(1+y)^{2\alpha}+2pq\left(1-y^2\right)^{\alpha}\cos(\alpha\pi)}. \tag{68}$$

For $y>1$ we get $\phi(y)=0$. □

It is also worth to mention here that for d-dimensional rotationally invariant LW when d is odd, the densities of the limit processes are also given by elementary functions [20, 21].

4. Potentials of non-symmetric α-stable processes

It turns out that the results for densities we laboriously derived in the previous section can be applied to get a formula for potential of the α-stable process $(M_\alpha(t), U_\alpha(t))$ defined in equation (6). Before we proceed further let us recall some basic facts from potential theory. The potential measure $U(\mathrm{d}x)$ of the d-dimensional stochastic process $W(t)$ is defined as

$$U(B)=\int_0^\infty \mathbf{P}(W(t)\in B)\,\mathrm{d}t=\mathbf{E}\int_0^\infty \mathbf{1}_B(W(s))\,\mathrm{d}s,\quad B\in\mathcal{B}\left(\mathbb{R}^d\right).$$

For a symmetric (rotationally invariant) stable process $W(t)$ with Fourier transform

$$\mathbf{E}\ \exp\left(-i\langle k, W(t)\rangle\right)=\exp\left(-t\|k\|^{\alpha}\right)$$

the potential measure $U(\mathrm{d}x)$ is absolutely continuous with respect to Lebesgue measure and its density $u(x)$ is given by Riesz potential [28]

$$u(x)=C_{\alpha,d}\|x\|^{\alpha-d},$$

where $C_{\alpha,d}$ is a constant which depends on α and d. When $W(t)$ is non-symmetric, the situation is more complicated. In a general case Fourier transform of a d-dimensional α-stable process $W(t)$ with $\alpha\in(0,1)$ is given by [28, 33]

$$\mathbf{E}\ \exp\left(-i\langle k, X(t)\rangle\right)=\exp\left(\int_S\lambda(\mathrm{d}\xi)\int_0^\infty\left(\mathrm{e}^{\mathrm{i}\langle -k,r\xi\rangle}-1\right)r^{-1-\alpha}\,\mathrm{d}r\right), \tag{69}$$

where λ is a finite measure on the unit sphere $S=\{x\in\mathbb{R}^d:\|x\|=1\}$. It is the spherical part of the Lévy measure corresponding to $W(t)$. When λ is a uniform measure we recover the symmetric case. In [33] potential theoretic properties of $W(t)$ were studied under the assumption that λ has a density with respect to the surface measure which is bounded and bounded away from zero. In this context estimates for the density of the potential measure were obtained. Here we focus on the process $(M_\alpha(t), U_\alpha(t))$ for which λ is an atomic measure

$$\lambda\left(\mathrm{d}x\right) = pD\delta_{\left(\sqrt{2}/2,\sqrt{2}/2\right)}\left(x\right) + qD\delta_{\left(-\sqrt{2}/2,\sqrt{2}/2\right)}\left(x\right),$$

where the constant D equals $D = C\left(\sqrt{2}\right)^{\alpha}\alpha = \frac{(\sqrt{2})^{\alpha}\alpha}{\Gamma(1-\alpha)}$. This corresponds to the Lévy measure given by equation (7). Interestingly, it turns out that in this setting it is possible to obtain explicit expressions for the potential density $u(x)$ and the results are non-comparable with the symmetric Riesz potential as it was the case in [33]. Subsection 4.2 is devoted to the case where $M_\alpha(t)$ is a d-dimensional rotation invariant α-stable process. We apply similar methods to derive the potential density of the $d+1$-dimensional process $(M_\alpha(t), U_\alpha(t))$ which is associated with d-dimensional Lévy walk (see [16]). In this scenario the spherical part λ of Lévy measure from equation (69) equals

$$\lambda\left(dx\right) = D\nu\left(d\left(x_1,\ldots,x_d\right)\right)\delta_{\sqrt{2}/2}\left(x_{d+1}\right), \tag{70}$$

where $x = (x_1,\ldots,x_d,x_{d+1})$ and ν is a uniform measure on a hypersphere $S = \{y \in \mathbb{R}^d : \|y\| = \sqrt{2}/2\}$. The potential density has a particularly elegant form when $d = 3$ and $\alpha = 1/2$ as corollary 3 shows.

4.1. Two-Dimensional process $(M_\alpha(t), U_\alpha(t))$

Let us begin with the two-dimensional process $(M_\alpha(t), U_\alpha(t))$ and denote its potential density as $u(x,t)$.

Theorem 7. *The potential density density $u(x,t)$ is given by the formula*

$$u\left(x,t\right) = \frac{1}{t^{2-\alpha}}\psi\left(\frac{x}{t}\right),$$

where the scaling function $\psi(y)$ is given by

$$\psi\left(y\right) = \begin{cases} \dfrac{1}{y}\displaystyle\int_y^1 \eta_+\left(w\right)\left(1-\frac{y}{w}\right)^{-1+\alpha}\mathrm{d}w & \text{for } 1 > y > 0 \\ \dfrac{1}{-y}\displaystyle\int_{-y}^1 \eta_-\left(w\right)\left(1+\frac{y}{w}\right)^{-1+\alpha}\mathrm{d}w & \text{for } -1 < y < 0 \\ 0 & \text{otherwise}. \end{cases}$$

and integrands η_+ and η_- are elementary functions

$$\eta_+\left(y\right) = \frac{p\sin\left(\alpha\pi\right)\alpha}{\pi\Gamma\left(\alpha\right)}\frac{q^2\left(3y-1\right)\left(1+y\right)^{2\alpha-1} - p^2(1-y)^{2\alpha} - 2pq(1-y)^{1+\alpha}(1+y)^{\alpha-1}\cos\left(a\pi\right)}{(1-y)^{1-\alpha}y^{1-\alpha}\left(p^2(1-y)^{2\alpha} + q^2(1+y)^{2\alpha} + 2pq\left(1-y^2\right)^{\alpha}\cos\left(\alpha\pi\right)\right)^2}$$

and

$$\eta_-\left(y\right) = \frac{q\sin\left(\alpha\pi\right)\alpha}{\pi\Gamma\left(\alpha\right)}\frac{p^2\left(3y-1\right)\left(1+y\right)^{2\alpha-1} - q^2(1-y)^{2\alpha} - 2pq(1-y)^{1+\alpha}(1+y)^{\alpha-1}\cos\left(a\pi\right)}{(1-y)^{1-\alpha}y^{1-\alpha}\left(q^2(1-y)^{2\alpha} + p^2(1+y)^{2\alpha} + 2pq\left(1-y^2\right)^{\alpha}\cos\left(\alpha\pi\right)\right)^2}.$$

Proof. The main idea behind finding the potential density is its relation with the density $p(x,t)$ of the wait-first limit process $X_p(t)$. More precisely the density $p(x,t)$ can be expressed in terms of the density $u(x,t)$ of the potential measure U in the following way [25]

$$p(x,t) = \int_0^t u(x,t-v)\mu_{(M_\alpha,U_\alpha)}(\mathbb{R}\times[v,\infty))\mathrm{d}v,$$

where $\mu_{(M_\alpha,U_\alpha)}$ is Lévy measure of the process $(M_\alpha(t), U_\alpha(t))$ and is given by equation (7). Thus

$$p(x,t) = \int_0^t u(x,v)\frac{(t-v)^{-\alpha}}{\Gamma(1-\alpha)}\,\mathrm{d}v = {}_0I_t^{1-\alpha}u(x,t),$$

where ${}_0I_t^\beta$ is the left Riemann–Liouville fractional integral of the order β [12]

$${}_0I_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{f(s)}{(t-s)^{1-\beta}}\,\mathrm{d}s.$$

The potential density $u(x,t)$ can be recovered after applying the left Riemann–Liouville fractional derivative to both sides of the above equation

$$u(x,t) = {}_0D_t^{1-\alpha}p(x,t).$$

The left fractional derivative ${}_0D_t^\beta$ of order β, $0<\beta<1$ is defined by the formula

$${}_0D_t^\beta f(t) = \frac{\mathrm{d}}{\mathrm{d}t}{}_0I_t^{1-\beta}f(t).$$

Since $u(x,0)=0$ the Riemann–Liouville derivative ${}_0D_t^{1-\alpha}$ is equivalent to the Caputo's definition of the fractional derivative ${}_0^CD_t^{1-\alpha}$ [12]

$${}_0D_t^{1-\alpha}p(x,t) = {}_0^CD_t^{1-\alpha}p(x,t) = {}_0I_t^\alpha\frac{\partial}{\partial t}p(x,t) = {}_0I_t^\alpha\frac{\partial}{\partial t}\left(\frac{1}{t}\phi\left(\frac{x}{t}\right)\right).$$

After differentiation we get

$$u(x,t) = \begin{cases} \int_x^t \frac{1}{v^2}\eta_+\left(\frac{x}{v}\right)(t-v)^{\alpha-1}\mathrm{d}v & \text{for } t>x>0\\ \int_{-x}^t \frac{1}{v^2}\eta_-\left(\frac{-x}{v}\right)(t-v)^{\alpha-1}\mathrm{d}v & \text{for } -t<x<0\\ 0 & \text{otherwise.}\end{cases} \tag{71}$$

The integrands η_+ and η_-—the results of differentiation of $\frac{1}{t}\phi\left(\frac{x}{t}\right)$—are given by elementary functions

$$\eta_+(y) = \frac{p\sin(\alpha\pi)\,\alpha}{\pi\Gamma(\alpha)}\frac{q^2(3y-1)(1+y)^{2\alpha-1} - p^2(1-y)^{2\alpha} - 2pq(1-y)^{1+\alpha}(1+y)^{\alpha-1}\cos(a\pi)}{(1-y)^{1-\alpha}y^{1-\alpha}\left(p^2(1-y)^{2\alpha} + q^2(1+y)^{2\alpha} + 2pq\left(1-y^2\right)^\alpha\cos(\alpha\pi)\right)^2}$$

and

$$\eta_-(y) = \frac{q\sin(\alpha\pi)\,\alpha}{\pi\Gamma(\alpha)}\frac{p^2(3y-1)(1+y)^{2\alpha-1} - q^2(1-y)^{2\alpha} - 2pq(1-y)^{1+\alpha}(1+y)^{\alpha-1}\cos(a\pi)}{(1-y)^{1-\alpha}y^{1-\alpha}\left(q^2(1-y)^{2\alpha} + p^2(1+y)^{2\alpha} + 2pq\left(1-y^2\right)^\alpha\cos(\alpha\pi)\right)^2}.$$

The only difference between ψ_+ and ψ_- is a swap between p and q parameters in the formula. Finally we substitute $\frac{x}{v}=w$ in equation (71) which gives us the scaling function ψ and ends the proof. □

It turns out that when $\alpha = 1/2$ after the integration the scaling function ψ takes the form

$$\psi(y) = \frac{pq}{\sqrt{2\pi}}\left(p^2(1-y) + q^2(1+y)\right)^{-3/2}\mathbf{1}_{(-1,1)}(y),$$

which was also obtained using different methods in [19]. In the symmetric case $p = q = 1/2$ (and $\alpha = 1/2$) the function ψ is particularly simple $\psi(y) = \frac{1}{\sqrt{\pi}}\mathbf{1}_{(-1,1)}(y)$ and the potential equals

$$u(x,t) = \frac{1}{2\sqrt{\pi}t^{3/2}}\mathbf{1}_{(-t,t)}(x).$$

The scaling function $\psi(y)$ is supported on the interval $(-1,1)$. It turns out that ψ displays an interesting behavior near the endpoints -1 and 1 which depends heavily on the value of α.

Proposition 7. *The scaling function $\psi(y)$ has the following asymptotics*

$$\psi(y) \approx \begin{cases} \dfrac{2^{1-4\alpha}p\alpha\sin(\alpha\pi)}{q^2\sqrt{\pi}\Gamma\left(\alpha+1/2\right)}(1-y)^{-1+2\alpha}, & \textit{when } y \to 1^- \\ \dfrac{2^{1-4\alpha}q\alpha\sin(\alpha\pi)}{p^2\sqrt{\pi}\Gamma\left(\alpha+1/2\right)}(1+y)^{-1+2\alpha}, & \textit{when } y \to -1^+ \end{cases}.$$

Proof. Due to the symmetry $\psi_c(y) = \psi_{1-c}(-y)$ (c in the subscript denotes here the value of the parameter p) it is enough to prove to asymptotic behavior when $y \to 1^-$. From theorem 7 we have for $0 < y < 1$

$$\psi(y) = \frac{1}{y}\int_y^1 \eta_+(w)\left(1-\frac{y}{w}\right)^{-1+\alpha} \mathrm{d}w,$$

where

$$\eta_+(y) = \frac{p\sin(\alpha\pi)\,\alpha}{\pi\Gamma(\alpha)}\frac{q^2(3y-1)(1+y)^{2\alpha-1} - p^2(1-y)^{2\alpha} - 2pq(1-y)^{1+\alpha}(1+y)^{\alpha-1}\cos(a\pi)}{(1-y)^{1-\alpha}y^{1-\alpha}\left(p^2(1-y)^{2\alpha} + q^2(1+y)^{2\alpha} + 2pq\left(1-y^2\right)^{\alpha}\cos(\alpha\pi)\right)^2}.$$

We have the following equivalence

$$\psi(y) \approx \int_y^1 w^{-1+\alpha}\eta_+(w)\left(1-\frac{y}{w}\right)^{-1+\alpha} \mathrm{d}w = \int_y^1 \eta_+(w)(w-y)^{-1+\alpha}\,\mathrm{d}w \quad \text{when } y \to 1^-.$$

Let

$$g(w) = \frac{2^{-2\alpha}p\alpha\sin(\alpha\pi)}{q^2\pi\Gamma(\alpha)}(1-w)^{-1+\alpha}.$$

We can notice that $\eta_+(w) \approx g(w)$ when $w \to 1^-$. Therefore

$$\begin{aligned}\psi(y) &\approx \int_y^1 g(w)(w-y)^{-1+\alpha}\,\mathrm{d}w \\ &= \frac{2^{-2\alpha}p\alpha\sin(\alpha\pi)}{q^2\pi\Gamma(\alpha)}\int_y^1 (1-w)^{-1+\alpha}(w-y)^{-1+\alpha}\,\mathrm{d}w \\ &= \frac{2^{1-4\alpha}p\alpha\sin(\alpha\pi)}{q^2\sqrt{\pi}\Gamma\left(\alpha+1/2\right)}(1-y)^{-1+2\alpha}.\end{aligned}$$

□

The left panel of figure 2 presents a shape of the scaling function $\psi(y)$ for different values of α. Clearly $\alpha = 0.5$ is the borderline case. The behavior of ψ changes drastically when $\alpha > 0.5$ and $\alpha < 0.5$. The right panel shows the plot of the potential $u(x,t)$ as a function of 2

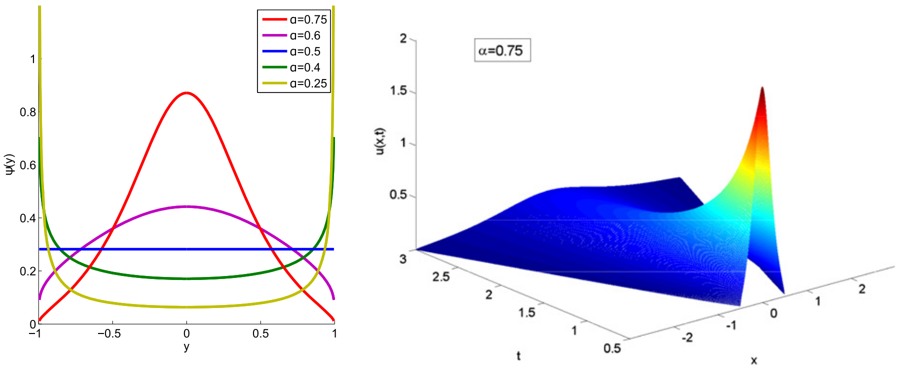

Figure 2. Left panel: the scaling function $\psi(y)$ for the potentials $u(x,t)$ for $p = 1/2$ and different α. Right panel: the potential $u(x,t)$ for $\alpha = 0.75$ plotted for $t \in (0.5, 3)$ and $x \in \mathbb{R}$.

variables. Recall that $u(x,t) = t^{\alpha-2}\psi\left(\frac{x}{t}\right)$. We can notice how the shape of ψ is propagated when t increases.

Remark 2. In [19] the potential density of $(M_\alpha(t), U_\alpha(t)$ was obtained using Meier G-function. Combining the two different formulas yields an interesting integral identity.

4.2. *d*+1-dimensional process $(M_\alpha(t), U_\alpha(t))$

We now proceed to a case where the process $M_\alpha(t)$ is d-dimensional with $d > 1$ and rotationally invariant. We denote the coordinate processes as $M_\alpha(t) = \left(M_{\alpha,1}(t), M_{\alpha,2}(t), \ldots, M_{\alpha,d}(t)\right)$. In this scenario the analyzed process $(M_\alpha(t), U_\alpha(t))$ is $d+1$-dimensional. The main idea for finding the potential measure is the same as in the previous section. The Lévy measure $\mu_{(M_\alpha,U_\alpha)}$ of $(M_\alpha(t), U_\alpha(t))$ in the hyperspherical coordinates equals

$$\mu_{(M_\alpha,U_\alpha)}(\mathrm{d}r, \mathrm{d}\phi, \mathrm{d}y) = \frac{\alpha}{\Gamma(1-\alpha)} y^{-1-\alpha}\mathrm{d}y\,\delta_y(\mathrm{d}r)\,\nu(\mathrm{d}\phi), \tag{72}$$

where (r, ϕ) are hyperspherical coordinates of $x \in R^d$, $\nu(\mathrm{d}\phi)$ is a uniform measure on a hypersphere in $\mathbb{R}^d$ and $y \in \mathbb{R}$. Let $u(x,t)$ be a density function of the potential measure U of our process. Due to the symmetry we have

$$u(x,t) = \frac{\Gamma(d/2)}{2\pi^{d/2}\|x\|^{d-1}} u_R(\|x\|, t),$$

where u_R is a density of a potential measure U_R of the process $(\| M_\alpha(t) \|, U_\alpha(t))$. Thus to determine $u(x,t)$ it suffices to calculate $u_R(r,t)$. Before we state the theorem for $u_R(r,t)$ let us recall definitions of right Riemann–Liouville fractional integral and derivative for $\beta \in \mathbb{R}^+$ which will be useful here. The right fractional integral ${}_tI^\beta$ is defined as

$${}_tI^\beta\{f(t)\}(x) = \frac{1}{\Gamma(\beta)}\int_x^\infty \frac{f(t)}{(t-x)^{1-\beta}}\,\mathrm{d}t$$

and the right fractional derivative ${}_tD^\beta$ as

$${}_tD^\beta\{f(t)\}(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left({}_tI^{n-\beta}\{f(t)\}(x)\right),$$

where $n = \lfloor \beta \rfloor + 1$. To express ϕ_R one also needs the hypergeometric function ${}_2F_1(a, b, c; x)$ with parameters a, b and c. For $|x| < 1$ it is given by the series

$$ {}_2F_1(a, b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, $$

with $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. When $|x| \geqslant 1$ the function is defined as an analytic continuation with respect to x. Let us now move to the theorem which gives the formula for the potential density $u_R(r, t)$.

Theorem 8. *The potential density $u_R(r, t)$ equals*

$$ u_R(r, t) = {}_0^C D_t^{1-\alpha} \left(\frac{1}{t} \phi_R \left(\frac{r}{t} \right) \right), $$

where

$$ \phi_R(r) = \frac{2\sqrt{\pi}}{\Gamma(d/2)} r^{d-1} {}_t D^{\frac{d-1}{2}} \{ \phi_1(\sqrt{t}) \} (r^2) \mathbf{1}_{(0,1)}(r) $$

Here $\phi_1(x)$ is a PDF of the random variable $M_{\alpha,1}(1)$ and is given by the formula involving a hypergeometric function

$$ \phi_1(x) = -\frac{1}{\pi |x|} \ \mathrm{Im} \frac{1}{{}_2F_1\left(-\alpha/2, (1-\alpha)/2; d/2; \frac{1}{x^2}\right)} \mathbf{1}_{(-1,1)}(x). $$

Proof. The limit process of the corresponding d-dimensional undershooting Lévy walk is $Y(t) = M_\alpha^-(S_\alpha(t))$ [16–18]. The PDF of $Y(t)$ due to symmetry takes the form

$$ p(x, t) = \frac{\Gamma(d/2)}{2\pi^{d/2} \|x\|^{d-1}} p_R(\|\mathbf{x}\|, t) $$

with p_R being a PDF of the one-dimensional process $\| Y(t) \|$. Moreover

$$ p_R(\|x\|, t) = \frac{1}{t} \phi_R \left(\frac{\|x\|}{t} \right). $$

The scaling function $\phi_R(r)$ equals [20, 21]

$$ \phi_R(r) = \frac{2\sqrt{\pi}}{\Gamma(d/2)} r^{d-1} {}_t D^{\frac{d-1}{2}} \{ \phi_1(\sqrt{t}) \} (r^2) \mathbf{1}_{(0,1)}(r), \tag{73} $$

where

$$ \phi_1(x) = -\frac{1}{\pi |x|} \ \mathrm{Im} \frac{1}{{}_2F_1\left(-\alpha/2, (1-\alpha)/2; d/2; \frac{1}{x^2}\right)} \mathbf{1}_{(-1,1)}(x). $$

The function ϕ_1 is a PDF of the random variable $M_{\alpha,1}(1)$ Another formula for $p(x, t)$ involves the density u of our potential measure [19, 25], similarly as in the previous section

$$ p(x, t) = \int_0^t u(x, t - v) \mu(\mathbb{R}^d \times [v, \infty)) \mathrm{d}v, $$

where μ is Lévy measure of the process $(M_\alpha(t), U_\alpha(t))$ as in equation (72). After integration we get $\mu(\mathbb{R}^d \times [v, \infty)) = \frac{1}{\Gamma(1-\alpha)} v^{-\alpha}$ and therefore

$$p(x,t) = \int_0^t u(x,v) \frac{(t-v)^{-\alpha}}{\Gamma(1-\alpha)}\,\mathrm{d}v = {}_0I_t^{1-\alpha} u(x,t).$$

Analogously as in the previous section we get

$$u(x,t) = {}_0^C D_t^{1-\alpha} p(x,t).$$

and finally

$$u_R(r,t) = {}_0^C D_t^{1-\alpha} \left[\frac{1}{t} \phi_R \left(\frac{r}{t} \right) \right].$$

□

Interestingly, when d is odd, ϕ_1 is an elementary function [21]. The right fractional derivative ${}_tD^{\frac{d-1}{2}}$ becomes then a classical derivative, that is ${}_tD^{\frac{d-1}{2}} = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac{d-1}{2}}$. Thus in this case the function ϕ_R which appears in theorem 8 is also elementary. The next corollary deals with an interesting special case $d = 3$ and $\alpha = 1/2$.

Corollary 3. *When $d = 3$ and $\alpha = 1/2$ we obtain a particularly elegant expression for potential density of the process $(M_\alpha(t), U_\alpha(t))$:*

$$u(x,t) = \frac{9}{8\pi^{3/2}} \frac{\sqrt{4t^5 - 5t(t^2 - 3\|x\|^2)^2 + (t^2 + 3\|x\|^2)^{5/2}}}{\|x\|(t^2 + 3\|x\|^2)^{5/2}} \mathbf{1}_{(-t,t)}(x).$$

Proof. The scaling function ϕ_1 from theorem 8 now equals

$$\phi_1(x) = \frac{3(1-x)^{3/2}}{2\pi\sqrt{x}(1+3x^2)} \mathbf{1}_{(-1,1)}(x)$$

and thus for ϕ_R we get from equation (73)

$$\phi_R(r) = \frac{3\sqrt{1-r}(1 + r(2 + 3(5-2r)r))}{2\pi\sqrt{r}(1+3r^2)^2} \mathbf{1}_{(0,1)}(r).$$

In order to calculate $u_R(r,t)$ we first take the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{t} \phi_R \left(\frac{r}{t} \right) \right) = \frac{3(-t^5 - 4t^4 r - 46t^3 r^2 + 144t^2 r^3 + 63tr^4 - 108r^5)}{4\pi\sqrt{(t-r)r}(t^2 + 3r^2)^3}.$$

for $t > r > 0$. Therefore

$$\begin{aligned} u_R(r,t) &= \frac{1}{\Gamma(1/2)} \int_r^t \frac{1}{(t-s)^{1/2}} \\ &\quad \times \frac{3(-s^5 - 4s^4 r - 46s^3 r^2 + 144s^2 r^3 + 63tr^4 - 108r^5)}{4\pi\sqrt{(s-r)r}(s^2+3r^2)^3}\,\mathrm{d}s \\ &= \frac{9r\sqrt{4t^5 - 5t(t^2 - 3r^2)^2 + (t^2 + 3r^2)^{5/2}}}{2\sqrt{\pi}(t^2 + 3r^2)^{5/2}} \end{aligned}$$

when $r \in (0,t)$ and $u_R(r,t) = 0$ otherwise. □

Acknowledgments

The research of TZ was partially supported by the National Science Center, Poland Preludium Grant No. 2017/27/N/ST1/01946. The research of MM was partially supported by the NCN HARMONIA 8 Grant No. UMO-2016/22/M/ST1/00233.

ORCID iDs

Marcin Magdziarz https://orcid.org/0000-0002-2239-7155

References

[1] Ariel G, Rabani A, Benisty S, Partridge J D, Harshey R M and Be'er A 2015 Swarming bacteria migrate by Lévy Walk *Nat. Commun.* **6** 8396
[2] Barthelemy P, Bertolotti J and Wiersma D S 2008 A Lévy flight for light *Nature* **453** 495–8
[3] Becker-Kern P, Meerschaert M M and Scheffler H P 2004 Limit theorems for coupled continuous time random walks *Ann. Probab.* **32** 730–56
[4] Bertoin J 2000 Subordinators: examples and applications *Lectures on Probability Theory and Statistics* ed P Bernard (Berlin: Springer)
[5] Buchanan M 2008 Ecological modelling: the mathematical mirror to animal nature *Nature* **453** 714
[6] Cisternas J, Albers T and Radons G 2018 Normal and anomalous random walks of 2-d solitons *Chaos* **28** 075505
[7] Fogedby H C 1994 Langevin equations for continuous time Lévy flights *Phys. Rev.* E **50** 1657
[8] Froemberg D, Schmiedeberg M, Barkai E and Zaburdaev V 2015 Asymptotic densities of ballistic Lévy walks *Phys. Rev.* E **91** 022131
[9] Godrèche C and Luck J M 2001 Statistics of the occupation time of renewal processes *J. Stat. Phys.* **104** 489
[10] Jakubowski A 1986 On the Skorokhod topology *Ann. Inst. Henri Poincare* **22** 263–85
[11] Jara M, Komorowski T and Olla S 2015 Superdiffusion of energy in a chain of harmonic oscillators with noise *Commun. Math. Phys.* **339** 407–53
[12] Kilbas A A, Srivastava A A and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (Amsterdam: North-Holland)
[13] Klafter J and Sokolov I M 2011 *First Steps in Random Walks: From Tools to Applications* (Oxford: Oxford University Press)
[14] Klafter J, Blumen A and Shlesinger M F 1987 Stochastic pathway to anomalous diffusion *Phys. Rev.* A **35** 3081
[15] LePage R 1990 Multidimensional infinitely divisible variables and processes Part I: stable case ed S Cambanis and A Weron *Probability Theory on Vector Spaces IV* (New York: Springer) pp 153–63
[16] Magdziarz M, Scheffler H P, Straka P and Zebrowski P 2015 Limit theorems and governing equations for Lévy walks *Stoch. Process. Their Appl.* **125** 4021–38
[17] Magdziarz M, Szczotka W and Żebrowski P 2012 *J. Stat. Phys.* **147** 74–96
[18] Magdziarz M and Teuerle M 2015 Asymptotic properties and numerical simulation of multidimensional Lévy walks *Commun. Nonlinear Sci. Numer. Simul.* **20** 489–505
[19] Magdziarz M and Zorawik T 2016 Densities of scaling limits of coupled continuous time random walks *Fract. Calc. Appl. Anal.* **19** 1488–506
[20] Magdziarz M and Zorawik T 2016 Explicit densities of multi-dimensional ballistic Lévy walks *Phys. Rev.* E **94** 022130
[21] Magdziarz M and Zorawik T 2017 Method of calculating densities for isotropic ballistic Lévy walks *Commun. Nonlinear Sci. Numer. Simul.* **48** 462–73
[22] Margolin G and Barkai E 2005 Nonergodicity of blinking nanocrystals and other Lévy-walk processes *Phys. Rev. Lett.* **94** 080601
[23] Marklof J and Tóth B 2016 Superdiffusion in the periodic Lorentz gas *Commun. Math. Phys.* **347** 933–81
[24] Meerschaert M M and Sikorskii A 2012 *Stochastic Models for Fractional Calculus, Studies in Mathematics* vol 43 (Berlin: de Gruyter)

[25] Meerschaert M M and Straka P 2014 Semi-Markov approach to continuous time random walk limit processes *Ann. Probab.* **42** 1699–723

[26] Rogers L C G and Williams D 2000 *Diffusions, Markov Processes and Martingales* 2nd edn vol 2 (Cambridge: Cambridge University Press)

[27] Samorodnitsky G and Taqqu M S 1994 *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance* (London: Chapman and Hall)

[28] Sato K-I 1999 *Lévy Processes and Infinitely Divisible Distributions* (Cambridge: Cambridge University Press)

[29] Shlesinger M F, Klafter J and Wong Y M 1982 Random walks with infinite spatial and temporal moments *J. Stat. Phys.* **27** 499–512

[30] Sokolov I M and Klafter J 2005 *Chaos* **15** 026103

[31] Song M S, Moon H C, Jeon J-H and Park H Y 2018 Neuronal messenger ribonucleoprotein transport follows an aging Lévy walk *Nat. Commun.* **9** 344

[32] Vladimirov V S 1971 *Equations of Mathematical Physics* (New York: Dekker)

[33] Vondraček Z 2002 Basic potential theory of certain nonsymmetric strictly α-stable processes *Glas. Mat.* **37** 193–215

[34] Zaburdaev V, Denisov S and Klafter J 2015 Lévy walks *Rev. Mod. Phys.* **87** 483–530