Solutions Manual for

Functions of One Complex Variable I, Second Edition[1]

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PREFACE

Most of the exercises I solved were assigned homeworks in the graduate courses Math 713 and Math 714 ”Complex Analysis I and II” at the University of Wisconsin – Milwaukee taught by Professor Dashan Fan in Fall 2008 and Spring 2009.
The solutions manual is intented for all students taking a graduate level Complex Analysis course. Students can check their answers to homework problems assigned from the excellent book “Functions of One Complex Variable I”, Second Edition by John B. Conway. Furthermore students can prepare for quizzes, tests, exams and final exams by solving additional exercises and check their results. Maybe students even study for preliminary exams for their doctoral studies.
However, I have to warn you not to copy straight of this book and turn in your homework, because this would violate the purpose of homeworks. Of course, that is up to you.
I strongly encourage you to send me solutions that are still missing to **kleefeld@tu-cottbus.de** (LaTeXpreferred but not mandatory) in order to complete this solutions manual. Think about the contribution you will give to other students.
If you find typing errors or mathematical mistakes pop an email to **kleefeld@tu-cottbus.de**. The recent version of this solutions manual can be found at
http://www.math.tu-cottbus.de/INSTITUT/kleefeld/Files/Solution.html.
The goal of this project is to give solutions to all of the 452 exercises.

CONTRIBUTION

I thank (without special order)

Christopher T. Alvin
Martin J. Michael
David Perkins

for contributions to this book.

Contents

Chapter 1

The Complex Number System

1.1 The real numbers

No exercises are assigned in this section.

1.2 The field of complex numbers

Exercise 1. *Find the real and imaginary parts of the following:*

$$\frac{1}{z};\ \frac{z-a}{z+a}(a\in\mathbb{R});\ z^3;\ \frac{3+5i}{7i+1};\ \left(\frac{-1+i\sqrt{3}}{2}\right)^3;$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^6;\ i^n;\ \left(\frac{1+i}{\sqrt{2}}\right)^n \quad \mathit{for} \quad 2\le n\le 8.$$

Solution. *Let $z=x+iy$. Then*
a)

$$\begin{aligned} Re\left(\frac{1}{z}\right) &= \frac{x}{x^2+y^2} = \frac{Re(z)}{|z|^2} \\ Im\left(\frac{1}{z}\right) &= -\frac{y}{x^2+y^2} = -\frac{Im(z)}{|z|^2} \end{aligned}$$

b)

$$\begin{aligned} Re\left(\frac{z-a}{z+a}\right) &= \frac{x^2+y^2-a^2}{x^2+y^2+2ax+a^2} = \frac{|z|^2-a^2}{|z|^2+2aRe(z)+a^2} \\ Im\left(\frac{z-a}{z+a}\right) &= \frac{2ya}{x^2+y^2+2xa+a^2} = \frac{2Im(z)a}{|z|^2+2aRe(z)+a^2} \end{aligned}$$

c)

$$\begin{aligned} Re\left(z^3\right) &= x^3-3xy^2 = Re(z)^3-3Re(z)Im(z)^2 \\ Im\left(z^3\right) &= 3x^2y-y^3 = 3Re(z)^2Im(z)-Im(z)^3 \end{aligned}$$

d)

$$\begin{aligned} Re\left(\frac{3+5i}{7i+1}\right) &= \frac{19}{25} \\ Im\left(\frac{3+5i}{7i+1}\right) &= -\frac{8}{25} \end{aligned}$$

e)

$$\begin{aligned} Re\left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right) &= 1 \\ Im\left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right) &= 0 \end{aligned}$$

f)

$$\begin{aligned} Re\left(\left(\frac{-1-i\sqrt{3}}{2}\right)^6\right) &= 1 \\ Im\left(\left(\frac{-1-i\sqrt{3}}{2}\right)^6\right) &= 0 \end{aligned}$$

g)

$$\begin{aligned} Re\left(i^n\right) &= \begin{cases} 0, & n \text{ is odd} \\ 1, & n \in \{4k : k \in \mathbb{Z}\} \\ -1, & n \in \{2+4k : k \in \mathbb{Z}\} \end{cases} \\ Im\left(i^n\right) &= \begin{cases} 0, & n \text{ is even} \\ 1, & n \in \{1+4k : k \in \mathbb{Z}\} \\ -1, & n \in \{3+4k : k \in \mathbb{Z}\} \end{cases} \end{aligned}$$

h)

$$Re\left(\left(\frac{1+i}{\sqrt{2}}\right)^n\right) = \begin{cases} 0, & n=2 \\ -\frac{\sqrt{2}}{2}, & n=3 \\ -1, & n=4 \\ -\frac{\sqrt{2}}{2}, & n=5 \\ 0, & n=6 \\ \frac{\sqrt{2}}{2}, & n=7 \\ 1, & n=8 \end{cases}$$

$$Im\left(\left(\frac{1+i}{\sqrt{2}}\right)^n\right) = \begin{cases} 1, & n=2 \\ \frac{\sqrt{2}}{2}, & n=3 \\ 0, & n=4 \\ -\frac{\sqrt{2}}{2}, & n=5 \\ -1, & n=6 \\ -\frac{\sqrt{2}}{2}, & n=7 \\ 0, & n=8 \end{cases}$$

Exercise 2. *Find the absolute value and conjugate of each of the following:*

$$-2+i;\, -3;\, (2+i)(4+3i);\, \frac{3-i}{\sqrt{2}+3i};\, \frac{i}{i+3};\, (1+i)^6;\, i^{17}.$$

Solution. *It is easy to calculate:*

a)

$$z=-2+i, \qquad |z| = \sqrt{5}, \qquad \overline{z} = -2-i$$

b)

$$z=-3, \qquad |z| = 3, \qquad \overline{z} = -3$$

c)

$$z=(2+i)(4+3i)=5+10i, \qquad |z| = 5\sqrt{5}, \qquad \overline{z} = 5-10i$$

d

$$z=\frac{3-i}{\sqrt{2}+3i}, \qquad |z| = \frac{1}{11}\sqrt{110}, \qquad \overline{z} = \frac{3+i}{\sqrt{2}-3i}$$

e)

$$z=\frac{i}{i+3}=\frac{1}{10}+\frac{3}{10}i, \qquad |z| = \frac{1}{10}\sqrt{10}, \qquad \overline{z} = \frac{1}{10}-\frac{3}{10}i$$

f)

$$z=(1+i)^6=-8i, \qquad |z| = 8, \qquad \overline{z} = 8i$$

g)

$$i^{17}=i, \qquad |z| = 1, \qquad \overline{z} = -i$$

Exercise 3. *Show that z is a real number if and only if $z = \overline{z}$.*

Solution. *Let $z = x + iy$.*
$\Rightarrow$: *If z is a real number, then $z = x$ ($y = 0$). This implies $\overline{z} = x$ and therefore $z = \overline{z}$.*
$\Leftrightarrow$: *If $z = \overline{z}$, then we must have $x + iy = x - iy$ for all $x, y \in \mathbb{R}$. This implies $y = -y$ which is true if $y = 0$ and therefore $z = x$. This means that z is a real number.*

Exercise 4. *If z and w are complex numbers, prove the following equations:*

$$\begin{aligned}
|z+w|^2 &= |z|^2 + 2Re(z\overline{w}) + |w|^2. \\
|z-w|^2 &= |z|^2 - 2Re(z\overline{w}) + |w|^2. \\
|z+w|^2 &+ |z-w|^2 = 2\left(|z|^2 + |w|^2\right).
\end{aligned}$$

Solution. *We can easily verify that $\overline{\overline{z}} = z$. Thus*

$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{z}w = |z|^2 + |w|^2 + z\overline{w} + \overline{z\overline{\overline{w}}} \\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{z\overline{w}} = |z|^2 + |w|^2 + 2\frac{z\overline{w} + \overline{z\overline{w}}}{2} \\
&= |z|^2 + |w|^2 + 2Re(z\overline{w}) = |z|^2 + 2Re(z\overline{w}) + |w|^2.
\end{aligned}$$

$$\begin{aligned}
|z-w|^2 &= (z-w)\overline{(z-w)} = (z-w)(\overline{z}-\overline{w}) = z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} \\
&= |z|^2 + |w|^2 - z\overline{w} - \overline{z}w = |z|^2 + |w|^2 - z\overline{w} - \overline{z\overline{\overline{w}}} \\
&= |z|^2 + |w|^2 - z\overline{w} - \overline{z\overline{w}} = |z|^2 + |w|^2 - 2\frac{z\overline{w} + \overline{z\overline{w}}}{2} \\
&= |z|^2 + |w|^2 - 2Re(z\overline{w}) = |z|^2 - 2Re(z\overline{w}) + |w|^2.
\end{aligned}$$

$$\begin{aligned}
|z+w|^2 + |z-w|^2 &= |z|^2 + Re(z\overline{w}) + |w|^2 + |z|^2 - Re(z\overline{w}) + |w|^2 \\
&= |z|^2 + |w|^2 + |z|^2 + |w|^2 = 2|z|^2 + 2|w|^2 = 2\left(|z|^2 + |w|^2\right).
\end{aligned}$$

Exercise 5. *Use induction to prove that for $z = z_1 + \ldots + z_n$; $w = w_1 w_2 \ldots w_n$:*

$$|w| = |w_1| \ldots |w_n|; \overline{z} = \overline{z}_1 + \ldots + \overline{z}_n; \overline{w} = \overline{w}_1 \ldots \overline{w}_n.$$

Solution. *Not available.*

Exercise 6. *Let $R(z)$ be a rational function of z. Show that $\overline{R(z)} = R(\overline{z})$ if all the coefficients in $R(z)$ are real.*

Solution. *Let $R(z)$ be a rational function of z, that is*

$$R(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \ldots a_0}{b_m z^m + b_{m-1} z^{m-1} + \ldots b_0}$$

where n, m are nonnegative integers. Let all coefficients of $R(z)$ be real, that is

$$a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m \in \mathbb{R}.$$

Then

$$\begin{aligned}
\overline{R(z)} &= \overline{\frac{a_n z^n + a_{n-1} z^{n-1} + \ldots a_0}{b_m z^m + b_{m-1} z^{m-1} + \ldots b_0}} = \frac{\overline{a_n z^n + a_{n-1} z^{n-1} + \ldots a_0}}{\overline{b_m z^m + b_{m-1} z^{m-1} + \ldots b_0}} \\
&= \frac{a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \ldots a_0}{b_m \overline{z^m} + b_{m-1} \overline{z^{m-1}} + \ldots b_0} = \frac{a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \ldots a_0}{b_m \overline{z}^m + b_{m-1} \overline{z}^{m-1} + \ldots b_0} = R(\overline{z}).
\end{aligned}$$

1.3 The complex plane

Exercise 1. *Prove (3.4) and give necessary and sufficient conditions for equality.*

Solution. *Let z and w be complex numbers. Then*

$$\begin{aligned} ||z| - |w|| &= ||z - w + w| - |w|| \\ &\leq ||z - w| + |w| - |w|| \\ &= ||z - w|| \\ &= |z - w| \end{aligned}$$

Notice that $|z|$ *and* $|w|$ *is the distance from z and w, respectively, to the origin while* $|z - w|$ *is the distance between z and w. Considering the construction of the implied triangle, in order to guarantee equality, it is necessary and sufficient that*

$$\begin{aligned} & ||z| - |w|| = |z - w| \\ \iff & (|z| - |w|)^2 = |z - w|^2 \\ \iff & (|z| - |w|)^2 = |z|^2 - 2Re(z\overline{w}) + |w|^2 \\ \iff & |z|^2 - 2|z||w| + |w|^2 = |z|^2 - 2Re(z\overline{w}) + |w|^2 \\ \iff & |z||w| = Re(z\overline{w}) \\ \iff & |z\overline{w}| = Re(z\overline{w}) \end{aligned}$$

Equivalently, this is $z\overline{w} \geq 0$. *Multiplying this by* $\frac{w}{w}$, *we get* $z\overline{w} \cdot \frac{w}{w} = |w|^2 \cdot \frac{z}{w} \geq 0$ *if* $w \neq 0$. *If* $t = \frac{z}{w} = \left(\frac{1}{|w|^2}\right) \cdot |w|^2 \cdot \frac{z}{w}$. *Then* $t \geq 0$ *and* $z = tw$.

Exercise 2. *Show that equality occurs in (3.3) if and only if* $z_k/z_l \geq 0$ *for any integers k and l,* $1 \leq k, l \leq n$, *for which* $z_l \neq 0$.

Solution. *Not available.*

Exercise 3. *Let* $a \in \mathbb{R}$ *and* $c > 0$ *be fixed. Describe the set of points z satisfying*

$$|z - a| - |z + a| = 2c$$

for every possible choice of a and c. Now let a be any complex number and, using a rotation of the plane, describe the locus of points satisfying the above equation.

Solution. *Not available.*

1.4 Polar representation and roots of complex numbers

Exercise 1. *Find the sixth roots of unity.*

Solution. *Start with* $z^6 = 1$ *and* $z = r\text{cis}(\theta)$, *therefore* $r^6\text{cis}(6\theta) = 1$. *Hence* $r = 1$ *and* $\theta = \frac{2k\pi}{6}$ *with* $k \in \{-3, -2, -1, 0, 1, 2\}$. *The following table gives a list of principle values of arguments and the corresponding value of the root of the equation* $z^6 = 1$.

$\theta_0 = 0$	$z_0 = 1$
$\theta_1 = \frac{\pi}{3}$	$z_1 = \text{cis}(\frac{\pi}{3})$
$\theta_2 = \frac{2\pi}{3}$	$z_2 = \text{cis}(\frac{2\pi}{3})$
$\theta_3 = \pi$	$z_3 = \text{cis}(\pi) = -1$
$\theta_4 = \frac{-2\pi}{3}$	$z_4 = \text{cis}(\frac{-2\pi}{3})$
$\theta_5 = \frac{-\pi}{3}$	$z_5 = \text{cis}(\frac{-\pi}{3})$

Exercise 2. *Calculate the following:*

a) the square roots of i

b) the cube roots of i

c) the square roots of $\sqrt{3}+3i$

Solution. *c) The square roots of $\sqrt{3}+3i$.*
Let $z=\sqrt{3}+3i$. Then $r=|z|=\sqrt{\left(\sqrt{3}\right)^2+3^2}=\sqrt{12}$ and $\alpha=\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)=\frac{\pi}{3}$. So, the 2 distinct roots of z are given by $\sqrt[2]{r}\left(\cos\frac{\alpha+2k\pi}{n}+i\sin\frac{\alpha+2k\pi}{n}\right)$ where $k=0,1$. Specifically,

$$\sqrt{z}=\sqrt[4]{12}\left(\cos\frac{\frac{\pi}{3}+2k\pi}{2}+i\sin\frac{\frac{\pi}{3}+2k\pi}{2}\right).$$

Therefore, the square roots of z, z_k, are given by

$$z_0=\sqrt[4]{12}\left(\cos\tfrac{\pi}{6}+i\sin\tfrac{\pi}{6}\right)=\sqrt[4]{12}\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right)$$
$$z_1=\sqrt[4]{12}\left(\cos\tfrac{7\pi}{6}+i\sin\tfrac{7\pi}{6}\right)=\sqrt[4]{12}\left(-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right).$$

So, in rectangular form, the second roots of z are given by $\left(\frac{\sqrt[4]{108}}{2},\frac{\sqrt[4]{12}}{2}\right)$ and $\left(-\frac{\sqrt[4]{108}}{2},-\frac{\sqrt[4]{12}}{2}\right)$.

Exercise 3. *A primitive nth root of unity is a complex number a such that $1,a,a^2,...,a^{n-1}$ are distinct nth roots of unity. Show that if a and b are primitive nth and mth roots of unity, respectively, then ab is a kth root of unity for some integer k. What is the smallest value of k? What can be said if a and b are nonprimitive roots of unity?*

Solution. *Not available.*

Exercise 4. *Use the binomial equation*

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k,$$

where

$$\binom{n}{k}=\frac{n!}{k!(n-k)!},$$

and compare the real and imaginary parts of each side of de Moivre's formula to obtain the formulas:

$$\begin{aligned}\cos n\theta &= \cos^n\theta-\binom{n}{2}\cos^{n-2}\theta\sin^2\theta+\binom{n}{4}\cos^{n-4}\theta\sin^4\theta-\ldots\\ \sin n\theta &= \binom{n}{1}\cos^{n-1}\theta\sin\theta-\binom{n}{3}\cos^{n-3}\theta\sin^3\theta+\ldots\end{aligned}$$

Solution. *Not available.*

Exercise 5. *Let $z=\operatorname{cis}\frac{2\pi}{n}$ for an integer $n\geq 2$. Show that $1+z+\ldots+z^{n-1}=0$.*

Solution. *The summation of the finite geometric sequence* $1, z, z^2, \ldots, z^{n-1}$ *can be calculated as* $\sum_{j=1}^{n} z^{j-1} = \frac{z^n-1}{z-1}$. *We want to show that* z^n *is an* n^{th} *root of unity. So, using de Moivre's formula,* $z^n = \left(cis\left(\frac{2\pi}{n}\right)\right)^n = cis\left(n \cdot \frac{2\pi}{n}\right) = cis(2\pi) = 1$. *It follows that* $1 + z + z^2 + ... + z^{n-1} = \frac{z^n-1}{z-1} = \frac{1-1}{z-1} = 0$ *as required.*

Exercise 6. *Show that* $\varphi(t) = \operatorname{cis} t$ *is a group homomorphism of the additive group* $\mathbb{R}$ *onto the multiplicative group* $T = \{z : |z| = 1\}$.

Solution. *Not available.*

Exercise 7. *If* $z \in \mathbb{C}$ *and* $Re(z^n) \leq 0$ *for every positive integer n, show that z is a non-negative real number.*

Solution. *Let n be an arbitrary but fixed positive integer and let* $z \in \mathbb{C}$ *and* $Re(z^n) \geq 0$. *Since* $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$, *we have*

$$Re(z^n) = r^n \cos(n\theta) \geq 0.$$

If $z = 0$, *then we are done, since* $r = 0$ *and* $Re(z^n) = 0$. *Therefore, assume* $z \neq 0$, *then* $r > 0$. *Thus*

$$Re(z^n) = r^n \cos(n\theta) \geq 0$$

implies $\cos(n\theta) \geq 0$ *for all n. This implies* $\theta = 0$ *as we will show next. Clearly,* $\theta \notin [\pi/2, 3\pi/2]$. *If* $\theta \in (0, \pi/2)$, *then there exists a* $k \in \{2, 3, \ldots\}$ *such that* $\frac{\pi}{k+1} \leq \theta < \frac{\pi}{k}$. *If we choose* $n = k + 1$, *we have*

$$\pi \leq n\theta < \frac{(k+1)\pi}{k}$$

which is impossible since $\cos(n\theta) \geq 0$. *Similarly, we can derive a contradiction if we assume* $\theta \in (3\pi/2, 2\pi)$. *Then* $2\pi - \pi/k \leq \theta < 2\pi - \pi/(k+1)$ *for some* $k \in \{2, 3, \ldots\}$.

1.5 Lines and half planes in the complex plane

Exercise 1. *Let C be the circle* $\{z : |z - c| = r\}$, $r > 0$*; let* $a = c + r\operatorname{cis}\alpha$ *and put*

$$L_\beta = \left\{z : Im\left(\frac{z-a}{b}\right) = 0\right\}$$

where $b = \operatorname{cis}\beta$. *Find necessary and sufficient conditions in terms of* β *that* L_β *be tangent to C at a.*

Solution. *Not available.*

1.6 The extended plane and its spherical representation

Exercise 1. *Give the details in the derivation of (6.7) and (6.8).*

Solution. *Not available.*

Exercise 2. *For each of the following points in* $\mathbb{C}$, *give the corresponding point of S:* $0, 1 + i, 3 + 2i$.

Solution. *We have*

$$\Phi(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

If $z_1 = 0$, *then* $|z_1| = 0$ *and therefore*

$$\Phi(z_1) = (0, 0, -1).$$

Thus, $z_1 = 0$ corresponds to the point $(0, 0, -1)$ on the sphere S.
If $z_2 = 1 + i$, then $|z_2| = \sqrt{2}$ and therefore

$$\Phi(z_2) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

Thus, $z_2 = 1 + i$ corresponds to the point $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ on the sphere S.
If $z_3 = 3 + 2i$, then $|z_3| = \sqrt{13}$ and therefore

$$\Phi(z_3) = \left(\frac{3}{7}, \frac{2}{7}, \frac{6}{7}\right).$$

Thus, $z_3 = 3 + 2i$ corresponds to the point $\left(\frac{3}{7}, \frac{2}{7}, \frac{6}{7}\right)$ on the sphere S.

Exercise 3. *Which subsets of S correspond to the real and imaginary axes in $\mathbb{C}$.*

Solution. *If z is on the real axes, then $z = x$ which implies $|z|^2 = x^2$. Thus*

$$\Phi(z) = \left(\frac{2x}{x^2 + 1}, 0, \frac{x^2 - 1}{x^2 + 1}\right).$$

Therefore the set

$$\left\{\left(\frac{2x}{x^2 + 1}, 0, \frac{x^2 - 1}{x^2 + 1}\right) | x \in \mathbb{R}\right\} \subset S$$

corresponds to the real axes in $\mathbb{C}$. That means the unit circle $x^2 + z^2 = 1$ lying in the $xz-$plane corresponds to the real axes in $\mathbb{C}$.
If z is on the imaginary axes, then $z = iy$ which implies $|z|^2 = y^2$. Thus

$$\Phi(z) = \left(0, \frac{2y}{y^2 + 1}, \frac{y^2 - 1}{y^2 + 1}\right).$$

Therefore the set

$$\left\{\left(0, \frac{2y}{y^2 + 1}, \frac{y^2 - 1}{y^2 + 1}\right) | y \in \mathbb{R}\right\} \subset S$$

corresponds to the imaginary axes in $\mathbb{C}$. That means the unit circle $y^2 + z^2 = 1$ lying in the $yz-$plane corresponds to the imaginary axes in $\mathbb{C}$.

Exercise 4. *Let Λ be a circle lying in S. Then there is a unique plane P in $\mathbb{R}^3$ such that $P \cap S = \Lambda$. Recall from analytic geometry that*

$$P = \{(x_1, x_2, x_3) : x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = l\}$$

where $(\beta_1, \beta_2, \beta_3)$ is a vector orthogonal to P and l is some real number. It can be assumed that $\beta_1^2 + \beta_2^2 + \beta_3^2 = 1$. Use this information to show that if Λ contains the point N then its projection on $\mathbb{C}$ is a straight line. Otherwise, Λ projects onto a circle in $\mathbb{C}$.

Solution. *Not available.*

Exercise 5. *Let Z and Z' be points on S corresponding to z and z' respectively. Let W be the point on S corresponding to $z + z'$. Find the coordinates of W in terms of the coordinates of Z and Z'.*

Solution. *Not available.*

Chapter 2

Metric Spaces and the Topology of $\mathbb{C}$

2.1 Definitions and examples of metric spaces

Exercise 1. *Show that each of the examples of metric spaces given in (1.2)-(1.6) is, indeed, a metric space. Example (1.6) is the only one likely to give any difficulty. Also, describe $B(x; r)$ for each of these examples.*

Solution. *Not available.*

Exercise 2. *Which of the following subsets of $\mathbb{C}$ are open and which are closed: (a) $\{z : |z| < 1\}$; (b) the real axis; (c) $\{z : z^n = 1$ for some integer $n \geq 1\}$; (d) $\{z \in \mathbb{C}$ is real and $0 \leq z < 1\}$; (e) $\{z \in \mathbb{C} : z$ is real and $0 \leq z \leq 1\}$?*

Solution. *We have*

(a) $A := \{z \in \mathbb{C} : |z| < 1\}$

Let $z \in A$ and set $\varepsilon_z = \frac{1-|z|}{2}$, then $B(z, \varepsilon_z) \subset A$ is open and therefore $A = \bigcup_{z \in A} B(z, \varepsilon_z)$ is open also. A cannot be closed, otherwise A and $\mathbb{C} - A$ were both closed and open sets yet $\mathbb{C}$ is connected.

(b) $B := \{z \in \mathbb{C} : z = x + iy, y = 0\}$ *(the real axis)*

Let $z \in \mathbb{C} - B$, then $\operatorname{Im} z \neq 0$. Set $\varepsilon_z = \frac{|\operatorname{Im} z|}{2}$, then $B(z, \varepsilon_z) \subset \mathbb{C} - B$.Hence B is closed since its complement is open.

For any real x and any $\varepsilon > 0$ the point $x + i\frac{\varepsilon}{2} \in B(x, \varepsilon)$ but $x + i\frac{\varepsilon}{2} \in \mathbb{C} - \mathbb{R}$. Thus B is not open.

(c) $C := \{z \in \mathbb{C} : z^n = 1$ *for some integer* $n \geq 1\}$

Claim: C is neither closed nor open.

C is not open because if $z^n = 1$ then $z = r\operatorname{cis}(\theta)$ with $r = 1$ and any ε-ball around z would contain an element $z' := (1 + \frac{\varepsilon}{4})\operatorname{cis}(\theta)$.

To show that C cannot be closed, note that for $\frac{p}{q} \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ the number $z := \operatorname{cis}\left(\frac{p}{q}2\pi\right) \in C$ since

$$z^q = \operatorname{cis}(p2\pi) = \cos(p2\pi) + i\sin(p2\pi) = 1.$$

Now fix $x \in \mathbb{R} - \mathbb{Q}$ and let $\{x_n\}_n$ be a rational sequence that converges to x. Let m be any natural number. Now $z^m = 1$ implies that $\sin(mx2\pi) = 0$ which in turn means that $mx \in \mathbb{Z}$ contradicting the choice $x \in \mathbb{R} - \mathbb{Q}$. We have constructed a sequence of elements of C that converges to a point that is not element of C. Hence C is not closed.

(d) $D := \{z \in \mathbb{C} : z \text{ is real and } 0 \le z < 1\}$

The set cannot be open by the argument given in 2.1) and it is not closed because $z_n := 1 - \frac{1}{n}$ *is a sequence in D that converges to a point outside of D.*

(e) $E := \{z \in \mathbb{C} : z \text{ is real and } 0 \le z \le 1\}$

This set E is not open by the observation in 2.1) and it is closed because its complement is open: If $z \neq E$ *and z real, then* $B(z, \frac{\min\{|x|,|x-1|\}}{2})$ *is contained in the complement of E, if z is imaginary then* $B(z, \frac{|\mathrm{Im} y|}{2})$ *is completely contained in the complement of E.*

Exercise 3. *If* (X, d) *is any metric space show that every open ball is, in fact, an open set. Also, show that every closed ball is a closed set.*

Solution. *Not available.*

Exercise 4. *Give the details of the proof of (1.9c).*

Solution. *Not available.*

Exercise 5. *Prove Proposition 1.11.*

Solution. *Not available.*

Exercise 6. *Prove that a set* $G \subset X$ *is open if and only if* $X - G$ *is closed.*

Solution. *Not available.*

Exercise 7. *Show that* $(\mathbb{C}_\infty, d)$ *where d is given (I. 6.7) and (I. 6.8) is a metric space.*

Solution. *To show* $(\mathbb{C}_\infty, d)$ *with* $d(z, z') = \frac{2|z-z'|}{[(1+|z|^2)(1+|z'|^2)]^{1/2}}$ *for* $z, z' \in \mathbb{C}$ *and* $d(z, \infty) = \frac{2}{(1+|z|^2)^{1/2}}$, $z \in \mathbb{C}$ *is a metric space.*
i) $d(z, z') \ge 0$.
Since $|z-z'| \ge 0$, *we have* $d(z, z') = \frac{2|z-z'|}{[(1+|z|^2)(1+|z'|^2)]^{1/2}} \ge 0$ *for all* $z, z' \in \mathbb{C}$ *(the denominator is always positive). Obviously,* $d(z, \infty) = \frac{2}{(1+|z|^2)^{1/2}} \ge 0$ *for all* $z \in \mathbb{C}$.
ii) $d(z, z') = 0$ *iff* $z = z'$.
We have $2|z - z'| = 0$ *iff* $z = z'$. *Therefore,* $d(z, z') = \frac{2|z-z'|}{[(1+|z|^2)(1+|z'|^2)]^{1/2}} = 0$ *iff* $z = z'$. $d(\infty, \infty) = \lim_{z\to\infty} \frac{2}{(1+|z|^2)^{1/2}} = 0$. *Thus,* $d(\infty, 0) = 0$ *iff* $z = \infty$.
iii) $d(z, z') = d(z', z)$.
We have $d(z, z') = \frac{2|z-z'|}{[(1+|z|^2)(1+|z'|^2)]^{1/2}} = \frac{2|z'-z|}{[(1+|z'|^2)(1+|z|^2)]^{1/2}} = d(z', z)$ *for all* $z, z' \in \mathbb{C}$. *Also* $d(z, \infty) = \frac{2}{(1+|z|^2)^{1/2}} = d(\infty, z)$ *by the symmetry of* $d(z, z')$ *in general.*
iv) $d(z, z') \le d(z, x) + d(x, z')$.
We have

$$\begin{aligned} d(z, z') &= \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{1/2}} \\ &\le \frac{2|z - x|}{[(1 + |z|^2)(1 + |x|^2)]^{1/2}} + \frac{2|x - z'|}{[(1 + |x|^2)(1 + |z'|^2)]^{1/2}} \\ &= d(z, x) + d(x, z') \end{aligned}$$

for all $x, z, z' \in \mathbb{C}$. *The first inequality will be shown next.*
First, we need the following equation

$$(z - z')(1 + \bar{x}x) = (z - x)(1 + \bar{x}z') + (x - z')(1 + \bar{x}z), \tag{2.1}$$

which follows from a simple computation

$$\begin{aligned}(z-x)(1+\bar{x}z')+(x-z')(1+\bar{x}z) &= z-x+\bar{x}z'z+x\bar{x}z'+x-z'+x\bar{x}z-\bar{x}z'z\\ &= z+x\bar{x}z-z'-x\bar{x}z'\\ &= (z-z')(1+\bar{x}x).\end{aligned}$$

Using (2.1) and taking norms gives

$$\begin{aligned}|(z-z')(1+\bar{x}x)| &= |(z-z')(1+|x|^2)| = |(z-z')|(1+|x|^2)\\ &= |(z-x)(1+\bar{x}z')+(x-z')(1+\bar{x}z)|\\ &\le |z-x|(1+|x|^2)^{1/2}(1+|z'|^2)^{1/2}+|x-z'|(1+|x|^2)^{1/2}(1+|z|^2)^{1/2}\end{aligned} \tag{2.2}$$

where the last inequality follows by

$$\begin{aligned}&|z+\bar{x}z'| \le (1+|x|^2)^{1/2}(1+|z'|^2)^{1/2}\\ \Leftrightarrow\quad &(1+\bar{x}z')(1+x\overline{z'}) \le (1+|x|^2)(1+|z'|^2)\end{aligned}$$

and

$$\begin{aligned}&|z+\bar{x}z| \le (1+|x|^2)^{1/2}(1+|z|^2)^{1/2}\\ \Leftrightarrow\quad &(1+\bar{x}z)(1+x\bar{z}) \le (1+|x|^2)(1+|z|^2)\end{aligned}$$

since

$$\begin{aligned}&(1+\bar{x}z)(1+x\bar{z}) \le (1+|x|^2)(1+|z|^2)\\ \Leftrightarrow\quad &\bar{x}z+x\bar{z} \le |x|^2+|y|^2\\ \Leftrightarrow\quad &2Re(x\bar{z}) \le |x|^2+|y|^2\end{aligned}$$

which is true by Exercise 4 part 2 on page 3. Thus, dividing (2.2) by $(1+|x|^2)^{1/2}$ *yields*

$$|z-z'|(1+|x|^2)^{1/2} \le |z-x|(1+|z'|^2)^{1/2}+|x-z'|(1+|z|^2)^{1/2}.$$

Multiplying this by

$$\frac{2}{(1+|x|^2)^{1/2}(1+|z'|^2)^{1/2}(1+|z|^2)^{1/2}}$$

gives the assertion above.

Exercise 8. *Let* (X,d) *be a metric space and* $Y\subset X$. *Suppose* $G\subset X$ *is open; show that* $G\cap Y$ *is open in* (Y,d). *Conversely, show that if* $G_1\subset Y$ *is open in* (Y,d), *there is an open set* $G\subset X$ *such that* $G_1=G\cap Y$.

Solution. *Set* $G_1=G\cap Y$, *let* G *be open in* (X,d) *and* $Y\subset X$. *In order to show that* G_1 *is open in* (Y,d), *pick an arbitrary point* $p\in G_1$. *Then* $p\in G$ *and since* G *is open, there exists an* $\epsilon>0$ *such that*

$$B^X(p;\epsilon)\subset G.$$

But then

$$B^Y(p;\epsilon)=B^X(p;\epsilon)\cap Y\subset G\cap Y=G_1$$

which proves that p *is an interior point of* G_1 *in the metric* d. *Thus* G_1 *is open in* Y *(Proposition 1.13a).*

Let G_1 *be an open set in* Y. *Then, for every* $p\in G_1$, *there exists an* $\epsilon-$*ball*

$$B^Y(p:\epsilon)\subset G_1.$$

Thus

$$G_1 = \bigcup_{p \in G_1} B^Y(p;\epsilon).$$

Since we can write

$$B^Y(p;\epsilon) = B^X(p;\epsilon) \cap Y,$$

we get

$$G_1 = \bigcup_{p \in G_1} B^Y(p;\epsilon) = \bigcup_{p \in G_1} \left(B^X(p;\epsilon) \cap Y\right) = \bigcup_{p \in G_1} B^X(p;\epsilon) \cap Y = G \cap Y$$

where $G = \bigcup_{p \in G_1} B^X(p;\epsilon)$ *is open in* (X, d). *(Proposition 1.9c since each* $B^X(p;\epsilon)$ *is open).*

Exercise 9. *Do Exercise 8 with "closed" in place of "open."*

Solution. *Not available.*

Exercise 10. *Prove Proposition 1.13.*

Solution. *Not available.*

Exercise 11. *Show that* $\{\text{cis } k : k$ *a non-negative integer* $\}$ *is dense in* $T = \{z \in \mathbb{C} : |z| = 1\}$. *For which values of* θ *is* $\{\text{cis}(k\theta) : k$ *a non-negative integer* $\}$ *dense in* T*?*

Solution. *Not available.*

2.2 Connectedness

Exercise 1. *The purpose of this exercise is to show that a connected subset of* $\mathbb{R}$ *is an interval.*

(a) *Show that a set* $A \subset \mathbb{R}$ *is an interval iff for any two points* a *and* b *in* A *with* $a < b$, *the interval* $[a, b] \subset A$.

(b) *Use part (a) to show that if a set* $A \subset \mathbb{R}$ *is connected then it is an interval.*

Solution. *Not available.*

Exercise 2. *Show that the sets* S *and* T *in the proof of Theorem 2.3 are open.*

Solution. *Not available.*

Exercise 3. *Which of the following subsets* X *of* $\mathbb{C}$ *are connected; if* X *is not connected, what are its components: (a)* $X = \{z : |z| \leq 1\} \cup \{z : |z - 2| < 1\}$. *(b)* $X = [0, 1) \cap \left\{1 + \frac{1}{n} : n \geq 1\right\}$. *(c)* $X = \mathbb{C} - (A \cap B)$ *where* $A = [0, \infty)$ *and* $B = \{z = r \text{ cis } \theta : r = \theta, 0 \leq \theta \leq \infty\}$*?*

Solution. *a) Define* $X = \{z : |z| \leq 1\} \cup \{z : |z - 2| < 1\} := A \cup B$. *It suffices to show that* X *is path-connected. Obviously* A *is path-connected and* B *is path-connected. Next, we will show that* X *is path-connected. Recall that a space is path-connected if for any two points* x *and* y *there exists a continuous function* f *from the interval* $[0, 1]$ *to* X *with* $f(0) = x$ *and* $f(1) = y$ *(this function* f *is called the path from* x *to* y*).*
Let $x \in A$ *and* $y \in B$ *and define the function* $f(t) : [0, 1] \to X$ *by*

$$f(t) = \begin{cases} (1 - 3t)x + 3tRe(x), & 0 \leq t \leq \frac{1}{3} \\ (2 - 3t)Re(x) + (3t - 1)Re(y), & \frac{1}{3} < t \leq \frac{2}{3} \\ (3 - 3t)Re(y) + (3t - 2)y, & \frac{2}{3} < t \leq 1. \end{cases}$$

This function is obviously continuous, since $f(1/3) = \lim_{t\to\frac{1}{3}^-} f(t) = \lim_{t\to\frac{1}{3}^+} f(t) = Re(x) \in X$ *and* $f(2/3) = \lim_{t\to\frac{2}{3}^-} f(t) = \lim_{t\to\frac{2}{3}^+} f(t) = Re(y) \in X$. *In addition, we have* $f(0) = x$ *and* $f(1) = y$. *Therefore* X *is path-connected and hence* X *is connected.*
b) There is no way to connect $\{2\}$ *and* $\{\frac{3}{2}\}$. *Therefore* X *is not connected. The components are* $[0,1)$, $\{2\}$, $\{\frac{3}{2}\}, \{\frac{4}{3}\}, \ldots, \{1+\frac{1}{n}\}$.
c) X *is not connected, since there is no way to connect* $(2,1)$ *and* $(1,-2)$. *The* $k-th$ *component is given by* $\mathbb{C} - \{A \cap B\}$ *where*

$$A = [2\pi k - 2\pi, 2\pi k)$$

and

$$B = \{z = r \operatorname{cis} \theta : r = \theta, 2\pi k - 2\pi \le \theta < 2\pi k\}, \quad k \in \{1,2,3,\ldots\}.$$

Exercise 4. *Prove the following generalization of Lemma 2.6. If* $\{D_j : j \in J\}$ *is a collection of connected subsets of* X *and if for each* j *and* k *in* J *we have* $D_j \cap D_k \neq \square$ *then* $D = \bigcup\{D_j : j \in J\}$ *is connected.*

Solution. *Let* $D = \bigcup_{j\in J} D_j$ *and* $\mathcal{C} = \{D_j : j \in J\}$. *If* D *is connected, we could write* D *as the disjoint union* $A \cup B$ *where* A *and* B *are nonempty subsets of* X. *Thus, for each* $C \in \mathcal{C}$ *either* $C \subset A$ *or* $C \subset B$.
We have $C \subset A\ \forall C \in \mathcal{C}$ *or* $C \subset B\ \forall C \in \mathcal{C}$. *If not, then there exist* $E, F \in \mathcal{C}$ *such that* $E \subset A$ *and* $F \subset B$. *But, we assume* $A \cup B$ *is disjoint and thus* $E \cup F$ *is disjoint which contradicts the assumption* $E \cap F \neq \emptyset$.
Therefore, all members of $\mathcal{C}$ *are contained in either* A *or all* B. *Thus, either* $D = A$ *and* $B = \emptyset$ *or* $D = B$ *and* $A = \emptyset$. *Both contradicting the fact that* A, B *are assumed to be nonempty. Hence,*

$$D = \bigcup_{j\in J} D_j$$

is connected.

Exercise 5. *Show that if* $F \subset X$ *is closed and connected then for every pair of points* a, b *in* F *and each* $\epsilon > 0$ *there are points* $z_0, z_1, \ldots, z_n$ *in* F *with* $z_0 = a$, $z_n = b$ *and* $d(z_{k-1}, z_k) < \epsilon$ *for* $1 \le k \le n$. *Is the hypothesis that* F *be closed needed? If* F *is a set which satisfies this property then* F *is not necessarily connected, even if* F *is closed. Give an example to illustrate this.*

Solution. *Not available.*

2.3 Sequences and completeness

Exercise 1. *Prove Proposition 3.4.*

Solution. *a) A set is closed iff it contains all its limit points.*
Let $S \subset X$ *be a set.*
"$\Leftarrow$*": Assume* S *contains all its limit points. We have to show that* S *is closed or that* S^c *is open. Let* $x \in S^c$. *By assumption* x *is not a limit point and hence there exists an open* $\epsilon-$*ball around* x, $B(x;\epsilon)$ *such that* $B(x;\epsilon) \cap S = \emptyset$ *(negation of Proposition 1.13f). So* $B(x;\epsilon) \subset S^c$ *and therefore* S^c *is open.*
"$\Rightarrow$*": Let* S *be closed and* x *be a limit point. We claim* $x \in S$. *If not,* S^c *(open) would be an open neighborhood of* x, *that does not intersect* S. *(Proposition 1.13f again) which contradicts the fact that* x *is a limit point of* S.
b) If $A \subset X$, *then* $A^- = A \cup \{x : x \text{ is a limit point of } A\} := A \cup A'$.
Let $A \subset X$ *be a set.*
"$\subseteq$*": Let* $x \in \bar{A}$. *We want to show* $x \in A \cup A'$. *If* $x \in A$, *then obviously* $x \in A \cup A'$. *Suppose* $x \notin A$. *Since* $x \in \bar{A}$, *we have (Proposition 1.13f) that for every* $\epsilon > 0$, $B(x;\epsilon) \cap A \neq \emptyset$. *Because* $x \notin A$, $B(x;\epsilon)$ *must*

intersect A in a point different from x. In particular, for every integer n there is a point x_n *in* $B(x; \frac{1}{n}) \cap A$. *Thus* $d(x, x_n) < \frac{1}{n}$ *which implies* $x_n \to x$ *(see Proof of Proposition 3.2). Then* $x \in A'$, *so* $x \in A \cup A'$.
"$\supseteq$*": To show* $A \cup A' \subseteq A^-$. *Let* $x \in A \cup A'$. *If* $x \in A$, *then* $x \in A^-$ $(A \subset A^-)$. *Now, assume* $x \in A'$ *but not in A. Then there exists* $\{x_n\} \subset A$ *with* $\lim_{n\to\infty} x_n = x$. *It follows,* $\forall \epsilon > 0$ $B(x; \epsilon) \cap A \neq \emptyset$ *since* $\{x_n\} \subset A$. *By Proposition 1.13f we get* $x \in A^-$.

Exercise 2. *Furnish the details of the proof of Proposition 3.8.*

Solution. *Not available.*

Exercise 3. *Show that diam A = diam* A^-.

Solution. *Not available.*

Exercise 4. *Let* z_n, z *be points in* $\mathbb{C}$ *and let d be the metric on* $\mathbb{C}_\infty$. *Show that* $|z_n - z| \to 0$ *if and only if* $d(z_n, z) \to 0$. *Also show that if* $|z_n| \to \infty$ *then* $\{z_n\}$ *is Cauchy in* $\mathbb{C}_\infty$. *(Must* $\{z_n\}$ *converge in* $\mathbb{C}_\infty$*?)*

Solution. *First assume that* $|z_n - z| \to 0$, *then*

$$d(z_n, z) = \frac{2}{\sqrt{(1 + |z_n|^2)(1 + |z|^2)}}|z_n - z| \to 0$$

since the denominator $\sqrt{(1 + |z_n|^2)(1 + |z|^2)} \geq 1$ *is bounded below away from 0. To see the converse, let* $d(z_n, z) \to 0$ *or equivalently* $d^2(z_n, z) \to 0$. *We need to show that if* $z_n \to z$ *in the d-norm, then* $|z_n| \nrightarrow \infty$ *because otherwise the denominator grows without bounds. In fact we will show that if* $|z_n| \to \infty$, *then* $d(z_n, z) \nrightarrow 0$ *for any* $z \in \mathbb{C}$. *Then*

$$\begin{aligned} d^2(z_n, z) &= \frac{4\left(|z_n|^2 - z_n\overline{z} - \overline{z}_n z + |z|^2\right)}{(1 + |z_n|^2)(1 + |z|^2)} \\ &= \frac{4\left(|z_n|^2 - 2Re(z_n\overline{z}) + |z|^2\right)}{(1 + |z|^2)|z_n|^2 + 1 + |z|^2} \\ &= \frac{4\left(1 - \frac{2Re(z_n\overline{z})}{|z_n|^2} + \frac{|z|^2}{|z_n|^2}\right)}{1 + |z|^2 + \frac{1+|z|^2}{|z_n|^2}} \qquad \text{if } |z_n| \neq 0 \end{aligned}$$

in particular if $|z_n| \to \infty$, $d(z_n, z) \to \frac{4}{1+|z|^2} \neq 0$. *This shows that the denominator remains bounded as* $d(z_n, z) \to 0$ *and therefore the numerator* $2|z_n - z| \to 0$. *Hence convergence in d-norm implies convergence in* $|\cdot|$*-norm for numbers* $z_n, z \in \mathbb{C}$. *Next assume that* $|z_n| \to \infty$. *Then clearly also* $\sqrt{1 + |z_n|^2} \to \infty$ *and therefore* $d(z_n, \infty) = \frac{2}{\sqrt{1+|z_n|^2}} \to 0$. *The last thing to show is that if* $z_n \to \infty$ *in* (C_∞, d), *also* $|z_n| \to \infty$. *But* $d(z_n, \infty) \to 0$ *implies* $\sqrt{1 + |z_n|^2} \to \infty$ *which is equivalent to* $|z_n| \to \infty$.

Exercise 5. *Show that every convergent sequence in* (X, d) *is a Cauchy sequence.*

Solution. *Let* $\{x_n\}$ *be a convergent sequence with limit x. That is, given* $\epsilon > 0$ $\exists N$ *such that* $d(x_n, x) < \frac{\epsilon}{2}$ *if* $n > N$ *and* $d(x, x_m) < \frac{\epsilon}{2}$ *if* $m > N$. *Thus*

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n, m \geq N$$

and therefore $\{x_n\}$ *is a Cauchy sequence.*

Exercise 6. *Give three examples of non complete metric spaces.*

Solution. *Example 1: Let $X = C[-1,1]$ and the metric $d(f,g) = \sqrt{\int_{-1}^{1}[f(x)-g(x)]^2\,dx}$, $f,g \in X$. Consider the Cauchy sequence*

$$f_n(x) = \begin{cases} 0, & -1 \le x \le 0 \\ nx, & 0 < x \le \frac{1}{n} \\ 1, & \frac{1}{n} < x \le 1 \end{cases}.$$

It is obvious that the limit function f is discontinuous. Hence, the metric space (X,d) is not complete.
Example 2: Let $X = (0,1]$ with metric $d(x,y) = |x-y|$, $x,y \in X$. The sequence $\{\frac{1}{n}\}$ is Cauchy, but converges to 0, which is not in the space. Thus, the metric space is not complete.
Example 3: Let $X = \mathbb{Q}$, the rationals, with metric $d(x,y) = |x-y|$, $x,y \in X$. The sequence defined by $x_1 = 1$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ is a Cauchy sequence of rational numbers. The limit $\sqrt{2}$ is not a rational number. Therefore, the metric space is not complete.

Exercise 7. *Put a metric d on $\mathbb{R}$ such that $|x_n - x| \to 0$ if and only if $d(x_n,x) \to 0$, but that $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R},d)$ when $|x_n| \to \infty$. (Hint: Take inspiration from $\mathbb{C}_\infty$.)*

Solution. *Not available.*

Exercise 8. *Suppose $\{x_n\}$ is a Cauchy sequence and $\{x_{n_k}\}$ is a subsequence that is convergent. Show that $\{x_n\}$ must be convergent.*

Solution. *Since $\{x_{n_k}\}$ is convergent, there is a x such that $x_{n_k} \to x$ as $k \to \infty$. We have to show that $x_n \to x$ as $n \to x$. Let $\epsilon > 0$. Then we have $\exists N \in \mathbb{N}$ such that $d(x_n,x_m) < \frac{\epsilon}{2}$ $\forall n,m \ge N$ since $\{x_n\}$ is Cauchy and $\exists M \in \mathbb{N}$ such that $d(x_{n_k},x) < \frac{\epsilon}{2}$ $\forall n_k \ge M$ since $x_{n_k} \to x$ as $k \to \infty$. Now, fix $n_{k_0} > M + N$, then*

$$d(x_n,x) \le d(x_n,x_{n_{k_0}}) + d(x_{n_{k_0}},x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n \ge N.$$

Thus, $x_n \to x$ as $n \to \infty$ and therefore $\{x_n\}$ is convergent.

2.4 Compactness

Exercise 1. *Finish the proof of Proposition 4.4.*

Solution. *Not available.*

Exercise 2. *Let $p = (p_1,\ldots,p_n)$ and $q = (q_1,\ldots,q_n)$ be points in $\mathbb{R}^n$ with $p_k < q_k$ for each k. Let $R = [p_1,q_1] \times \ldots \times [p_n,q_n]$ and show that*

$$diam\ R = d(p,q) = \left[\sum_{k=1}^{n}(q_k - p_k)^2\right]^{\frac{1}{2}}.$$

Solution. *By definition*

$$diam(R) = \sup_{x\in R, y\in R} d(x,y).$$

Obviously, R is compact, so we have

$$diam(R) = \max_{x\in R, y\in R} d(x,y).$$

Let $x = (x_1, \ldots, x_n) \in R$ and $y = (y_1, \ldots, y_n) \in R$. Then clearly, $p_i \leq x_i \leq q_i$ and $p_i \leq y_i \leq q_i$ for all $i = 1, \ldots, n$. We also have

$$(y_i - x_i)^2 \leq (q_i - p_i)^2, \qquad \forall i = 1, \ldots, n. \tag{2.3}$$

Therefore, by (2.3) we obtain

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} \leq \sqrt{\sum_{i=1}^{n} (q_i - p_i)^2} = d(p,q).$$

Note that we get equality if for example $x_i = p_i$ and $y_i = q_i$ for all $i = 1, \ldots, n$. Thus the maximum distance is obtained, so

$$diam(R) = \sup_{x \in R, y \in R} d(x,y) = \sqrt{\sum_{i=1}^{n} (q_i - p_i)^2} = d(p,q).$$

Exercise 3. *Let $F = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$ and let $\epsilon > 0$; use Exercise 2 to show that there are rectangles $R_1, \ldots, R_m$ such that $F = \bigcup_{k=1}^{m} R_k$ and diam $R_k < \epsilon$ for each k. If $x_k \in R_k$ then it follows that $R_k \subset B(x_k; \epsilon)$.*

Solution. *Not available.*

Exercise 4. *Show that the union of a finite number of compact sets is compact.*

Solution. *Let $K = \bigcup_{i=1}^{n} K_i$ be a finite union of compact sets. Let $\{G_\lambda\}_{\lambda \in \Gamma}$ be an open cover of K, that is*

$$K \subset \bigcup_{\lambda \in \Gamma} G_\lambda.$$

Of course, $\{G_\lambda\}_{\lambda \in \Gamma}$ is an open cover of each K_i, $i = 1, \ldots, n$, that is

$$K_i \subset \bigcup_{\lambda \in \Gamma} G_\lambda, \qquad \forall i = 1, \ldots, n.$$

Since each K_i is compact,

$$K_i \subset \bigcup_{j=1}^{k_i} G_{i,j}$$

for each $i = 1, \ldots, n$. Thus

$$K = \bigcup_{j=1}^{n} K_i \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} G_{i,j}$$

which is a finite union and therefore K is compact.

Exercise 5. *Let X be the set of all bounded sequences of complex numbers. That is, $\{x_n\} \in X$ iff $\sup\{|x_n| : n \geq 1\} \leq \infty$. If $x = \{x_n\}$ and $y = \{y_n\}$, define $d(x,y) = \sup\{|x_n - y_n| : n \geq 1\}$. Show that for each x in X and $\epsilon > 0$, $\overline{B}(x;\epsilon)$ is not totally bounded although it is complete. (Hint: you might have an easier time of it if you first show that you can assume $x = (0, 0, \ldots)$.)*

Solution. *Not available.*

Exercise 6. *Show that the closure of a totally bounded set is totally bounded.*

Solution. *Let* (X, d) *be a given metric space and let* $S \subset X$ *be totally bounded. Let* $\epsilon > 0$. *Since* S *is totally bounded, we have by Theorem 4.9 d) p. 22 that there exist a finite number of points* $x_1, \ldots, x_n \in S$ *such that*

$$S \subseteq \bigcup_{k=1}^{n} B(x_k; \epsilon/2).$$

Taking the closure, gives the desired result

$$\bar{S} \subseteq \bigcup_{k=1}^{n} B(x_k; \epsilon).$$

2.5 Continuity

Exercise 1. *Prove Proposition 5.2.*

Solution. *Not available.*

Exercise 2. *Show that if* f *and* g *are uniformly continuous (Lipschitz) functions from* X *into* $\mathbb{C}$ *then so is* $f + g$.

Solution. *The distance in* $\mathbb{C}$ *is* $\rho(x, y) = |x - y|$. *The distance in* X *is* $d(x, y)$. *Let* $f : X \to \mathbb{C}$ *be Lipschitz, that is, there is a constant* $M > 0$ *such that*

$$\rho(f(x), f(y)) = |f(x) - f(y)| \leq Md(x, y), \qquad \forall x, y \in X$$

and $g : X \to \mathbb{C}$ *be Lipschitz, that is, there is a constant* $N > 0$ *such that*

$$\rho(g(x), g(y)) = |g(x) - g(y)| \leq Nd(x, y), \qquad \forall x, y \in X.$$

We have

$$\begin{aligned} \rho(f(x) + g(x), f(y) + g(y)) &= |f(x) + g(x) - f(y) - g(y)| = |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq Md(x, y) + Nd(x, y) = (M + N)d(x, y), \qquad \forall x, y \in X. \end{aligned}$$

Since there is a constant $K = M + N > 0$ *such that*

$$\rho(f(x) + g(x), f(y) + g(y)) \leq Kd(x, y), \qquad \forall x, y \in X,$$

we have that $f + g$ *is Lipschitz.*

Now, let f, g *be both uniformly continuous, that is,*

$$\forall \epsilon_1 > 0 \; \exists \delta_1 > 0 \text{ such that } \rho(f(x), f(y)) = |f(x) - f(y)| < \epsilon_1 \text{ whenever } d(x, y) < \delta_1$$

and

$$\forall \epsilon_2 > 0 \; \exists \delta_2 > 0 \text{ such that } \rho(f(x), f(y)) = |f(x) - f(y)| < \epsilon_2 \text{ whenever } d(x, y) < \delta_2.$$

We have

$$\begin{aligned} \rho(f(x) + g(x), f(y) + g(y)) &= |f(x) + g(x) - f(y) - g(y)| = |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq \epsilon_1 + \epsilon_2, \end{aligned}$$

whenever $d(x, y) < \delta_1$ *and* $d(x, y) < \delta_2$*, that is, whenever* $d(x, y) < \min(\delta_1, \delta_2)$*. So, choosing* $\epsilon = \epsilon_1 + \epsilon_2$ *and* $\delta = \min(\delta_1, \delta_2)$*, we have shown that*

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \rho(f(x), f(y)) = |f(x) - f(y)| < \epsilon \text{ whenever } d(x, y) < \delta.$$

Thus $f + g$ *is uniformly continuous.*

Exercise 3. *We say that* $f : X \to \mathbb{C}$ *is bounded if there is a constant* $M > 0$ *with* $|f(x)| \leq M$ *for all* x *in* X*. Show that if* f *and* g *are bounded uniformly continuous (Lipschitz) functions from* X *into* $\mathbb{C}$ *then so is* fg*.*

Solution. *Let* f *be bounded, that is, there exists a constant* $M_1 > 0$ *with* $|f(x)| < M_1$ *for all* $x \in X$ *and let* g *be bounded, that is, there exists a constant* $M_2 > 0$ *with* $|g(x)| < M_2$ *for all* $x \in X$*. Obviously* fg *is bounded, because*

$$|f(x)g(x)| \leq |f(x)|\,|g(x)| \leq M_1 \cdot M_2 \qquad \forall x \in X.$$

So, there exists a constant $M = M_1 M_2$ *with*

$$|f(x)g(x)| \leq M \qquad \forall x \in X.$$

Let f *be Lipschitz, that is, there exists a constant* $N_1 > 0$ *such that*

$$\rho(f(x), f(y)) \leq N_1 d(x, y) \qquad \forall x, y \in X$$

and let g *be Lipschitz, that is, there exists a constant* $N_2 > 0$ *such that*

$$\rho(g(x), g(y)) \leq N_1 d(x, y) \qquad \forall x, y \in X.$$

Now,

$$\begin{aligned}
\rho(f(x)g(x), f(y)g(y)) &= |f(x)g(x) - f(y)g(y)| = \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)|\,|g(x) - g(y)| + |f(x)|\,|g(y) - g(y)| \\
&\leq M_1 N_2 d(x, y) + M_2 N_1 d(x, y) = (M_1 N_2 + M_2 N_1) d(x, y) \qquad \forall x, y \in X.
\end{aligned}$$

Thus, there exists a constant $K = M_1 N_2 + M_2 N_1 > 0$ *with*

$$\rho(f(x)g(x), f(y)g(y)) \leq K d(x, y) \qquad \forall x, y \in X,$$

so fg *is Lipschitz and bounded.*

Now, let f *and* g *be both bounded and uniformly continuous, that is*

$$\begin{aligned}
&\exists \quad M_1 \text{ such that } |f(x)| \leq M_1 \qquad \forall x \in X \\
&\exists \quad M_2 \text{ such that } |g(x)| \leq M_2 \qquad \forall x \in X \\
&\forall \quad \epsilon_1 > 0 \ \exists \delta_1 > 0 \text{ such that } \rho(f(x), f(y)) = |f(x) - f(y)| < \epsilon_1 \text{ whenever } d(x, y) < \delta_1 \\
&\forall \quad \epsilon_2 > 0 \ \exists \delta_2 > 0 \text{ such that } \rho(f(x), f(y)) = |f(x) - f(y)| < \epsilon_2 \text{ whenever } d(x, y) < \delta_2.
\end{aligned}$$

It remains to verify that fg is uniformly continuous, since we have already shown that fg is bounded. We have

$$\begin{aligned}
\rho(f(x)g(x), f(y)g(y)) &= |f(x)g(x) - f(y)g(y)| = \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)|\,|g(x) - g(y)| + |f(x)|\,|g(y) - g(y)| \\
&\leq M_1\epsilon_2 + M_2\epsilon_1,
\end{aligned}$$

whenever $d(x,y) < \min(\delta_1, \delta_2)$. So choosing $\epsilon = M_1\epsilon_2 + M_2\epsilon_1$ and $\delta = \min(\delta_1, \delta_2)$, we have $\forall\epsilon > 0\ \exists\delta > 0$ such that

$$|f(x)g(x) - f(y)g(y)| < \epsilon$$

whenever $d(x,y) < \delta$. Thus, fg is uniformly continuous and bounded.

Exercise 4. *Is the composition of two uniformly continuous (Lipschitz) functions again uniformly continuous (Lipschitz)?*

Solution. *Not available.*

Exercise 5. *Suppose $f : X \to \Omega$ is uniformly continuous; show that if $\{x_n\}$ is a Cauchy sequence in X then $\{f(x_n)\}$ is a Cauchy sequence in Ω. Is this still true if we only assume that f is continuous? (Prove or give a counterexample.)*

Solution. *Assume $f : X \to \Omega$ is uniformly continuous, that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ if $d(x,y) < \delta$. If $\{x_n\}$ is a Cauchy sequence in X, we have, for every $\epsilon_1 > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon_1$ for all $n, m \geq N$. But then, by the uniform continuity, we have that*

$$\rho(f(x_n), f(x_m)) < \epsilon \qquad \forall n, m \geq N$$

whenever $d(x_n, x_m) < \delta$ which tells us that $\{f(x_n)\}$ is a Cauchy sequence in Ω.
If f is continuous, the statement is not ***true****. Here is a counterexample: Let $f(x) = \frac{1}{x}$ which is continuous on $(0,1)$. The sequence $x_n = \frac{1}{n}$ is apparently convergent and therefore a Cauchy sequence in X. But $\{f(x_n)\} = \{f(\frac{1}{n})\} = \{n\}$ is obviously not Cauchy. Note that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$. To see that pick $\epsilon = 1$. Then there is no $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Assume there exists such a δ. WLOG assume $\delta < 1$ since the interval $(0,1)$ is considered. Let $y = x + \delta/2$ and set $x = \delta/2$, then*

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{y-x}{xy} = \frac{\delta/2}{\delta/2\cdot\delta} = \frac{1}{\delta} > 1,$$

that is no matter what $\delta < 1$ we choose, we always obtain $|f(x) - f(y)| > 1$. Therefore $f(x) = \frac{1}{x}$ cannot be uniformly continuous.

Exercise 6. *Recall the definition of a dense set (1.14). Suppose that Ω is a complete metric space and that $f : (D, d) \to (\Omega; \rho)$ is uniformly continuous, where D is dense in (X, d). Use Exercise 5 to show that there is a uniformly continuous function $g : X \to \Omega$ with $g(x) = f(x)$ for every x in D.*

Solution. *Not available.*

Exercise 7. *Let G be an open subset of $\mathbb{C}$ and let P be a polygon in G from a to b. Use Theorems 5.15 and 5.17 to show that there is a polygon $Q \subset G$ from a to b which is composed of line segments which are parallel to either the real or imaginary axes.*

Solution. *Not available.*

Exercise 8. *Use Lebesgue's Covering Lemma (4.8) to give another proof of Theorem 5.15.*

Solution. *Suppose $f : X \to \Omega$ is continuous and X is compact. To show f is uniformly continuous. Let $\epsilon > 0$. Since f is continuous, we have for all $x \in X$ there is a $\delta_x > 0$ such that $\rho(f(x), f(y)) < \epsilon/2$ whenever $d(x, y) < \delta_x$. In addition,*

$$X = \bigcup_{x \in X} B(x; \delta_x)$$

is an open cover of X. Since X is by assumption compact (it is also sequentially compact as stated in Theorem 4.9 p. 22), we can use Lebesgue's Covering Lemma 4.8 p. 21 to obtain a $\delta > 0$ such that $x \in X$ implies that $B(x, \delta) \subset B(z; \delta_z)$ for some $z \in X$. More precisely, $x, y \in B(z; \delta_z)$ and therefore

$$\rho(f(x), f(z)) \leq \rho(f(x), f(z)) + \rho(f(z), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and hence f is uniformly continuous on X.

Exercise 9. *Prove the following converse to Exercise 2.5. Suppose (X, d) is a compact metric space having the property that for every $\epsilon > 0$ and for any points a, b in X, there are points $z_0, z_1, \ldots, z_n$ in X with $z_0 = a$, $z_n = b$, and $d(z_{k-l}, z_k) < \epsilon$ for $1 \leq k \leq n$. Then (X, d) is connected. (Hint: Use Theorem 5.17.)*

Solution. *Not available.*

Exercise 10. *Let f and g be continuous functions from (X, d) to (Ω, p) and let D be a dense subset of X. Prove that if $f(x) = g(x)$ for x in D then $f = g$. Use this to show that the function g obtained in Exercise 6 is unique.*

Solution. *Not available.*

2.6 Uniform convergence

Exercise 1. *Let $\{f_n\}$ be a sequence of uniformly continuous functions from (X, d) into (Ω, ρ) and suppose that $f = u - - \lim f_n$ exists. Prove that f is uniformly continuous. If each f_n is a Lipschitz function with constant M_n and $\sup M_n < \infty$, show that f is a Lipschitz function. If $\sup M_n = \infty$, show that f may fail to be Lipschitz.*

Solution. *Not available.*

Chapter 3

Elementary Properties and Examples of Analytic Functions

3.1 Power series

Exercise 1. *Prove Proposition 1.5.*

Solution. *Not available.*

Exercise 2. *Give the details of the proof of Proposition 1.6.*

Solution. *Not available.*

Exercise 3. *Prove that* $\limsup(a_n+b_n) \leq \limsup a_n + \limsup b_n$ *and* $\liminf(a_n+b_n) \geq \liminf a_n + \liminf b_n$ *for bounded sequences of real numbers* $\{a_n\}$ *and* $\{b_n\}$*.*

Solution. *Let* $r > \limsup_{n\to\infty} a_n$ *(we know there are only finitely many by definition) and let* $s > \limsup_{n\to\infty}$ *(same here, there are only finitely many by definition). Then* $r + s > a_n + b_n$ *for all but finitely many* n*'s. This however, implies that*

$$r + s \geq \limsup_{n\to\infty}(a_n + b_n).$$

Since this holds for any $r > \limsup_{n\to\infty} a_n$ *and* $s > \limsup_{n\to\infty} b_n$*, we have*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Let $r < \liminf_{n\to\infty} a_n$ *(we know there are only finitely many by definition) and let* $s < \liminf_{n\to\infty}$ *(same here, there are only finitely many by definition). Then* $r + s < a_n + b_n$ *for all but finitely many* n*'s. This however, implies that*

$$r + s \leq \liminf_{n\to\infty}(a_n + b_n).$$

Since this holds for any $r < \liminf_{n\to\infty} a_n$ *and* $s < \liminf_{n\to\infty} b_n$*, we have*

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Exercise 4. *Show that* $\liminf a_n \leq \limsup a_n$ *for any sequence in* $\mathbb{R}$*.*

Solution. *Let $m = \liminf_{n\to\infty} a_n$ and $b_n = \inf\{a_n, a_{n+1}, \ldots\}$. Let $M = \limsup_{n\to\infty} a_n$. Take any $s > M$. Then, by definition of the $\limsup_{n\to\infty} a_n = M$, we obtain that $a_n < s$ for infinitely many n's which implies that $b_n < s$ for all n and hence $\limsup_{n\to\infty} b_n = m < s$. This holds for all $s > M$. But the infimum of all these s's is M. Therefore $m \le M$ which is*

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

Exercise 5. *If $\{a_n\}$ is a convergent sequence in $\mathbb{R}$ and $a = \lim a_n$, show that $a = \liminf a_n = \limsup a_n$.*

Solution. *Suppose that $\{a_n\}$ is a convergent sequence in $\mathbb{R}$ with limit $a = \lim_{n\to\infty} a_n$. Then by definition, we have: $\forall \epsilon > 0\ \exists N > 0$ such that $\forall n \ge N$, we have $|a_n - a| \le \epsilon$, that is $a - \epsilon \le a_n \le a + \epsilon$. This means that all but finitely many a_n's are $\le a + \epsilon$ and $\ge a - \epsilon$. This shows that*

$$a - \epsilon \le \underbrace{\liminf_{n\to\infty} a_n}_{=:m} \le a + \epsilon$$

and

$$a - \epsilon \le \underbrace{\limsup_{n\to\infty} a_n}_{=:M} \le a + \epsilon.$$

By the previous Exercise 4, we also have

$$a - \epsilon \le m \le M \le a + \epsilon.$$

Hence,

$$0 \le M - m \le 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain $m = M$ and further

$$a - \epsilon \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le a + \epsilon,$$

we obtain

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a.$$

Exercise 6. *Find the radius of convergence for each of the following power series: (a) $\sum_{n=0}^{\infty} a^n z^n$, $a \in \mathbb{C}$; (b) $\sum_{n=0}^{\infty} a^{n^2} z^n$, $a \in \mathbb{C}$; (c) $\sum_{n=0}^{\infty} k^n z^n$, k an integer $\neq 0$; (d) $\sum_{n=0}^{\infty} z^{n!}$.*

Solution. *a) We have $\sum_{n=0}^{\infty} a^n z^n = \sum_{k=0}^{\infty} b_k z^n$ with $b_k = a^k$, $a \in \mathbb{C}$. We also have,*

$$\limsup_{k\to\infty} |b_k|^{1/k} = \limsup_{k\to\infty} |a^k|^{1/k} = \limsup_{k\to\infty} |a| = |a|.$$

Therefore, $R = 1/|a|$, so

$$R = \begin{cases} \frac{1}{|a|}, & a \neq 0 \\ \infty, & a = 0 \end{cases}.$$

b) In this case, $b_n = a^{n^2}$ where $a \in \mathbb{C}$.

$$\begin{aligned} R &= \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a^{n^2}}{a^{(n+1)^2}}\right| = \lim_{n\to\infty} \left|\frac{a^{n^2}}{a^{n^2+2n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{a^{2n+1}}\right| \\ &= \lim_{n\to\infty} \frac{1}{|a|^{2n+1}} = \begin{cases} 0, & |a| > 1 \\ 1, & |a| = 1 \\ \infty, & |a| < 1 \end{cases}. \end{aligned}$$

c) Now, $b_n = k^n$, k is an integer $\neq 0$. We have

$$R = \limsup_{n\to\infty} |b_n|^{1/n} = \limsup_{n\to\infty} |k^n|^{1/n} = \limsup_{n\to\infty} |k| = |k|.$$

So

$$R = \frac{1}{|k|} = \begin{cases} \frac{1}{k}, & k > 0, k \text{ integer} \\ -\frac{1}{k}, & k < 0, k \text{ integer} \end{cases}.$$

d) We can write $\sum_{n=0}^{\infty} z^{n!} = \sum_{k=0}^{\infty} a_k z^k$ where

$$a_k = \begin{cases} 0, & k = 0 \\ 2, & k = 1 \\ 1, & k = n!, n \in \mathbb{N}, n > 1 \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} |1|^{1/k!} = 1.$$

Therefore $1/R = 1$ which implies $R = 1$.

Exercise 7. *Show that the radius of convergence of the power series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1, -1,$ and i. (Hint: The nth coefficient of this series is not $(-1)^n/n$.)

Solution. *Rewrite the power series in standard form, then*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{n=1}^{\infty} a_k z^k \text{ with } a_k = \begin{cases} \frac{(-1)^n}{n} & \text{if } \exists\, n \in \mathbb{N} \text{ s.t. } k = n(n+1) \\ 0 & \text{else} \end{cases}.$$

To find the radius of convergence we use the root criterion and therefore need the estimates

$$1 \leq \sqrt[n(n+1)]{n} \leq \sqrt[n]{n} \qquad \text{for } n \in \mathbb{N}.$$

The first inequality is immediate from the fact that $n \geq 1$ and hence $n^{\frac{1}{n(n+1)}} \geq 1$. For the second inequality note that

$$\begin{aligned} & n \leq n^{n+1} \\ \Leftrightarrow \quad & n^{\frac{1}{n(n+1)}} \leq n^{\frac{1}{n}} \\ \Leftrightarrow \quad & \sqrt[n(n+1)]{n} \leq \sqrt[n]{n} \quad . \end{aligned}$$

Using this one obtains

$$\sqrt[n(n+1)]{\left|\frac{(-1)^n}{n}\right|} = \frac{1}{\sqrt[n(n+1)]{n}} \leq 1$$

and

$$\sqrt[n(n+1)]{\left|\frac{(-1)^n}{n}\right|} = \frac{1}{\sqrt[n(n+1)]{n}} \geq \frac{1}{\sqrt[n]{n}}.$$

Vague memories of calculus classes tell me that $\sqrt[n]{n} \to 1$*, thus* $\frac{1}{R} = \limsup \sqrt[n(n+1)]{a_n} = 1$*, i.e.* $R = 1$*.*

If $z = 1$ *the series reduces to* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *which converges with the Leibniz Criterion.*

If $z = -1$ *we note that the exponents* $n(n+1)$ *are always even integers and therefore the series is the same as in the previous case of* $z = 1$*.*

Now let $z = i$*. The expression* $i^{n(n+1)}$ *will always be real, so if the series converges at* $z = i$*, it converges to a real number. We also note that formally*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} i^{n(n+1)} = \sum_{n=1}^{\infty} c_n \text{ with } \begin{cases} \frac{1}{n} & \text{if} n \mod 4 \in \{0,1\} \\ -\frac{1}{n} & \text{if } n \mod 4 \in \{2,3\} \end{cases}.$$

Define the partial sums $S_k := \sum_{n=0}^{k} c_k$*. We claim that the following chain of inequalities holds*

$$0 \le S_{4k+3} \overset{a)}{<} S_{4k} \overset{b)}{<} S_{4k+4} \overset{c)}{<} S_{4k+2} \overset{d)}{<} S_{4k+1} \le 1.$$

To verify this, note that

$$\begin{aligned}
S_{4k+3} - S_{4k} &= c_{4k+1} + c_{4k+2} + c_{4k+3} = -\frac{16k^2 + 8k - 1}{(4k+1)(4k+2)(4k+3)} < 0, \text{ hence a)} \\
S_{4k+4} - S_{4k} &= c_{4k+3} + c_{4k+4} = -\frac{1}{4k+3} + \frac{1}{4k+4} < 0, \text{ hence c)} \\
S_{4k+2} - S_{4k+1} &= c_{4k+2} < 0, \text{ hence d).}
\end{aligned}$$

Relation b) is obvious and so are the upper bound $c_1 = 1$ *and the non-negativity constraint. We remark that* $\{S_{4k+l}\}_{k \ge 1}, l \in \{0,1,2,3\}$ *describe bounded and monotone subsequences that converge to some point. Now that* $|c_n| \searrow 0$ *the difference between* S_{4k+l} *and* $S_{4k+m}, l, m \in 0,1,2,3$ *tends to zero, i.e. all subsequences converge to the same limit. Therefore the power series converges also in the case of* $z = i$*.*

3.2 Analytic functions

Exercise 1. *Show that* $f(z) = |z|^2 = x^2 + y^2$ *has a derivative only at the origin.*

Solution. *The derivative of* f *at* z *is given by*

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \quad h \in \mathbb{C}$$

provided the limit exist. We have

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\bar{z} + \bar{h}) - z\bar{z}}{h} = \frac{z\bar{z} + h\bar{z} + z\bar{h} + h\bar{h} - z\bar{z}}{h} \\
&= \bar{z} + \bar{h} + z\frac{\bar{h}}{h} =: D.
\end{aligned}$$

If the limit of D *exists, it may be found by letting the point* $h = (x, y)$ *approach the origin (0,0) in the complex plane* $\mathbb{C}$ *in any manner.*

1.) Take the path along the real axes, that is $y = 0$*. Then* $\bar{h} = h$ *and thus*

$$D = \bar{z} + h + z\frac{h}{h} = \bar{z} + h + z$$

and therefore, if the limit of D exists, its value has to be $\bar{z} + z$.
2.) Take the path along the imaginary axes, that is $x = 0$. *Then* $\bar{h} = -h$ *and thus*

$$D = \bar{z} - h - z\frac{h}{h} = \bar{z} - h - z$$

and therefore, if the limit of D exists, its value has to be $\bar{z} - z$.
Because of the uniqueness of the limit of D, we must have

$$\bar{z} + z = \bar{z} - z \iff z = -z \iff z = 0,$$

if the limit of D exists. It remains to show that the limit of D exists at $z = 0$. *Since* $z = 0$, *we have that* $D = \bar{h}$ *and thus the limit of D is 0.*
In summary, the function $f(z) = |z|^2 = x^2 + y^2$ *has a derivative only at the origin with value 0.*

Exercise 2. *Prove that if* b_n, a_n *are real and positive and* $0 < b = \lim b_n < \infty$, $a = \limsup a_n$ *then* $ab = \limsup(a_n b_n)$. *Docs this remain true if the requirement of positivity is dropped?*

Solution. *Let* $a = \limsup_{n\to\infty} a_n < \infty$. *Then there exists a monotonic subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *that converges to a. Since* $\lim_{k\to\infty} b_{n_k} = b$, $\lim_{n\to\infty} a_{n_k} b_{n_k} = ab$. *Hence,* $\{a_{n_k} b_{n_k}\}$ *is a subsequence of* $a_n b_n$ *that converges to ab. So* $ab \le \limsup_{n\to\infty} a_n b_n$. *Hence,* $\limsup_{n\to\infty} a_n b_n \ge b \limsup_{n\to\infty} a_n$.

Now, let $a = \limsup_{n\to\infty} a_n = \infty$. *Then there exists a subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *such that* $\lim_{k\to\infty} a_{n_k} = a > 0$. *And, since* $\lim_{n\to\infty} b_n > 0$, $\lim_{k\to\infty} a_{n_k} b_{n_k} = \infty$. *Hence* $\limsup_{n\to\infty} a_n b_n = \infty$. *In this second case,* $\limsup_{n\to\infty} a_n b_n \ge b \limsup_{n\to\infty} a_n$.

In both cases, we have established that $ab \le \limsup_{n\to\infty} a_n b_n$. *Now, since for all* $n \in \mathbb{N}$, $a_n > 0$ *and* $b_n > 0$, *consider* $\lim_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{b}$. *Applying the inequality we have established replacing* b_n *with* $\frac{1}{b_n}$ *and* a_n *replaced with* $a_n b_n$:

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \frac{1}{b_n}(a_n b_n) \ge \frac{1}{b} \limsup_{n\to\infty} a_n b_n.$$

Rearranging,

$$\limsup_{n\to\infty} a_n b_n \le b \limsup_{n\to\infty} a_n.$$

It follows that $ab = \limsup_{n\to\infty} a_n b_n$ *as required.*

Now, consider the case where we drop the positivity requirement. Let $b_n = 0, -\frac{1}{2}, 0, -\frac{1}{3}, 0, -\frac{1}{4}, \ldots$ *and note* $0 = b = \lim_{n\to\infty} b_n < \infty$. *Also let* $a_n = 0, -2, 0, -3, 0, -4 \ldots$ *and note* $\limsup_{n\to\infty} a_n = a = 0$. *In this case,* $ab = 0 \ne 1 = \limsup_{n\to\infty} a_n b_n$.

Exercise 3. *Show that* $\lim n^{1/n} = 1$.

Solution. *Let $n \in \mathbb{N}$. Also let $a = n^{1/n}$. Then*

$$\begin{aligned} & a = n^{1/n} \\ \iff & \log a = \log n^{1/n} \\ \iff & \log a = \frac{\log n}{n} \\ \iff & \lim_{n\to\infty} \log a = \lim_{n\to\infty} \frac{\log n}{n} \end{aligned}$$

Now, let $f(x) = \dfrac{\log(x)}{x}$. Then $\lim_{n\to\infty} f(x) = 0$ by L'Hopital's Rule. Thus, $\lim_{n\to\infty} \dfrac{\log n}{n} = \lim_{n\to\infty} f(n) = 0$ also. So $\lim_{n\to\infty} a = \lim_{n\to\infty} n^{1/n} = 1$.

Exercise 4. *Show that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$.*

Solution. *We have by definition*

$$(\cos z)' = \left(\frac{1}{2}\left(e^{iz} + e^{-iz}\right)\right)' = \frac{i}{2}\left(e^{iz} - e^{-iz}\right) = -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

and similarly

$$(\sin z)' = \left(\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right)' = \frac{i}{2i}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z.$$

Exercise 5. *Derive formulas (2.14).*

Solution. *Not available.*

Exercise 6. *Describe the following sets: $\{z : e^z = i\}$, $\{z : e^z = -1\}$, $\{z : e^z = -i\}$, $\{z : \cos z = 0\}$, $\{z : \sin z = 0\}$.*

Solution. *Using the definition we obtain*

$$\{z : e^z = i\} = \left\{\left(\frac{1}{2} + 2k\right)\pi i\right\}, \qquad \{z : e^z = -1\} = \{(1 + 2k)\,\pi i\},$$

$$\{z : e^z = -i\} = \left\{\left(-\frac{1}{2} + 2k\right)\pi i\right\}, \qquad \{z : \cos z = 0\} = \left\{\left(\frac{1}{2} + k\right)\pi\right\},$$

and

$$\{z : \sin z = 0\} = \{k\pi\}$$

where $k \in \mathbb{Z}$.

Exercise 7. *Prove formulas for $\cos(z + w)$ and $\sin(z + w)$.*

Solution. *We have*

$$\begin{aligned} \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i}. \end{aligned}$$

1. Claim: $\cos(z)\cos(w) - \sin(z)\sin(w) = \cos(z+w)$.
Proof:

$$\begin{aligned}
\cos(z)\cos(w) - \sin(z)\sin(w) &= \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}+e^{-iw}}{2} - \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{1}{4}\left(e^{iz}e^{iw} + e^{-iz}e^{-iw}\right) + \frac{1}{4}\left(e^{iz}e^{iw} + e^{-iz}e^{-iw}\right) \\
&= \frac{1}{2}e^{iz}e^{iw} + \frac{1}{2}e^{-iz}e^{-iw} \\
&= \frac{1}{2}\left(e^{iz}e^{iw} + e^{-iz}e^{-iw}\right) \\
&= \cos(z+w).
\end{aligned}$$

2. Claim: $\sin(z)\cos(w) + \cos(z)\sin(w) = \sin(z+w)$.
Proof:

$$\begin{aligned}
\sin(z)\cos(w) + \cos(z)\sin(w) &= \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}+e^{-iw}}{2} + \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{1}{4i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right) + \frac{1}{4i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right) \\
&= \frac{1}{2i}e^{iz}e^{iw} - \frac{1}{2}e^{-iz}e^{-iw} \\
&= \frac{1}{2i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right) \\
&= \sin(z+w).
\end{aligned}$$

Exercise 8. *Define* $\tan z = \frac{\sin z}{\cos z}$*; where is this function defined and analytic?*

Solution. *Since both* $\sin z$ *and* $\cos z$ *are analytic in the entire complex plane, it follows from the discussion in the text following Definition 2.3 that* $\tan z = \frac{\sin z}{\cos z}$ *is analytic wherever* $\cos z \neq 0$. *Now,* $\cos z = 0$ *implies that z is real and equal to an odd multiple of* $\frac{\pi}{2}$. *Thus let*

$$G \equiv \left\{ \frac{(2k+1)\pi}{2} \;\middle|\; k \in \mathbb{Z} \right\}.$$

Then $\tan z$ *is defined and analytic on* $\mathbb{C} - G$. *If* $z \in G$*, then* $\cos z = 0$ *so* $\tan z$ *is undefined on the non-extended complex plane.*

Exercise 9. *Suppose that* $z_n, z \in G = \mathbb{C} - \{z : z \leq 0\}$ *and* $z_n = r_n e^{i\theta_n}, z = re^{i\theta}$ *where* $-\pi < \theta, \theta_n < \pi$. *Prove that if* $z_n \to z$ *then* $\theta_n \to \theta$ *and* $r_n \to r$.

Solution. *Not available.*

Exercise 10. *Prove the following generalization of Proposition 2.20. Let G and* Ω *be open in* $\mathbb{C}$ *and suppose* f *and* h *are functions defined on* G, $g : \Omega \to \mathbb{C}$ *and suppose that* $f(G) \subset \Omega$. *Suppose that g and h are analytic,* $g'(\omega) \neq 0$ *for any* ω*, that* f *is continuous, h is one-one, and that they satisfy* $h(z) = g(f(z))$ *for z in G. Show that* f *is analytic. Give a formula for* $f'(z)$.

Solution. *Not available.*

Exercise 11. *Suppose that* $f : G \to \mathbb{C}$ *is a branch of the logarithm and that n is an integer. Prove that* $z^n = \exp(nf(z))$ *for all z in G.*

Solution. *Let $f(z)$ be a branch of the logarithm so that $e^{f(z)} = z$. Let $n \in \mathbb{Z}$ and consider $e^{nf(z)}$. In the following cases we shall apply several of the properties of the complex exponential function developed in the discussion on page 38 in the text.*
CASE 1: Assume $n > 0$. Then we note that

$$\begin{aligned} e^{nf(z)} &= e^{f(z)+f(z)+\cdots+f(z)\ (n\ times)} \\ &= e^{f(z)}e^{f(z)}\cdots e^{f(z)}\ (n\ times) \\ &= \left(e^{f(z)}\right)^n \\ &= x^n. \end{aligned}$$

CASE 2: Assume $n < 0$. Then let $m = -n$ so that $m > 0$ and

$$e^{nf(z)} = \frac{1}{e^{mf(z)}} = \frac{1}{z^m} = z^n,$$

the middle step following from Case 1.
CASE 3: Assume $n = 0$. Then $e^{nf(z)} = e^0 = 1 = z^0 = z^n$.

Exercise 12. *Show that the real part of the function $z^{\frac{1}{2}}$ is always positive.*

Solution. *We know that we can write $z = re^{i\theta} \neq 0$, $-\pi < \theta < \pi$ and $Log(z) = ln(r) + i\theta$. Thus*

$$z^{\frac{1}{2}} = e^{Log z^{\frac{1}{2}}} = e^{\frac{1}{2}Log z} = e^{\frac{1}{2}(ln r + i\theta)} = e^{\frac{1}{2}ln r}e^{\frac{1}{2}\theta i} = e^{\frac{1}{2}ln r}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right).$$

Hence,

$$Re(z) = e^{\frac{1}{2}ln r}\cos\left(\frac{\theta}{2}\right) > 0,$$

since $e^{\frac{1}{2}ln r} > 0$ and $\cos\left(\frac{\theta}{2}\right) > 0$ since $-\pi < \theta < \pi$. Thus, the real part of the function $z^{\frac{1}{2}}$ is always positive.

Exercise 13. *Let $G = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$ and let n be a positive integer. Find all analytic functions $f : G \to \mathbb{C}$ such that $z = (f(z))^n$ for all $z \in G$.*

Solution. *Let $Log(z)$ be the principal branch, then $\log(z) = Log(z) + 2k\pi i$ for some $k \in \mathbb{Z}$. Thus, we can write*

$$f(z) = z^{1/n} = e^{\log(z)/n} = e^{(Log(z)+2k\pi i)/n} = e^{Log(z)/n}\cdot e^{2k\pi i/n}.$$

We know that the latter factor are the n-th roots of unity and depend only on k and n. They correspond to the n distinct powers of the expression $\zeta = e^{2\pi i/n}$. Therefore, the branches of $z^{1/n}$ on the set U are given by

$$f(z) = \zeta^k\cdot e^{Log(z)/n},$$

where $k = 0, \ldots, n-1$ and therefore they are all constant multiples of each other.

Exercise 14. *Suppose $f : G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.*

Solution. *First of all, we can write $f : G \to \mathbb{C}$ as*

$$f(z) = u(z) + iv(z)$$

where u, v are real-valued functions. Since $f : G \to \mathbb{C}$ is analytic, that is f is continuously differentiable (Definition 2.3), we have that u and v have continuous partial derivatives. By Theorem 2.29, this implies that u, v satisfy the Cauchy-Riemann equations. That is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3.1}$$

Since $f(z)$ is real $\forall z \in G$, this implies

$$v(z) \equiv 0$$

and therefore $f(z) = u(z)$. So, since $v(z) = 0$, we have

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and by (3.1) we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

and thus $u'(z) = 0$ (see reasoning of equation 2.22 and 2.23 on page 41). Hence $f'(z) = 0$. Since G is connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0\ \forall z \in G$, we have that f is constant.

Exercise 15. *For $r > 0$ 1et $A = \left\{\omega : \omega = \exp\left(\frac{1}{z}\right) \text{ where } 0 < |z| < r\right\}$; determine the set A.*

Solution. *Define the set $S = \{z : 0 < |z| < r\}$ where $r > 0$. The image of this set under $1/z$ is clearly*

$$T = \left\{z : \frac{1}{r} < |z|\right\}.$$

To find the image of A is the same as finding the image of T under e^z.
Claim: The image of A is $\mathbb{C} - \{0\}$ (thus not depending on r).
To proof the claim, we need to show that for $w \neq 0$, the equation $e^z = w$ has a solution $z \in T$. Using polar coordinates we can write $w = |w|e^{i\theta}$. We have to find a complex number $z = x + iy$ such that $x^2 + y^2 > \frac{1}{r}$ and $e^x e^{iy} = w = |w|e^{i\theta}$. We want $e^x = |w|$ and $y = \theta + 2k\pi$, for some $k \in \mathbb{Z}$. Using $x = \log|w|$ and $k \gg 0$ such that $(\log|w|)^2 + (\theta + 2k\pi)^2 > \frac{1}{r}$, then we found $z = x + iy$.

Exercise 16. *Find an open connected set $G \subset \mathbb{C}$ and two continuous functions f and g defined on G such that $f(z)^2 = g(z)^2 = 1 - z^2$ for all z in G. Can you make G maximal? Are f and g analytic?*

Solution. *Not available.*

Exercise 17. *Give the principal branch of $\sqrt{1-z}$.*

Solution. *Not available.*

Exercise 18. *Let $f : G \to \mathbb{C}$ and $g : G \to \mathbb{C}$ be branches of z^a and z^b respectively. Show that fg is a branch of z^{a+b} and f/g is a branch of z^{a-b}. Suppose that $f(G) \subset G$ and $g(G) \subset G$ and prove that both $f \circ g$ and $g \circ f$ are branches of z^{ab}.*

Solution. *Not available.*

Exercise 19. *Let G be a region and define $G^* = \{z : \overline{z} \in G\}$. If $f : G \to \mathbb{C}$ is analytic prove that $f^* : G^* \to \mathbb{C}$, defined by $f^*(z) = \overline{f(\overline{x})}$, is also analytic.*

Solution. *Let $z = x + iy$ and let $f(z) = u(x, y) + iv(x, y)$. By assumption f is analytic and therefore u and v have continuous partial derivatives. In addition the Cauchy-Riemann Equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. Since $f^*(z) = \overline{f(\overline{x})}$, we have $f^*(z) = u^*(x, y) + iv^*(x, y)$ where $u^*(x, y) = u(x, -y)$ and $v^*(x, y) = -v(x, -y)$. Hence, we have $u^*_x(x, y) = u_x(x, -y) = v_y(x, -y)$, $u^*_y(x, y) = -u_y(x, -y) = v_x(x, -y)$, $v^*_x(x, y) = -v_x(x, -y)$ and $v^*_y(x, y) = v_y(x, -y)$ and therefore $u^*_x = v^*_y$ and $u^*_y = -v^*_x$ so f^* is analytic.*

Exercise 20. *Let $z_1, z_2, \ldots, z_n$ be complex numbers such that Re $z_k > 0$ and Re$(z_l \ldots z_k) > 0$ for $1 \le k \le n$. Show that $\log(z_1 \ldots z_n) = \log z_1 + \ldots + \log z_n$, where $\log z$ is the principal branch of the logarithm. If the restrictions on the z_k are removed, does the formula remain valid?*

Solution. *Let $z_1, \ldots, z_n \in \mathbb{C}$ such that Re$(z_j) > 0$ and Re$(z_1 \cdots z_j) > 0$ for $1 \le j \le n$. The proof will be by induction. Consider first the case where $n = 2$. Let $z_1, z_2 \in \mathbb{C}$ as above. Then $-\frac{\pi}{2} < \arg z_1 < \frac{\pi}{2}$, $-\frac{\pi}{2} < \arg z_2 < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg(z_1 z_2) < \frac{\pi}{2}$. But note $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ implies that $-\pi < \arg z_1 + \arg z_2 < \pi$. Now, recall*

$$\begin{aligned}\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2)\\ &= \ln|z_1| + \ln|z_2| + i\arg(z_1) + i\arg(z_2)\\ &= \log(z_1) + \log(z_2)\end{aligned}$$

Assume this formula is true for $n = k - 1$. Let $z_1, \ldots, z_k \in \mathbb{C}$ as above. Then

$$\begin{aligned}\log(z_1 \ldots z_k) &= \log((z_1 \cdots z_{k-1}) z_k)\\ &= \log(z_1 \cdots z_{k-1}) + \log z_k \ \ \textit{by the case when } n = 2\\ &= \log z_1 + \log z_2 + \ldots + \log z_{k-1} + \log z_k \ \textit{by the case when } n = k - 1\end{aligned}$$

Hence, this is true for all n such that z_i, $1 \le i \le n$, satisfy the restrictions.

If the restriction on the z_k are removed, does the formula remain valid? Consider the complex numbers $z_1 = -1 + 2i$, $z_2 = -2 + i$. Then $z_1 z_2 = 0 - 5i$. Clearly, z_1, z_2, and $z_1 z_2$ do not meet the restrictions as stated above. Now,

$$\begin{aligned}\log(z_1) &= \ln|z_1| + i\arg z_1\\ \log(z_2) &= \ln|z_2| + i\arg z_2\end{aligned}$$

Thus $\log z_1 + \log z_2 = \ln\sqrt{5} + \ln\sqrt{5} + i(\frac{3\pi}{2}) = \ln 5 + i(\frac{3\pi}{2})$, but note that $\frac{3\pi}{2} \notin (-\pi, \pi)$ and $\log(z_1 z_2) = \ln 5 + i(-\frac{\pi}{2})$. Thus $\log z_1 + \log z_2 \neq \log(z_1 z_2)$ where $\log z$ is the principal branch of the logarithm. Hence, the formula is invalid.

Exercise 21. *Prove that there is no branch of the logarithm defined on $G = \mathbb{C} - \{0\}$. (Hint: Suppose such a branch exists and compare this with the principal branch.)*

Solution. *Define the subset $\hat{G}$ of G by $\hat{G} = \mathbb{C} - \{z \in \mathbb{R} : z \le 0\}$. We use the notation Log for the principal part of the* log *on $\hat{G}$, that is*

$$Log(z) = \log|z| + i\arg(z)$$

where $\arg(z) \in (-\pi, \pi)$. We will prove the statement above by contradiction.
Assume $f(z)$ is a branch of the logarithm defined on G. Then restricting f to $\hat{G}$ gives us a branch of the log *on $\hat{G}$. Therefore its only difference to the principal branch is $2\pi ik$ for some $k \in \mathbb{Z}$. This yields*

$$f(z) = \log|z| + i\arg(z) + 2\pi ik$$

where $z \in \hat{G}$. Since f is analytic in G, it is continuous at -1. But we can check that this is not the case, thus we have derived a contradiction

3.3 Analytic functions as mappings. Möbius transformations

Exercise 1. *1. Find the image of* $\{z : Re\, z < 0, |Im\, z| < \pi\}$ *under the exponential function.*

Solution. *We have*

$$\{z : Re\, z < 0, |Im\, z| < \pi\} = \{z = x + iy : x < 0, -\pi < y < \pi\}.$$

The image of $\{z = x + iy : x < 0, -\pi < y < \pi\}$ *under the exponential function is given by*

$$\begin{aligned} \{e^{x+iy} : x < 0, -\pi < y < \pi\} &= \{e^x e^{iy} : x < 0, -\pi < y < \pi\} \\ &= \{e^x(\cos(y) + i\sin(y)) : x < 0, -\pi < y < \pi\} \\ &= \{r(\cos(y) + i\sin(y)) : 0 < r < 1, -\pi < y < \pi\}. \end{aligned}$$

If $0 < r < 1$ *is fixed, then* $r(\cos(y) + i\sin(y))$ *describes a circle with radius r without the point* $(-r, 0)$ *centered at* $(0, 0)$.
Since r varies between 0 and 1, we get that the image is a solid circle with radius 1, where the boundary does not belong to it, the negative x-axis does not belong to it and the origin does not belong to is.

Exercise 2. *Do exercise 1 for the set* $\{z : |Im\, z| < \pi/2\}$.

Solution. *We have*

$$\{z : |Im\, z| < \pi/2\} = \{z = x + iy : x \in \mathbb{R}, -\frac{\pi}{2} < y < \frac{\pi}{2}\}.$$

The image of $\{z = x + iy : x \in \mathbb{R}, -\frac{\pi}{2} < y < \frac{\pi}{2}\}$ *under the exponential function is given by*

$$\begin{aligned} \{e^{x+iy} : x \in \mathbb{R}, -\frac{\pi}{2} < y < \frac{\pi}{2}\} &= \{e^x(\cos(y) + i\sin(y)) : x \in \mathbb{R}, -\frac{\pi}{2} < y < \frac{\pi}{2}\} \\ &= \{r(\cos(y) + i\sin(y)) : r > 0, -\frac{\pi}{2} < y < \frac{\pi}{2}\}. \end{aligned}$$

If $r > 0$ *is fixed, then* $r(\cos(y) + i\sin(y))$ *describes a half circle with radius r centered at* $(0, 0)$ *lying in the right half plane not touching the imaginary axis.*
Since r varies between 0 and infinity, we get that the image is the right half plane without the imaginary axis.

Exercise 3. *Discuss the mapping properties of* $\cos z$ *and* $\sin z$.

Solution. *Not available.*

Exercise 4. *Discuss the mapping properties of* z^n *and* $z^{1/n}$ *for* $n \geq 2$. *(Hint: use polar coordinates.)*

Solution. *Not available.*

Exercise 5. *Find the fixed points of a dilation, a translation and the inversion on* $\mathbb{C}_\infty$.

Solution. *In general, we have*

$$S(z) = \frac{az + b}{cz + d}.$$

To get a dilation $S(z) = az$, *we have that* $b = 0, c = 0, d = 1$ *and* $a > 0$. *To find a fixed point, we have to find all z such that* $S(z) = z$. *In this case* $az = z$. *Obviously* $z = 0$ *is a fixed point. Also* $z = \infty$ *is a fixed point, since* $a\cdot\infty = \infty (a > 0)$ *or* $S(\infty) = \frac{a}{c} = \frac{a}{0} = \infty$.
To get a translation $S(z) = z + b$, *we have that* $a = 1, c = 0, d = 1$ *and* $b \in \mathbb{R}$. *To find a fixed point, we have to find all z such that* $S(z) = z$. *In this case* $z + b = z$, *which is true if* $z = \infty$. *To see that* $S(\infty) = \frac{a}{c} = \frac{1}{0} = \infty$.
To get a inversion $S(z) = \frac{1}{z}$, *we have that* $a = 0, b = 1, c = 1$ *and* $d = 0$. *To find a fixed point, we have to find all z such that* $S(z) = z$. *In this case* $\frac{1}{z} = z$ *which is equivalent to* $z^2 = 1$ *and thus* $z = 1$ *and* $z = -1$ *are fixed points.*

Exercise 6. *Evaluate the following cross ratios: (a)* $(7+i,1,0,\infty)$ *(b)* $(2,1-i,1,1+i)$ *(c)* $(0,1,i,-1)$ *(d)* $(i-1,\infty,1+i,0)$.

Solution. *We have*

$$S(z) = \frac{z-z_3}{z-z_4}\Big/\frac{z_2-z_3}{z_2-z_4}, \ \text{if } z_2,z_3,z_4 \in \mathbb{C} \tag{3.2}$$

$$S(z) = \frac{z-z_3}{z-z_4}, \ \text{if } z_2=\infty \tag{3.3}$$

$$S(z) = \frac{z-z_3}{z_2-z_3}, \ \text{if } z_4=\infty. \tag{3.4}$$

a) By (3.4) we get

$$(7+i,1,0,\infty) = \frac{7+i-0}{1-0} = 7+i.$$

b) By (3.2) we get

$$(2,1-i,1,1+i) = \frac{2-1}{2-1-i}\Big/\frac{1-i-1}{1-i-1-i} = \frac{1}{1-i}\Big/\frac{-i}{-2i} = \frac{2}{1-i} = 2\frac{1+i}{(1-i)(i+1)} = 2\frac{1+i}{2} = 1+i.$$

c) By (3.2) we get

$$(0,1,i,-1) = \frac{0-i}{0+1}\Big/\frac{1-i}{1+1} = -i\Big/\frac{1-i}{2} = -i\frac{2}{1-i}\frac{1+i}{1+i} = -i\frac{2}{2}(1+i) = -i-i^2 = 1-i.$$

d) By (3.3) we get

$$(i-1,\infty,1+i,0) = \frac{i-1-1-i}{i-1-0} = \frac{-2}{i-1}\frac{i+1}{i+1} = \frac{-2}{-2}(i+1) = 1+i.$$

Exercise 7. *If* $Tz = \frac{az+b}{cz+d}$, *find* z_2, z_3, z_4 *(in terms of a, b, c, d) such that* $Tz = (z,z_2,z_3,z_4)$.

Solution. *The inverse of* T *is given by*

$$T^{-1}(z) = \frac{dz-b}{-cz+a}$$

as shown on p. 47. We have $T^{-1}(1) = \frac{d-b}{a-c}$, $T^{-1}(0) = -\frac{b}{a}$ *and* $T^{-1}(\infty) = -\frac{d}{c}$. *Set* $z_2 = \frac{d-b}{a-c}$, $z_3 = -\frac{b}{a}$ *and* $z_4 = -\frac{d}{c}$ *to obtain* $Tz = (z,z_2,z_3,z_4)$.

Exercise 8. *If* $Tz = \frac{az+b}{cz+d}$ *show that* $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ *iff we can choose* a,b,c,d *to be real numbers.*

Solution. *Let* $Tz = \dfrac{az+b}{cz+d}$. *Assume* $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. *Then let* $z_0 \in \mathbb{R}_\infty$ *such that* $Tz_0 = 0$. *Observe that this implies* $az_0 = -b$, *so* $-\dfrac{b}{a} \in \mathbb{R}_\infty$ *and* $r_1 \equiv \dfrac{b}{a} \in \mathbb{R}_\infty$ *as well. Likewise, if* $z_\infty \in \mathbb{R}_\infty$ *such that* $Tz_\infty = \infty$, *then*

$r_2 \equiv \dfrac{d}{c} \in \mathbb{R}_\infty$. Now let $z_1 \in \mathbb{R}_\infty$ such that $Tz_1 = 1$. Then

$$\begin{aligned}
\frac{az_1 + b}{cz_1 + d} &= 1 \\
az_1 + b &= cz_1 + d \\
z_1\left(1 - \frac{c}{a}\right) &= \frac{d}{a} - \frac{b}{a} \\
\frac{z_1}{c} - \frac{z_1}{a} - \frac{r_2}{a} + \frac{r_1}{c} &= 0 \\
\frac{z_1 + r_1}{c} &= \frac{z_1 + r_2}{a} \\
\frac{z_1 + r_1}{z_1 + r_2} &= \frac{c}{a} \in \mathbb{R}_\infty.
\end{aligned}$$

Let $r_3 = \dfrac{c}{a}$. Then $\dfrac{d}{a} = \dfrac{d}{c} \times \dfrac{c}{a} = r_2 r_3 \in \mathbb{R}_\infty$. Thus

$$Tz = \frac{az + b}{cz + d} = \frac{z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} = \frac{z + r_1}{r_3 z + r_2 r_3}$$

and we have thus found real coefficients for T.

Now to prove the converse, assume $T(\mathbb{R}_\infty) \neq \mathbb{R}_\infty$. Then recognizing that $\mathbb{R}_\infty$ is a circle in $\mathbb{C}_\infty$, by Theorem 3.14 we conclude that $T(\mathbb{R}_\infty)$ is some other circle in $\mathbb{C}_\infty$. In particular, this means that $T(\mathbb{R}_\infty) \cap (\mathbb{C}_\infty - \mathbb{R}_\infty) \neq \emptyset$. In other words, there must be some value $z_c \in \mathbb{R}_\infty$ for which $Tz_c \notin \mathbb{R}_\infty$.

Now, suppose by way of contradiction that there is some representation of T in which a, b, c, and d are all real. Then

$$Tz_c = \frac{az_c + b}{cz_c + d}$$

is clearly an element of $\mathbb{R}_\infty$, contradicting the observation that Tz_c is not real. We see, therefore, that there is a real choice for a, b, c, and d simultaneously if and only if $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$.

Exercise 9. *If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z : |z| = 1\}$.*

Solution. *We want if $z\bar{z} = 1$, then $T(z)\overline{T(z)} = 1$.*

$$\begin{aligned}
T(z)\overline{T(z)} = 1 \quad &\iff \quad \frac{az+b}{cz+d}\,\overline{\frac{az+b}{cz+d}} = 1 \iff \frac{(az+b)(\overline{az}+\bar{b})}{(cz+d)(\overline{cz}+\bar{d})} = 1 \\
&\iff \quad az\overline{az} + b\overline{az} + az\bar{b} + b\bar{b} = cz\overline{cz} + d\overline{cz} + \bar{d}cz + d\bar{d} \\
&\iff \quad z\bar{z}(a\bar{a} - c\bar{c}) + z(a\bar{b} - c\bar{d}) + \bar{z}(b\bar{a} + d\bar{c}) - d\bar{d} = 0.
\end{aligned}$$

This is the same as the equation $z\bar{z} - 1 = 0$ if

$$\begin{aligned}
a\bar{b} - c\bar{d} &= 0 \\
b\bar{a} - d\bar{c} &= 0
\end{aligned}$$

and

$$\begin{aligned}
a\bar{a} - c\bar{c} &= 1 \\
b\bar{b} - d\bar{d} &= -1
\end{aligned}$$

which is equivalent to $|a|^2 - |c|^2 = 1$ *and* $1 = |d|^2 - |b|^2$. *Hence, we have the two conditions*

$$a\overline{b} - c\overline{d} = 0 \qquad \text{and} \qquad |a|^2 + |b|^2 = |c|^2 + |d|^2.$$

These are the sufficient conditions. Let $c = \lambda\overline{b}$, *then* $a\overline{b} - c\overline{d} = 0$ *yields* $a\overline{b} - \lambda\overline{b}\overline{d} = 0$ *iff* $a = \lambda\overline{d}$ *iff* $d = \frac{\overline{a}}{\lambda}$. *Insert this into* $|c|^2 + |d|^2 = |a|^2 + |b|^2$ *yields*

$$|\lambda|^2|b|^2 + \frac{|a|^2}{|\lambda|^2} = |a|^2 + |b|^2,$$

and therefore we can take $|\lambda| = 1$. *Then* $\frac{1}{\lambda} = \lambda$, *and so the form of the Möbius transformation is*

$$T(z) = \frac{az+b}{\lambda(\overline{b}z + \overline{a})}, \qquad \text{where } |\lambda| = 1$$

or

$$T(z) = \overline{\lambda}\frac{az+b}{\overline{b}z+\overline{a}}, \qquad \text{where } |\lambda| = 1.$$

Take $\lambda = e^{-i\theta}$, *then*

$$T(z) = e^{i\theta}\frac{az+b}{\overline{b}z+\overline{a}}, \qquad \text{for some } \theta.$$

This mapping transforms $|z| = 1$ *into* $|T(z)| = 1$.

Exercise 10. *Let* $D = \{z : |z| < 1\}$ *and find all Möbius transformations* T *such that* $T(D) = D$.

Solution. *Take an* $\alpha \in D = \{z : |z| < 1\}$ *such that* $T(\alpha) = 0$. *Its symmetric point with respect to the unit circle is*

$$\alpha^* = \frac{1}{\overline{\alpha}}$$

since

$$z^* - a = \frac{R^2}{\overline{z} - \overline{a}}$$

(see page 51) *with* $a = 0$ *and* $R = 1$ *(the unit circle). Therefore* $T(\alpha^*) = \infty$. *Thus* T *looks like*

$$T(z) = K\frac{z-\alpha}{\overline{\alpha}z - 1}$$

where K *is a constant. (It is easy to check that* $T(\alpha) = 0$ *and* $T(\alpha^*) = T(\frac{1}{\overline{\alpha}}) = \infty$*). Finally, we are going to choose the constant* K *in such a way that* $|T(z_0)| = 1$ *where* $z_0 = e^{i\theta}$. *We have*

$$T(z_0) = K\frac{e^{i\theta} - \alpha}{\overline{\alpha}e^{i\theta} - 1}$$

and therefore

$$1 = |T(z_0)| = |K|\frac{|e^{i\theta} - \alpha|}{|\overline{\alpha}e^{i\theta} - 1|} = |K|\frac{\left(e^{i\theta} - \alpha\right)\left(e^{-i\theta} - \overline{\alpha}\right)}{|e^{i\theta}|\cdot|\overline{\alpha} - e^{-i\theta}|} = |K|\frac{\left(e^{i\theta} - \alpha\right)\left(e^{-i\theta} - \overline{\alpha}\right)}{\left(\overline{\alpha} - e^{-i\theta}\right)\left(\alpha - e^{i\theta}\right)} = |K|.$$

So $|K| = 1$ *implies* $K = e^{i\theta}$ *for some real* θ. *We arrive at*

$$T(z) = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z - 1}, \qquad \text{for some real } \theta.$$

Exercise 11. *Show that the definition of symmetry (3.17) does not depend on the choice of points* z_2, z_3, z_4*. That is, show that if* $\omega_2, \omega_3, \omega_4$ *are also in* Γ *then equation (3.18) is satisfied iff* $(z^*, \omega_2, \omega_3, \omega_4) = \overline{(z, \omega_2, \omega_3, \omega_4)}$*. (Hint: Use Exercise 8.)*

Solution. *Let* Γ *be a circle in* $\mathbb{C}_\infty$ *containing points* z_2*,* z_3*,* z_4*,* w_2*,* w_3*, and* w_4*, with all* z_i *distinct and all* w_i *distinct. Let* $z \in \mathbb{C}_\infty$ *and consider* z^**, as defined in the text and established by the points* z_i*. We know that*

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$$

Recall that the above left-hand cross-ratio implies a unique Möbius transformation T *for which* $Tz_2 = 1$*,* $Tz_3 = 0$*, and* $Tz_4 = \infty$*. By Proposition 3.10,* T *maps* Γ *to* $\mathbb{R}_\infty$*. In fact, the definition of symmetry may be rewritten as*

$$Tz^* = \overline{Tz}.$$

or

$$z^* = T^{-1}\overline{Tz} \tag{3.5}$$

Likewise, there is some transformation S *for which* $Sw_2 = 1$*,* $Sw_3 = 0$*, and* $Sw_4 = \infty$*. Again,* S *maps* Γ *to* $\mathbb{R}_\infty$*. We wish to show that*

$$z^* = S^{-1}\overline{Sz}. \tag{3.6}$$

Now, we proceed by showing that the right hand sides of (3.5) and (3.6) must be equal:

$$T^{-1}\overline{Tz} = T^{-1}\overline{TS^{-1}Sz}$$

Observe here that TS^{-1} *must have real coefficients by Exercise 8 because* $\mathbb{R}_\infty \mapsto \mathbb{R}_\infty$ *under* TS^{-1}*. Thus we observe that* $\overline{TS^{-1}} = TS^{-1}$ *so*

$$T^{-1}\overline{Tz} = T^{-1}TS^{-1}\overline{Sz} = S^{-1}\overline{Sz}.$$

Hence we have the desired result.

Exercise 12. *Prove Theorem 3.4.*

Solution. *Not available.*

Exercise 13. *Give a discussion of the mapping* $f(z) = \frac{1}{2}(z + 1/z)$*.*

Solution. *The function* $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$ *can be defined for all* $z \in \mathbb{C} - \{0\}$ *and therefore also in the punctured disk* $0 < |z| < 1$*. To see that in this domain the function is injective, let* z_1, z_2 *be two numbers in the domain of* f *with the same image, then*

$$0 = f(z_1) - f(z_2) = \frac{z_1^2 + 1}{z_1} - \frac{z_2^2 + 1}{z_2} = \frac{z_1^2 z_2 + z_2 - z_1 z_2^2 - z_1}{z_1 z_2} = \frac{(z_1 z_2 - 1)(z_1 - z_2)}{z_1 z_2}.$$

With the assumption $0 < |z_i| < 1, i = 1, 2$ *the factor* $z_1 z_2 - 1$*, is always nonzero and we conclude that* $z_1 = z_2$*; hence* $f(z)$ *is injective.*
The range of the function is $\mathbb{C} - \{z \in \mathbb{C} \mid \Re z \in [-1, 1] \text{ and } \operatorname{Im} z = 0\}$*. To see this, write* $z = re^{i\theta}$ *in polar coordinates and let* $f(z) = w = a + ib, a, b \in \mathbb{R}$*.*

$$f(z) = f(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = \frac{1}{2}\left[\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right].$$

For the real and imaginary part of w the following equations must hold

$$\begin{aligned} a &= \tfrac{1}{2}\left(r + \tfrac{1}{r}\right)\cos\theta \\ b &= \tfrac{1}{2}\left(r - \tfrac{1}{r}\right)\sin\theta. \end{aligned} \tag{3.7}$$

If $f(z) = w$ has imaginary part $b = 0$ then $\sin\theta = 0$ and $|\cos\theta| = 1$. Therefore points of the form $a + ib, a \in [-1, 1], b = 0$ cannot be in the range of f. For all other points the equations (3.7) can be solved for r and θ uniquely (after restricting the argument to $[-\pi, \pi)$).
Given any value of $r \in (0, 1)$, the graph of $f(re^{i\theta})$ as a function of θ looks like an ellipse. In fact from formulas (3.7) we see that $\left(\frac{a}{\frac{1}{2}(r+\frac{1}{r})}\right)^2 + \left(\frac{b}{\frac{1}{2}(1-\frac{1}{r})}\right)^2 = 1$.
If we fix the argument θ and let r vary in $(0, 1)$ it follows from from equation (3.7) that the graph of $f(re^{i\theta})$ is a hyperbola and it degenerates to rays if z is purely real or imaginary. In the case $\theta \in \{(2k + 1)\pi \mid k \in \mathbb{Z}\}$ the graph of f in dependence on r is on the imaginary axis and for $\theta \in \{2k\pi \mid k \in \mathbb{Z}\}$ the graph of $f(re^{i\theta})$ is either $(-\infty, -1)$ or $(1, \infty)$. If $\cos\theta \neq 0$ and $\sin\theta \neq 0$ then $\left(\frac{a}{\cos\theta}\right)^2 - \left(\frac{b}{\sin\theta}\right)^2 = 1$.

Exercise 14. *Suppose that one circle is contained inside another and that they are tangent at the point a. Let G be the region between the two circles and map G conformally onto the open unit disk. (Hint: first try $(z - a)^{-1}$.)*

Solution. *Using the hint, define the Möbius transformation $T(z) = (z - a)^{-1}$ which sends the region G between two lines. Afterward applying a rotation followed by a translation, it is possible to send this region to any region bounded by two parallel lines we want. Hence, choose $S(z) = cz + d$ where $|c| = 1$ such that*

$$S(T(G)) = \left\{x + iy : 0 < y < \frac{\pi}{2}\right\}.$$

Applying the exponential function to this region yields the right half plane

$$\exp(S(T(G))) = \{x + iy : x > 0\}.$$

Finally, the Möbius transformation

$$R(z) = \frac{z-1}{z+1}$$

maps the right half plane onto the unit disk (see page 53). Hence the function f defined by $R(\exp(S(T(z))))$ maps G onto D and is the desired conformal mapping (f is a composition of conformal mappings). Doing some simplifications, we obtain

$$f(z) = \frac{e^{\frac{c}{z-a}+d} - 1}{e^{\frac{c}{z-a}+d} + 1}$$

where the constants c and d will depend on the circle location.

Exercise 15. *Can you map the open unit disk conformally onto $\{z : 0 < |z| < 1\}$?*

Solution. *Not available.*

Exercise 16. *Map $G = \mathbb{C} - \{z : -1 \leq z \leq 1\}$ onto the open unit disk by an analytic function f. Can f be one-one?*

Solution. *Not available.*

Exercise 17. *Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.*

Solution. *Not available.*

Exercise 18. *Let $-\infty < a < b < \infty$ and put $Mz = \frac{z-ia}{z-ib}$. Define the lines $L_1 = \{z : Im\, z = b\}$, $L_2 = \{z : Im\, z = a\}$ and $L_3 = \{z : Re\, z = 0\}$. Determine which of the regions A, B, C, D, E, F in Figure 1, are mapped by M onto the regions U, V, W, X, Y, Z in Figure 2.*

Solution. *We easily see that we have $M(ia) = 0$. Therefore the region B, C, E and F which touch the line ia are mapped somehow to the region U, V, X and Y which touch 0. Similarly we have $M(ib) = \infty$ and therefore the region B and E which touch the line ib are mapped somehow to the region U and X which touch ∞. Thus we conclude that C or F goes to either V or Y. Let us find out. Let x, y be small positive real numbers such that the poin $z = x + iy + ia \in E$. Thus, the imaginary part of Mz is a positive number multiplied by $x(b - a)$ and therefore also positive. Therefore we conclude that M maps E to U and B to X. Because B and C meet at the line ia, we conclude that X and $M(C)$ do, too. Hence, M maps C to Y and F to V. By a similar argument, we obtain that M maps A to Z and D to W.*

Exercise 19. *Let a, b, and M be as in Exercise 18 and let* log *be the principal branch of the logarithm.*
(a) Show that $\log(Mz)$ is defined for all z except $z = ic$, $a \le c \le b$; and if $h(z) = Im\,[\log Mz]$ then $0 < h(z) < \pi$ for $Re\, z > 0$.
(b) Show that $\log(z - ic)$ is defined for $Re\, z > 0$ and any real number c; also prove that $|Im \log(z - ic)| < \frac{\pi}{2}$ if $Re\, z > 0$.
(c) Let h be as in (a) and prove that $h(z) = Im\,[\log(z - ia) - \log(z - ib)]$.
(d) Show that

$$\int_a^b \frac{dt}{z - it} = i[\log(z - ib) - \log(z - ia)]$$

(Hint: Use the Fundamental Theorem of Calculus.)
(e) Combine (c) and (d) to get that

$$h(x + iy) = \int_a^b \frac{x}{x^2 + (y - t)^2}\, dt = arctan\left(\frac{y - a}{x}\right) - arctan\left(\frac{y - b}{x}\right)$$

(f) Interpret part (e) geometrically and show that for $Re\, z > 0$, $h(z)$ is the angle depicted in the figure.

Solution. *Not available.*

Exercise 20. *Let $Sz = \frac{az+b}{cz+d}$ and $Tz = \frac{\alpha z+\beta}{\gamma z+\delta}$, show that $S = T$ iff there is a non zero complex number λ such that $\alpha = \lambda a, \beta = \lambda b, \gamma = \lambda c, \delta = \lambda d$.*

Solution. $\Leftarrow$: *Let $\lambda \neq 0$ be a complex number such that*

$$\begin{aligned} \alpha &= \lambda a \\ \beta &= \lambda b \\ \gamma &= \lambda c \\ \delta &= \lambda d. \end{aligned}$$

Then

$$T(z) = \frac{\alpha z + \beta}{\gamma z + \delta} = \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{\lambda(az + b)}{\lambda(cz + d)} = \frac{(az + b)}{(cz + d)} = S(z).$$

Thus, $S = T$.
$\Rightarrow$: *Let $S = T$, that is $S(z) = T(z)$. Then $S(0) = T(0)$, $S(1) = T(1)$ and $S(\infty) = T(\infty)$ which is equivalent*

to

$$\frac{b}{d} = \frac{\beta}{\delta} = \lambda_1 \iff b = \lambda_1 d, \beta = \lambda_1 \delta \tag{3.8}$$

$$\frac{a+b}{c+d} = \frac{\alpha+\beta}{\gamma+\delta} \tag{3.9}$$

$$\frac{a}{c} = \frac{\alpha}{\gamma} = \lambda_3 \iff a = \lambda_3 c, \alpha = \lambda_3 \gamma. \tag{3.10}$$

Insert $b = \lambda_1 d, \beta = \lambda_1 \delta$, $a = \lambda_3 c$ and $\alpha = \lambda_3 \gamma$ into (3.9), then

$$\begin{aligned}
& \frac{\lambda_3 c + \lambda_1 d}{c+d} = \frac{\lambda_3 \gamma + \lambda_1 \delta}{\gamma + \delta} \\
\iff\ & \lambda_3 c\gamma + \lambda_1 d\gamma + \lambda_3 c\delta + \lambda_1 d\delta = \lambda_3 \gamma c + \lambda_1 \delta c + \lambda_3 \gamma d + \lambda_1 \delta d \\
\iff\ & \lambda_1 d\gamma + \lambda_3 c\delta - \lambda_1 \delta c - \lambda_3 \gamma d = 0 \\
\iff\ & \lambda_1 (d\gamma - \delta c) - \lambda_3 (d\gamma - \delta c) = 0 \\
\iff\ & (\lambda_1 - \lambda_3)(d\gamma - \delta c) = 0.
\end{aligned}$$

Thus, either $\lambda_1 - \lambda_3 = 0$ or $d\gamma - \delta c = 0$.
If $\lambda_1 - \lambda_3 = 0$, then from (3.8) and (3.9), we get $\lambda_3 = \frac{a}{c} = \frac{b}{d} = \lambda_1$ which implies $ad - bc = 0$ a contradiction (Not possible, otherwise we do not have a Möbius transformation).
Thus, $d\gamma - \delta c = 0 \iff \frac{c}{d} = \frac{\gamma}{\delta} \iff \frac{\gamma}{c} = \frac{\delta}{d} := \lambda$, $\lambda \neq 0$ or

$$\gamma = c\lambda \tag{3.11}$$

$$\delta = d\lambda. \tag{3.12}$$

Insert (3.11) and (3.12) into (3.8) to obtain

$$\frac{b}{d} = \frac{\beta}{\delta} \Rightarrow \frac{b}{d} = \frac{\beta}{d\lambda} \Rightarrow \beta = \lambda b.$$

Insert (3.11) and (3.12) into (3.10) to obtain

$$\frac{a}{c} = \frac{\alpha}{\gamma} \Rightarrow \frac{a}{c} = \frac{\alpha}{c\lambda} \Rightarrow \alpha = \lambda a.$$

Therefore, we have

$$\alpha = \lambda a, \beta = \lambda b, \gamma = \lambda c, \delta = \lambda d.$$

Exercise 21. *Let T be a Möbius transformation with fixed points z_1 and z_2. If S is a Möbius transformation show that $S^{-1}TS$ has fixed points $S^{-1}z_1$, and $S^{-1}z_2$.*

Solution. *To show*

$$S^{-1}TS(S^{-1}z_1) = S^{-1}z_1.$$

We have

$$S^{-1}TS(S^{-1}z_1) = S^{-1}TSS^{-1}z_1 = S^{-1}Tz_1 = S^{-1}z_1,$$

where the first step follows by the associativity of compositions, the second step since $SS^{-1} = id$ and the last step by $Tz_1 = z_1$ since z_1 is a fixed point of T.
To show

$$S^{-1}TS(S^{-1}z_2) = S^{-1}z_2.$$

We have
$$S^{-1}TS(S^{-1}z_2) = S^{-1}TSS^{-1}z_2 = S^{-1}Tz_2 = S^{-1}z_2,$$
where the first step follows by the associativity of compositions, the second step since $SS^{-1} = id$ and the last step by $Tz_2 = z_2$ since z_2 is a fixed point of T.
Note that $S^{-1}TS$ is a well defined Möbius transformation, since S and T are Möbius transformation. The composition of Möbius transformation is a Möbius transformation.

Exercise 22. *(a) Show that a Möbius transformation has* 0 *and* ∞ *as its only fixed points iff it is a dilation, but not the identity.*
(b) Show that a Möbius transformation has ∞ *as its only fixed point iff it is a translation, but not the identity.*

Solution. *a) Let M be a Möbius transformation with exactly two fixed points, at* 0 *and* ∞*. We know that*
$$Mz = \frac{az+b}{cz+d}$$
with $a, b, c, d \in \mathbb{C}$. We know that $M(0) = 0$ so $\frac{b}{d} = 0$, whence we conclude that $b = 0$. But we also know that $M(\infty) = \infty$ so $\frac{a}{c} = \infty$, meaning $c = 0$. Thus $Mz = \frac{az}{d} = \alpha z$ for some $\alpha \in \mathbb{C}$. Since both b and c were 0*, we know that both a and d are nonzero so α is likewise nonzero and finite. Suppose by way of contradiction that $\alpha = 1$. Then for any $\tilde{z} \in \mathbb{C}_\infty$, $M\tilde{z} = \tilde{z}$ and M has infinitely many fixed points, a contradiction. Thus $\alpha \neq 1$ and M is a (complex) dilation not equal to the identity.*

On the other hand, assume M is a dilation not equal to the identity. In this case, we are free to express M as $Mz = \alpha z$ with $\alpha \neq 1$. It is clear from this representation that M has fixed points at 0 *and* ∞ *and, of course, at no other points.*

b) Let M be a Möbius transformation with exactly one fixed point at ∞*. Again, we know that*
$$Mz = \frac{az+b}{cz+d}$$
and we can see that $c = 0$ as before. This time, however, we also examine the lack of a second fixed point. Recall that if $\tilde{z}$ is a fixed point of M, then $\tilde{z}$ satisfies $c\tilde{z}^2 + (d-a)\tilde{z} - b = 0$, which in the case of $c = 0$ reduces to $(d-a)\tilde{z} - b = 0$ or $\tilde{z} = \frac{b}{d-a}$. We note that $b \neq 0$ because 0 *is not a fixed point of M. Suppose that $d \neq a$. In such a case, there is a finite value of $\tilde{z}$ that is a fixed point of M, a contradiction. Thus, it must be the case that $d = a$ and*
$$Mz = z + \frac{b}{d}.$$
Thus M is a translation.

Now assume M is a translation. Then we can express M as $Mz = z + \beta$. Here, in the convention of the text, we have $a = 1$, $b = \beta$, $c = 0$, and $d = 1$. The fixed points $\tilde{z}$ of M satisfy $c\tilde{z}^2 + (d-a)\tilde{z} - b = 0$, but no finite $\tilde{z}$ satisfies this equation for the given coefficients. Clearly, however, M has ∞ *as a fixed point, so* ∞ *is the only fixed point of M.*

Exercise 23. *Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ iff $Tz = az^{-1}$ for some a in $\mathbb{C}$.*

Solution. *Not available.*

Exercise 24. *Let T be a Möbius transformation, $T \neq$ the identity. Show that a Möbius transformation S commutes with T if S and T have the same fixed points. (Hint: Use Exercises 21 and 22.)*

Solution. *Let T and S have the same fixed points. To show $TS = ST$, $T \neq id$.*

$\star$: *Suppose T and S have two fixed points, say z_1 and z_2. Let M be a Möbius transformation with $M(z_1) = 0$ and $M(z_2) = \infty$. Then*

$$MSM^{-1}(0) = MSM^{-1}Mz_1 = MSz_1 = Mz_1 = 0$$

and

$$MSM^{-1}(\infty) = MSM^{-1}Mz_2 = MSz_2 = Mz_2 = \infty.$$

Thus MSM^{-1} is a dilation by exercise 22 a) since MSM^{-1} has 0 and ∞ as its only fixed points. Similar, we obtain $MTM^{-1}(0) = 0$ and $MTM^{-1}(\infty) = \infty$ and therefore is also a dilation. It is easy to check that dilations commute (define $C(z) = az$, $a > 0$ and $D(z) = bz$, $b > 0$, then $CD(z) = abz = baz = DC(z)$), thus

$$\begin{aligned} & (MTM^{-1})(MSM^{-1}) = (MSM^{-1})(MTM^{-1}) \\ \Rightarrow \quad & MTSM^{-1} = MSTM^{-1} \\ \Rightarrow \quad & TS = ST. \end{aligned}$$

$\star$: *Suppose T and S have one fixed points, say z. Let M be a Möbius transformation with $M(z) = \infty$. Then*

$$MSM^{-1}(\infty) = MSM^{-1}Mz = MSz = Mz = \infty.$$

Thus MSM^{-1} is a translation by exercise 22 b) since MSM^{-1} has ∞ as its only fixed point. Similar, we obtain $MTM^{-1}(\infty) = \infty$ and therefore is also a translation. It is easy to check that translations commute (define $C(z) = z + 1$ and $D(z) = z + b$, $b > 0$, then $CD(z) = z + a + b = z + b + a = DC(z)$), thus

$$\begin{aligned} & (MTM^{-1})(MSM^{-1}) = (MSM^{-1})(MTM^{-1}) \\ \Rightarrow \quad & MTSM^{-1} = MSTM^{-1} \\ \Rightarrow \quad & TS = ST. \end{aligned}$$

Exercise 25. *Find all the abelian subgroups of the group of Möbius transformations.*

Solution. *Not available.*

Exercise 26. *26. (a) Let $GL_2(\mathbb{C})$ = all invertible 2×2 matrices with entries in $\mathbb{C}$ and let $\mathcal{M}$ be the group of Möbius transformations. Define $\varphi : GL_2(\mathbb{C}) \to \mathcal{M}$ by $\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}$. Show that φ is a group homomorphism of $GL_2(\mathbb{C})$ onto $\mathcal{M}$. Find the kernel of φ.*
(b) Let $SL_2(\mathbb{C})$ be the subgroup of $GL_2(\mathbb{C})$ consisting of all matrices of determinant 1. Show that the image of $SL_2(\mathbb{C})$ under φ is all of $\mathcal{M}$. What part of the kernel of φ is in $SL_2(\mathbb{C})$?

Solution. *a) We have to check that if*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$$

then $\varphi(AB) = \varphi(A) \circ \varphi(B)$. A simple calculation shows that this is true. To find the kernel of the group homomorphism we have to find all z such that $\frac{az+b}{cz+d} = z$. This is equivalent to $az + b = cz^2 + dz$ and by comparing coefficients we obtain $b = c = 0$ and $a = d$. Therefore, the kernel is given by $N = ker(\varphi) = \{\lambda I : \lambda \in \mathbb{C}^\times\}$. Note that the kernel is a normal subgroup of $GL_2(\mathbb{C})$.
b) Restricting φ to $SL_2(\mathbb{C})$ still yields a surjective map since for any matrix $A \in GL_2(\mathbb{C})$ both A and the modification $M = \frac{1}{\sqrt{\det A}}A$ have the same image and the modification matrix M has by construction determinant 1. The kernel of the restriction is simply $N \cap SL_2(\mathbb{C}) = \{\pm I\}$.

Exercise 27. *If $\mathcal{G}$ is a group and $\mathcal{N}$ is a subgroup then $\mathcal{N}$ is said to be a normal subgroup of $\mathcal{G}$ if $S^{-1}TS \in \mathcal{N}$ whenever $T \in \mathcal{N}$ and $S \in \mathcal{G}$. $\mathcal{G}$ is a simple group if the only normal subgroups of $\mathcal{G}$ are $\{I\}$ (I = the identity of $\mathcal{G}$) and $\mathcal{G}$ itself. Prove that the group $\mathcal{M}$ of Möbius transformations is a simple group.*

Solution. *Not available.*

Exercise 28. *Discuss the mapping properties of $(1-z)^i$.*

Solution. *Not available.*

Exercise 29. *For complex numbers α and β with $|\alpha|^2 + |\beta|^2 = 1$*

$$u_{\alpha,\beta} = \frac{\alpha z - \overline{\beta}}{\beta z + \overline{\alpha}} \qquad \text{and let} \qquad U = \{u_{\alpha,\beta} : |\alpha|^2 + |\beta|^2 = 1\}$$

(a) Show that U is a group under composition.
(b) If SU_2 is the set of all unitary matrices with determinant 1, show that SU_2 is a group under matrix multiplication and that for each A in SU_2 there are unique complex numbers α and β with $|\alpha|^2 + |\beta|^2 = 1$ and

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

(c) Show that $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mapsto u_{\alpha,\beta}$ is an isomorphism of the group SU_2 onto U. What is its kernel?
(d) If $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ let H_l = all the polynomials of degree $\le 2l$. For $u_{\alpha,\beta} = u$ in U define $T_u^{(l)} : H_l \to H_l$, by $(T_u^{(l)} f)(z) = (\beta z + \overline{\alpha})^{2l} f(u(z))$. Show that $T_u^{(l)}$ is an invertible linear transformation on H_l, and $u \mapsto T_u^{(l)}$ is an injective homomorphism of U into the group of invertible linear transformations of H_l, onto H_l.

Solution. *Not available.*

Exercise 30. *For $|z| < l$ define $f(z)$ by*

$$f(z) = \exp\left\{-i\log\left[i\left(\frac{1+z}{1-z}\right)\right]^{1/2}\right\}$$

(a) Show that f maps $D = \{z : |z| < 1\}$ conformally onto an annulus G.
(b) Find all Möbius transformations $S(z)$ that map D onto D and such that $f(S(z)) = f(z)$ when $|z| < 1$.

Solution. *Not available.*

Chapter 4

Complex Integration

4.1 Riemann-Stieltjes integrals

Exercise 1. *Let $\gamma : [a, b] \to \mathbb{R}$ be non decreasing. Show that γ is of bounded variation and $V(\gamma) = \gamma(b) - \gamma(a)$.*

Solution. *Let $\gamma : [a, b] \to \mathbb{R}$ be a monotone, non-decreasing function. For a given partition of $[a, b]$, $\{a = t_0 < t_1 < \ldots < t_m = b\}$, for all $n \in \mathbb{N}$, $\gamma(t_n) \geq \gamma(t_{n-1})$. Therefore, $|\gamma(t_n) - \gamma(t_{n-1})| = \gamma(t_n) - \gamma(t_{n-1}) \geq 0$. Let $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$, any partition of $[a, b]$. Then,*

$$\begin{aligned}\sum_{n=1}^{m} |\gamma(t_n) - \gamma(t_{n-1})| &= [\gamma(t_m) - \gamma(t_{m-1})] + [\gamma(t_{m-1}) - \gamma(t_{m-2})] + \ldots + [\gamma(t_2) - \gamma(t_1)] + [\gamma(t_1) - \gamma(t_0)] \\ &= [\gamma(b) - \gamma(t_{m-1})] + [\gamma(t_{m-1}) - \gamma(t_{m-2})] + \ldots + [\gamma(t_2) - \gamma(t_1)] + [\gamma(t_1) - \gamma(a)] \\ &= \gamma(b) - \gamma(a)\end{aligned}$$

Since $\gamma(a), \gamma(b) \in \mathbb{R}$, $\gamma(b) - \gamma(a) < \infty$. It follows that γ is of bounded variation for any partition P. And also $v(\gamma; P) = sup\{v(\gamma; P) : P$ a partition of $[a, b]\} = V(\gamma) = \gamma(b) - \gamma(a)$.

Exercise 2. *Prove Proposition 1.2.*

Solution. *Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation.*
(a) If P and Q are partitions of $[a, b]$ and $P \subset Q$ then $v(\gamma; P) \leq v(\gamma; Q)$.
Proof will be by induction. Let $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$. Suppose $Q = P \cup \{x\}$, where $t_k < x < t_{k+1}$. Then,

$$\begin{aligned}v(\gamma; P) &= \sum_{i \neq k+1} |\gamma(t_i) - \gamma(t_{i-1})| + |\gamma(t_{k+1}) - \gamma(t_k)| \\ &\leq \sum_{i \neq k+1} |\gamma(t_i) - \gamma(t_{i-1})| + |\gamma(x) - \gamma(t_k)| + |\gamma(t_{k+1}) - \gamma(x)| \\ &= v(\gamma; Q)\end{aligned}$$

Now consider the general case on the number of elements of $Q \setminus P$. The smallest partition set for $Q \setminus P$ contains the trivial partition composed of $\{a, b\}$. Using the case defined above, we have $v(\gamma; Q) \geq v(\gamma; P) \geq v(\gamma; Q \setminus P) \geq |\gamma(b) - \gamma(a)|$, whenever $P \subset Q$.
(b) If $\sigma : [a, b] \to \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$, then $\alpha\gamma + \beta\sigma$ is of bounded variation and

$V(\alpha\gamma+\beta\sigma) \le |\alpha|V(\gamma)+|\beta|V(\sigma)$.

Let $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ *be a partition. Since* γ *and* σ *are of bounded variation for* $[a,b] \subset \mathbb{R}$, *there exists constants* $M_\gamma > 0$ *and* $M_\sigma > 0$, *respectively, such that* $v(\gamma;P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le M_\gamma$ *and* $v(\gamma;P) = \sum_{k=1}^{m} |\sigma(t_k) - \sigma(t_{k-1})| \le M_\sigma$. *Then,*

$$
\begin{aligned}
v(\alpha\gamma+\beta\sigma;P) &= \sum_{k=1}^{m} |(\alpha\gamma+\beta\sigma)(t_k) - (\alpha\gamma+\beta\sigma)(t_{k-1})| \\
&= \sum_{k=1}^{m} |\alpha\gamma(t_k)+\beta\sigma(t_k)-\alpha\gamma(t_{k-1})-\beta\sigma(t_{k-1})| \\
&= \sum_{k=1}^{m} |(\alpha\gamma(t_k)-\alpha\gamma(t_{k-1}))+(\beta\sigma(t_k)-\beta\sigma(t_{k-1}))| \\
&\le \sum_{k=1}^{m} |\alpha\gamma(t_k)-\alpha\gamma(t_{k-1})|+|\beta\sigma(t_k)-\beta\sigma(t_{k-1})| \\
&= \sum_{k=1}^{m} |\alpha\gamma(t_k)-\alpha\gamma(t_{k-1})|+\sum_{k=1}^{m}|\beta\sigma(t_k)-\beta\sigma(t_{k-1})| \\
&= \sum_{k=1}^{m} |\alpha|\,|(\gamma(t_k)-\gamma(t_{k-1}))|+\sum_{k=1}^{m}|\beta|\,|(\sigma(t_k)-\sigma(t_{k-1}))| \\
&= |\alpha|\sum_{k=1}^{m} |\gamma(t_k)-\gamma(t_{k-1})|+|\beta|\sum_{k=1}^{m}|\sigma(t_k)-\sigma(t_{k-1})| \\
&= |\alpha|v(\gamma;P)+|\beta|v(\sigma;P) \\
&\le |\alpha| sup\{v(\gamma;P) : P \text{ a partition of } [a,b]\}+|\beta| sup\{v(\sigma;P) : P \text{ a partition of } [a,b]\} \\
&= |\alpha|V(\gamma)+|\beta|V(\sigma) \\
&< \infty
\end{aligned}
$$

The result is two-fold, $V(\alpha\gamma+\beta\sigma) \le |\alpha|V(\gamma)+|\beta|V(\sigma)$ *and it follows* $\alpha\gamma+\beta\sigma$ *is of bounded variation by* $|\alpha|V(\gamma)+|\beta|V(\sigma)$.

Exercise 3. *Prove Proposition 1.7.*

Solution. *Not available.*

Exercise 4. *Prove Proposition 1.8 (Use induction).*

Solution. *Not available.*

Exercise 5. *Let* $\gamma(t) = \exp((-1+i)/t^{-1})$ *for* $0 < t \le 1$ *and* $\gamma(0) = 0$. *Show that* γ *is a rectifiable path and find* $V(\gamma)$. *Give a rough sketch of the trace of* γ.

Solution. *Not available.*

Exercise 6. *Show that if* $\gamma : [a,b] \to \mathbb{C}$ *is a Lipschitz function then* γ *is of bounded variation.*

Solution. *Let $\gamma : [a,b] \to \mathbb{C}$ be Lipschitz, that is $\exists$ a constant $C > 0$ such that*

$$|\gamma(x) - \gamma(y)| \le C|x - y|, \quad \forall x, y \in [a,b].$$

Then, for any partition $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a,b]$, we have

$$v(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le \sum_{k=1}^{m} C|t_k - t_{k-1}| = C \sum_{k=1}^{m} |t_k - t_{k-1}| = C(b-a) =: M > 0.$$

Hence, the function $\gamma : [a,b] \to \mathbb{C}$, for $[a,b] \subset \mathbb{R}$, is of bounded variation, since there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a,b]$

$$v(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le M.$$

Exercise 7. *Show that $\gamma : [0,1] \to \mathbb{C}$, defined by $\gamma(t) = t + it \sin \frac{1}{t}$ for $t \neq 0$ and $\gamma(0) = 0$, is a path but is not rectifiable. Sketch this path.*

Solution. *Not available.*

Exercise 8. *Let γ and σ be the two polygons $[1, i]$ and $[1, 1+i, i]$. Express γ and σ as paths and calculate $\int_\gamma f$ and $\int_\sigma f$ where $f(z) = |z|^2$.*

Solution. *For γ we have $\gamma(t) = (1-t) + it$ for $0 \le t \le 1$. So $\gamma'(t) = -1 + i$. Therefore,*

$$\begin{aligned}
\int_\gamma |z|^2 dz &= \int_0^1 ((1-t)^2 + t^2)(-1+i)dt \\
&= (-1+i) \int_0^1 (2t^2 - 2t + 1)dt \\
&= (-1+i) \left[\frac{2}{3}t^3 - t^2 + t \right]_0^1 \\
&= (-1+i) \left(\frac{2}{3} - 1 + 1 \right) \\
&= -\frac{2}{3} + i\frac{2}{3}.
\end{aligned}$$

Since σ is composed of multiple lines, the path is given by $\{\gamma_1(t) = 1 + it : 0 \le t \le 1; \gamma_2(t) = (1-t) + i :$

$0 \le t \le 1\}$. *So* $\gamma_1'(t) = i$ *and* $\gamma_2'(t) = -1$. *Therefore,*

$$\begin{aligned}
\int_\gamma f &= \int_{\gamma_1} f + \int_{\gamma_2} f \\
&= \int_{\gamma_1} |z|^2 dz + \int_{\gamma_2} |z|^2 dz \\
&= \int_0^1 (t^2+1)(i)dt + \int_0^1 ((1-t)^2+1)(-1)dt \\
&= i\int_0^1 (t^2+1)dt - \int_0^1 (t^2-2t+2)dt \\
&= i\left[\frac{t^3}{3}+t\right]_0^1 - \left[\frac{t^3}{3}-t^2+2t\right]_0^1 \\
&= \frac{4}{3}i - \left(\frac{1}{3}-1+2\right) \\
&= \frac{4}{3}i - \frac{4}{3} \\
&= \frac{4}{3}(-1+i)
\end{aligned}$$

Exercise 9. *Define* $\gamma : [0, 2\pi] \to \mathbb{C}$ *by* $\gamma(t) = \exp(int)$ *where n is some integer (positive, negative, or zero). Show that* $\int_\gamma \frac{1}{z}\, dz = 2\pi in$.

Solution. *Clearly,* $\gamma(t) = e^{int}$ *is continuous and smooth on* $[0, 2\pi]$. *Thus*

$$\int_\gamma z^{-1}\, dz = \int_0^{2\pi} e^{-int} ine^{int}\, dt = \int_0^{2\pi} in\, dt = in(2\pi - 0) = 2\pi in.$$

Exercise 10. *Define* $\gamma(t) = e^{it}$ *for* $0 \le t \le 2\pi$ *and find* $\int_\gamma z^n\, dz$ *for every integer n.*

Solution. *Clearly,* $\gamma(t) = e^{int}$ *is continuous and smooth on* $[0, 2\pi]$ *(It is the unit circle).*
Case 1: $n = -1$

$$\int_\gamma z^{-1}\, dz = \int_0^{2\pi} e^{-it} ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

Case 2: $n \ne -1$

$$\begin{aligned}
\int_\gamma z^n\, dz &= \int_0^{2\pi} e^{int} ie^{it}\, dt = i\int_0^{2\pi} e^{i(n+1)t}\, dt = t\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi} \\
&= \frac{1}{n+1}\left[e^{i(n+1)t}\right]_0^{2\pi} = \frac{1}{n+1}\left[e^{i(n+1)2\pi} - 1\right] \\
&= \frac{1}{n+1}\left[\cos((n+1)2\pi) - i\sin((n+1)2\pi)1\right] = \frac{1}{n+1}\left[1 - i\cdot 0 - 1\right] \\
&= 0.
\end{aligned}$$

Exercise 11. *Let* γ *be the closed polygon* $[1-i, 1+i, -1+i, -1-i, 1-i]$. *Find* $\int_\gamma \frac{1}{z}\, dz$.

Solution. *Define*

$$\begin{array}{rcccccccc}\gamma_1(t) &=& 1+it, & t \text{ from } -1 \text{ to } 1 & \to & \gamma_1'(t) &=& i\\ \gamma_2(t) &=& t+i, & t \text{ from } 1 \text{ to } -1 & \to & \gamma_2'(t) &=& 1\\ \gamma_3(t) &=& -1+it, & t \text{ from } 1 \text{ to } -1 & \to & \gamma_3'(t) &=& i\\ \gamma_4(t) &=& t-i, & t \text{ from } -1 \text{ to } 1 & \to & \gamma_4'(t) &=& 1.\end{array}$$

Thus

$$\begin{aligned}&\int_\gamma \frac{1}{z}\,dz\\ =\ & \int_{-1}^{1}\frac{1}{1+it}i\,dt+\int_{1}^{-1}\frac{1}{t+i}\,dt+\int_{1}^{-1}\frac{1}{-1+it}i\,dt+\int_{-1}^{1}\frac{1}{t-i}\,dt\\ =\ & \int_{-1}^{1}\left(\frac{-1}{t+i}+\frac{1}{t-i}\right)dt+i\int_{-1}^{1}\left(\frac{1}{1+it}-\frac{1}{-1+it}\right)dt\\ =\ & \int_{-1}^{1}\frac{-t+i+t+i}{(t+i)(t-i)}\,dt+i\int_{-1}^{1}\frac{-1+it-1-it}{(1+it)(-1+it)}\,dt\\ =\ & 2i\int_{-1}^{1}\frac{1}{t^2+1}\,dt-2i\int_{-1}^{1}\frac{1}{-1-t^2}\,dt\\ =\ & 2i\int_{-1}^{1}\frac{1}{t^2+1}\,dt+2i\int_{-1}^{1}\frac{1}{t^2+1}\,dt\\ =\ & 4i\int_{-1}^{1}\frac{1}{t^2+1}\,dt=4i\,[arctan(z)]_{-1}^{1}=4i\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=4i\frac{\pi}{2}=2\pi i.\end{aligned}$$

Exercise 12. *Let* $I(r)=\int_\gamma \frac{e^{iz}}{z}\,dz$ *where* $\gamma:[0,\pi]\to\mathbb{C}$ *is defined by* $\gamma(t)=re^{it}$. *Show that* $\lim_{r\to\infty} I(r)=0$.

Solution. *To show* $\lim_{r\to\infty} I(r)=0$ *which is equivalent to* $\forall\epsilon>0$, $\exists n\in\mathbb{N}$ *such that if* $r>N$, *then* $|I(r)|<\epsilon$. *Let* $\epsilon>0$, *then*

$$|I(r)-0|=|I(r)|=\left|\int_\gamma \frac{e^{iz}}{z}\,dz\right|=\left|\int_0^\pi \frac{e^{ire^{it}}}{re^{it}}rie^{it}\,dt\right|=\left|i\int_0^\pi e^{ire^{it}}\,dt\right|\le\int_0^\pi\left|e^{ire^{it}}\right|\,dt.$$

Now, recall that $|e^z|=e^{Re(z)}$ *(p. 38 eq. 2.13), so*

$$\left|e^{ire^{it}}\right|=e^{Re(ire^{it})}=e^{Re(ir\cos(t)+ir\cdot i\sin(t))}=e^{Re(ir\cos(t)-r\sin(t))}=e^{-r\sin(t)}.$$

Hence,

$$\int_0^\pi\left|e^{ire^{it}}\right|\,dt=\int_0^\pi e^{-r\sin(t)}\,dt=\int_0^{\delta_1}e^{-r\sin(t)}\,dt+\int_{\delta_1}^{\pi-\delta_2}e^{-r\sin(t)}\,dt+\int_{\pi-\delta_2}^{\pi}e^{-r\sin(t)}\,dt$$

for $\delta_1>0$ *and* $\delta_2>0$. *Since* $e^{-r\sin(t)}$ *is continuous, so* $\delta_1>0$ *can be chosen such that*

$$\int_0^{\delta_1}e^{-r\sin(t)}\,dt<\frac{\epsilon}{3},\qquad \forall r>0.$$

Similar, $\delta_2>0$ *can be chosen such that*

$$\int_{\pi-\delta_2}^{\pi}e^{-r\sin(t)}\,dt<\frac{\epsilon}{3},\qquad \forall r>0.$$

Finally, because of the Lebesgue's dominated convergence Theorem

$$\lim_{r\to\infty}\int_{\delta_1}^{\pi-\delta_2} e^{-r\sin(t)}\,dt = \int_{\delta_1}^{\pi-\delta_2}\lim_{r\to\infty} e^{-r\sin(t)}\,dt$$

we have

$$\lim_{r\to\infty}\int_{\delta_1}^{\pi-\delta_2} e^{-r\sin(t)}\,dt < \frac{\epsilon}{3}.$$

Hence,

$$\lim_{r\to\infty} I(r) = 0.$$

Exercise 13. *Find* $\int_\gamma z^{-\frac{1}{2}}\,dz$ *where: (a)* γ *is the upper half of the unit circle from* $+1$ *to* -1*: (b)* γ *is the lower half of the unit circle from* $+1$ *to* -1.

Solution. *Not available.*

Exercise 14. *Prove that if* $\varphi : [a,b] \to [c,d]$ *is continuous and* $\varphi(a) = c$, $\varphi(b) = d$ *then* φ *is one-one iff* φ *is strictly increasing.*

Solution. *Not available.*

Exercise 15. *Show that the relation in Definition 1.16 is an equivalence relation.*

Solution. *Not available.*

Exercise 16. *Show that if* γ *and* σ *are equivalent rectifiable paths then* $V(\gamma) = V(\sigma)$.

Solution. *Not available.*

Exercise 17. *Show that if* $\gamma : [a,b] \to \mathbb{C}$ *is a path then there is an equivalent path* $\sigma : [0,1] \to \mathbb{C}$.

Solution. *Not available.*

Exercise 18. *Prove Proposition 1.17.*

Solution. *Not available.*

Exercise 19. *Let* $\gamma = 1 + e^{it}$ *for* $0 \le t \le 2\pi$ *and find* $\int_\gamma (z^2-1)^{-1}\,dz$.

Solution. *Not available.*

Exercise 20. *Let* $\gamma = 2e^{it}$ *for* $-\pi \le t \le \pi$ *and find* $\int_\gamma (z^2-1)^{-1}\,dz$.

Solution. *Not available.*

Exercise 21. *Show that if* F_1 *and* F_2 *are primitives for* $f : G \to \mathbb{C}$ *and* G *is open end connected then there is a constant* c *such that* $F_1(z) = c + F_2(z)$ *for each* z *in* G.

Solution. *Let* F_1 *and* F_2 *be primitives for* $f : G \to \mathbb{C}$*. Recall then that* $F_1' = f = F_2'$*. Let* $h = F_1 - F_2$ *for all* $z \in G$*. Then for all* $z \in G$,

$$h'(z) = F_1'(z) - F_2'(z) = f(z) - f(z) = 0$$

Since G *is open and connected with* $h'(z) = 0$ *for all* $z \in G$*, then* h *is a constant, say* c*:* $h = c$*. So* $h = F_1 - F_2 = c$ *and* $F_1(z) = F_2(z) + c$ *for all* $z \in G$.

Exercise 22. *Let γ be a closed rectifiable curve in an open set G and $a \notin G$. Show that for $n \geq 2$, $\int_\gamma (z-a)^{-n}\, dz = 0$.*

Solution. *Let $f(z) = (z-a)^{-n}$ for all $z \in G$. Then f has the antiderivative $F(z) = \dfrac{(z-a)^{1-n}}{1-n}$ on G. Since $F'(z) = f(z)$ where $n \geq 2$ (note F is undefined for $n = 1$). Now, let γ be a closed rectifiable curve in G with endpoints $z_1, z_2 \in \mathbb{C}$. Since $z_1 = z_2$, $F(z_1) = F(z_2)$. So, $\displaystyle\int_\gamma (z-a)^{-n} dz = F(z_2) - F(z_1) = 0$.*

Exercise 23. *Prove the following integration by parts formula. Let f and g be analytic in G and let γ be a rectifiable curve from a to b in G. Then $\int_\gamma fg' = f(b)g(b)f(a)g(a) - \int_\gamma f'g$.*

Solution. *Let f, g be analytic in G and let γ be a rectifiable curve from a to b in G. Then*

$$\begin{aligned}
\int_\gamma fg' + \int_\gamma f'g \quad &\underset{(def)}{=} \quad \int_a^b f(\gamma(t))g'(\gamma(t))\, d\gamma(t) + \int_a^b f'(\gamma(t))g(\gamma(t))\, d\gamma(t) \\
&\underset{\text{Prop. 1.7 (a)}}{=} \quad \int_a^b f(\gamma(t))g'(\gamma(t)) + f'(\gamma(t))g(\gamma(t))\, d\gamma(t) \\
&\underset{(def)}{=} \quad \int_\gamma fg' + f'g = \int_\gamma (fg)'
\end{aligned}$$

Since $(fg)'$ is continuous (f and g are analytic) by Theorem 1.18 we obtain

$$\int_\gamma (fg)' = (fg)(b) - (fg)(a) = f(b)g(b) - f(a)g(a).$$

Hence,

$$\int_\gamma fg' = f(b)g(b) - f(a)g(a) - \int_\gamma f'g.$$

4.2 Power series representation of analytic functions

Exercise 1. *Show that the function defined by (2.2) is continuous.*

Solution. *Not available.*

Exercise 2. *Prove the following analogue of Leibniz.s rule (this exercise will be frequently used in the later sections.) Let G be an open set and let γ be a rectifiable curve in $\mathbb{C}$. Suppose that $\varphi : \{y\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by*

$$g(z) = \int_\gamma \varphi(w, z)\, dw$$

then g is continuous. If $\frac{\partial\varphi}{\partial z}$ exists for each (w, z) in $\{\gamma\} \times G$ and is continuous then g is analytic and

$$g'(z) = \int_\gamma \frac{\partial\varphi}{\partial z}(w, z)\, dw.$$

Solution. *Not available.*

Exercise 3. *Suppose that γ is a rectifiable curve in $\mathbb{C}$ and φ is defined and continuous on $\{\gamma\}$. Use Exercise 2 to show that*

$$g(z) = \int_\gamma \frac{\varphi(w)}{w-z}\, dw$$

is analytic on $\mathbb{C} - \{\gamma\}$ and

$$g^{(n)}(z) = n! \int_\gamma \frac{\varphi(w)}{(w-z)^{n+1}}\, dw.$$

Solution. *Not available.*

Exercise 4. *(a) Prove Abel's Theorem: Let $\sum a_n(z-a)^n$ have radius of convergence 1 and suppose that $\sum a_n$ converges to A. Prove that*

$$\lim_{r\to 1^-} \sum a_n r^n = A.$$

(Hint: Find a summation formula which is the analogue of integration by parts.)
(b) Use Abel's Theorem to prove that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$.

Solution. *Not available.*

Exercise 5. *Give the power series expansion of $\log z$ about $z = i$ and find its radius of convergence.*

Solution. *Assume $a \in C$ is not zero, then*

$$\frac{1}{z} = \frac{1}{a+z-a} = \frac{1}{a}\frac{1}{\frac{a+z-a}{a}} = \frac{1}{a}\frac{1}{1+\frac{z-a}{a}} = \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{a^n}(z-a)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}(z-a)^n,$$

where the radius of convergence is $|a|$. Let $a = i$, then integrate term by term yields

$$\log(z) = \frac{\pi}{2}i + \sum_{n=0}^{\infty}\frac{\left(i^2\right)^n}{(n+1)i^{n+1}}(z-i)^{n+1} + \frac{\pi}{2}i - \sum_{n=0}^{\infty}\frac{i^{n+1}}{n+z}(z-i)^{n+1} = \frac{\pi}{2}i - \sum_{n=1}^{\infty}\frac{(1+iz)^n}{n},$$

since $Log(i) = \frac{\pi}{2}i$. The convergence radius is $|i| = 1$.

Exercise 6. *Give the power series expansion of $\sqrt{z}$ about $z = 1$ and find its radius of convergence.*

Solution. *From the definition of numbers with rational exponents $z^a = \exp(a \log z)$ we see that the square root cannot be analytic on all of $\mathbb{C}$. In fact, taking the first derivatives we see that $f^{(n)}(z) = 2^{-n}\prod_{k=0}^{n-1}(1 - 2k)z^{-n+\frac{1}{2}}$ and therefore*

$$a_n = \frac{1}{n!}f^{(n)}(1) = \frac{1}{n!2^n}\prod_{k=0}^{n-1},$$

and with these a_n,

$$f(z) = \sqrt{z} = \sum_{n=0}^{\infty} a_n(z-1)^z = \sum_{k=0}^{\infty}\frac{1}{n!2^n}\prod_{k=0}^{n-1}(1-2k)(z-1)^n.$$

Note that the logarithm is undefined at zero. Thus, of all branch cuts of the logarithm, the best radius of convergence we can achieve is 1.

Exercise 7. *Use the results of this section to evaluate the following integrals:*
(a)

$$\int_\gamma \frac{e^{iz}}{z^2}\,dz, \qquad \gamma(t) = e^{it}, \qquad 0 \le t \le 2\pi;$$

(b)

$$\int_\gamma \frac{dz}{z-a}, \qquad \gamma(t) = a + re^{it}, \qquad 0 \le t \le 2\pi;$$

(c)

$$\int_\gamma \frac{\sin(z)}{z^3}\,dz, \qquad \gamma(t) = e^{it}, \qquad 0 \le t \le 2\pi;$$

(d)

$$\int_\gamma \frac{\log z}{z^n}\,dz, \qquad \gamma(t) = 1 + \frac{1}{2}e^{it}, \qquad 0 \le t \le 2\pi \text{ and } n \ge 0.$$

Solution. *a) Let $a = 0$, $r = 1$, $n = 1$, $f(z) = e^{iz}$ and $f'(z) = ie^{iz}$. Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\bar{B}(0;1) \subset \mathbb{C}$. Now use Corollary 2.13.*

$$f'(0) = \frac{1!}{2\pi i}\int_\gamma \frac{e^{iz}}{(z-0)^2}\,dz \iff i = \frac{1}{2\pi i}\int_\gamma \frac{e^{iz}}{z^2}\,dz \iff \int_\gamma \frac{e^{iz}}{z^2}\,dz = -2\pi.$$

b) Let $a = a$, $r = r$, $n = 0$, and $f(z) = 1$. Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\bar{B}(a;r) \subset \mathbb{C}$. Now use Corollary 2.13.

$$f^{(0)}(a) = \frac{0!}{2\pi i}\int_\gamma \frac{1}{(z-a)^1}\,dz \iff 1 = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz \iff \int_\gamma \frac{1}{z-a}\,dz = 2\pi i.$$

c) Let $a = 0$, $r = 1$, $n = 2$ and $f(z) = \sin(z)$. Then $f'(z) = \cos(z)$ and $f''(z) = -\sin(z)$. Clearly $f(z)$ is analytic on $\mathbb{C}$ (p. 38) and $\bar{B}(0;1) \subset \mathbb{C}$. Now use Corollary 2.13.

$$f''(0) = \frac{2!}{2\pi i}\int_\gamma \frac{\sin(z)}{(z-0)^3}\,dz \iff -\sin(0) = \frac{1}{\pi i}\int_\gamma \frac{\sin(z)}{z^3}\,dz \iff 0 = \frac{1}{\pi i}\int_\gamma \frac{\sin(z)}{z^3}\,dz$$
$$\iff \int_\gamma \frac{\sin(z)}{z^3}\,dz = 0.$$

d) We have that

$$f(z) = \frac{\log(z)}{z^n}$$

is analytic in the disk $B(1;\frac{1}{2})$ where $n \ge 0$. Furthermore, γ is a closed rectifiable curve ($\gamma(t)$ is a circle with radius $r = 1/2$ and center $(1,0)$ in $\mathbb{C}$). Obviously $\gamma(t) = 1 + \frac{1}{2}e^{it} \subseteq B(1;\frac{1}{2})$. Hence, by Proposition 2.15, f has a primitive and therefore $\int_\gamma f = 0$. Therefore,

$$\int_\gamma \frac{\log(z)}{z^n}\,dz = 0.$$

Exercise 8. *Use a Möbuis transformation to show that Proposition 2.15 holds if the disk $B(a;R)$ is replaced by a half plane.*

Solution. *Not available.*

Exercise 9. *Evaluate the following integrals:*

(a)

$$\int_\gamma \frac{e^z - e^{-z}}{z^n}\, dz \text{ where } n \text{ is a positive integer and } \gamma(t) = e^{it}, 0 \le t \le 2\pi;$$

(b)

$$\int_\gamma \frac{dz}{\left(z - \frac{1}{2}\right)^n} \text{ where } n \text{ is a positive integer and } \gamma(t) = \frac{1}{2} + e^{it}, 0 \le t \le 2\pi;$$

(c)

$$\int_\gamma \frac{dz}{z^2+1} \text{ where } \gamma(t) = 2e^{it}, 0 \le t \le 2\pi. \text{ (Hint: expand } (z^2+1)^{-1} \text{ by means of partial fractions);}$$

(d)

$$\int_\gamma \frac{\sin z}{z}\, dz \text{ where } \gamma(t) = e^{it}, 0 \le t \le 2\pi;$$

(e)

$$\int_\gamma \frac{z^{1/m}}{(z-1)^m}\, dz \text{ where } \gamma(t) = 1 + \frac{1}{2}e^{it}, 0 \le t \le 2\pi;$$

Solution. *a) Let $a = 0$, $r = 1$, $n = m - 1$ and*

$$\begin{aligned} f(z) &= e^z - e^{-z} \\ f'(z) &= e^z + e^{-z} \\ f''(z) &= e^z - e^{-z} \\ &\vdots \\ f^{(m-1)}(z) &= e^z - (-1)^{m+1}e^{-z}. \end{aligned}$$

Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\overline{B}(0;1) \subset \mathbb{C}$. Now use Corollary 2.13.

$$\begin{aligned} & f^{(m-1)}(0) = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{e^z - e^{-z}}{(z-0)^m}\, dz \\ \iff\quad & \frac{(m-1)!}{2\pi i} \int_\gamma \frac{e^z - e^{-z}}{z^m}\, dz = \begin{cases} 0 & , m \text{ odd} \\ 2 & , m \text{ even} \end{cases} \\ \iff\quad & \int_\gamma \frac{e^z - e^{-z}}{z^m}\, dz = \begin{cases} 0 & , m \text{ odd} \\ \frac{4\pi i}{(m-1)!} & , m \text{ even} \end{cases} \qquad (m > 0). \end{aligned}$$

b) Let $a = 1/2$, $r = 1$, $n = m - 1$ and

$$\begin{aligned} f(z) &= 1 \\ f'(z) &= 0 \\ f''(z) &= 0 \\ &\vdots \\ f^{(m-1)}(z) &= 0. \end{aligned}$$

Clearly $f(z)$ *is analytic on* $\mathbb{C}$ *and* $\overline{B}(\frac{1}{2};1) \subset \mathbb{C}$*. Now use Corollary 2.13.*

$$
\begin{aligned}
& f^{(m-1)}\left(\frac{1}{2}\right) = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{1}{\left(z-\frac{1}{2}\right)^m}\, dz \\
\iff \quad & \frac{(m-1)!}{2\pi i} \int_\gamma \frac{1}{\left(z-\frac{1}{2}\right)^m}\, dz = \begin{cases} 1 & , m = 1 \\ 0 & , m > 1 \end{cases} \\
\iff \quad & \int_\gamma \frac{1}{\left(z-\frac{1}{2}\right)^m}\, dz = \begin{cases} 2\pi i & , m = 1 \\ 0 & , m > 1 \end{cases}.
\end{aligned}
$$

c) Using partial fraction decomposition, we obtain

$$\frac{1}{z^2+1} \stackrel{!}{=} \frac{A}{z-i} + \frac{B}{z+i} = \frac{Az + Ai + Bz - Bi}{z^2+1}$$

which implies

$$
\begin{aligned}
A + B &= 0 && (4.1) \\
Ai - Bi &= 1. && (4.2)
\end{aligned}
$$

From (4.1), we get $A = -B$ *inserted into (4.2) yields* $-2Bi = 1$ *and therefore* $B = -\frac{1}{2i}$ *and* $A = \frac{1}{2i}$*. Thus*

$$\int_\gamma \frac{dz}{z^2+1} = \frac{1}{2i}\int_\gamma \frac{dz}{z-i} - \frac{1}{2i}\int_\gamma \frac{dz}{z+i}.$$

For the first integral on the right side, use Proposition 2.6. Let $a = 0$, $r = 2$ *and* $f(w) = 1$, $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$*. Clearly* $f(z)$ *is analytic on* $\mathbb{C}$*. We also have* $\overline{B}(0;2) \subset \mathbb{C}$*. Then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-i}\, dw \iff 1 = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-i}\, dw \iff \int_\gamma \frac{1}{z-i}\, dz = 2\pi i,$$

where $|-i-0| < 2 \iff 1 < 2$.
For the second integral on the right side, use Proposition 2.6. Let $a = 0$, $r = 2$ *and* $f(w) = 1$, $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$*. Clearly* $f(z)$ *is analytic on* $\mathbb{C}$*. We also have* $\overline{B}(0;2) \subset \mathbb{C}$*. Then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w+i}\, dw \iff 1 = \frac{1}{2\pi i}\int_\gamma \frac{1}{w+i}\, dw \iff \int_\gamma \frac{1}{z+i}\, dz = 2\pi i,$$

where $|i| < 2 \iff 1 < 2$*. Hence*

$$\int_\gamma \frac{dz}{z^2+1} = \frac{1}{2i}\int_\gamma \frac{dz}{z-i} - \frac{1}{2i}\int_\gamma \frac{dz}{z+i} = \frac{1}{2i}2\pi i - \frac{1}{2i}2\pi i = \pi - \pi = 0.$$

So

$$\int_\gamma \frac{dz}{z^2+1} = 0.$$

d) Let $a = 0$, $r = 1$, $n = 0$ *and* $f(z) = \sin(z)$*. Clearly* $f(z)$ *is analytic on* $\mathbb{C}$ *and* $\overline{B}(0;1) \subset \mathbb{C}$*. Now use Corollary 2.13.*

$$f^{(0)}(0) = \frac{0!}{2\pi i}\int_\gamma \frac{\sin(z)}{(z-0)^1}\, dz \iff 0 = \frac{1}{2\pi i}\int_\gamma \frac{\sin(z)}{z}\, dz \iff \int_\gamma \frac{\sin(z)}{z}\, dz = 0.$$

e) Let $a = 1$, $r = 1/2$, $n = m - 1$ and

$$\begin{aligned}
f(z) &= z^{1/m} \\
f'(z) &= \frac{1}{m} z^{1/m-1} \\
f''(z) &= \frac{1}{m}\left(\frac{1}{m} - 1\right) z^{1/m-2} \\
&\vdots \\
f^{(m-1)}(z) &= \frac{1}{m}\left(\frac{1}{m} - 1\right) \cdot \ldots \cdot \left(\frac{1}{m} - (m-2)\right) z^{1/m-(m-1)}.
\end{aligned}$$

Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\overline{B}(1; \frac{1}{2}) \subset \mathbb{C}$. Now use Corollary 2.13.

$$\begin{aligned}
& f^{(m-1)}(1) = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{z^{1/m}}{(z-1)^m}\, dz \\
\iff\quad & \frac{1}{m}\left(\frac{1}{m} - 1\right) \cdot \ldots \cdot \left(\frac{1}{m} - (m-2)\right) \cdot 1 = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{z^{1/m}}{(z-1)^m}\, dz \\
\iff\quad & \int_\gamma \frac{z^{1/m}}{(z-1)^m}\, dz = \frac{2\pi i}{(m-1)!} \frac{1}{m} \left(\frac{1-m}{m}\right)\left(\frac{1-2m}{m}\right) \cdot \ldots \cdot \left(\frac{1-(m-1)m}{m}\right) \\
\iff\quad & \int_\gamma \frac{z^{1/m}}{(z-1)^m}\, dz = \frac{2\pi i}{(m-1)! m^{m-1}} \prod_{i=0}^{m-2} (1 - im).
\end{aligned}$$

Exercise 10. *Evaluate $\int_\gamma \frac{z^2+1}{z(z^2+4)}\, dz$ where $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$, for all possible values of r, $0 < r < 2$ and $2 < r < \infty$.*

Solution. *Let $f(z) = \dfrac{z^2+1}{z(z^2+4)}$. Then $f(z)$ can be decomposed using partial fractions as*

$$f(z) = \frac{z^2+1}{z(z^2+4)} = \frac{1/4}{z} + \frac{3/8}{z-2i} + \frac{3/8}{z+2i}.$$

It follows, $\displaystyle\int_\gamma \frac{z^2+1}{z(z^2+4)} dz = \frac{1}{4}\int_\gamma \frac{1}{z} dz + \frac{3}{8}\int_\gamma \frac{1}{z-2i} dz + \frac{3}{8}\int_\gamma \frac{1}{z+2i} dz$. By Proposition 2.6, $\displaystyle\int_\gamma \frac{1}{z} dz = 2\pi i f(0) = 2\pi i$ where $f \equiv 1$ as $f \equiv 1$ is analytic on $\overline{B}(0,2)$ and $\gamma = 0 + re^{it}$ with $0 < r < 2$. Now, note $\frac{1}{z-2i}$ is analytic on $B(0,2)$ and γ for $0 < r < 2$ is in $B(0,2)$. Thus, by Proposition 2.15, $\displaystyle\int_\gamma \frac{1}{z-2i} dz = 0$. Similarly, $\displaystyle\int_\gamma \frac{1}{z+2i} dz = 0$. It follows for $0 < r < 2$, $\displaystyle\int_\gamma \frac{z^2+1}{z(z^2+4)} dz = \frac{1}{4} 2\pi i = \frac{\pi}{2} i$. Now, for $2 < r < \infty$, note $\gamma(t) = re^{it}$ so $\gamma'(t) = ire^{it}$. Calculating the integrals in turn,

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = i \int_0^{2\pi} dt = 2\pi i.$$

For the second integral, we use Proposition 2.6 with $a = 0$, $2 < r < \infty$, $f(w) = 1$, and $\gamma(t) = 0 + re^{it}$ for

$0 \le t \le 2\pi$. Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\overline{B}(0; 2 < r < \infty) \subset \mathbb{C}$. Then,

$$\begin{aligned} f(-2i) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w+2i}dw \iff 1 &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w+2i}dw \\ &\iff \int_\gamma \frac{1}{z+2i}dz = 2\pi i \end{aligned}$$

if $2 < r < \infty$. Similarly,

$$f(2i) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-2i}dw \iff \int_\gamma \frac{1}{z-2i}dz = 2\pi i.$$

It follows that,

$$\begin{aligned} \int_\gamma \frac{z^2+1}{z(z^2+4)}dz &= \frac{1}{4}\int_\gamma \frac{1}{z}dz + \frac{3}{8}\int_\gamma \frac{1}{z-2i}dz + \frac{3}{8}\int_\gamma \frac{1}{z+2i}dz \\ &= \frac{1}{4}(2\pi i) + \frac{3}{8}(2\pi i) + \frac{3}{8}(2\pi i) \\ &= 2\pi i \end{aligned}$$

Exercise 11. *Find the domain of analyticity of*

$$f(z) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right);$$

also, show that $\tan f(z) = z$ *(i.e., f is a branch of arctan z). Show that*

$$f(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{2k+1}, \quad \textit{for} \quad |z| < 1$$

(Hint: see Exercise III. 3.19.)

Solution. *Not available.*

Exercise 12. *Show that*

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!}z^{2k}$$

for some constants $E_2, E_4, \cdots$. These numbers are called Euler's constants. What is the radius of convergence of this series? Use the fact that $1 = \cos z \sec z$ to show that

$$E_{2n} - \binom{2n}{2n-2}E_{2n-2} + \binom{2n}{2n-4}E_{2n-4} + \cdots + (-1)^{n-1}\binom{2n}{2}E_2 + (-1)^n = 0.$$

Evaluate E_2, E_4, E_6, E_8. ($E_{10} = 50521$ and $E_{12} = 2702765$).

Solution. *It is easily seen that*

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!}z^{2k}, \quad \textit{(for some } E_2, E_4, \ldots),$$

since $\sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1$ *and* $\sec z = \frac{1}{\cos z}$ *is an even function. The radius of convergence is the closest distance from 0 to the nearest non-analytic point which is precisely* $\frac{\pi}{2}$.
Since $1 = \cos z \sec z$, *we can multiply the series for* $\sec z$ *and* $\cos z$ *together to obtain*

$$1 = \sum_{i=0}^{\infty}\left[\sum_{k=0}^{n}(-1)^{i-k}\frac{E_{2k}}{(2k)!(2i-2k)!}\right]z^{2i}.$$

Finally, we can compare the coefficients of z^{2i} *and multiply by* $(2i)!$ *to obtain the formula*

$$\sum_{i=0}^{n}(-1)^{n-k}E_{2k}\left(\begin{array}{c}2n\\2k\end{array}\right) = 0$$

which is equivalent to

$$E_{2n} - \left(\begin{array}{c}2n\\2n-2\end{array}\right)E_{2n-2} + \left(\begin{array}{c}2n\\2n-4\end{array}\right)E_{2n-4} + \cdots + (-1)^{n-1}\left(\begin{array}{c}2n\\2\end{array}\right)E_2 + (-1)^n = 0.$$

In addition, we have $E_2 = 1$, $E_4 = 5$, $E_6 = 61$ *and* $E_8 = 1285$.

Exercise 13. *Find the series expansion of* $\frac{e^z-1}{z}$ *about zero and determine its radius of convergence. Consider* $f(z) = \frac{z}{e^z-1}$ *and let*

$$f(z) = \sum_{k=0}^{\infty}\frac{a_k}{k!}z^k$$

be its power series expansion about zero. What is the radius of convergence? Show that

$$0 = a_0 + \left(\begin{array}{c}n+1\\1\end{array}\right)a_1 + \cdots + \left(\begin{array}{c}n+1\\n\end{array}\right)a_n.$$

Using the fact that $f(z) + \frac{1}{2}z$ *is an even function show that* $a_k = 0$ *for* k *odd and* $k > 1$. *The numbers* $B_{2n} = (-1)^{n-1}a_{2n}$ *are called the Bernoulli numbers for* $n \geq 1$. *Calculate* $B_2, B_4, \cdots, B_{l0}$.

Solution. *The Bernoulli numbers* B_n *are defined by*

$$B_n = \lim_{z\to\infty}\frac{\mathrm{d}^n\frac{z}{e^z-1}}{\mathrm{d}z^n}, \qquad n = 0, 1, 2, \ldots$$

We determine the largest R *such that*

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}z^k \qquad \textit{for } 0 < |z| < R.$$

The function $f(z) = \frac{z}{e^z-1}$ *has a removable singularity at* $z = 0$, *because it can be extended to* 0 *by an analytic function* g *on a small neighborhood* $B(0, r)$ *of* 0, *setting*

$$g(z) := \begin{cases}\frac{z}{e^z-1} & \textit{if } 0 < |z| < r\\ 1 & \textit{if } x = 0\end{cases}.$$

Furthermore the function f *has essential singularities exactly at* $z = 2\pi k, k \in \mathbb{Z} - \{0\}$. *Therefore* $f(z)$ *has a Laurent series expansion on the punctured disk* $ann(0; 0, 2\pi)$ *and the negative indices disappear because the singularity at the origin is a removable one. The uniqueness of the Laurent expression together with the*

formula for the coefficients in the expansion (Theorem 2.8, p. 72) give the desired result.
First we compute $B_0 = \lim_{z\to 0} \frac{z}{e^z-1} = 1$ *and note that the result in question does not hold for* B_0*. Also compute* B_1*. Taking the limits,* $B_1 = \lim_{z\to 0} \frac{(1-z)e^z-1}{(e^z-1)^2} = -\frac{1}{2}$*. Consider the equation* $\frac{z}{e^z-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$ *for* $z \in ann(0; 0, 2\pi)$*. This is equivalent to*

$$z = \left(\sum_{l=1}^{\infty} \frac{z^l}{l!}\right)\left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k\right).$$

Divide by $z \neq 0$*, then*

$$\begin{aligned} 1 &= \left(\sum_{l=1}^{\infty} \frac{z^{l-1}}{l!}\right)\left(\sum_{k=0}^{\infty} \frac{B_k}{k!}\right) = \left(\sum_{l=0}^{\infty} \frac{z^l}{(l+1)!}\right)\left(\sum_{k=0}^{\infty} \frac{B_k}{k!}\right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{B_m}{m!} z^m \frac{z^{n-m}}{(n+1-m)!} \\ &= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} \frac{B_m}{m!(n+1-m)!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \sum_{m=0}^{n} \binom{n+1}{m} B_m \\ &= B_0 + \sum_{n=1}^{\infty} \frac{z^n}{(n+1)!} \sum_{m=0}^{n} \binom{n+1}{m} B_m. \end{aligned}$$

Since $B_0 = 0$ *conclude that*

$$0 = \sum_{n=1}^{\infty} \frac{z^n}{(n+1)!} \sum_{m=0}^{n} \binom{n+1}{m} B_m$$

for all nonzero $z \in ann(0; 0, 2\pi)$*. This can only happen if the inner summation equals zero for all* $n \in \mathbb{N}$*. This prove the first part of the statement. For the second claim, one only needs to show that* $f(z) + \frac{1}{2}z$ *is an even function. It is true that*

$$f(z) + \frac{1}{2}z = \frac{2z + (e^z - 1)z}{2(e^z - 1)} = \frac{z(e^z + 1)}{2(e^z - 1)}.$$

as well as t'

$$f(-z) - \frac{1}{2}z = \frac{-z(e^z + 1)}{e^{-z} - 1} = \frac{-z(1 + e^z)}{2(1 - e^z)} = \frac{z(1 + e^z)}{2(e^z - 1)} = f(z) + \frac{1}{2}z.$$

With the power series expansion of f *and recalling that* $B_1 = -\frac{1}{2}$ *this is the same as saying*

$$1 + \sum_{k=2}^{\infty} \frac{B_k}{k!}(-z)^k = 1 + \sum_{k=2}^{\infty} \frac{B_k}{k!}(z)^k$$

or, yet in other terms,

$$B_k = -B_k = 0 \qquad \text{for } k \text{ odd }, k > 1.$$

In part b) the values of $B_0 = 1$ *and* $B_1 = -\frac{1}{2}$ *have been computed already. Also* $B_3 = B_5 = B_7 = 0$ *by the second statement of part b). It remains to compute* B_2, B_4, B_6 *and* B_8*. The formula involving binomial coefficients in the part above can be solved for* B_n *and*

$$B_n = -\frac{1}{n+1}\left(\sum_{k=0}^{n-1}\binom{n+1}{k}B_k\right).$$

This gives

$$\begin{aligned} B_2 &= -\frac{1}{3}(B_0 + 3B_1) = -\frac{1}{3}\left(1 - \frac{3}{2}\right) = \frac{1}{6}, \\ B_4 &= -\frac{1}{5}(B_0 + 5B_1 + 10B_2) = -\frac{1}{30}, \\ B_6 &= -\frac{1}{7}(B_0 + 7B_1 + 21B_2 + 35B_4) = \frac{1}{42}, \end{aligned}$$

and

$$B_8 = -\frac{1}{9}(B_0 + 9B_1 + 36B_2 + 126B_4 + 84B_6) = -\frac{1}{30}.$$

Exercise 14. *Find the power series expansion of* $\tan z$ *about* $z = 0$*, expressing the coefficients in terms of Bernoulli numbers. (Hint: use Exercise 13 and the formula* $\cot 2z = \frac{1}{2}\cot z - \frac{1}{2}\tan z$*.)*

Solution. *To find the power series expansion of* $\tan z$ *about* $z = 0$*, we replace* z *by* $2iz$ *in the function*

$$f(z) = \frac{z(e^z+1)}{2(e^z-1)}$$

which we obtained in the previous problem. We obtain

$$f(2iz) = \frac{2iz(e^{2iz}+1)}{2(e^{2iz}-1)} = \frac{iz(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}} = z\frac{\cos(z)}{\sin(z)} = z\cot(z).$$

Next, we replace z *by* $2iz$ *of the power series expansion for* $f(z)$ *to obtain*

$$z\cot(z) = 1 - \sum_{n=1}^{\infty}\frac{4^n B_{2n}}{(2n)!}z^{2n}.$$

Multiplying the formula $\cot 2z = \frac{1}{2}\cot z - \frac{1}{2}\tan z$ *with* $2z$ *gives*

$$z\tan z = z\cot z - 2z\cot 2z$$

and therefore

$$z\tan z = \left[1 - \sum_{n=1}^{\infty}\frac{4^n B_{2n}}{(2n)!}z^{2n}\right] - \left[1 - \sum_{n=1}^{\infty}\frac{4^{2n} B_{2n}}{(2n)!}z^{2n}\right] = \sum_{n=1}^{\infty}\frac{4^n(4^n-1)B_{2n}}{(2n)!}z^{2n}.$$

Hence,

$$\tan(z) = \sum_{n=1}^{\infty}\frac{4^n(4^n-1)B_{2n}}{(2n)!}z^{2n-1}.$$

4.3 Zeros of an analytic function

Exercise 1. *Let f be an entire function and suppose there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.*

Solution. *Since f is an entire function, that is $f \in A(\mathbb{C})$, we have by Proposition 3.3 p. 77, that f has a power series expansion*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \text{where} a_k = \frac{f^{(k)}(0)}{k!}$$

with infinite radius of convergence. We have to show that $f^{(k)}(0) = 0$ for $k \geq n+1$, then we are done. Let $r > R$ and let $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$. Note that γ is a closed and rectifiable curve. Then

$$|f^{(k)}(0)| \underset{\text{Corollary 2.13 p.73}}{=} \left| \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{z^{k+1}}\, dz \right| \leq \frac{k!}{2\pi} \int_\gamma \frac{|f(z)|}{|z|^{k+1}}\, |dz|.$$

Since $r > R$, $|\gamma(t)| = |re^{it}| = r > R$, $0 \leq t \leq 2\pi$. So $|f(z)| \leq M|z|^n$, $z \in \{\gamma\}$, we get

$$\begin{aligned}
\frac{k!}{2\pi} \int_\gamma \frac{|f(z)|}{|z|^{k+1}}\, |dz| \quad &\leq \quad \frac{k!}{2\pi} \int_\gamma \frac{M|z|^n}{|z|^{k+1}}\, |dz| = \frac{k!M}{2\pi} \int_\gamma |z|^{n-k-1}\, |dz| \\
&\underset{|z|=r}{=} \frac{k!M}{2\pi} r^{n-k-1} \int_\gamma |dz| = \frac{k!M}{2\pi} r^{n-k-1} 2\pi r \\
&= \quad k!Mr^{n-k} = k!M\frac{1}{r^{k-n}} \to 0
\end{aligned}$$

as $r \to \infty$ since $k \geq n+1 \iff k-n \geq 1$. Hence, $|f^{(k)}(0)| = 0$ $\forall k \geq n+1$ and thus

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k;$$

a polynomial of degree less or equal n.

Exercise 2. *Give an example to show that G must be assumed to be connected in Theorem 3.7.*

Solution. *Not available.*

Exercise 3. *Find all entire functions f such that $f(x) = e^x$ for x in $\mathbb{R}$.*

Solution. *Let $f(z) = e^z$. Clearly f is analytic on $\mathbb{C}$ since it is entire. Then for all $x \in \mathbb{R}$, $f(x) = e^x$. Now let g be an entire function such that $g(x) = e^x$ for all $x \in \mathbb{R}$. Now, set $H = \{z \in G : f(z) = g(z)\}$. So, $\mathbb{R} \subseteq H$. Clearly 0 is a limit point of H and $0 \in G$. Hence, by Corollary 3.8, $f(z) \equiv g(z)$. So $f(z) = e^z$ is the set of all entire functions f such that $f(x) = e^x$ for all $x \in \mathbb{R}$.*

Exercise 4. *Prove that $e^{z+a} = e^z e^a$ by applying Corollary 3.8.*

Solution. *Not available.*

Exercise 5. *Prove that $\cos(a+b) = \cos a \cos b - \sin a \sin b$ by applying Corollary 3.8.*

Solution. *Not available.*

Exercise 6. *Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic and $a \in G$ such that $|f(a)| \leq |f(z)|$ for all z in G. Show that either $f(a) = 0$ or f is constant.*

Solution. *Let $f : G \to \mathbb{C}$ be analytic, where G is a region. Let $a \in G$ such that*

$$|f(a)| \leq |f(z)| \qquad \forall z \in G. \tag{4.3}$$

Obviously, $f(a)$ satisfies (4.3) since $0 \leq |f(z)|\ \forall z \in G$ is true.
Now, suppose $f(a) \neq 0\ \forall a \in G$. Define

$$g(z) = \frac{1}{f(z)}.$$

Clearly, $g \in A(G)$ (since $f(a) \neq 0\ \forall a \in G$) and

$$|g(z)| \underset{(def)}{=} \frac{1}{|f(z)|} \underset{(4.3)}{\leq} \frac{1}{|f(a)|} \underset{(def)}{=} |g(a)|, \qquad \forall z \in G.$$

According to the Maximum Modulus Theorem 3.11 p. 79, g is constant and thus f is constant. Thus, we conclude: Either $f(a) = 0$ or f is constant.

Exercise 7. *Give an elementary proof of the Maximum Modulus Theorem for polynomials.*

Solution. *Not available.*

Exercise 8. *Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all z in G. Show that either $f \equiv 0$ or $g \equiv 0$.*

Solution. *It is easy to verify the following claim:*
If $f(z)$ =constant in $B(a; R) \subset G$, then $f(z)$ =same constant for all $z \in G$.
Proof: Suppose $f(z) = c$ in $B(a; R)$. Define $g(z) = f(z) - c$, then $g(z) \in A(G)$. $g^{(n)}(a) = 0$ for all $n = 0, 1, 2, \ldots$. So $g(z) = f(z) - c \equiv 0$ on G.

Let G be a region and let $f, g \in A(G)$ such that

$$f(z)g(z) = 0, \qquad \forall z \in G.$$

Assume $f(z) \neq 0$, $\forall z \in G$. Let $a \in G$ such that $f(a) \neq 0$, then (since f is continuous, because $f \in A(G)$) there is an $R > 0$ such that $f(z) \neq 0$ whenever $z \in B(a; R)$. Hence $g(z) = 0\ \forall z \in B(a; R)$ to fulfill $f(z)g(z) = 0$ $\forall z \in B(a; R)$. By the claim above we obtain that

$$g(z) = 0, \qquad \forall z \in G.$$

So $g \equiv 0$.
Similar, assume $g(z) \neq 0\ \forall z \in G$, then by the same reasoning, we obtain $f \equiv 0$. Thus, we conclude: Either $f \equiv 0$ or $g \equiv 0$.

Exercise 9. *Let $U : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all z in $\mathbb{C}$; prove that U is constant.*

Solution. *There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that*

$$U(z) = Re(f(z)),$$

since $U : \mathbb{C} \to \mathbb{R}$ is assumed to be harmonic. Hence, $Re(f(z)) \geq 0\ \forall z \in \mathbb{C}$ by assumption. For all $z \in \mathbb{C}$, define $g(z) = e^{-f(z)} \in A(\mathbb{C})$. Then

$$|g(z)| \underset{(def)}{=} |e^{-f(z)}| = e^{Re(-f(z))} = e^{-Re(f(z))} \leq 1, \qquad \forall z \in \mathbb{C},$$

where the last step follows since $Re(f(z)) \geq 0$ by assumption. Thus, $g(z)$ is bounded and an entire function, and therefore g is constant according to Liouville's Theorem 3.4 p. 77 and hence $f(z)$ is constant, too. It follows that U is constant.

Exercise 10. *Show that if f and g are analytic functions on a region G such that $\bar{f}g$ is analytic then either f is constant or $g \equiv 0$.*

Solution. *Not available.*

4.4 The index of a closed curve

Exercise 1. *Prove Proposition 4.3.*

Solution. *Not available.*

Exercise 2. *Give an example of a closed rectifiable curve γ in $\mathbb{C}$ such that for any integer k there is a point $a \notin \{\gamma\}$ with $n(\gamma; a) = k$.*

Solution. *Not available.*

Exercise 3. *Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}, 0 \leq t \leq 2\pi$, show that $\int_\gamma \frac{p'(z)}{p(z)}\, dz = 2\pi in$.*

Solution. *Since $p(z)$ is a polynomial of degree n, we can write*

$$p(z) = c\prod_{k=1}^{n}(z - a_k)$$

for some constant c (see Corollary 3.6 p. 77) where $a_1, \ldots, a_n$ are the zeros of $p(z)$. By the product rule we obtain

$$p'(z) = c\sum_{k=1}^{n}(z - a_k)'\prod_{\substack{l=1\\ l\neq k}}^{n}(z - a_l) = c\sum_{k=1}^{n}\prod_{\substack{l=1\\ l\neq k}}^{n}(z - a_l).$$

Thus

$$\begin{aligned}
\int_\gamma \frac{p'(z)}{p(z)}\, dz &= \int_\gamma \frac{c\sum_{k=1}^{n}\prod_{\substack{l=1\\ l\neq k}}^{n}(z - a_l)}{c\prod_{k=1}^{n}(z - a_k)}\, dz \\
&= \int_\gamma \left(\frac{1}{z - a_1} + \frac{1}{z - a_2} + \ldots + \frac{1}{z - a_n}\right) dz \\
&= \int_\gamma (z - a_1)^{-1}\, dz + \int_\gamma (z - a_2)^{-1}\, dz + \ldots + \int_\gamma (z - a_n)^{-1}\, dz \\
&= n(\gamma, a_1)2\pi i + n(\gamma, a_2)2\pi i + \ldots + n(\gamma, a_n)2\pi i = \sum_{k=1}^{n} 2\pi i = 2\pi in,
\end{aligned}$$

since by Definition 4.2 p. 81 $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$ or $\gamma(t) = Re^{2\pi it}$, $0 \leq t \leq 1$ is a closed and rectifiable curve and $a_1, \ldots, a_n \notin \{\gamma\}$ since $R > 0$ sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. In addition, we have $n(\gamma; a_i) = 1$ $\forall 1 \leq i \leq n$. Hence,

$$\int_\gamma \frac{p'(z)}{p(z)}\, dz = 2\pi in.$$

Exercise 4. *Fix $w = re^{i\theta} \neq 0$ and let γ be a rectifiable path in $\mathbb{C} - \{0\}$ from 1 to w. Show that there is an integer k such that $\int_\gamma z^{-1}\, dz = \log r + i\theta + 2\pi ik$.*

Solution. *Not available.*

4.5 Cauchy's Theorem and Integral Formula

Exercise 1. *Suppose $f : G \to \mathbb{C}$ is analytic and define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z, w) = [f(z) - f(w)](z - w)^{-1}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Prove that φ is continuous and for each fixed w, $z \to \varphi(z, w)$ is analytic.*

Solution. *Not available.*

Exercise 2. *Give the details of the proof of Theorem 5.6.*

Solution. *Not available.*

Exercise 3. *Let $B_\pm = \bar{B}(\pm 1; \frac{1}{2})$, $G = B(0; 3) - (B_+ \cup B_-)$. Let $\gamma_1, \gamma_2, \gamma_3$ be curves whose traces are $|z - 1| = 1$, $|z + 1| = 1$, and $|z| = 2$, respectively. Give γ_1, γ_2, and γ_3 orientations such that $n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0$ for all w in $\mathbb{C} - G$.*

Solution. *Not available.*

Exercise 4. *Show that the Integral Formula follows from Cauchy's Theorem.*

Solution. *Assume the conditions of Cauchy's Theorem, that is: Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ $\forall w \in G - \mathbb{C}$, then $\int_\gamma f = 0$.*
Now, let $a \in G$, where $a \notin \{\gamma\}$. Define

$$g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a}, & z \neq a \\ f'(a), & z = a \end{cases}.$$

Clearly, $g \in A(G)$. (Check the power series of $f(z)$ like in the proof of Theorem 3.7 c) $\Rightarrow$ b) p. 78) By Cauchy's Theorem, we obtain

$$\begin{aligned} \int_\gamma g = 0 \quad & \underset{a \notin \gamma}{\Rightarrow} \quad \int_\gamma \frac{f(z) - f(a)}{z - a}\, dz = 0 \\ & \Rightarrow \quad \int_\gamma \frac{f(z)}{z - a}\, dz - \int_\gamma \frac{f(a)}{z - a}\, dz = 0 \\ & \Rightarrow \quad \int_\gamma \frac{f(z)}{z - a}\, dz = f(a) \int_\gamma \frac{1}{z - a}\, dz \\ & \underset{\text{Def. 4.2 p.81}}{\Rightarrow} \quad \int_\gamma \frac{f(z)}{z - a}\, dz = f(a) n(\gamma; a) 2\pi i \\ & \Rightarrow \quad \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz = n(\gamma; a) f(a). \end{aligned}$$

This is exactly Cauchy's Integral Formula.

Exercise 5. *Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$ $\int_\gamma (z - a)^{-n}\, dz = 0$.*

Solution. *Since γ is a closed rectifiable curve in $\mathbb{C}$, $\{\gamma\}$ is bounded. So there exists $R > 0$ such that $\{\gamma\} \subset B(a; R)$. Hence $\mathbb{C} \backslash B(a; R)$ belongs to the unbounded component of $B(a; R)$. Therefore $n(\gamma; w) = 0$ $\forall w \in \mathbb{C} \backslash B(a; R)$ by Theorem 4.4 p.82. We can apply Corollary 5.9 p.86 if we let $G = B(a; R)$ the open set and $f : G \to \mathbb{C}$ given by $f(z) = 1$ is analytic. Further γ is a closed and rectifiable curve in G such that*

$$n(\gamma; w) = 0, \qquad \forall w \in \mathbb{C} \backslash G,$$

then for a in $B(a;R)\backslash\{\gamma\}$

$$\begin{aligned} f^{(n-1)}(a)n(\gamma;a) &= \frac{(n-1)!}{2\pi i}\int_\gamma \frac{1}{(z-a)^n}\,dz \\ \Rightarrow \quad 0\cdot n(\gamma;a) &= \frac{(n-1)!}{2\pi i}\int_\gamma \frac{1}{(z-a)^n}\,dz \end{aligned}$$

since $f^{(n-1)}(a) = 0$ *if* $n \geq 2$. *Hence*

$$\int_\gamma (z-a)^{-n}\,dz = 0, \qquad \text{for } n \geq 2.$$

Exercise 6. *Let* f *be analytic on* $D = B(0;1)$ *and suppose* $|f(z)| \leq 1$ *for* $|z| < 1$. *Show* $|f'(0)| \leq 1$.

Solution. *We can directly apply Cauchy's estimate 2.14 p. 73 with* $a = 0$, $R = 1$, $M = 1$ *and* $N = 1$, *that is by assumption* f *is analytic on* $B(0;1)$ *and* $|f(z)| \leq 1$ *for all* z *in* $B(0;1)$ *(since* $|z| < 1$*). Then*

$$|f'(0)| \leq \frac{1!\cdot 1}{1^1} = 1.$$

So $|f'(0)| \leq 1$.

Exercise 7. *Let* $\gamma(t) = 1 + e^{it}$ *for* $0 \leq t \leq 2\pi$. *Find* $\int_\gamma \left(\frac{z}{z-1}\right)^n\,dz$ *for all positive integers n.*

Solution. *Let* $G = B(1;R)$ *(open) where* $1 < R < \infty$ *and let* $f : G \to \mathbb{C}$ *be given by* $f(z) = z^n$ *which is analytic on G. By Theorem 4.4 p. 82,* $n(\gamma;w) = 0$ $\forall w \in \mathbb{C}\backslash G$ *since* $\mathbb{C}\backslash G$ *belongs to the unbounded component of G and* γ *is a closed and rectifiable curve in* $\mathbb{C}$.
Now, we can apply Corollary 5.9 with G and f *defined above such that* $n(\gamma;w) = 0$ $\forall w \in \mathbb{C}\backslash G$ *and* γ *is a closed rectifiable curve in G, then for* $a = 1$ *in* $G\backslash\{\gamma\}$

$$\begin{aligned} f^{(n-1)}(1)n(\gamma;a) &= \frac{(n-1)!}{2\pi i}\int_\gamma \frac{z^n}{(z-1)^n}\,dz = \frac{(n-1)!}{2\pi i}\int_\gamma \left(\frac{z}{z-1}\right)^n\,dz \\ \Rightarrow \quad n! &= \frac{(n-1)!}{2\pi i}\int_\gamma \left(\frac{z}{z-1}\right)^n\,dz \\ \Rightarrow \quad \int_\gamma \left(\frac{z}{z-1}\right)^n\,dz &= 2\pi i n, \end{aligned}$$

since $f^{(n-1)}(1) = n!$ *and* $n(\gamma;a) = 1$.

Exercise 8. *Let G be a region and suppose* $f_n : G \to \mathbb{C}$ *is analytic for each* $n \geq 1$. *Suppose that* $\{f_n\}$ *converges uniformly to a function* $f : G \to \mathbb{C}$ *Show that* f *is analytic.*

Solution. *Let G be a region and suppose* $f_n : G \to \mathbb{C}$ *is analytic for each* $n \geq 1$, *that us* f_n *is continuous for each* $n \geq 1$.
Let T be an arbitrary triangular path in G. Then T is closed and rectifiable. We also have, that $n(T;w) = 0$ $\forall w \in \mathbb{C}\backslash G$, *since w is an element of the unbounded component of G (Theorem 4.4 p. 82).*
By Cauchy's Theorem (First Version), p. 85 with $m = 1$, *we have*

$$\int_T f_n = 0, \qquad \forall n \in \mathbb{N}, \tag{4.4}$$

since we suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$ and T is closed and rectifiable. We also have

$$\int_T f = \lim_{n\to\infty} \int_T f_n \underset{(4.4)}{=} \lim_{n\to\infty} 0 = 0$$

The first equality follows by Lemma 2.7 p. 71 since T is closed and rectifiable and since each $\{f_n\} \in A(G)$, we have f_n is continuous for each f_n and by the uniform convergence to a function f, we get f is continuous (Theorem 6.1 p. 29).
Finally, we can apply Morera's Theorem 5.10 p. 86 to obtain that f is analytic in G, since f is continuous and $\int_\gamma f = 0$ for every triangular path T in G.

Exercise 9. *Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$ then f is an entire function.*

Solution. *By Morera's Theorem, it suffice to show that for every triangular path T in $\mathbb{C}$, we have $\int_T f = 0$, since we already assume f is continuous. There are several cases to consider:*
Case 1: A triangle does not intersect I. In this case, we obtain by Cauchy's Theorem, that $\int_T f = 0$, since we can find an open neighborhood G containing T (is closed and rectifiable) such that $f \in A(G)$ (by assumption) and $n(T; w) = 0$ $\forall w \in \mathbb{C}\backslash G$ (by Theorem 4.4 p. 82).
Case 2: A triangle touches I exactly at one point P. This single point of intersection P is a removable singularity, since f is continuous. Again apply Cauchy's Theorem to obtain $\int_T f = 0$. (Procedure: Translate the triangle by $\pm\epsilon i$ depending if it lies above or below the x-axis. By Cauchy's Theorem $\int_T f = 0$ over the translated triangle and let $\epsilon \to 0$ since f is continuous).
Case 3: One edge of the triangle touches I. Like in case 2 we can translate the triangle by $\pm\epsilon$ We can also argue this way: Let G be an open neighborhood containing the triangle T. Let $\{T_n\}$ be a sequence of triangles that are not intersecting I, but whose limit is the given T. Then by the continuity of f, we get

$$\int_T f = \lim_{n\to\infty} \int_{T_n} f = \lim_{n\to\infty} 0 = 0,$$

where the first equality follows by Lemma 2.7 and the second one by case 1.
Case 4: A triangle T is cut into 2 parts by I. Then, we can always decompose the triangle T in three parts. Two triangles are of the kind explained in case 3 and one is of the kind explained in case 2 (a sketch might help). Thus, $\int_T f = 0$ again.
Case 5: A triangle T contains parts of I (also here a sketch might help). In this instance, we can decompose the triangle into 5 parts. Two triangles are of the kind explained in case 3 and the other three triangles are of the kind explained in case 2. Hence, $\int_T f = 0$.
Summary: Since $\int_T f = 0$ for every triangular path in $\mathbb{C}$ and f is continuous, we get by Morera's Theorem: $f \in A(\mathbb{C})$, that is f is an entire function.

Exercise 10. *Use Cauchy's Integral Formula to prove the Cayley–Hamilton Theorem: If A is an $n \times n$ matrix over $\mathbb{C}$ and $f(z) = \det(z - A)$ is the characteristic polynomial of A then $f(A) = 0$. (This exercise was taken from a paper by C. A. McCarthy, Amer. Math. Monthly, 82 (1975), 390–391).*

Solution. *Not available.*

4.6 The homotopic version of Cauchy's Theorem and simple connectivity

Exercise 1. *Let G be a region and let $\sigma_1, \sigma_2 : [0, 1] \to G$ be the constant curves $\sigma_1(t) \equiv a$, $\sigma_2(t) \equiv b$. Show that if γ is closed rectifiable curve in G and $\gamma \sim \sigma_1$, then $\gamma \sim \sigma_2$. (Hint: connect a and b by a curve.)*

Solution. *Assume γ is a closed and rectifiable curve in G and $\gamma \sim \sigma_1$. Since $\sigma_1(t) \equiv a$ and $\sigma_2(t) \equiv b$, we surely have that σ_1 and σ_2 are closed and rectifiable curves in G. If we can show that $\sigma_1 \sim \sigma_2$, then we get $\gamma \sim \sigma_2$, since $\sim$ is an equivalence relation, that is:*

$$\gamma \sim \sigma_1 \quad \text{and} \quad \sigma_1 \sim \sigma_2 \Rightarrow \gamma \sim \sigma_2.$$

So, now we show $\sigma_1 \sim \sigma_2$, that is, we have to find a continuous function $\Gamma : [0,1] \times [0,1] \to G$ such that

$$\Gamma(s,0) = \sigma_1(s) = a \quad \text{and} \quad \Gamma(s,1) = \sigma_2(s) = b, \quad \text{for} \quad 0 \le s \le 1$$

and $\Gamma(0,t) = \Gamma(1,t)$ for $0 \le t \le 1$. Let $\Gamma(s,t) = tb + (1-t)a$. Clearly Γ is a continuous function. (it is constant with respect to s and a line with respect to t). In addition, it satisfies $\Gamma(s,0) = a$ $\forall 0 \le s \le 1$ and $\Gamma(s,1) = b$ $\forall 0 \le s \le 1$ and $\Gamma(0,t) = tb + (1-t)a = \Gamma(1,t)$ $\forall 0 \le t \le 1$. So $\sigma_1 \sim \sigma_2$.

Exercise 2. *Show that if we remove the requirement "$\Gamma(0,t) = \Gamma(1,t)$ for all t" from Definition6.1 then the curve $\gamma_0(t) = e^{2\pi it}$, $0 \le t \le 1$, is homotopic to the constant curve $\gamma_1(t) \equiv 1$ in the region $G = \mathbb{C} - \{0\}$.*

Solution. *Not available.*

Exercise 3. *3. Let C = all rectifiable curves in G joining a to b and show that Definition 6.11 gives an equivalence relation on C.*

Solution. *Not available.*

Exercise 4. *Let $G = \mathbb{C} - \{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.*

Solution. *Not available.*

Exercise 5. *Evaluate the integral $\int_\gamma \frac{dz}{z^2+1}$ where $\gamma(\theta) = 2|\cos 2\theta| e^{i\theta}$ for $0 \le \theta \le 2\pi$.*

Solution. *A sketch shows that the two zeros of z^2+1 (they are $\pm i$) are inside of the closed and rectifiable curve γ (The region looks like a clover with four leaves). Using partial fraction decompositions gives*

$$\begin{aligned}
\int_\gamma \frac{dz}{z^2+1} &= \frac{1}{2i}\int_\gamma \frac{1}{z-i}\,dz - \frac{1}{2i}\int_\gamma \frac{1}{z+i}\,dz \\
&= \pi\frac{1}{2\pi i}\int_\gamma \frac{1}{z-i}\,dz - \pi\frac{1}{2i}\int_\gamma \frac{1}{z+i}\,dz \\
&\underset{\text{Def 4.2 p. 81}}{=} \pi\left(n(\gamma;i) - n(\gamma;-i)\right) \\
&= \pi \cdot 0 = 0,
\end{aligned}$$

since $n(\gamma;i) = n(\gamma;-i)$ since i and $-i$ are contained in the region generated by γ. Hence

$$\int_\gamma \frac{dz}{z^2+1} = 0.$$

Exercise 6. *Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \le \theta \le 4\pi$. Evaluate $\int_\gamma \frac{dz}{z^2+\pi^2}$.*

Solution. *A sketch reveals that the zero $-\pi i$ is inside the region and the zero πi is outside the region. Using partial fraction decomposition yields*

$$\begin{aligned}
\int_\gamma \frac{dz}{z^2+\pi^2} &= \frac{1}{2\pi i}\int_\gamma \frac{1}{z-i\pi}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{1}{z+i\pi}\,dz \\
&\underset{\text{Def 4.2 p. 81}}{=} n(\gamma;i\pi) - n(\gamma;-i\pi) = 0 - 1 = -1,
\end{aligned}$$

since $i\pi$ is not contained in the region generated by γ, so $n(\gamma; i\pi) = 0$ and $-i\pi$ is contained in the region, so $n(\gamma; -i\pi) = 1$. Hence,

$$\int_\gamma \frac{dz}{z^2 + \pi^2} = -1.$$

Exercise 7. *Let $f(z) = [(z-\frac{1}{2}-i)\cdot(z-1-\frac{3}{2}i)\cdot(z-1-\frac{i}{2})\cdot(z-\frac{3}{2}-i)]^{-1}$ and let γ be the polygon $[0, 2, 2+2i, 2i, 0]$. Find $\int_\gamma f$.*

Solution. *Not available.*

Exercise 8. *Let $G = \mathbb{C} - \{a, b\}$, $a \neq b$, and let γ be the curve in the figure below.*
(a) Show that $n(\gamma; a) = n(\gamma; b) = 0$.
(b) Convince yourself that γ is not homotopic to zero. (Notice that the word is "convince" and not "prove". Can you prove it?) Notice that this example shows that it is possible to have a closed curve γ in a region such that $n(\gamma; z) = 0$ for all z not in G without γ being homotopic to zero. That is, the converse to Corollary 6.10 is false.

Solution. *Let γ be the path depicted on p. 96. We can write it as a sum of 6 paths. Two of them will be closed and have a and b in their unbounded component. Therefore, two integrals will be zero and we will have another 2 pair of non-closed paths. The first pair begins at the leftmost crossing pair and goes around a in opposite direction. The second pair begins at the middle crossing pair and goes around b in opposite direction. They also meet at the rightmost crossing point. If we integrate over the path around a is equivalent to evaluate $\int_{\gamma_1-\gamma_2} \frac{1}{z-a}\, dz$ where $\gamma_1(t) = a + re^{it}$, $0 \leq t \leq \pi$ and $\gamma_2(t) = a + re^{it}$, $\pi \leq t \leq 2\pi$ for some $r > 0$. It is easily seen that we have*

$$\int_{\gamma_1} \frac{1}{z-a}\, dz = \int_{\gamma_2} \frac{1}{z-a}\, dz = \pi i$$

and therefore $n(\gamma; a) = 0$ and similarly we obtain $n(\gamma; b) = 0$.

Exercise 9. *Let G be a region and let γ_0 and γ_1 be two closed smooth curves in G. Suppose $\gamma_0 \sim \gamma_1$, and Γ satisfies (6.2). Also suppose that $\gamma_t(s) = \Gamma(s, t)$ is smooth for each t. If $w \in \mathbb{C} - G$ define $h(t) = n(\gamma_t; w)$ and show that $h : [0, 1] \to \mathbb{Z}$ is continuous.*

Solution. *Not available.*

Exercise 10. *Find all possible values of $\int_\gamma \frac{dz}{1+z^2}$, where γ is any closed rectifiable curve in $\mathbb{C}$ not passing through $\pm i$.*

Solution. *Let γ be a closed rectifiable curve in $\mathbb{C}$ not passing through $\pm i$. Using partial fraction decomposition and the definition of the winding number we obtain*

$$\int_\gamma \frac{dz}{1+z^2} = \frac{1}{2i}\left(\int_\gamma \frac{1}{z-1}\, dz - \int_\gamma \frac{1}{z+i}\, dz\right) = \frac{1}{2i}\left(2\pi i n(\gamma; i) - 2\pi i n(\gamma; -i)\right) = \pi\left(n(\gamma; i) - n(\gamma; -i)\right).$$

Exercise 11. *Evaluate $\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz$ where γ is one of the curves depicted below. (Justify your answer.)*

Solution. *Using Corollary 5.9 p. 86 we have $\forall a \in G - \{\gamma\}$*

$$\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\, dz = 2\pi i f^{(k)}(a) n(\gamma; a) \frac{1}{k!},$$

where γ is a closed rectifiable curve in G (open) such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - G$ and $f : G \to \mathbb{C}$ is analytic.
In our case $a = 0$ and γ satisfies the above assumptions for exercise a), b) and c). Let $f(z) = e^z - e^{-z}$ which is clearly analytic on any region G. We have $f^{(3)}(0) = e^0 + e^{-0} = 2$ and using the formula above for $k = 3$ we obtain

$$\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz = 2\pi i \cdot 2n(\gamma; 0)\frac{1}{6} = \frac{2\pi i}{3} n(\gamma; 0).$$

Hence, $n(\gamma; 0) = 1$ for part a) and thus the result is $\frac{2\pi i}{3}$. We have $n(\gamma; 0) = 1$ for part b) and thus the result is $\frac{4\pi i}{3}$. We obtain the same result for part c).

4.7 Counting zeros; the Open Mapping Theorem

Exercise 1. *Show that if $f : G \to \mathbb{C}$ is analytic and γ is a rectifiable curve in G then $f \circ \gamma$ is also a rectifiable curve.*

Solution. *Not available.*

Exercise 2. *Let G be open and suppose that γ is a closed rectifiable curve in G such that $\gamma \approx 0$. Set $r = d(\{\gamma\}, \partial G)$ and $H = \{z \in \mathbb{C} : n(\gamma; z) = 0\}$. (a) Show that $\{z : d(z, \partial G) < \frac{1}{2}r\} \subset H$. (b) Use part (a) to show that if $f : G \to \mathbb{C}$ is analytic then $f(z) = \alpha$ has at most a finite number of solutions z such that $n(\gamma; z) \neq 0$.*

Solution. *Not available.*

Exercise 3. *Let f be analytic in $B(a; R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = \ldots = f(a) = 0$ and $f^{(a)}(a) \neq 0$.*

Solution. $\Leftarrow$*: Let $f^{(n)}(a) = 0$ for all $n = 1, 2, \ldots, m-1$ (we also have $f(a) = 0$) and $f^{(m)}(a) \neq 0$. Since $f \in A(B(a; R))$, we can write*

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k (z-a)^k \quad \text{where} \quad a_k = \frac{f^{(k)}(a)}{k!} \text{ and } |z-a| < R \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = \sum_{k=m}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k \\ &= (z-a)^m \underbrace{\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^{k-m}}_{g(z)} \underset{g(a)\neq 0,\ \text{since } f^{(m)}\neq 0}{=} (z-a)^m g(z) \end{aligned}$$

where $g \in A(B(a; R))$. Hence, a is a zero of multiplicity m.
$\Rightarrow$: *Suppose a is a zero of multiplicity $m \geq 1$ of f. By definition, there exists $g \in A(B(0; R))$ such that*

$$f(z) = (z-a)^m g(z), \qquad g(a) \neq 0 \text{ (p. 97)}.$$

Then after a lengthy calculation, we obtain

$$f^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{m!}{(m-(n-k))!}(z-a)^{m-(n-k)} g^{(k)}(z), \qquad n = 1, 2, \ldots, m.$$

If $z = a$, then clearly $f^{(n)}(a) = 0$ for $m - (n - k) > 0$ and $f^{(n)}(a) \neq 0$ for $m - (n - k) = 0$. If $k = 0$ and $m = n$, then $f^{(m)}(a) = f^{(n)}(a) = m!g(a) \neq 0$ since $g(a) \neq 0$ by assumption. Hence, $f^{(n)}(a) = 0$ for $n = 1, 2, \ldots, m-1$ and $f^{(n)}(a) \neq 0$ for $n = m$.

Exercise 4. *Suppose that $f : G \to \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any z in G.*

Solution. *Assume $f'(a) = 0$ for some $a \in G$. Then $f(a) = \alpha$ where α is a constant. Define*

$$F(z) = f(z) - \alpha.$$

Then $F(a) = f(a) - \alpha = \alpha - \alpha = 0$ and $F'(a) = f'(a) = 0$ by assumption. Hence, F has a as a root of multiplicity $m \geq 2$.
Then, we can use Theorem 7.4 p. 98 to argue that there is an $\epsilon > 0$ and $\delta > 0$ such that for $0 < |\xi - a| < \delta$, the equation $f(z) = \xi$ has exactly m simple roots in $B(a; \epsilon)$, since f is analytic in $B(a; R)$ and $\alpha = f(a)$. Also $f(z) - \alpha$ has a zero of order $m \geq 2$ at $z = a$.
Since $f(z) = \xi$ has exactly m (our case $m \geq 2$) simple roots in $B(a; \epsilon)$, we can find at least two distinct points $z_1, z_2 \in B(a; \epsilon) \subset G$ such that

$$f(z_1) = \xi = f(z_2)$$

contradicting that f is $1 - 1$. So $f'(z) \neq 0$ for any z in G.

Exercise 5. *Let X and Ω be metric spaces and suppose that $f : X \to \Omega$, is one-one and onto. Show that f is an open map iff f is a closed map. (A function f is a closed map if it takes closed sets onto closed sets.)*

Solution. *Not available.*

Exercise 6. *Let $P : \mathbb{C} \to \mathbb{R}$ be defined by $P(z) = Re\ z$; show that P is an open map but is not a closed map. (Hint: Consider the set $F = \{z : Im\ z = (Re\ z)^{-1}$ and $Re\ z \neq 0\}$.)*

Solution. *Not available.*

Exercise 7. *Use Theorem 7.2 to give another proof of the Fundamental Theorem of Algebra.*

Solution. *Not available.*

4.8 Goursat's Theorem

No exercises are assigned in this section.

Chapter 5

Singularities

5.1 Classification of singularities

Exercise 1. *Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity determine $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small values of δ.*

a)

$$f(z) = \frac{\sin z}{z};$$

b)

$$f(z) = \frac{\cos z}{z};$$

c)

$$f(z) = \frac{\cos z - 1}{z};$$

d)

$$f(z) = \exp(z^{-1});$$

e)

$$f(z) = \frac{\log(z+1)}{z^2};$$

f)

$$f(z) = \frac{\cos(z^{-1})}{z^{-1}};$$

g)

$$f(z) = \frac{z^2+1}{z(z-1)};$$

h)

$$f(z) = (1 - e^z)^{-1};$$

i)

$$f(z) = z \sin \frac{1}{z};$$

j)

$$f(z) = z^n \sin\frac{1}{z}.$$

Solution. *a) According to Theorem 1.2 p.103, $z = 0$ is a removable singularity, since*

$$\lim_{z\to\infty}(z-0)\frac{\sin(z)}{z} = \lim_{z\to\infty} z\frac{\sin(z)}{z} = \lim_{z\to\infty}\sin(z) = 0.$$

If we define,

$$f(z) = \begin{cases} \frac{\sin(z)}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases},$$

then f is analytic at $z = 0$. A simple computation using L'Hospital's rule yields $\lim_{z\to 0}\frac{\sin(z)}{z} = 1$.
b) We have (see p. 38)

$$\cos(z) = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}$$

and therefore

$$\frac{\cos(z)}{z} = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k-1}}{(2k)!} = \frac{1}{z} + \text{ analytic part.}$$

Thus $\frac{\cos z}{z}$ has a simple pole at $z = 0$.
c) According to Theorem 1.2 p.103, $z = 0$ is a removable singularity, since

$$\lim_{z\to\infty}(z-0)\frac{\cos(z)-1}{z} = \lim_{z\to\infty} z\frac{\cos(z)-1}{z} = \lim_{z\to\infty}\cos(z)-1 = 1-1 = 0.$$

If we define,

$$f(z) = \begin{cases} \frac{\cos(z)-1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases},$$

then f is analytic at $z = 0$. A simple computation using L'Hospital's rule yields $\lim_{z\to 0}\frac{\cos(z)-1}{z} = 0$.
e) We know

$$\log(1+z) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}.$$

So we can write

$$\frac{\log(1+z)}{z^2} = \frac{1}{z^2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^{k-2}}{k} = \frac{1}{z} - \frac{1}{2} + \sum_{k=3}^{\infty}(-1)^{k+1}\frac{z^{k-2}}{k}$$

and hence $f(z) = \frac{\log(1+z)}{z^2}$ has a pole of order 1 at $z = 0$ and the singular part is $\frac{1}{z}$ (by Equation 1.7 and Definition 1.8 p. 105).
g) First, we simplify $\frac{z^2+1}{z^2-z} = 1 + \frac{z+1}{z(z-1)}$. Now partial fraction decomposition of $\frac{z+1}{z(z-1)}$ yields

$$\frac{z+1}{z(z-1)} = -\frac{1}{z} + \frac{2}{z-1}$$

so

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1}$$

and hence $f(z) = \frac{z^2+1}{z^2-z}$ *has poles of order 1 at* $z = 0$ *and* $z = 1$ *and the singular part is* $\frac{2}{z-1} - \frac{1}{z}$ *(by Equation 1.7 and Definition 1.8 p. 105).*
i) We know (see p. 38)

$$\sin(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

so

$$f(z) = z\sin\frac{1}{z} = z\sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{1}{z}\right)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{1}{z}\right)^{2k}}{(2k+1)!} = 1 - \frac{1}{3!z^2} + \frac{1}{5!z^4} - \frac{1}{7!z^6} + \dots$$

Therefore $f(z) = z\sin\frac{1}{z}$ *has an essential singularity at* $z = 0$ *(it is neither a pole nor a removable singularity by Corollary 1.18c). We have*

$$\overline{f(\{z : 0 < |z| < \delta\})} = \overline{f[ann(0;0,\delta)]} = \mathbb{C}$$

by the Casorati-Weierstrass Theorem p. 109. But $f(\{z : 0 < |z| < \delta\}) = f[ann(0;0,\delta)]$ *is either* $\mathbb{C}$ *or* $\mathbb{C}\backslash\{a\}$ *for one* $a \in \mathbb{C}$ *by the Great Picard Theorem (p. 300). In this case,*

$$f(\{z : 0 < |z| < \delta\}) = f[ann(0;0,\delta)] = \mathbb{C}.$$

Exercise 2. *Give the partial fraction expansion of* $r(z) = \frac{z^2+1}{(z^2+z+1)(z-1)^2}$.

Solution. *Not available.*

Exercise 3. *Give the details of the derivation of (1.17) from (1.16).*

Solution. *Not available.*

Exercise 4. *Let* $f(z) = \frac{1}{z(z-1)(z-2)}$*; give the Laurent Expansion of* $f(z)$ *in each of the following annuli: (a)* $ann(0;0,1)$*; (b)* $ann(0;1,2)$*; (c)* $ann(0;2,\infty)$.

Solution. *a) We have* $0 < |z| < 1$ *and using partial fraction decomposition yields*

$$\begin{aligned}
f(z) &= \frac{1}{z(z-1)(z-2)} = \frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{-1}{z-1} + \frac{1}{z-2}\right) \\
&= \frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}\right) \underset{|z|<1 \, and \, |z/2|<1}{=} \frac{1}{z}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) \\
&= \sum_{n=0}^{\infty} z^{n-1} - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^{n-1} = \sum_{n=-1}^{\infty}\left(1 - \left(\frac{1}{2}\right)^{n+2}\right) z^n \\
&= \sum_{n=-1}^{\infty}\left(1 - 2^{-n-2}\right) z^n, \qquad (0 < |z| < 1).
\end{aligned}$$

b) We have $1 < |z| < 2$ *and thus*

$$\begin{aligned}
f(z) &= \frac{1}{z(z-1)(z-2)} \\
&= \frac{1}{z}\left(-\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}\right) \underset{|1/z|<1 \, and \, |z/2|<1}{=} \frac{1}{z}\left(-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) \\
&= \sum_{n=0}^{\infty} -\left(\frac{1}{z}\right)^{n+2} - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^{n-1} = \sum_{n=0}^{\infty} -z^{-n-2} - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^{n-1} \\
&= \sum_{n=-\infty}^{-2} -z^n - \sum_{n=-1}^{\infty}\left(\frac{1}{2}\right)^{n+2} z^n, \qquad (1 < |z| < 2).
\end{aligned}$$

c) We have $2 < |z| < \infty$ and thus

$$\begin{aligned}
f(z) &= \frac{1}{z(z-1)(z-2)} \\
&= \frac{1}{z}\left(-\frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{z}\frac{1}{1-\frac{2}{z}}\right) \underset{|1/z|<1 and\, |2/z|<1}{=} \frac{1}{z}\left(-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n\right) \\
&= \sum_{n=0}^{\infty} -\left(\frac{1}{z}\right)^{n+2} + \sum_{n=0}^{\infty} 2^n \left(\frac{1}{z}\right)^{n+2} = \sum_{n=0}^{\infty}(2^n - 1)z^{-n-2} \\
&= \sum_{n=-\infty}^{-2}(2^{-n-2} - 1)z^n, \qquad (2 < |z| < \infty).
\end{aligned}$$

Exercise 5. *Show that $f(z) = \tan z$ is analytic in $\mathbb{C}$ except for simple poles at $z = \frac{\pi}{2} + n\pi$, for each integer n. Determine the singular part of f at each of these poles.*

Solution. *Define $z_n = \frac{\pi}{2} + n\pi$. We know $\tan(z) = \frac{\sin(z)}{\cos(z)}$ and $(\cos(z))' = -\sin(z)$. Hence, $\tan(z)$ has a simple pole at each z_n with residue -1 and with singular part $-\frac{1}{z-z_n}$ at each z_n.*

Exercise 6. *If $f : G \to \mathbb{C}$ is analytic except for poles show that the poles of f cannot have a limit point in G.*

Solution. *Not available.*

Exercise 7. *Let f have an isolated singularity at $z = a$ and suppose $f(z) \not\equiv 0$. Show that if either (1.19) or (1.20) holds for some s in $\mathbb{R}$ then there is an integer m such that (1.19) holds if $s > m$ and (1.20) holds if $s < m$.*

Solution. *Let $\lim_{z\to a} |z-a|^s |f(z)| = 0$ hold, that is $\lim_{z\to a}(z-a)^s f(z) = 0$ for some s in $\mathbb{R}$. Then it holds obviously for any greater s's. Hence, there exist an integer $n > 0$ such that*

$$\lim_{z\to a}(z-a)^n f(z) = 0 \tag{5.1}$$

and

$$\lim_{z\to a}(z-a)^{n+1} f(z) = 0 \tag{5.2}$$

(In fact we are free to choose any positive integer greater than s). Now, $(z-a)^n f(z)$ thus has a removable singularity at $z = a$ (by Theorem 1.2 p. 103) applied to (5.2)) and its extended value is also zero by (5.2). Thus $(z-a)^n f(z)$ has a zero of finite order k at $z - a$. So

$$(z-a)^n f(z) = (z-a)^k h(z)$$

where $h(z)$ is analytic at a and $h(a) \neq 0$. Hence

$$\lim_{z\to a}(z-a)^s f(z) = \lim_{z\to a}(z-a)^{s-n}(z-a)^n f(z) = \lim_{z\to a}(z-a)^{s-n+k}h(z) = \begin{cases} 0, & s > n-k \\ \pm\infty, & s < n-k \\ h(a) \neq 0, & s = n-k. \end{cases}$$

Thus,

$$\lim_{z\to a}|z-a|^s |f(z)| = \begin{cases} 0, & s > m \\ \infty, & s < m \end{cases}$$

where $m = n - k$.
Now let $\lim_{z\to a} |z-a|^s |f(z)| = \infty$ for some s in $\mathbb{R}$. Then there exists an integer $n < 0$ so that

$$\lim_{z\to a} |z-a|^n |f(z)| = \infty \tag{5.3}$$

(In fact we are free to choose any negative integer less than s). Now $(z-a)^n f(z)$ thus has a pole at $z = a$ (by Definition 1.3 p.105 and (5.1)) and $(z-a)^n f(z)$ has a pole of finite order l at $z = a$ and so

$$(z-a)^n f(z) = (z-a)^{-l} k(z)$$

where $k(z)$ is analytic at a and $k(a) \neq 0$ (Proposition 1.4 p.105). Hence

$$\lim_{z\to a}(z-a)^s f(z) = \lim_{z\to a}(z-a)^{s-n+n} f(z) = \lim_{z\to a}(z-a)^{s-n-l} k(z) = \begin{cases} 0, & s > n+l \\ \pm\infty, & s < n+l \\ k(a) \neq 0, & s = n+l. \end{cases}$$

Thus,

$$\lim_{z\to a} |z-a|^s |f(z)| = \begin{cases} 0, & s > m \\ \infty, & s < m \end{cases}$$

where $m = n + l$.

Exercise 8. *Let f, a, and m be as in Exercise 7. Show: (a) $m = 0$ iff $z = a$ is a removable singularity and $f(a) \neq 0$; (b) $m < 0$ iff $z = a$ is a removable singularity and f has a zero at $z = a$ of order $-m$; (c) $m > 0$ iff $r = a$ is a pole of f of order m.*

Solution. *a) $\Rightarrow$: Let $m = 0$. Then by Exercise 7, we have*

$$\lim_{z\to a}(z-a)^2 f(z) = \begin{cases} 0, & s > 0 = m \\ \neq 0, & s = 0 = m \end{cases}. \tag{5.4}$$

Thus using $s = 1$, $f(z)$ has a removable singularity (by Theorem 1.2 p. 103) with the extended value equal to $f(a) = \lim_{z\to a} f(z) \neq 0$.
$\Leftarrow$: Let $z = a$ be a removable singularity and $f(a) = \lim_{z\to a} f(z) \neq 0$. Then $\lim_{z\to a}(z-a)f(z) = 0$ by Theorem 1.2 p.103. By Exercise 7, we obtain that there is an integer m such that

$$\lim_{z\to a} |z-a|^s |f(z)| = \begin{cases} 0, & s > m \\ \infty, & s < m \end{cases}.$$

But $m = 0$, since

$$\lim_{z\to a}(z-a)^s f(z) = \begin{cases} 0, & s > 0 \\ \infty, & s < 0 \end{cases},$$

which follows by (5.4).
b) $\Rightarrow$: Let the algebraic order be defined at $z = a$ and be equal to $-m$ ($m > 0$). Then by Exercise 7, we have

$$\lim_{z\to a} f(z) = 0 \quad \text{and} \quad \lim_{z\to a}(z-a) f(z) = 0.$$

Recall $\lim_{z\to a}(z-a)^s f(z) = 0$ if $s > -m$ so it is true of $s = 0$ and $s = 1$. Thus, $f(z)$ has a removable singularity at $z = a$ and also the extended function has the value $\lim_{z\to a} f(z) = 0$ at $z = a$ (p. 103 Theorem

1.2). Now $h(z) := (z-a)^{-m} f(z)$ *has a removable singularity at* $z = a$ *(since* $\lim_{z\to a}(z-a)^{-m+1} f(z) = 0$*) but* $h(z)$ *has extended value*

$$h(a) = \lim_{z\to a}(z-a)^{-m} f(z) \neq 0$$

(by Exercise 7). Therefore $f(z) = (z-a)^m h(z)$ *has a zero of order m at* $z = a$.
$\Leftarrow$*: Assume* $f(z)$ *has a removable singularity at* $z = a$ *and let* $z = a$ *be a zero of order* $m > 0$*. Then* $f(z) = (z-a)^m h(z)$ *where* $h(z)$ *is analytic in* $|z-a| < \delta$, $\delta > 0$ *and* $h(a) \neq 0$*. Then*

$$\lim_{z\to a} |z-a|^s |f(z)| = \lim_{z\to a} |z-a|^{s+m} |h(z)| = \begin{cases} 0, & s > -m \\ \infty, & s < -m \end{cases}$$

by Exercise 7. Hence, the algebraic order of f *at* $z = a$ *is equal to* $-m$ *(*$m > 0$*).*
c) $\Rightarrow$*: Let the algebraic order be* $m > 0$*. Then*

$$\lim_{z\to a}(z-a)^s f(z) = 0$$

for $s > m$ *by Exercise 7, that is*

$$\lim_{z\to a}(z-a)^{m+1} f(z) = 0 \quad \textit{and} \lim_{z\to a}(z-a)^m f(z) \neq 0$$

by Exercise 7. Then for $z \neq a$*, we have* $h(z) = (z-a)^m f(z)$ *has a removable singularity at* $z = a$ *with* $h(a) \neq 0$*. Now* $f(z) = (z-a)^{-m} h(z)$ *where* $h \in A(B(a;\delta))$ *and* $h(a) \neq 0$*. Thus* $f(z)$ *has a pole of order m at* $z = a$*.*
$\Leftarrow$*: Let* $f(z)$ *have a pole of order m at* $z = a$*. Then* $f(z) = (z-a)^{-m} g(z)$ *where* $g(z) \in A(B(a;\delta))$ *and* $g(a) \neq 0$*. Thus* $\lim_{z\to a}(z-a)^m f(z) = g(a) \neq 0$ *and*

$$\lim_{z\to a}(z-a)^{m+1} f(z) = \lim_{z\to a}(z-a) g(z) = 0.$$

So

$$\lim_{z\to a} |z-a|^s |f(z)| = \lim_{z\to a} |z-a|^{s-m} |z-a|^m |f(z)| = \begin{cases} 0, & s > m \\ \infty, & s < m \end{cases}$$

by Exercise 7. Therefore, $m > 0$ *is the algebraic order of* f *at* $z = a$*.*

Exercise 9. *A function* f *has an essential singularity at* $z = a$ *iff neither (1.19) nor (1.20) holds for any real number* s*.*

Solution. $\Rightarrow$*: We prove this direction by proving the contrapositive (*$P \to Q \iff \neg Q \Rightarrow \neg P$*). Assume (1.19) or (1.20) hold for some real number* s*. Then by Exercise 7, there exists an integer m such that (1.19) holds if* $s > m$ *and (1.20) holds if* $s < m$*. Then by Exercise 8,* $z = a$ *is either a removable singularity or a pole. Hence* $z = a$ *is not an essential singularity (by Definition 1.3 p.105).*
$\Leftarrow$*: Also this direction by is being proved by showing the contrapositive. Assume that* f *has an isolated singularity at* $z = a$ *which is not an essential singularity. Then either* $z = a$ *is a removable singularity or a pole (by Definition 1.3 p.105). Again by Problem 7 and 8, in either case there exists an m such that (1.19) or (1.20) holds. (If* $z = a$ *is removable, then either* $m = 0$ *or* $m < 0$*; if* $z = a$ *is a pole, then* $m > 0$ *by Exercise 8).*

Exercise 10. *Suppose that* f *has an essential singularity at* $z = a$*. Prove the following strengthened version of the Casorati–Weierstrass Theorem. If* $c \in \mathbb{C}$ *and* $\epsilon > 0$ *are given then for each* $\delta > 0$ *there is a number* α*,* $|c-\alpha| < \epsilon$*, such that* $f(z) = \alpha$ *has infinitely many solutions in* $B(a;\delta)$*.*

Solution. *Not available.*

Exercise 11. *Give the Laurent series development of $f(z) = \exp\left(\frac{1}{z}\right)$. Can you generalize this result?*

Solution. *Not available.*

Exercise 12. *(a) Let $\lambda \in \mathbb{C}$ and show that*

$$\exp\left\{\frac{1}{2}\lambda\left(z + \frac{1}{z}\right)\right\} = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

for $0 < |z| < \infty$, where for $n \geq 0$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt\ dt.$$

(b) Similarly, show

$$\exp\left\{\frac{1}{2}\lambda\left(z - \frac{1}{z}\right)\right\} = b_0 + \sum_{n=1}^{\infty} b_n\left(z^n + \frac{(-1)^n}{z^n}\right)$$

for $0 < |z| < \infty$, where

$$b_n = \frac{1}{\pi}\int_0^{\pi} \cos(nt - \lambda\sin t)\ dt.$$

Solution. *Not available.*

Exercise 13. *Let $R > 0$ and $G = \{z : |z| > R\}$; a function $f : G \to \mathbb{C}$ has a removable singularity, a pole, or an essential singularity at infinity if $f(z^{-1})$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞ then the order of the pole is the order of the pole of $f(z^{-1})$ at $z = 0$.*
(a) Prove that an entire function has a removable singularity at infinity iff it is a constant.
(b) Prove that an entire function has a pole at infinity of order m iff it is a polynomial of degree m.
(c) Characterize those rational functions which have a removable singularity at infinity.
(d) Characterize those rational functions which have a pole of order m at infinity.

Solution. *Not available.*

Exercise 14. *Let $G = \{z : 0 < |z| < 1\}$ and let $f : G \to \mathbb{C}$ be analytic. Suppose that γ is a closed rectifiable curve in G such that $n(\gamma; a) = 0$ for all a in $\mathbb{C} - G$. What is $\int_\gamma f$? Why?*

Solution. *Not available.*

Exercise 15. *Let f be analytic in $G = \{z : 0 < |z - a| < r\}$ except that there is a sequence of poles $\{a_n\}$ in G with $a_n \to a$. Show that for any ω in $\mathbb{C}$ there is a sequence $\{z_n\}$ in G with $a = \lim z_n$ and $\omega = \lim f(z_n)$.*

Solution. *Not available.*

Exercise 16. *Determine the regions in which the functions $f(z) = \left(\sin\frac{1}{z}\right)^{-1}$ and $g(z) = \int_0^1 (t - z)^{-1}\ dt$ are analytic. Do they have any isolated singularities? Do they have any singularities that are not isolated?*

Solution. *Not available.*

Exercise 17. *Let f be analytic in the region $G =$ann$(a; 0, R)$. Show that if $\iint_G |f(x + iy)|^2\ dx\ dy < \infty$ then f has a removable singularity at $z = a$. Suppose that $p > 0$ and $\iint_G |f(x + iy)|^p\ dx\ dy < \infty$; what can be said about the nature of the singularity at $z = a$?*

Solution. *Not available.*

5.2 Residues

Exercise 1. *Calculate the following integrals:*
a)

$$\int_0^\infty \frac{x^2\,dx}{x^4+x^2+1}$$

b)

$$\int_0^\infty \frac{\cos x - 1}{x^2}\,dx$$

c)

$$\int_0^\pi \frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2} \text{ where } a^2<1$$

d)

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} \text{ where } a>1.$$

Solution. *Note that*

$$\int_0^\infty \frac{x^2\,dx}{x^4+x^2+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{x^4+x^2+1}$$

since $\frac{x^2}{x^4+x^2+1}$ *is even. Let* $\gamma_R = [-R,R]\cup C_R$ *where* $C_R = Re^{it}$, $0\le t\le\pi$ *and* $R\gg 0$ *(draw a sketch).* $f(z) = \frac{z^2}{z^4+z^2+1}$ *has two simple poles at* $a_1 = e^{i\frac{\pi}{3}}$, $a_2 = e^{i\frac{2\pi}{3}}$ *enclosed by* γ_R. *By the residual Theorem, we have*

$$2\pi i(Res(f,a_1)+Res(f,a_2)) = \int_{\gamma_R} f = \int_{-R}^R \frac{x^2}{x^4+x^2+1}\,dx + \int_{C_R}\frac{z^2}{z^4+z^2+1}\,dz.$$

Thus,

$$\int_{-\infty}^\infty \frac{x^2\,dx}{x^4+x^2+1} = -\lim_{R\to\infty}\int_{C_R}\frac{z^2}{z^4+z^2+1}\,dz + 2\pi i\sum_{k=1}^2 Res(f,a_k).$$

We have

$$\begin{aligned}\int_{C_R} f(z)\,dz &= \int_{C_R}\frac{z^2}{z^4+z^2+1}\,dz \le \left|\int_{C_R}\frac{z^2}{z^4+z^2+1}\,dz\right| \le \int_{C_R}\frac{|z|^2}{|z^4+z^2+1|}\,|dz| \le \int_{C_R}\frac{|z|^2}{|z|^4-|z|^2-1}\,|dz|\\ &= \frac{R^2}{R^4-R^2-1}\int_{C_R}|dz| = \frac{\pi R^3}{R^4-R^2-1} \to 0 \text{ as } R\to\infty.\end{aligned}$$

Next, we calculate $Res(f,a_1)$.

$$\begin{aligned}Res(f,e^{i\frac{\pi}{3}}) &= \lim_{z\to e^{i\frac{\pi}{3}}}(z-e^{i\frac{\pi}{3}})\frac{z^2}{z^4+z^2+1} = e^{i\frac{2\pi}{3}}\lim_{z\to e^{i\frac{\pi}{3}}}(z-e^{i\frac{\pi}{3}})\frac{1}{z^4+z^2+1}\\ &= e^{i\frac{2\pi}{3}}\lim_{z\to e^{i\frac{\pi}{3}}}\frac{1}{4z^3+2z} = \frac{e^{i\frac{2\pi}{3}}}{4e^{i\frac{3\pi}{3}}+2e^{i\frac{\pi}{3}}} = \frac{e^{i\frac{\pi}{3}}}{4e^{i\frac{2\pi}{3}}+2}\\ &= \frac{\frac12+\frac12\sqrt{3}i}{4(-\frac12+\frac12\sqrt{3}i)+2} = \frac{\frac12+\frac12\sqrt{3}i}{2\sqrt{3}i} = \frac{1}{4\sqrt{3}i}+\frac14\\ &= \frac14 - \frac{1}{4\sqrt{3}}i.\end{aligned}$$

Finally, we calculate Res(f, a_2).

$$\begin{aligned}
Res(f, e^{i\frac{2\pi}{3}}) &= \lim_{z\to e^{i\frac{2\pi}{3}}}(z - e^{i\frac{2\pi}{3}})\frac{z^2}{z^4+z^2+1} = e^{i\frac{4\pi}{3}}\lim_{z\to e^{i\frac{2\pi}{3}}}(z - e^{i\frac{2\pi}{3}})\frac{1}{z^4+z^2+1} \\
&= e^{i\frac{4\pi}{3}}\lim_{z\to e^{i\frac{2\pi}{3}}}\frac{1}{4z^3+2z} = \frac{e^{i\frac{4\pi}{3}}}{4e^{i\frac{6\pi}{3}}+2e^{i\frac{2\pi}{3}}} = \frac{e^{i\frac{2\pi}{3}}}{4e^{i\frac{4\pi}{3}}+2} \\
&= \frac{-\frac{1}{2}+\frac{1}{2}\sqrt{3}i}{4(-\frac{1}{2}-\frac{1}{2}\sqrt{3}i)+2} = \frac{-\frac{1}{2}+\frac{1}{2}\sqrt{3}i}{-2\sqrt{3}i} = \frac{1}{4\sqrt{3}i} - \frac{1}{4} \\
&= -\frac{1}{4} - \frac{1}{4\sqrt{3}}i.
\end{aligned}$$

So

$$2\pi i\sum_{k=1}^{2} Res(f, a_k) = 2\pi i\left(\frac{1}{4} - \frac{1}{4\sqrt{3}}i + \frac{1}{4\sqrt{3}i} - \frac{1}{4}\right) = 2\pi i\left(-\frac{1}{2\sqrt{3}}i\right) = -\frac{\pi}{\sqrt{3}}i^2 = \frac{\pi}{\sqrt{3}}.$$

Hence,

$$\int_0^\infty \frac{x^2\,dx}{x^4+x^2+1} = \frac{1}{2}\frac{\pi}{\sqrt{3}}.$$

b) Let $f(z) = \frac{e^{iz}-1}{z^2}$ and $\gamma = [-R, -r] + \gamma_r + [r, R] + \gamma_R$ where $\gamma_R = Re^{it}$, $0 \le t \le \pi$ and $\gamma_r = re^{-it}$, $0 \le t \le \pi$ and $0 < r < R$ (A sketch might help to see that γ is a closed and rectifiable curve). Clearly $\int_\gamma f(z)\,dz = 0$ since f is analytic on the region enclosed by γ. Also

$$\int_\gamma f(z)\,dz = \int_{-R}^{-r}\frac{e^{ix}-1}{x^2}\,dx + \int_r^R \frac{e^{ix}-1}{x^2}\,dx + \int_{\gamma_r} f + \int_{\gamma_R} f.$$

We have

$$\int_{-R}^{-r}\frac{e^{ix}-1}{x^2}\,dx = \int_r^R\frac{e^{-ix}-1}{x^2}\,dx.$$

So

$$\begin{aligned}
\int_{-R}^{-r}\frac{e^{ix}-1}{x^2}\,dx + \int_r^R\frac{e^{ix}-1}{x^2}\,dx &= \int_r^R\frac{e^{-ix}+e^{-ix}-2}{x^2}\,dx = 2\int_r^R\frac{\frac{e^{-ix}+e^{-ix}}{2}-1}{x^2}\,dx \\
&= 2\int_r^R\frac{\cos(x)-1}{x^2}\,dx
\end{aligned}$$

Therefore,

$$\int_r^R\frac{\cos(x)-1}{x^2}\,dx = -\frac{1}{2}\int_{\gamma_R}\frac{e^{iz}-1}{z^2}\,dz - \frac{1}{2}\int_{\gamma_r}\frac{e^{iz}-1}{z^2}\,dz$$

which implies that

$$\int_0^\infty\frac{\cos(x)-1}{x^2}\,dx = \lim_{R\to\infty}\left(-\frac{1}{2}\int_{\gamma_R}\frac{e^{iz}-1}{z^2}\,dz\right) - \lim_{r\to 0}\left(\frac{1}{2}\int_{\gamma_r}\frac{e^{iz}-1}{z^2}\,dz\right)$$

The first term on the right hand side is zero, since

$$\begin{aligned}
\left|\int_{\gamma_R}\frac{e^{iz}-1}{z^2}\,dz\right| &\le \int_{\gamma_R}\frac{|e^{iz}-1|}{|z|^2}\,|dz| \le \frac{1}{R^2}\left(\int_{\gamma_R}|dz| + \int_{\gamma_R}|e^{iz}|\,|dz|\right) \\
&\le \frac{1}{R^2}(\pi R + \pi R) = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.
\end{aligned}$$

The second term on the right hand side is π, since

$$\lim_{r\to 0}\left(\int_{\gamma_r}\frac{e^{iz}-1}{z^2}\,dz\right)=-\frac{1}{2}2\pi i\,Res(f(z),0)=-\pi i^2=\pi$$

($f(z)$ can be written as $\frac{i}{z}$+analytic part and therefore $Res(f(z),0)=i$). Hence,

$$\int_0^\infty\frac{\cos(x)-1}{x^2}\,dx=-\frac{1}{2}\cdot 0-\frac{1}{2}\cdot\pi=-\frac{\pi}{2}.$$

c) If $a=0$, then

$$\int_0^\pi\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}=\int_0^\pi\cos 2\theta\,d\theta=\frac{1}{2}\left[\sin 2\theta\right]_0^\pi=\frac{1}{2}[0-0]=0.$$

Assume $a\neq 0$. We can write

$$\int_0^\pi\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}=\frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}=\frac{1}{2}\int_0^{2\pi}\frac{\frac{e^{2\theta i}+e^{2\theta i}}{2}}{1-2a\frac{e^{\theta i}+e^{\theta i}}{2}+a^2}\,d\theta.$$

Let $z=e^{i\theta}$. Then $dz=iz\,d\theta\implies d\theta=\frac{dz}{iz}$. Hence,

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\frac{\frac{e^{2\theta i}+e^{2\theta i}}{2}}{1-2a\frac{e^{\theta i}+e^{\theta i}}{2}+a^2}\,d\theta &= \frac{1}{2}\int_{|z|=1}\frac{\frac{z^2+\frac{1}{z^2}}{2}}{1-a\left(z+\frac{1}{z}\right)+a^2}\,\frac{dz}{iz}=\frac{1}{4i}\int_{|z|=1}\frac{z^4+1}{z^2\left(z-az^2-a+a^2z\right)}\,dz\\ &= -\frac{1}{4ia}\int_{|z|=1}\frac{z^4+1}{z^2\left(z^2-\frac{1+a^2}{a}z+1\right)}\,dz=-\frac{1}{4ia}\int_{|z|=1}\frac{z^4+1}{z^2\left(z-a\right)\left(z-\frac{1}{a}\right)}\,dz\end{aligned}$$

By the Residue Theorem and the fact that $|a|<1$, since $a^2<1$

$$\int_{|z|=1}\frac{z^4+1}{z^2\left(z-a\right)\left(z-\frac{1}{a}\right)}\,dz=2\pi i\left[Res f(f,0)+Res(f,a)\right]$$

where $f(z)=\frac{z^4+1}{z^2(z-a)\left(z-\frac{1}{a}\right)}$. It is easily obtained

$$\begin{aligned}Res(f,0) &= \lim_{z\to 0}\frac{d}{dz}\left[z^2\frac{z^4+1}{z^2\left(z-a\right)\left(z-\frac{1}{a}\right)}\right]=\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{\left(z-a\right)\left(z-\frac{1}{a}\right)}\right]\\ &= \lim_{z\to 0}\frac{-4z^3\left(z^2-az-\frac{1}{a}z+1\right)-\left(z^4-1\right)\left(2z-a-\frac{1}{a}\right)}{\left(z^2-az-\frac{1}{a}z+1\right)^2}=a+\frac{1}{a}\end{aligned}$$

and

$$Res(f,a)=\lim_{z\to a}(z-a)\frac{z^4+1}{z^2\left(z-a\right)\left(z-\frac{1}{a}\right)}=\lim_{z\to a}\frac{z^4+1}{z^2\left(z-\frac{1}{a}\right)}=\frac{a^4+1}{a^2\left(a-\frac{1}{a}\right)}=\frac{a^4+1}{a(a^2-1)}.$$

So

$$\int_{|z|=1}\frac{z^4+1}{z^2\left(z-a\right)\left(z-\frac{1}{a}\right)}\,dz=2\pi i\left(a+\frac{1}{a}+\frac{a^4+1}{a(a^2-1)}\right)=2\pi i\frac{2a^4}{a(a^2=1)}=-\frac{4\pi i a^3}{1-a^2}.$$

Finally,

$$\int_0^\pi\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}=-\frac{1}{4ia}\cdot\left(-\frac{4\pi i a^3}{1-a^2}\right)=\pi\frac{a^2}{1-a^2}.$$

Exercise 2. *Verify the following equations:*

a)

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a>0;$$

b)

$$\int_0^\infty \frac{(\log x)^3}{1+x^2}\, dx = 0;$$

c)

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}\, dx = \frac{\pi(a+1)e^{-a}}{4}, \quad \textit{if} \quad a>0;$$

d)

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \frac{\pi}{2\left[a(a+1)\right]^{\frac{1}{2}}}, \quad \textit{if} \quad a>0;$$

e)

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx = -\frac{\pi}{4};$$

f)

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2};$$

g)

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin a\pi}, \quad \textit{if} \quad 0<a<1;$$

h)

$$\int_0^{2\pi} \log\sin^2 2\theta\, d\theta = 4\int_0^{4\pi} \log\sin\theta\, d\theta = -4\pi\log 2.$$

Solution. *a) Let* $f(z) = \frac{1}{(z^2+a^2)^2}$ *and* $\gamma_R = [-R,R] + C_R$ *where* $C_R = Re^{it}$, $0 \le t \le \pi$. *So* f *has* $z_1 = ai$ *as a pole of order* 2 *enclosed by* γ_R. *So we have by the Residue Theorem*

$$\int_{\gamma_R} f(z)\, dz = 2\pi i\, Res(f, z_1)$$

where

$$\begin{aligned} Res(f,z_1) &= \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2}\right] = \lim_{z\to ai}\frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right] \\ &= \lim_{z\to ai} \frac{-2}{(z+ai)^3} = -\frac{2}{(2ai)^3} = \frac{1}{4a^3 i}. \end{aligned}$$

Also

$$2\pi i \frac{1}{4a^3 i} = \frac{\pi}{2a^3} = \int_{-R}^R \frac{dx}{(x^2+a^2)^2} + \int_{C_R} f(z)\, dz$$

where the latter integral on the right hand side is zero since

$$\begin{aligned} \left|\int_{C_R} \frac{1}{(z^2+a^2)^2}\, dz\right| &\le \int_{C_R} \frac{1}{|z^2+a^2|^2}\, |dz| \le \int_{C_R} \frac{1}{(|z|^2-|a|^2)^2}\, |dz| \le \frac{1}{(R^2-a^2)^2}\int_{C_R} |dz| \\ &= \frac{\pi R}{(R^2-a^2)^2} \to 0 \textit{ as } R\to\infty. \end{aligned}$$

Hence,

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\frac{\pi}{2a^3} = \frac{\pi}{4a^3}.$$

c) Let $f(z) = \frac{e^{iaz}}{(1+z^2)^2}$ *and* $\gamma_R = [-R,R] + C_R$ *where* $C_R = Re^{it}$, $0 \le t \le \pi$. *So* f *has* $z_1 = i$ *as a pole of order 2 enclosed by* γ_R. *By the Residue Theorem, we have*

$$\int_{\gamma_R} f(z)\,dz = 2\pi\, Res(f, z_1)$$

where

$$\begin{aligned} Res(f,z_1) &= \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{e^{iaz}}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{e^{iaz}}{(z+i)^2}\right] \\ &= \lim_{z\to i}\left[\frac{(z+i)^2 iae^{iaz} - 2(z+i)e^{iaz}}{(z+i)^4}\right] = \lim_{z\to i}\left[\frac{(z+i)iae^{iaz} - 2e^{iaz}}{(z+i)^3}\right] \\ &= \frac{2i^2ae^{i^2a} - 2e^{i^2a}}{(2i)^3} = \frac{-2ae^{-a} - 2e^{-a}}{-8i} = \frac{(a+1)e^{-a}}{4i}. \end{aligned}$$

Also

$$2\pi i\frac{(a+1)e^{-a}}{4i} = \frac{\pi(a+1)e^{-a}}{2} = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

where the latter integral on the right hand side is zero since

$$\begin{aligned} \left|\int_{C_R} f(z)\,dz\right| &= \left|\int_{C_R}\frac{e^{iaz}}{(1+z^2)^2}\,dz\right| \le \int_{C_R}\frac{|e^{iaz}|}{|1+z^2|^2}\,|dz| \le \int_{C_R}\frac{1}{(|z|^2-1)^2}\,|dz| \\ &= \frac{1}{(R^2-1)^2}R\pi \to 0 \text{ as } R\to\infty, \end{aligned}$$

since $|e^{iaz}| = e^{-Im(az)} \le 1$ *since* $Im(az) \ge 0$ *(*$a > 0$ *by assumption). Hence*

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos ax}{(1+x^2)^2}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos ax}{(1+x^2)^2}\,dx = \frac{1}{2}\frac{\pi(a+1)e^{-a}}{2} = \frac{\pi(a+1)e^{-a}}{4}$$

provided $a > 0$.
g) First of all substitute $u = e^x$, *so*

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx = \int_0^\infty \frac{u^a}{1+u}\frac{du}{u} = \int_0^\infty \frac{u^{a-1}}{1+u}\,du.$$

Now, let $u = t^2$, *then*

$$\int_0^\infty \frac{u^{a-1}}{1+u}\,du = 2\int_0^\infty \frac{t^{2a-2}t}{1+t^2}\,dt = 2\int_0^\infty \frac{t^{2a-1}}{1+t^2}\,dt.$$

Hence,

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx = 2\int_0^\infty \frac{x^{2a-1}}{1+x^2}\,dx.$$

Let $f(z) = \frac{z^{2a-1}}{1+z^2}$ *and* $\gamma = [-R,-r] + \gamma_r + [r,R] + \gamma_R$ $(0 < r < R)$. *Define* $\log z$ *on* $G = \{z \in \mathbb{C} : z \neq 0, -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}\}$ *with* $\log(z) = \log|z| + i\theta$, $\theta = \arg(z)$. $f(z)$ *has a simple pole* i *enclosed by* γ. *By the Residue*

Theorem

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi Res(f,i) = 2\pi i \lim_{z\to i}(z-i)\frac{z^{2a-1}}{(z-i)(z+i)} = 2\pi i\frac{i^{2a-1}}{2i}\\ &= \pi i^{2a-1} = \pi e^{(2a-1)\log i} = \pi e^{(2a-1)(0+i\frac{\pi}{2})} = \pi e^{a\pi i}e^{-i\frac{\pi}{2}} = -\pi i e^{a\pi i}.\end{aligned}$$

Also

$$\int_\gamma f(z)\,dz = \underbrace{\int_{-R}^{-r} f(x)\,dx}_{I} + \underbrace{\int_{\gamma_r} f(z)\,dz}_{II} + \underbrace{\int_r^R f(x)\,dx}_{III} + \underbrace{\int_{\gamma_R} f(z)\,dz}_{IV}.$$

I)

$$\begin{aligned}\int_{-R}^{-r} f(x)\,dx &= \int_r^R f(-x)\,dx = \int_r^R \frac{(-x)^{2a-1}}{1+x^2}\,dx = \int_r^R \frac{e^{(2a-1)(\log|-x|+i\pi)}}{1+x^2}\,dx\\ &= e^{(2a-1)i\pi}\int_r^R \frac{e^{(2a-1)\log x}}{1+x^2}\,dx = -e^{2a\pi i}\int_r^R \frac{e^{(2a-1)\log x}}{1+x^2}\,dx\\ &= -e^{2a\pi i}\int_r^R \frac{x^{2a-1}}{1+x^2}\,dx.\end{aligned}$$

IV)

$$\left|\int_{\gamma_R} f(z)\,dz\right| = \left|\int_{\gamma_R}\frac{z^{2a-1}}{1+z^2}\,dz\right| \le \int_{\gamma_R}\frac{|z|^{2a-1}}{|1+z^2|}\,|dz| \le \frac{R^{2a-1}}{R^2-1}\pi R = \frac{\pi R^{2a}}{R^2-1} \to 0 \text{ as } R\to\infty$$

since $a<1$.

II) Similar to IV)

$$\left|\int_{\gamma_r} f(z)\,dz\right| \le \frac{\pi r^{2a}}{r^2-1} \to 0 \text{ as } r\to 0$$

since $a>0$.

Hence, if $r\to 0$ *and* $R\to\infty$, *we get*

$$-\pi i e^{a\pi i} = \int_\gamma f(z)\,dz = -e^{2a\pi i}\int_0^\infty f(x)\,dx + 0 + \int_0^\infty f(x)\,dx + 0 = (1-e^{2a\pi i})\int_0^\infty f(x)\,dx.$$

Thus,

$$-\pi i e^{a\pi i} = (1-e^{2a\pi i})\int_0^\infty f(x)\,dx$$

and so

$$\int_0^\infty f(x)\,dx = \frac{-\pi i e^{a\pi i}}{1-e^{2a\pi i}} = \frac{-i\pi}{e^{-a\pi i}-e^{a\pi i}} = \frac{i\pi}{e^{a\pi i}-e^{-a\pi i}} = \frac{i\pi}{2i\sin(a\pi)} = \frac{\pi}{2\sin(a\pi)}.$$

Therefore,

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx = 2\int_0^\infty \frac{x^{2a-1}}{1+x^2}\,dx = 2\frac{\pi}{2\sin(a\pi)} = \frac{\pi}{\sin(a\pi)}$$

provided $0<a<1$.

Exercise 3. *Find all possible values of* $\int_\gamma \exp z^{-1}\,dz$ *where* γ *is any closed curve not passing through* $z=0$.

Solution. *Not available.*

Exercise 4. *Suppose that f has a simple pole at $z = a$ and let g be analytic in an open set containing a. Show that $Res(fg; a) = g(a)Res(f; a)$.*

Solution. *Not available.*

Exercise 5. *Use Exercise 4 to show that if G is a region and f is analytic in G except for simple poles at $a_1, \ldots, a_n$; and if g is analytic in G then*

$$\frac{1}{2\pi i}\int_\gamma fg = \sum_{k=1}^{n} n(\gamma; a_k)g(a_k)Res(f; a_k)$$

for any closed rectifiable curve γ not passing through $a_1, \ldots, a_n$ such that $\gamma \approx 0$ in G.

Solution. *Not available.*

Exercise 6. *Let γ be the rectangular path $[n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni, n + i + ni]$ and evaluate the integral $\int_\gamma \pi(z + a)^{-2} \cot \pi z\, dz$ for $a \neq$ an integer. Show that $\lim_{n\to\infty} \int_\gamma \pi(z + a)^{-2} \cot \pi z\, dz = 0$ and, by using the first part, deduce that*

$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a + n)^2}$$

(Hint: Use the fact that for $z = x + iy$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$ to show that $|\cot \pi z| \leq 2$ for z on γ if n is sufficiently large.)

Solution. *Not available.*

Exercise 7. *Use Exercise 6 to deduce that*

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}$$

Solution. *Not available.*

Exercise 8. *Let γ be the polygonal path defined in Exercise 6 and evaluate $\int_\gamma \pi(z^2 - a^2)^{-1} \cot \pi z\, dz$ for $a \neq$ an integer. Show that $\lim_{n\to\infty} \pi(z^2 - a^2)^{-1} \cot \pi z\, dz = 0$, and consequently*

$$\pi \cot \pi a = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$$

for $a \neq$ an integer.

Solution. *Not available.*

Exercise 9. *Use methods similar to those of Exercises 6 and 8 to show that*

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2(-1)^n a}{a^2 - n^2}$$

for $a \neq$ an integer.

Solution. *Not available.*

Exercise 10. *Let γ be the circle $|z| = 1$ and let m and n be non-negative integers. Show that*

$$\frac{1}{2\pi i}\int_\gamma \frac{(z^2 \pm 1)^m\, dz}{z^{m+n+1}} = \begin{cases} \frac{(\pm 1)^p (n+2p)!}{p!(n+p)!}, & \text{if } m = 2p+n, p \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution. *Not available.*

Exercise 11. *In Exercise 1.12, consider a_n and b_n as functions of the parameter λ and use Exercise 10 to compute power series expansions for $a_n(\lambda)$ and $b_n(\lambda)$. ($b_n(\lambda)$ is called a Bessel function.)*

Solution. *Not available.*

Exercise 12. *Let f be analytic in the plane except for isolated singularities at $a_1, a_2, \ldots, a_m$. Show that*

$$Res(f; \infty) = -\sum_{k=1}^{\infty} Res(f, a_k).$$

($Res(f;\infty)$ is defined as the residue of $-z^{-2} f(z^{-1})$ at $z = 0$. Equivalently, $Res(f;\infty) = -\frac{1}{2\pi i}\int f$ when $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, for sufficiently large R.) What can you say if f has infinitely many isolated singularities?

Solution. *Not available.*

Exercise 13. *Let f be an entire function and let $a, b \in \mathbb{C}$ such that $|a| < R$ and $|b| < R$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2pi$, evaluate $\int_\gamma [(z-a)(z-b)]^{-1} f(z)\ dz$. Use this result to give another proof of Liouville's Theorem.*

Solution. *Not available.*

5.3 The Argument Principle

Exercise 1. *Prove Theorem 3.6.*

Solution. *Not available.*

Exercise 2. *Suppose f is analytic on $\bar{B}(0;1)$ and satisfies $|f(z)| < 1$ for $|z| = l$. Find the number of solutions (counting multiplicities) of the equation $f(z) = z^n$ where n is an integer larger than or equal to 1.*

Solution. *Not available.*

Exercise 3. *Let f be analytic in $\bar{B}(0;R)$ with $f(0) = 0$, $f'(0) \neq 0$ and $f(z) \neq 0$ for $0 < |z| \leq R$. Put $\rho = \min\{|f(z)| : |z| = R\} > 0$. Define $g : B(0;\rho) \to \mathbb{C}$ by*

$$g(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{z f'(z)}{f(z) - \omega}\ dz$$

where γ is the circle $|z| = R$. Show that g is analytic and discuss the properties of g.

Solution. *Not available.*

Exercise 4. *If f is meromorphic on G and $\tilde{f} : G \to \mathbb{C}_\infty$ is defined by $\tilde{f}(z) = \infty$ when z is a pole of f and $\tilde{f}(z) = f(z)$ otherwise, show that $\tilde{f}$ is continuous.*

Solution. *Not available.*

Exercise 5. *Let f be meromorphic on the region G and not constant; show that neither the poles nor the zeros of f have a limit point in G.*

Solution. *Not available.*

Exercise 6. *Let G be a region and let $H(G)$ denote the set of all analytic functions on G. (The letter "H" stands for holomorphic. Some authors call a differentiable function holomorphic and call functions analytic if they have a power series expansion about each point of their domain. Others reserve the term "analytic" for what many call the complete analytic function, which we will not describe here.) Show that $H(G)$ is an integral domain; that is, $H(G)$ is a commutative ring with no zero divisors. Show that $M(G)$, the meromorphic functions on G, is a field.*
We have said that analytic functions are like polynomials; similarly, meromorphic functions are analogues of rational functions. The question arises, is every meromorphic function on G the quotient of two analytic functions on G? Alternately, is $M(G)$ the quotient field of $H(G)$? The answer is yes but some additional theory will be required before this answer can be proved.

Solution. *Not available.*

Exercise 7. *State and prove a more general version of Rouche's Theorem for curves other than circles in G.*

Solution. *Not available.*

Exercise 8. *Is a non-constant meromorphic function on a region G an open mapping of G into $\mathbb{C}$? Is it an open mapping of G into $\mathbb{C}_\infty$?*

Solution. *Not available.*

Exercise 9. *Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : Re\, z > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$?*

Solution. *Not available.*

Exercise 10. *Let f be analytic in a neighborhood of $D = \bar{B}(0;1)$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \le 1$ for $|z| = 1$, what can you say?*

Solution. *Let $g(z) = f(z) - z$ and $h(z) = z$. Clearly g, h are analytic in a neighborhood of $\bar{B}(0,1)$ since f is analytic. Note that g, h have no poles. Let $\gamma = \{z : |z| = 1\}$. We have*

$$|g(z) + h(z)| = |f(z) - z + z| = |f(z)| < 1 = |z| = |h(z)| \le |g(z)| + |h(z)|$$

on γ since $|f(z)| < 1$ for $|z| = 1$. By Roché's Theorem, we get

$$Z_g - P_g = Z_h - P_h.$$

But $P_g = P_h = 0$, so

$$Z_g = Z_h \qquad \text{(inside the unit circle)}.$$

Since $h(z) = z$ has only one zero ($z = 0$) inside the unit circle, we get that

$$g(z) = f(z) - z$$

has exactly one zero inside the unit circle. Hence, there is a unique z such that $f(z) - z = 0 \iff f(z) = z$ with $|z| < 1$.

If we choose $f(z) = z$, then $|f(z)| \le 1$ for $|z| = 1$ and therefore we have infinitely many fixed points.
If we choose $f(z) = 1$, then $|f(z)| \le 1$ for $|z| = 1$ and therefore we do not have any fixed points.

Chapter 6

The Maximum Modulus Theorem

6.1 The Maximum Principle

Exercise 1. *Prove the following Minimum Principle. If f is a non-constant analytic function on a bounded open set G and is continuous on G^-, then either f has a zero in G or $|f|$ assumes its minimum value on ∂G. (See Exercise IV. 3.6.)*

Solution. *Since $f \in C(\overline{G})$ we have $|f| \in C(\overline{G})$. Hence $\exists\, a \in \overline{G}$ such that $|f(a)| \le |f(z)|\ \forall z \in \overline{G}$.
If $a \in \partial G$, then $|f|$ assumes its minimum value on ∂G and we are done. Otherwise, if $a \notin \partial G$, then $a \in G$ and we can write $G = \bigcup A_i$ where A_i are the components of G, that is $a \in A_i$ for some i. But each A_i is a region, so we can use Exercise IV 3.6 which yields either $f(a) = 0$ or f is constant. But f was assumed to be non-constant, so f has to have a zero in G. Therefore, either f has a zero in G or $|f|$ assumes its minimum value on ∂G.*

Exercise 2. *Let G be a bounded region and suppose f is continuous on G^- and analytic on G. Show that if there is a constant $c \ge 0$ such that $|f(z)| = c$ for all z on the boundary of G then either f is a constant function or f has a zero in G.*

Solution. *Assume there is a constant $c \ge 0$ such that $|f(z)| = c$ for all $z \in \partial G$. According to the Maximum Modulus Theorem (Version 2), we get*

$$\max\{|f(z)| : z \in \overline{G}\} = \max\{|f(z)| : z \in \partial G\} = c.$$

So

$$|f(z)| \le c, \qquad \forall z \in \overline{G}. \tag{6.1}$$

Since $f \in C(G)$ which implies $|f| \in C(\overline{G})$ and hence there exists an $a \in G$ such that

$$|f(a)| \le |f(z)| \underset{(6.1)}{\le} c,$$

then by Exercise IV 3.6 either f is constant of f has a zero in G.

Exercise 3. *(a) Let f be entire and non-constant. For any positive real number c show that the closure of $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \le c\}$.
(b) Let p be a polynomial and show that each component of $\{z : |p(z)| < c\}$ contains a zero of p. (Hint: Use*

Exercise 2.)
(c) If p is a polynomial and $c > 0$ show that $\{z : |p(z)| = c\}$ is the union of a finite number of closed paths. Discuss the behavior of these paths as $c \to \infty$.

Solution. *Not available.*

Exercise 4. *Let $0 < r < R$ and put $A = \{z : r \leq |z| \leq R\}$. Show that there is a positive number $\epsilon > 0$ such that for each polynomial p,*

$$\sup\{|p(z) - z^{-1}| : z \in A\} \geq \epsilon$$

This says that z^{-1} is not the uniform limit of polynomials on A.

Solution. *Not available.*

Exercise 5. *Let f be analytic on $\bar{B}(0; R)$ with $|f(z)| \leq M$ for $|z| \leq R$ and $|f(0)| = a > 0$. Show that the number of zeros of f in $B(0; \frac{1}{3}R)$ is less than or equal to $\frac{1}{\log 2} \log\left(\frac{M}{a}\right)$. Hint: If $z_1, \ldots, z_n$ are the zeros of f in $B(0; \frac{1}{3}R)$; consider the function*

$$g(z) = f(z)\left[\prod_{k=1}^{n}\left(1 - \frac{z}{z_k}\right)\right]^{-1},$$

and note that $g(0) = f(0)$. (Notation: $\prod_{k=1}^{n} a_k = a_1 a_2 \ldots a_n$.)

Solution. *Not available.*

Exercise 6. *Suppose that both f and g are analytic on $\bar{B}(0; R)$ with $|f(z)| = |g(z)|$ for $|z| = R$. Show that if neither f nor g vanishes in $B(0; R)$ then there is a constant $\lambda, |\lambda| = 1$, such that $f = \lambda g$.*

Solution. *Not available.*

Exercise 7. *Let f be analytic in the disk $B(0; R)$ and for $0 \leq r < R$ define $A(r) = \max\{Re\ f(z) : |z| = r\}$. Show that unless f is a constant, $A(r)$ is a strictly increasing function of r.*

Solution. *Not available.*

Exercise 8. *Suppose G is a region, $f : G \to \mathbb{C}$ is analytic, and M is a constant such that whenever z is on $\partial_\infty G$ and $\{z_n\}$ is a sequence in G with $z = \lim z_n$ we have $\limsup |f(z_n)| \leq M$. Show that $|f(z)| \leq M$, for each z in G.*

Solution. *We need to show*

$$\limsup_{z \to a} |f(z)| \leq M \qquad \forall u \in \partial_\infty G.$$

Then we can use the Maximum Modulus Theorem (Version 3). Instead of showing that

$$\limsup |f(z_n)| \leq M \Rightarrow \limsup_{z \to a} |f(z)| \leq M,$$

we show the contrapositive, that is

$$\limsup_{z \to a} |f(z)| > M \Rightarrow \limsup |f(z_n)| > M.$$

So assume $\limsup_{z \to a} |f(z)| > M$. But this implies $\limsup_{z \to a} |f(z_n)| > M$ since $z_n \to z$ as $z \to \infty$, that is z_n gets arbitrarily close to z and since f is analytic, that is continuous, we have $f(z_n)$ gets arbitrarily close to $f(z)$.

6.2 Schwarz's Lemma

Exercise 1. *Suppose $|f(z)| \leq 1$ for $|z| < 1$ and f is a non-constant analytic function. By considering the function $g : D \to D$ defined by*

$$g(z) = \frac{f(z) - a}{1 - \overline{a} f(z)}$$

where $a = f(0)$, prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)|\,|z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)|\,|z|}$$

for $|z| < 1$.

Solution. *We claim that $|g(z)| \leq 1$ and $g(0) = 0$. It is easy to check that $g(0) = 0$ since*

$$g(0) = \frac{f(0) - a}{1 - \overline{a} f(0)} = \frac{a - a}{1 - \overline{a}a} = 0.$$

We also have

$$|g(z)| = \frac{|f(z) - a|}{|1 - \overline{a} f(z)|} \leq 1$$

since $|f(z)| \leq 1$. (Can you see it?) Now, apply Schwarz Lemma.

$$\begin{aligned}
& |g(z)| \leq |z| \text{ for } |z| < 1 \\
\iff \quad & |f(z) - a| \leq |z| \cdot |1 - \overline{a} f(z)| \\
\Rightarrow \quad & ||f(z)| - |a|| \leq |f(z) - a| \leq |z| \cdot |1 - \overline{a} f(z)| \leq |z| + |z| \cdot |a| \cdot |f(z)| \\
\Rightarrow \quad & ||f(z)| - |a|| \leq |z| + |z| \cdot |a| \cdot |f(z)| \\
\Rightarrow \quad & -|z| - |z| \cdot |a| \cdot |f(z)| \leq |f(z)| - |a| \leq |z| + |z| \cdot |a| \cdot |f(z)| \\
\Rightarrow \quad & \frac{|a| - |z|}{1 + |a|\,|z|} \leq |f(z)| \leq \frac{|a| + |z|}{1 - |a|\,|z|}.
\end{aligned}$$

Set $a = f(0)$ to obtain the desired result.

Exercise 2. *Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?*

Solution. *The answer is no. Assume there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$. According to the definition, we have*

$$E^a_\alpha = \{f \in A(D) : |f(z)| \leq 1, f(a) = \alpha\}.$$

In our case, we have $a = \frac{1}{2}$ and $\alpha = \frac{3}{4}$. We know,

$$\left|f'(a)\right| \leq \frac{1 - |\alpha|^2}{1 - |a|^2}$$

and therefore we must have

$$\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{1 - \left(\frac{3}{4}\right)^2}{1 - \left(\frac{1}{2}\right)^2} = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{16}\frac{3}{4} = \frac{7}{12} = 0.58\overline{3}.$$

But $f'(\frac{1}{2}) = \frac{2}{3} = 0.\overline{6}$ which is not possible.

Exercise 3. *Suppose $f : D \to \mathbb{C}$ satisfies Re $f(z) \geq 0$ for all z in D and suppose that f is analytic.*
(a) Show that Re $f(z) > 0$ for all z in D.
(b) By using an appropriate Möbius transformation, apply Schwarz's Lemma to prove that if $f(0) = 1$ then

$$|f(z)| \leq \frac{1+|z|}{1-|z|}$$

for $|z| < 1$. What can be said if $f(0) \neq 1$?
(c) Show that $f(0) = 1$, f also satisfies

$$|f(z)| \geq \frac{1-|z|}{1+|z|}$$

(Hint: Use part (a)).

Solution. *Not available.*

Exercise 4. *Prove Caratheodory's Inequality whose statement is as follows: Let f be analytic on $\bar{B}(0; R)$ and let $M(r) = \max\{|f(z)| : |z| = r\}$, $A(r) = \max\{\text{Re } f(z) : |z| = r\}$; then for $0 < r < R$, if $A(r) \geq 0$,*

$$M(r) \leq \frac{R+r}{R-r}[A(R) + |f(0)|]$$

(Hint: First consider the case where $f(0) = 0$ and examine the function $g(z) = f(Rz)[2A(R) + f(Rz)]^{-1}$ for $|z| < 1$.)

Solution. *Not available.*

Exercise 5. *Let f be analytic in $D = \{z : |z| < 1\}$ and suppose that $|f(z)| \leq M$ for all z in D. (a) If $f(z_k) = 0$ for $1 \leq k \leq n$ show that*

$$|f(z)| \leq M \prod_{k=1}^{n} \frac{|z - z_k|}{|1 - \bar{z}_k z|}$$

for $|z| < 1$. (b) If $f(z_k) = 0$ for $1 \leq k \leq n$, each $z_k \neq 0$, and $f(0) = Me^{ia}(z_1 z_2, \ldots, z_n)$, find a formula for f.

Solution. *Not available.*

Exercise 6. *Suppose f is analytic in some region containing $\bar{B}(0; 1)$ and $|f(z)| = 1$ where $|z| = 1$. Find a formula for f. (Hint: First consider the case where f has no zeros in $B(0; 1)$.)*

Solution. *Not available.*

Exercise 7. *Suppose f is analytic in a region containing $\bar{B}(0; 1)$ and $|f(z)| = 1$ when $|z| = 1$. Suppose that f has a zero at $z = \frac{1}{4}(1 + i)$ and a double zero at $z = \frac{1}{2}$. Can $f(0) = \frac{1}{2}$?*

Solution. *Not available.*

Exercise 8. *Is there an analytic function f on $B(0; 1)$ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = \frac{1}{2}$, and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?*

Solution. *Not available.*

6.3 Convex functions and Hadamard's Three Circles Theorem

Exercise 1. *Let $f : [a, b] \to \mathbb{R}$ and suppose that $f(x) > 0$ for all x and that f has a continuous second derivative. Show that f is logarithmically convex iff $f''(x)f(x) - [f'(x)]^2 \geq 0$ for all x.*

Solution. *Not available.*

Exercise 2. *Show that if $f : (a, b) \to \mathbb{R}$ is convex then f is continuous. Does this remain true if f is defined on the closed interval $[a, b]$?*

Solution. *Since f is convex on (a, b) and if $a < s < t < u < b$, we have*

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t} \tag{6.2}$$

which we show now. Let $a < s < t < u < b$, $t = \lambda s + (1 - \lambda)u$ with $\lambda = \frac{u-t}{u-s} \in (0, 1)$. Since f is convex on (a, b), we get

$$f(t) \leq \frac{u - t}{u - s} f(s) + \frac{t - s}{u - s} f(u)$$

which implies

$$(t - s)f(u) + (u - t)f(s) - (u - s)f(t) \geq 0$$

which is equivalent to

$$tf(u) - sf(u) + uf(s) - tf(s) - uf(t) + sf(t) \geq 0. \tag{6.3}$$

From (6.3), we get (rearrange terms and add/subtract a term)

$$\begin{aligned} & tf(u) - tf(s) - sf(u) + sf(s) - uf(t) - uf(s) + sf(t) - sf(s) \geq 0 \\ \iff \quad & (t - s)(f(u) - f(s)) - (u - s)(f(t) - f(s)) \geq 0 \\ \iff \quad & \frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}. \end{aligned}$$

Similar, from (6.3), we get

$$\begin{aligned} & uf(u) - sf(u) - uf(t) + sf(t) - uf(u) + uf(s) + tf(u) - tf(s) \geq 0 \\ \iff \quad & (u - s)(f(u) - f(t)) - (u - t)(f(u) - f(s)) \geq 0 \\ \iff \quad & \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}. \end{aligned}$$

Thus, we have proved (6.2).
Given $x \in (a, b)$ choose $\delta > 0$ such that $[x - \delta, x + \delta] \subset (a, b)$.
Claim:

$$\frac{f(x) - f(x - \delta)}{\delta} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(x + \delta) - f(x)}{\delta} \qquad \forall z \in (x - \delta, x + \delta). \tag{6.4}$$

If the claim is true, (6.4) is equivalent to

$$\frac{f(x) - f(x - \delta)}{\delta}(z - x) \leq f(z) - f(x) \leq \frac{f(x + \delta) - f(x)}{\delta}(z - x).$$

Taking the limit $z \to x$, we have that $\frac{f(x)-f(x-\delta)}{\delta}(z - x) \to 0$ and $\frac{f(x+\delta)-f(x)}{\delta}(z - x) \to 0$. Thus $f(z) - f(x) \to 0$, that is $|z - x| < \delta \Rightarrow |f(z) - f(x)| < \epsilon$ which shows that f is continuous.

Thus it remains to show the claim. The proof uses (6.2).
First, consider a point $z \in (x-\delta, x)$ and apply the second inequality in (6.2) gives

$$\frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(z)-f(x)}{z-x}$$

which gives the first inequality in (6.4).
Applying the outer inequality in (6.2) to the three points $z < x < x+\delta$ gives

$$\frac{f(x)-f(z)}{x-z} \leq \frac{f(x+\delta)-f(x)}{x+\delta-x} \iff \frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{\delta}$$

which gives the second inequality in (6.4).
Now, consider the case $x < z < x+\delta$. Then, the first inequality in (6.4) follows from the outer inequality in (6.2) applied to the three points $x-\delta, x, z$.

$$\frac{f(x)-f(x-\delta)}{x-x+\delta} \leq \frac{f(z)-f(x)}{z-x} \iff \frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(z)-f(x)}{z-x}.$$

The second inequality in (6.4) follows from the first inequality in (6.2) applied to $x, z, x+\delta$.

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{x+\delta-x} \iff \frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{\delta}.$$

Thus, we have proved the claim.
The statement is not true if f is defined on the closed interval $[a,b]$! Here is a counterexample: Define $f:[a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in [a,b) \\ 1, & x = b \end{cases}.$$

Clearly, f is convex on $[a,b]$, but f is not continuous at $x = b$ (but the function is continuous on (a,b)).

Exercise 3. *Show that a function $f:[a,b] \to \mathbb{R}$ is convex iff any of the following equivalent conditions is satisfied:*
(a) $a \leq x < u < y \leq b$ gives $\det \begin{pmatrix} f(u) & u & 1 \\ f(x) & x & 1 \\ f(y) & y & 1 \end{pmatrix} \geq 0$*;*
(b) $a \leq x < u < y \leq b$ gives $\frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(x)}{y-x}$;
(c) $a \leq x < u < y \leq b$ gives $\frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(u)}{y-u}$;
Interpret these conditions geometrically.

Solution. *Not available.*

Exercise 4. *Supply the details in the proof of Hadamard's Three Circle Theorem.*

Solution. *Not available.*

Exercise 5. *Give necessary and sufficient conditions on the function f such that equality occurs in the conclusion of Hadamard's Three Circle Theorem.*

Solution. *Not available.*

Exercise 6. *Prove Hardy's Theorem: If f is analytic on $\bar{B}(0;R)$ and not constant then*

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

is strictly increasing and $\log I(r)$ is a convex function of $\log r$. Hint: If $0 < r_1 < r < r_2$ find a continuous function $\varphi : [0,2\pi] \to \mathbb{C}$ such that $\varphi(\theta)f(re^{i\theta}) = |f(re^{i\theta})|$ and consider the function $F(z) = \int_0^{2\pi} f(ze^{i\theta})\varphi(\theta)\,d\theta$. (Note that r is fixed, so φ may depend on r.)

Solution. *Not available.*

Exercise 7. *Let f be analytic in ann$(0;R_1,R_2)$ and not identically zero; define*

$$I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta.$$

Show that $\log I_2(r)$ is a convex function of $\log r$, $R_1 < r < R_2$.

Solution. *Not available.*

6.4 The Phragmén-Lindelöf Theorem

Exercise 1. *In the statement of the Phragmén-Lindelöf Theorem, the requirement that G be simply connected is not necessary. Extend Theorem 4.1 to regions G with the property that for each z in $\partial_\infty G$ there is a sphere V in $\mathbb{C}_\infty$ centered at z such that $V \cap G$ is simply connected. Give some examples of regions that are not simply connected but have this property and some which don't.*

Solution. *Not available.*

Exercise 2. *In Theorem 4.1 suppose there are bounded analytic functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ on G that never vanish and $\partial_\infty G = A \cup B_1 \cup \ldots \cup B_n$ such that condition (a) is satisfied and condition (b) is also satisfied for each φ_k and B_k. Prove that $|f(z)| \le M$ for all z in G.*

Solution. *We have $\varphi_1, \ldots, \varphi_n \in A(G)$, bounded and $\varphi_k \not\equiv 0$ by assumption. Hence, $|\varphi_k(z)| \le \kappa_k\ \forall\, z$ in G for each $k = 1, \ldots, n$. Further a) for every a in A, $\limsup_{z\to a} |f(z)| \le M$ and b) for every b in B_k and $\eta_k > 0$, $\limsup_{z\to b} |f(z)||\varphi_k(z)|^{\eta_k} \le M\ \forall\ k = 1, \ldots, n$.*
Also because G is simply connected, there is an analytic branch of $\log \varphi_k(z)$ on G for each $k = 1, \ldots, n$ (Corollary IV 6.17). Hence, $g_k(z) = \exp\{\eta_k \log \varphi_k(z)\}$ is an analytic branch of $\varphi_k(z)^{\eta_k}$ for $\eta_k > 0$ and $|g_k(z)| = |\varphi_k(z)|^{\eta_k}\ \forall\ k = 1, \ldots, n$. Define $F : G \to \mathbb{C}$ by

$$F(z) = f(z)\prod_{k=1}^{n} g_k(z)\kappa_k^{-\eta_k};$$

then F is analytic on G and

$$|F(z)| \le |f(z)|\prod_{k=1}^{n} |\varphi_k(z)|^{\eta_k}\kappa_k^{-\eta_k} \underset{|\varphi_k(z)|\le\kappa_k\ \forall k}{\le} |f(z)|\prod_{k=1}^{n} \kappa_k^{\eta_k}\kappa_k^{-\eta_k}|\varphi_k(z)| = |f(z)| \quad \forall z \in G.$$

Hence,

$$\limsup_{z\to a} |f(z)| \underset{(a),(b)}{=} \begin{cases} M, & a \in A \\ M\kappa_k^{-\eta_k}, & a \in B_k \quad \forall k \end{cases}.$$

By the Maximum Modulus Theorem III, we have

$$|F(z)| \leq \max\{M, M\kappa_1^{-\eta_1}, \ldots, M\kappa_n^{-\eta_n}\} \quad \forall z \in G.$$

Thus,

$$|f(z)| = |F(z)| \prod_{k=1}^{n} |g_k(z)|^{-1}\kappa_k^{\eta_k} \leq \prod_{k=1}^{n} \left|\frac{\kappa_k}{\varphi_k(z)}\right|^{\eta_k} \max\{M, M\kappa_1^{-\eta_1}, \ldots, M\kappa_n^{-\eta_n}\}$$

for all z in G and for all $\eta_k > 0$. Letting $\eta_k \to 0^+$ $\forall k = 1, \ldots, n$ gives that $|f(z)| \leq M$ for all z in G.

Exercise 3. *Let $G = \{z : |Im\, z| < \frac{1}{2}\pi\}$ and suppose $f : G \to \mathbb{C}$ is analytic and $\limsup_{z\to w} |f(z)| \leq M$ for w in ∂G. Also, suppose $A < \infty$ and $a < 1$ can be found such that*

$$|f(z)| < \exp[A \exp(a|Re\, z|)]$$

for all z in G. Show that $|f(z)| \leq M$ for all z in G. Examine $\exp(\exp z)$ to see that this is the best possible growth condition. Can we take $a = 1$ above?

Solution. *Not available.*

Exercise 4. *Let $f : G \to \mathbb{C}$ be analytic and suppose M is a constant such that $\limsup |f(z_n)| \leq M$ for each sequence $\{z_n\}$ in G which converges to a point in $\partial_\infty G$. Show that $|f(z)| \leq M$. (See Exercise 1.8).*

Solution. *Claim: Let M be a constant such that $\limsup_{n\to\infty} |f(z_n)| \leq M$ for each sequence $\{z_n\}$ in G with*

$$z = \lim_{n\to\infty} z_n, \qquad \forall z \in \partial_\infty G \tag{6.5}$$

implies

$$\limsup_{z\to a} |f(z)| \leq M \quad \text{for all} \quad a \in \partial_\infty G. \tag{6.6}$$

If G would be a region together with the claim would give $|f(z)| \leq M$ $\forall z \in G$ by the Maximum Modulus Theorem III.
However, G is assumed to be an open set. But every open set can be written as a union of components (components are open and connected). So each component is a region.
Therefore, we can apply the Maximum Modulus Theorem III to each component once we have shown the claim and additionally we have to show that the boundary of G is the same as taking the boundaries of the components:
Clearly, since the components of G add up to G, we have that the boundary of the components must include the boundary of G.
So it remains to show that the boundary of the component does not include more than the boundary of G.
Assume it would be the case. Then there needs to be a boundary of a component lying inside G (since every component is lying inside G). But this would imply that it must be connected to another component of G contradicting the fact a component is a maximally connected subset.
Hence, the boundary of G equals the boundary of all its components.
It remains to show the claim which we will do by showing the contrapositive.
Proof of the claim: Assume not (6.6), that is

$$\forall M > 0\ \exists a \in \partial_\infty G : \limsup_{z\to a} |f(z)| > M$$

which is equivalent to

$$\forall M > 0\ \exists a \in \partial_\infty G : \lim_{r\to 0^+} \sup\{|f(z)| : z \in G \cap B(a; r)\} > M$$

by definition. Then, clearly $\exists\delta > 0$ *such that* $\lim_{r\to 0^+}\sup\{|f(z)| : z \in G \cap B(a;r)\} > M+\delta$. *Let* $\{z_n\}$ *be a sequence with* $\lim_{n\to\infty} = a$ *and let* $r_n = 2|z_n - a|$. *Obviously, we have* $\lim_{n\to\infty} r_n = 0$. *Next, consider the sequence* $\alpha_n = \sup\{|f(z)| : z \in F \cap B(a;r_n)\}$. *Then* $\lim_{n\to\infty}\alpha_n > M+\delta$. *In addition,* $\{\alpha_n\}$ *is a nonincreasing sequence, therefore* $\lim_{n\to\infty}\alpha_n > M+\delta$ *implies* $\alpha_n > M+\delta$ $\forall n$. *Thus,* $\exists\, y_n \in G \cap B(a;r_n)$ *such that* $|f(y_n)| > M+\delta$. *Taking* lim sup *in this inequality yields*

$$\limsup_{n\to\infty} |f(y_n)| \ge M+\delta,$$

that is

$$\limsup_{n\to\infty} |f(y_n)| \ge M$$

where $\lim_{n\to\infty} y_n = a$, *which gives "not (6.5)". Hence (6.5) implies (6.6).*

Exercise 5. *Let* $f : G \to \mathbb{C}$ *be analytic and suppose that* G *is bounded. Fix* z_0 *in* ∂G *and suppose that* $\limsup_{z\to w}|f(z)| \le M$ *for* w *in* ∂G, $w \ne z_0$. *Show that if* $\lim_{z\to z_0}|z - z_0|^\epsilon |f(z)| = 0$ *for every* $\epsilon > 0$ *then* $|f(z)| \le M$ *for every* z *in* ∂G. *(Hint: If* $a \notin G$, *consider* $\varphi(z) = (z - z_0)(z-a)^{-1}$.*)*

Solution. *Not available.*

Exercise 6. *Let* $G = \{z : Re\, z > 0\}$ *and let* $f : G \to \mathbb{C}$ *be an analytic function with* $\limsup_{z\to w}|f(z)| \le M$ *for* w *in* ∂G, *and also suppose that for every* $\epsilon > 0$,

$$\lim_{r\to\infty}\sup\{\exp(-\epsilon/r)|f(re^{i\theta})| : |\theta| < \frac{1}{2}\pi\} = 0.$$

Show that $|f(z)| \le M$ *for all* z *in* G.

Solution. *Not available.*

Exercise 7. *Let* $G = \{z : Re\, z > 0\}$ *and let* $f : G \to \mathbb{C}$ *be analytic such that* $f(1) = 0$ *and such that* $\limsup_{z\to w}|f(z)| \le M$ *for* w *in* ∂G. *Also, suppose that for every* δ, $0 < \delta < 1$, *there is a constant* P *such that*

$$|f(z)| \le P\exp(|z|^{1-\delta}).$$

Prove that

$$|f(z)| \le M\left[\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\right]^{\frac{1}{2}}.$$

Hint: Consider $f(z)\left(\frac{1+z}{1-z}\right)$.

Solution. *Not available.*

Chapter 7

Compactness and Convergence in the Space of Analytic Functions

7.1 The space of continuous functions $C(G, \Omega)$

Exercise 1. *Prove Lemma 1.5 (Hint: Study the function $f(t) = \frac{t}{1+t}$ for $t > -1$.)*

Solution. *To show that $\mu : S \times S \to \mathbb{R}$, $\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$ is a metric we proof that μ satisfies the conditions of a metric function.*
For $s, t \in S$ the value $\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$ is well defined in the nonnegative real numbers because d is a metric. Also $\mu(s,t) = 0$ if and only if $d(s,t) = 0$ and since d is a metric it follows that $s = t$. The metric d is symmetric and therefore μ is also. It remains to show the triangle inequality. Let therefore $s, t, u \in S$. The metric d satisfies the triangle inequality

$$d(s,u) \le d(s,t) + d(t,u). \tag{7.1}$$

The function $f(t) = \frac{t}{1+t}$ is a strictly increasing, concave function on the interval $[0, \infty)$. Thus if $d(s,u) \le \max\{d(s,t), d(t,u)\}$, then also $\mu(s,u) \le \max\{\mu(s,t), \mu(t,u)\}$ and by nonnegativity of μ also

$$\mu(s,u) \le \mu(s,t) + \mu(t,u).$$

If instead $\max\{d(s,t), d(t,u)\} < d(s,u)$ then

$$\frac{1}{1 + d(s,u)} < \min\left\{\frac{1}{1 + d(s,t)}, \frac{1}{1 + d(t,u)}\right\}.$$

Together with equation (7.1) we have

$$\begin{aligned}
\mu(s,u) = \frac{d(s,u)}{1 + d(s,u)} &\le \frac{d(s,t) + d(t,u)}{1 + d(s,u)} \\
&= \frac{d(s,t)}{1 + d(s,u)} + \frac{d(t,u)}{1 + d(s,u)} \le \frac{d(s,t)}{1 + d(s,t)} + \frac{d(t,u)}{1 + d(t,u)} \\
&= \mu(s,t) + \mu(t,u),
\end{aligned}$$

which gives the triangle inequality.
To show that the metrics d and μ are equivalent on S, let O be on open set in (S, d) and let $x \in O$. There is

$\varepsilon \in (0,1)$ such that $B_d(x,\varepsilon) \subset O$. Choose δ positive such that $\delta < \frac{\varepsilon}{1+\varepsilon}$, then $B_\mu(x,\delta) \subset B_d(x,\varepsilon)$. Similarly, if O is an open set in (S,μ) and δ is such that $x \in B_\mu(x,\delta) \subset O$ then choose $\varepsilon > 0$ with $\varepsilon \leq \frac{\delta}{1-\delta}$ which implies $B_d(x,\varepsilon) \subset B_\mu(x,\delta)$. Since this can be done for any element $x \in O$, open sets in (S,d) are open in (S,μ) and vice-versa.

The fact that exactly the open sets in (S,d) are open in (S,μ) leads to the statement about Cauchy sequences. Assume that $\{x_n\}_n$ is a Cauchy sequence in (S,d). To see that it is Cauchy in (S,μ), let $\varepsilon > 0$ be arbitrary but fixed. By the above there is a δ small enough that $B_d(x,\delta) \subset B_\mu(x,\varepsilon)$ and δ depends only on ε, but not on $x \in S$. Since $\{x_n\}_n$ is Cauchy in (S,d) there is $N \in \mathbb{N}$ such that $d(x_m,x_n) < \delta$ whenever $m,n \geq N$ and therefore also $\mu(x_m,x_n) < \varepsilon$ for $m,n \geq N$. The opposite statement holds with a similar argument.

Exercise 2. *Find the sets K_n obtained in Proposition 1.2 for each of the following choices of G: (a) G is an open disk; (b) G is an open annulus; (c) G is the plane with n pairwise disjoint closed disks removed; (d) G is an infinite strip; (e) $G = \mathbb{C} - \mathbb{Z}$.*

Solution. a) *G is an open disk,*

Let $a \in \mathbb{C}$ and $r > 0$ such that $G = B(a,r)$. Set

$$K_n = \overline{B\left(a, r - \frac{1}{n}\right)}$$

where a ball of negative radius is to be understood as the empty set. Depending on the size of r, a finite number of K_n may be empty, not violating the condition $K_n \subset \operatorname{int} K_{n+1}$, and obviously $G = \bigcup_{n=1}^{\infty}$.

b) *G is an open annulus,*

Let $a \in \mathbb{C}$ be the center of the annulus with radii $r, R, 0 \leq r < R < \infty$ so that $G = \operatorname{ann}(a,r,R)$. Define

$$K_n = \overline{\operatorname{ann}\left(a, r + \frac{1}{n}, R - \frac{1}{n}\right)},$$

again with the interpretation that an annulus with inner radius larger than the outer radius is the empty set.

d) $G = \{z \in \mathbb{C} : |\mathrm{Im} z| < 1\}$,

For this open set G define

$$K_n = \{z = x + iy \in C : x, y \in \mathbb{R}, |x| \leq n\} \cap \left\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R}|y| \leq 1 - \frac{1}{n}\right\}.$$

Although G is unbounded each K_n is bounded and closed and hence compact and $G = \bigcup_{n=1}^{\infty} K_n$.

e) $G = \mathbb{C} - \mathbb{Z}$.

Here we can define

$$K_n = \{z \in \mathbb{C} : |z| \leq n\} \cap \left(\bigcup_{j=-n}^{n} B\left((j, \frac{1}{n}\right)\right)^C,$$

an intersection of closed set, one of which is bounded, hence K_n is compact. Also $K_n \subset \operatorname{int} K_{n+1}$ by definition and $G = \bigcup_{n=1}^{\infty}$.

Exercise 3. *Supply the omitted details in the proof of Proposition 1.18.*

Solution. *Not available.*

Exercise 4. *Let F be a subset of a metric space (X, d) such that F^- is compact. Show that F is totally bounded.*

Solution. *Not available.*

Exercise 5. *Suppose $\{f_n\}$ is a sequence in $C(G, \Omega)$ which converges to f and $\{z_n\}$ is a sequence in G which converges to a point z in G. Show $\lim f_n(z_n) = f(z)$.*

Solution. *Not available.*

Exercise 6. *(Dini's Theorem) Consider $C(G, \mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G, \mathbb{R})$ which is monotonically increasing (i.e., $f_n(z) \leq f_{n+1}(z)$ for all z in G) and $\lim f_n(z) = f(z)$ for all z in G where $f \in C(G, \mathbb{R})$. Show that $f_n \to f$.*

Solution. *By Proposition 1.2 we can find a sequence $\{K_n\}$ of compact subsets of G such that $\bigcup_{n=1}^{\infty} K_n$. Thus, WLOG we can assume that G is compact. Define $g_n = f - f_n \ \forall n$. Clearly g_n is continuous $\forall n$, since f and f_n are continuous by assumption. Since f_n is monotonically increasing, g_n is monotonically decreasing. Since f_n converges to f, g_n converges pointwise to zero, that is*

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall x \in G : 0 \leq g_N(x) < \frac{\epsilon}{2}.$$

Since g_N is continuous, there is an open neighborhood $B(x; r)$ around x (with $r < \infty$). For any $y \in B(x; r)$ we have $g_N(y) < \epsilon$, that is

$$\forall \epsilon > 0 \ \forall x \in G \ \exists N \in \mathbb{N} \ \exists B(x; r) : \ \forall y \in B(x; r) : g_N(y) < \epsilon.$$

Since g_N is monotonically decreasing, we surely get $g_n(y) < \epsilon$ for every $n \geq N$ and for every $y \in B(x; r)$. These open neighborhoods will cover G. Since G is compact, we can find finitely many $x_1, \ldots, x_k$ such that $B(x_i; r)$ cover G for every value of ϵ. Note that the values of N can change the chosen center x of each those neighborhoods, say N_i. Therefore, let $M = \max_i N_i$. Because of the monotonicity of g_n, we have that $g_n(y) < \epsilon \ \forall \ n \geq M$ in every open neighborhood $B(x_i; r)$, that is, for every value in G. Hence,

$$\forall \epsilon > 0 \ \exists M \in \mathbb{N}, \forall n \geq M \ \forall x \in G : |f_n(x) - f(x)| < \epsilon,$$

which shows $f_n \to f$.

Exercise 7. *Let $\{f_n\} \subset C(G, \Omega)$ and suppose that $\{f_n\}$ is equicontinuous at each point of G. If $f \in C(G, \Omega)$ and $f(z) = \lim f_n(z)$ for each z then show that $f_n \to f$.*

Solution. *By Proposition 1.2 we can find a sequence $\{K_n\}$ of compact subsets of G such that $\bigcup_{n=1}^{\infty} K_n$. Thus, WLOG we can assume that G is compact. Since f_n is equicontiuous at each point of G, we have that*

$$\forall \epsilon > 0 \ \exists \delta > 0 : d(f_n(x), f_n(y)) < \frac{\epsilon}{3} \text{ whenever } |x - y| < \delta. \tag{7.2}$$

Taking the limit as $n \to \infty$, we also get

$$d(f(x), f(y)) < \frac{\epsilon}{3} \text{ whenever } |x - y| < \delta. \tag{7.3}$$

by the pointwise convergence ($f(z) = \lim f_n(z) \ \forall z \in G$ by assumption). As in the previous exercise, we can find finitely many points $y_1, \ldots, y_k \in G$ such that the open neighborhoods $B(y_i; \delta)$ cover G, since G is *compact. To be more precise: For every point x in G, we can find an open neighborhood $B(y_i;\delta)$ for some $i = 1,\dots,k$. Since $f_n \to f$ pointwise, we have*

$$d(f_n(y_i), f(y_i)) < \frac{\epsilon}{3} \quad \forall n \geq N_i.$$

Set $N = \max_i N_i$, then we have

$$d(f_n(y_i), f(y_i)) < \frac{\epsilon}{3} \quad \forall n \geq N \quad \forall i = 1,\dots,k. \tag{7.4}$$

Finally, for an arbitrary $x \in G$ and $n \geq N$ we have

$$d(f_n(x), f(x)) \leq d(f_n(x), f_n(y_i)) + d(f_n(y_i), f(y_i)) + d(f(y_i), f(x)) \underset{(7.2),(7.3),(7.4)}{<} \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

where we choose y_i such that $|x - y_i| < \delta$. This shows $f_n \to f$.

Exercise 8. *(a) Let f be analytic on $B(0;R)$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$. If $f_n(z) = \sum_{k=0}^{n} a_k z^k$, show that $f_n \to f$ in $C(G;\mathbb{C})$.*
(b) Let $G = ann(0;0,R)$ and let f be analytic on G with Laurent series development $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Put $f_n(z) = \sum_{k=-\infty}^{n} a_k z^k$ and show that $f_n \to f$ in $C(G;\mathbb{C})$.

Solution. *a) Let $G = B(0;R)$. This result is proved with the Arzela-Ascoli Theorem. With the given function f define an analytic function*

$$g : G \to \mathbb{C}, \qquad g(z) := \sum_{k=0}^{\infty} |a_k| z^k. \tag{7.5}$$

Let $z \in G$ and define $r := \frac{|z|+R}{2} (< R)$ then $z \in B(0;r)$. Let $\mathcal{F} = \{f_n\}_n$. Then for this fixed z and for any $n \in \mathbb{N}$ we have

$$|f_n(z)| = |\sum_{k=0}^{n} a_k z^k| \leq \sum_{k=0}^{n} |a_k||z|^k = g(|z|) < \infty.$$

Thus for a given $z \in G$, $\{f_n(z) : f_n \in \mathcal{F}\}$ is bounded and therefore has compact closure.
Next, we want to show equicontinuity at each point of G. Fix $z_0 \in G$ and let ε be positive. Let $r > 0$ be such that $B(z_0;r) \subset G$. With the function $g(z)$ defined in equation (7.5) and the computations from above we have for all $w \in \partial B(z_0;r)$ and for all $n \in \mathbb{N}$

$$|f_n(w)| \leq g(|w|).$$

The boundary of the disk with radius r is compact and so is $g(\partial B(z_0;r))$. Thus there is a positive number M such that $|g(\partial B(z_0,r))| \leq M$.
Choose $\delta \in \left(0, \min\{\frac{r}{2}, \frac{\varepsilon r}{2M}\}\right)$ and let $|z - z_0| < \delta$ then as in the proof of Montel's theorem the following estimate holds

$$\begin{aligned}|f_n(z) - f_n(z_0)| &= |\frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f_n(w)(z-z_0)}{(w-z)(w-z_0)} \, \mathrm{d}w| \\ &\leq \frac{2M}{r}\delta \leq \varepsilon.\end{aligned}$$

By Arzela-Ascoli the sequence $\{f_n\}$ is normal in $C(G)$ so there is a subsequence $\{f_{n_k}\}$ that converges in $C(G)$. Since f is analytic and $\{f_n\}$ is the sequence of partial sums of f, $\{f_n\}$ itself is an analytic Cauchy sequence in $C(G)$. Hence every subsequence converges to the same limit, which shows that $f_n \to f$ in $C(G)$. As a last step recall that $H(G)$ is complete as a subspace of $C(G)$ (Corollary 2.3) and $\{f_n\} \subset H(G)$ thus also the limit function $f \in H(G)$.

7.2 Spaces of analytic functions

Exercise 1. *Let $f, f_1, f_2, \ldots$ be elements of $H(G)$ and show that $f_n \to f$ iff for each closed rectifiable curve γ in G, $f_n(z) \to f(z)$ uniformly for z in $\{\gamma\}$.*

Solution. *Not available.*

Exercise 2. *Let G be a region, let $a \in \mathbb{R}$, and suppose that $f : [a, \infty] \times G \to \mathbb{C}$ is a continuous function. Define the integral $F(z) = \int_a^\infty f(t,z)\, dt$ to be uniformly convergent on compact subsets of G if $\lim_{b\to\infty} \int_a^b f(t,z)\, dt$ exists uniformly for z in any compact subset of G. Suppose that this integral does converge uniformly on compact subsets of G and that for each t in (a, ∞), $f(t, \cdot)$ is analytic on G. Prove that F is analytic and*

$$F^{(k)}(z) = \int_a^\infty \frac{\partial^k f(t,z)}{\partial z^k}\, dt$$

Solution. *Not available.*

Exercise 3. *The proof of Montel's Theorem can be broken up into the following sequence of definitions and propositions: (a) Definition. A set $\mathcal{F} \subset C(G, \mathbb{C})$ is locally Lipschitz if for each a in G there are constants M and $r > 0$ such that $|f(z) - f(a)| \le M|z - a|$ for all f in $\mathcal{F}$ and $|z - a| < r$. (b) If $\mathcal{F} \in C(G, \mathbb{C})$ is locally Lipschitz then $\mathcal{F}$ is equicontinuous at each point of G. (c) If $\mathcal{F} \subset H(G)$ is locally bounded then $\mathcal{F}$ is locally Lipschitz.*

Solution. *Not available.*

Exercise 4. *Prove Vitali's Theorem: If G is a region and $\{f_n\} \subset H(G)$ is locally bounded and $f \in H(G)$ that has the property that $A = \{z \in G : \lim f_n(z) = f(z)\}$ has a limit point in G then $f_n \to f$.*

Solution. *The sequence $\{f_n\}$ is a locally bounded sequence in $H(G)$ and by Montel's Theorem then $\{f_n\}$ is normal, so there is a subsequence $\{f_{n_k}\} \subset \{f_n\}$ that converges to f in $H(G)$.*
Suppose that $f_n \nrightarrow f$ in $H(G)$ then there must be a compact set $K \subset G$ the convergence is not uniform on K. In other words there must be an $\varepsilon > 0$ so that for all $n \in \mathbb{N}$ there is $z_n \in K$ with $|f_n(z_n) - f(z_n)| \ge \varepsilon$. By compactness of K extract a subsequence of $\{z_n\}$, say $\{z_{n_m}\}$ that converges to a point $z_0 \in K$.
The sequence $\{f_n\}$ is locally bounded, so in particular the subsequence $\{f_{n_m}\}$ with indices corresponding to $\{z_{n_m}\}$ is, and again by Montel's Theorem now applied to the subsequence and again the completeness of $H(G)$ there is an analytic function g such that $f_{n_m} \to g$ in $H(G)$. On the set A of points of pointwise convergence $f_n(z) \to f$ and $f_n(z) \to g$.
Now G is a region and A has a limit point in G so by the Identity Theorem (Chapter 4, Corollary 3.8) already $f = g$ on G which gives a contradiction on the set K and the point z_0 because

$$|f_{n_m}(z_{n_m}) - f(z_{n_m})| \to |g(z_0) - f(z_0)| \ge \varepsilon.$$

Hence we can conclude that $f_n \to f$ in $H(G)$.

Exercise 5. *Show that for a set $\mathcal{F} \subset H(G)$ the following are equivalent conditions:*
(a) $\mathcal{F}$ is normal;
(b) For every $\epsilon > 0$ there is a number $c > 0$ such that $\{cf : f \in \mathcal{F}\} \subset B(0;\epsilon)$ (here $B(0;\epsilon)$ is the ball in $H(G)$ with center at 0 and radius ϵ).

Solution. *Not available.*

Exercise 6. *Show that if $\mathcal{F} \subset H(G)$ is normal then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also normal. Is the converse true? Can you add something to the hypothesis that $\mathcal{F}'$ is normal to insure that $\mathcal{F}$ is normal?*

Solution. *Not available.*

Exercise 7. *Suppose $\mathcal{F}$ is normal in $H(G)$ and Ω is open in $\mathbb{C}$ such that $f(G) \subset \Omega$ for every f in $\mathcal{F}$. Show that if g is analytic on Ω and is bounded on bounded sets then $\{g \circ f : f \in \mathcal{F}\}$ is normal.*

Solution. *Not available.*

Exercise 8. *Let $D = \{z : |z| < 1\}$ and show that $\mathcal{F} \subset H(D)$ is normal iff there is a sequence $\{M_n\}$ of positive constants such that $\limsup \sqrt[n]{M_n} \leq 1$ and if $f(z) = \sum_0^\infty a_n z^n$ is in $\mathcal{F}$ then $|a_n| \leq M_n$ for all n.*

Solution. *Not available.*

Exercise 9. *Let $D = B(0;1)$ and for $0 < r < 1$ let $\gamma_r(t) = re^{2\pi i t}$, $0 \leq t \leq 1$. Show that a sequence $\{f_n\}$ in $H(D)$ converges to f iff $\int_{\gamma_r} |f(r) - f_n(z)|\,|dz| \to 0$ as $n \to \infty$ for each r, $0 < r < 1$.*

Solution. $\Rightarrow$: *Assume $\lim_{n\to\infty} f_n(z) = f(z)\ \forall z \in D$. Then*

$$\begin{aligned}\int_{\gamma_r} |f(z) - f_n(z)|\,|dz| &\leq \sup_{z\in\gamma_r} |f(z) - f_n(z)| \int_{\gamma_r} |dz| \\ &\underset{\gamma_r = re^{2\pi i t}, 0\leq t\leq 1}{=} \sup_{z\in\gamma_r} |f(z) - f_n(z)| \cdot 2\pi r, \quad \forall 0 < r < 1.\end{aligned}$$

But $\sup_{z\in\gamma_r} |f(z) - f_n(z)| \to$ as $n \to \infty$ since $\lim_{n\to\infty} f_n(z) = f(z)\ \forall\, z \in D$. Therefore,

$$\int_{\gamma_r} |f(z) - f_n(z)|\,|dz| \to 0$$

as $n \to \infty$ for each r, $0 < r < 1$.
$\Leftarrow$: *Assume $\int_{\gamma_r} |f(z) - f_n(z)|\,|dz| \to 0$ as $n \to \infty$ for each r, $0 < r < 1$. Let K be an arbitrary compact subset of $D = B(0;1)$ and choose $0 < r < 1$ such that $K \subset B(0;r)$ and denote the shortest distance between an arbitrary point in K and an arbitrary point on ∂B by $\delta > 0$. By the assumption, we have*

$$\int_{\partial B} |f(z) - f_n(z)|\,|dz| < 2\pi\epsilon\delta. \tag{7.6}$$

Let $a \in K$, then by Cauchy's Integral formula, we have

$$f(a) - f_n(a) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z) - f_n(z)}{z - a}\, dz.$$

Thus,

$$|f(a) - f_n(a)| \leq \frac{1}{2\pi} \int_{\partial B} \frac{|f(z) - f_n(z)|}{|z - a|}\,|dz| \leq \frac{1}{2\pi\delta} \int_{\partial B} |f(z) - f_n(z)|\,|dz| \underset{(7.6)}{<} \frac{2\pi\epsilon\delta}{2\pi\delta} = \epsilon.$$

Hence, $f_n(a) \to f(z)$ uniformly $\forall a \in K$. Since it converges uniformly on all compact subsets of D, Proposition 1.10 b) yields that $\{f_n\}$ converges to $f\ \forall\, z \in D$.

Exercise 10. *Let $\{f_n\} \subset H(G)$ be a sequence of one-one functions which converge to f. If G is a region, show that either f is one-one or f is a constant function.*

Solution. *Assume $\{f_n\} \subset H(G)$ is a sequence of one-one functions which converge to f where G is a region (open and connected). Suppose f is not the constant function. Then, we have to show that f is one-one. Choose an arbitrary point $z_0 \in G$ and define the sequence $g_n(z) = f_n(z) - f_n(z_0)$. Clearly, the sequence $\{g_n\}$ converges to $g(z) = f(z) - f(z_0)$ on the open connected set $G\backslash\{z_0\}$, which is again a region. In addition, we*

have that g_n never vanishes on $G\backslash\{z_0\}$, since f_n are assumed to be one-one functions.
Therefore, the sequence $\{g_n\}$ satisfies the conditions of the Corollary 2.6 resulting from Hurwitz's Theorem, where the region is $G\backslash\{z_0\}$. Thus, either $g \equiv 0$ or g never vanishes (g has no zeros). Since $g(z) = f(z) - f(z_0)$ is not identically zero on $G\backslash\{z_0\}$, we must have $g(z) = f(z) - f(z_0)$ has no zeros in $G\backslash\{z_0\}$. This implies

$$f(z) \neq f(z_0) \qquad \forall z \in G\backslash\{z_0\}.$$

Since z_0 was an arbitrary point in G, we have shown that f is one-one on G.

Exercise 11. *Suppose that $\{f_n\}$ is a sequence in $H(G)$, f is a non-constant function, and $f_n \to f$ in $H(G)$. Let $a \in G$ and $\alpha = f(a)$; show that there is a sequence $\{a_n\}$ in G such that: (i) $a = \lim a_n$; (ii) $f_n(a_n) = \alpha$ for sufficiently large n.*

Solution. *Not available.*

Exercise 12. *Show that $\lim \tan nz = -i$ uniformly for z in any compact subset of $G = \{z : Im\, z > 0\}$.*

Solution. *Not available.*

Exercise 13. *(a) Show that if f is analytic on an open set containing the disk $\bar{B}(a; R)$ then*

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r\, dr\, d\theta.$$

(b) Let G be a region and let M be a fixed positive constant. Let $\mathcal{F}$ be the family of all functions f in $H(G)$ such that $\iint_G |f(z)|^2\, dx\, dy \leq M$. Show that $\mathcal{F}$ is normal.

Solution. *a) Let $\bar{B}(a; R) \subset G$ and $f : G \to \mathbb{C}$ be analytic. Let $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$, then by Cauchy's Integral Formula we have*

$$g(a) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w - a}\, dw = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt.$$

Thus, letting $g(z) = f(z)^2$, we obtain

$$f^2(a) = \frac{1}{2\pi} \int_0^{2\pi} \left[f(a + re^{it}) \right]^2\, dt.$$

Multiply by r, we get

$$f^2(a) r = \frac{1}{2\pi} \int_0^{2\pi} \left[f(a + re^{i\theta}) \right]^2 r\, d\theta \tag{7.7}$$

and then integrate with respect to r yields

$$\int_0^R f^2(a) r\, dr = f^2(a) \frac{R^2}{2} \underset{(7.7)}{=} \int_0^R \frac{1}{2\pi} \int_0^{2\pi} \left[f(a + re^{i\theta}) \right]^2 r\, d\theta\, dr = \frac{1}{2\pi} \int_0^{2\pi} \int_0^R \left[f(a + re^{i\theta}) \right]^2 r\, dr\, d\theta$$

which implies

$$f^2(a) = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left[f(a + re^{i\theta}) \right]^2 r\, dr\, d\theta.$$

So

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left| f(a + re^{i\theta}) \right|^2 r\, dr\, d\theta.$$

b) Let G be a region and M be a fixed constant. Let $\mathcal{F}$ be the family of all functions f in $H(G)$ such that

$$\iint_G |f(z)|^2 \, dx \, dy \leq M. \tag{7.8}$$

To show that $\mathcal{F}$ is normal.
According to Montel's Theorem, it suffices to show that $\mathcal{F}$ is locally bounded, that is $\exists\, r > 0$ such that $\sup\{|f(z)| : |z-a| < r, f \in \mathcal{F}\} < \infty$ (p. 153). We will prove this fact by contradiction:
Assume $\mathcal{F}$ is not locally bounded. Then there is a compact set $K \subset G$ such that $\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = \infty$. That is, there is a sequence $\{f_n\}$ in $\mathcal{F}$ such that $\sup\{|f_n(z)| : z \in K\} \geq n$. Therefore, $\exists\, z_n \in K$ such that

$$|f_n(z_n)| \geq \frac{n}{2}. \tag{7.9}$$

Every sequence has a convergence subsequence, say $\{z_{n_k}\}$, with $z_{n_k} \to z_0 \in K$ since K is compact. WLOG, we write $\{z_{n_k}\} = \{z_n\}$. Clearly $z_0 \in G$ since $z_0 \in K$ ($K \subset G$), hence $\exists\, R > 0$ such that $B(z_0; R) \subset G$ (since G is open). If we pick n large enough, we can get $z_n \in B(z_0; \frac{R}{2})$ (a picture might help). In addition, we can find an $\hat{R} > 0$ such that $\overline{B(z_n; \hat{R})} \subset B(z_0; \frac{R}{2})$ because $B(z_0; \frac{R}{2})$ is open. Apply part a) to the ball $\overline{B(z_n; \hat{R})}$ to obtain:

$$|f_n(z_n)|^2 \leq \frac{1}{\pi \hat{R}^2} \int_0^{2\pi} \int_0^{\hat{R}} \left| f_n(z_n + re^{i\theta}) \right|^2 r \, dr \, d\theta \underset{B(z_n;\hat{R}) \subset G}{\leq} \frac{1}{\pi \hat{R}^2} \iint_G |f_n(z)|^2 \, dx \, dy \underset{(7.8)}{\leq} \frac{1}{\pi \hat{R}^3} M < \infty.$$

So

$$|f_n(z_n)| \leq \sqrt{\frac{M}{\pi}} \frac{1}{\hat{R}} < \infty.$$

But by (7.9) we have

$$|f_n(z_n)| \geq \frac{n}{2}.$$

This gives a contradiction if we let $n \to \infty$. Thus, $\mathcal{F}$ is locally bounded and therefore normal.

7.3 Spaces of meromorphic functions

Exercise 1. *Prove Proposition 3.3.*

Solution. *a) Let $a \in \mathbb{C}$ and $r > 0$. To show: There is a number $\delta > 0$ such that $B_\infty(a; \delta) \subset B(a; r)$. Let $R > 0$ such that $B(a; r) \subset B(0; R)$ and $B_\infty(a; \delta) \subset B(0; R)$.*
Claim: Pick

$$\delta = \frac{2r}{\sqrt{1+R^2}\sqrt{1+|a|^2}}$$

to obtain

$$B_\infty(a; \delta) \subset B(a; R).$$

Proof of the claim: Let $z \in B_\infty(a; \delta)$. This is equivalent to

$$\begin{aligned}
& \frac{2|z-a|}{\sqrt{1+|z|^2}\sqrt{1+|a|^2}} < \delta \\
\iff \quad & \frac{2|z-a|}{\sqrt{1+|z|^2}\sqrt{1+|a|^2}} < \frac{2r}{\sqrt{1+R^2}\sqrt{1+|a|^2}} \\
\iff \quad & |z-a| < \frac{r}{\sqrt{1+R^2}} \sqrt{1+|z|^2}
\end{aligned}$$

which implies ($z \in B_\infty(a;\delta) \Rightarrow z \in B(0;R)$, that is $|z| < R$)

$$\begin{aligned} & |z-a| < \frac{r}{\sqrt{1+R^2}}\sqrt{1+R^2} \\ \iff \quad & |z-a| < r. \end{aligned}$$

So $z \in B(a;r)$. Therefore

$$B_\infty(a;\delta) \subset B(a;r).$$

b) Let $\delta > 0$ and $a \in \mathbb{C}$. To show: There is a number $r > 0$ such that $B(a;r) \subset B_\infty(a;\delta)$.
Claim: Pick $r < \frac{\delta}{2}$ to obtain

$$B(a;r) \subset B_\infty(a;\delta).$$

Proof of the claim: Let $z \in B(a;r)$. This is equivalent to

$$\begin{aligned} & |z-a| < r \\ \underset{r<\frac{\delta}{2}}{\Rightarrow} \quad & |z-a| < \frac{\delta}{2} \\ \iff \quad & |z-a| < \frac{\delta}{2}\cdot 1 \\ \underset{1\leq\sqrt{1+|z|^2}\sqrt{1+|a|^2}}{\Rightarrow} \quad & |z-a| < \frac{\delta}{2}\sqrt{1+|z|^2}\sqrt{1+|a|^2} \\ \iff \quad & \frac{2|z-a|}{\sqrt{1+|z|^2}\sqrt{1+|a|^2}} < \delta \\ \iff \quad & d(z,a) < \delta. \end{aligned}$$

So $z \in B_\infty(a;\delta)$. Therefore,

$$B(a;r) \subset B_\infty(a;\delta).$$

c) Let $\delta > 0$. To show: There is a compact set $K \subset \mathbb{C}$ such that $\mathbb{C}_\infty - K \subset B_\infty(\infty;\delta)$.
Claim: Choose $K = \overline{B(0;r)}$ where $r > \sqrt{\frac{4}{\delta^2}-1}$ to obtain

$$\mathbb{C}_\infty - K \subset B_\infty(\infty;\delta).$$

(Clearly K is compact and $r > 0$ since $\delta \leq 2$.)

Proof of the claim:

$$\begin{aligned}
\mathbb{C}_\infty - K &= \mathbb{C}_\infty - \overline{B(0;r)} \\
&= \{\mathbb{C} - \overline{B(0;r)}\} \cup \{\infty\} \\
&= \{\mathbb{C} - \{z \in \mathbb{C} : |z| \le r\}\} \cup \{\infty\} \\
&\underset{r>\sqrt{\frac{4}{\delta^2}-1}}{\subset} \left\{\mathbb{C} - \left\{z \in \mathbb{C} : |z| \le \sqrt{\frac{4}{\delta^2} - 1}\right\}\right\} \cup \{\infty\} \\
&= \left\{z \in \mathbb{C} : |z| > \sqrt{\frac{4}{\delta^2} - 1}\right\} \cup \{\infty\} \\
&= \left\{z \in \mathbb{C} : \sqrt{\frac{4}{\delta^2} - 1} < |z|\right\} \cup \{\infty\} \\
&= \left\{z \in \mathbb{C} : \frac{2}{\sqrt{1+|z|^2}} < \delta\right\} \cup \{\infty\} \\
&= \{z \in \mathbb{C} : d(\infty, z) < \delta\} \cup \{\infty\} \\
&= B_\infty(\infty; \delta).
\end{aligned}$$

Therefore,

$$\mathbb{C}_\infty - K \subset B_\infty(\infty; \delta).$$

d) Let $K \subset \mathbb{C}$ where K is compact. To show: There is a number $\delta > 0$ such that $B_\infty(\infty; \delta) \subset \mathbb{C}_\infty - K$. Let $B(0; r)$ such that $B(0; r) \supset K$ where $r > 0$. Clearly

$$\mathbb{C}_\infty - B(0; r) \subset \mathbb{C}_\infty - K.$$

So it suffices to show: There is a number $\delta > 0$ such that

$$B_\infty(\infty; \delta) \subset \mathbb{C}_\infty - B(0; r)$$

for a given $r > 0$.
Claim: Choose $\delta \le \frac{2}{\sqrt{r^2+1}}$ to obtain

$$B_\infty(\infty; \delta) \subset \mathbb{C}_\infty - B(0; r)$$

and hence

$$B_\infty(\infty; \delta) \subset \mathbb{C}_\infty - K.$$

Proof of the claim:

$$\begin{aligned}
B_\infty(\infty; \delta) &= \{z \in \mathbb{C} : d(\infty, z) < \delta\} \cup \{\infty\} \\
&= \left\{z \in \mathbb{C} : \frac{2}{\sqrt{1+|z|^2}} < \delta\right\} \cup \{\infty\} \\
&\underset{\delta \le \frac{2}{\sqrt{r^2+1}}}{\subset} \left\{z \in \mathbb{C} : \frac{2}{\sqrt{1+|z|^2}} < \frac{2}{\sqrt{r^2+1}}\right\} \cup \{\infty\} \\
&= \{z \in \mathbb{C} : r \le |z|\}\} \cup \{\infty\} \\
&= \{\mathbb{C} \cup \{z \in \mathbb{C} : |z| < r\} \cup \{\infty\} \\
&= \mathbb{C}_\infty - \{z \in \mathbb{C} : |z| < r\} \\
&= \mathbb{C}_\infty - B(0; r).
\end{aligned}$$

Therefore,

$$B_\infty(\infty;\delta) \subset \mathbb{C}_\infty - B(0;r).$$

Exercise 2. *Show that if $\mathcal{F} \subset M(G)$ is a normal family in $C(G,\mathbb{C}_\infty)$ then $\mu(\mathcal{F})$ is locally bounded.*

Solution. *Not available.*

7.4 The Riemann Mapping Theorem

Exercise 1. *Let G and Ω be open sets in the plane and let $f : G \to \Omega$ be a continuous function which is one-one, onto, and such that $f^{-1} : \Omega \to G$ is also continuous (a homeomorphism). Suppose $\{z_n\}$ is a sequence in G which converges to a point z in ∂G; also suppose that $w = \lim f(z_n)$ exists. Prove that $w \in \partial\Omega$.*

Solution. *Not available.*

Exercise 2. *(a) Let G be a region, let $a \in G$ and suppose that $f : (G - \{a\}) \to \mathbb{C}$ is an analytic function such that $f(G - \{a\}) = \Omega$ is bounded. Show that f has a removable singularity at $z = a$. If f is one-one, show that $f(a) \in \partial\Omega$.*
(b) Show that there is no one-one analytic function which maps $G = \{z : 0 < |z| < 1\}$ onto an annulus $\Omega = \{z : r < |z| < R\}$ where $r > 0$.

Solution. *Not available.*

Exercise 3. *Let G be a simply connected region which is not the whole plane and suppose that $\overline{z} \in G$ whenever $z \in G$. Let $a \in G \cap \mathbb{R}$ and suppose that $f : G \to D = \{z : |z| < 1\}$ is a one-one analytic function with $f(a) = 0$, $f'(a) > 0$ and $f(G) = D$. Let $G_+ = \{z \in G : Im\, z > 0\}$. Show that $f(G_+)$ must lie entirely above or entirely below the real axis.*

Solution. *Not available.*

Exercise 4. *Find an analytic function f which maps $\{z : |z| < 1, Re\, z > 0\}$ onto $B(0;1)$ in a one-one fashion.*

Solution. *Not available.*

Exercise 5. *Let f be analytic on $G = \{z : Re\, z > 0\}$, one-one, with $Re\, f(z) > 0$ for all z in G, and $f(a) = a$ for some real number a. Show that $|f'(a)| \le 1$.*

Solution. *Clearly G is a simply connected region (not the whole plane). Let $a \in G$. Then, there is a unique analytic function $g : G \to D$ having the properties*
a) $g(a) = 0$ and $g'(a) > 0$
b) g is 1-1
c) $g(G) = D$
(by the Riemann Mapping Theorem). Since, we have $f : G \xrightarrow{1-1} G$ (since $Re\, f(z) > 0$) and $g : \xrightarrow[onto]{1-1} D$ so $g^{-1} : D \xrightarrow[onto]{1-1} G$ we can define $h(z) = g(f(g^{-1}(z)))$. Clearly $h(z)$ is analytic and one-one, since f and g are analytic and one-one, and we have

$$h(D) = D$$

by construction. We have

$$h(0) = g(f(g^{-1}(0))) \underset{g^{-1}(0)=a}{=} g(f(a)) \underset{f(a)=a}{=} g(a) = 0.$$

We also have

$$|h(z)| \le 1 \qquad \forall z \in D$$

since $h : D \to D$*. Thus, the hypothesis of Schwarz's Lemma are satisfied, and hence, we get*

$$|h'(0)| \le 1.$$

We have

$$\begin{aligned} h'(z) &= \left[g(f(g^{-1}(z)))\right]' = g'(f(g^{-1}(z)))f'(g^{-1}(z))\left(g^{-1}(z)\right)' \\ &= g'(f(g^{-1}(z)))f'(g^{-1}(z))\frac{1}{g'(g^{-1}(z))} \end{aligned}$$

where the last step follows from Proposition 2.20 provided $g'(g^{-1}(z)) \neq 0$*. So*

$$\begin{aligned} h'(0) &= g'(f(g^{-1}(0)))f'(g^{-1}(0))\frac{1}{g'(g^{-1}(0))} \\ &= g'(f(a))f'(a)\frac{1}{g'(a)} = g'(a)f'(a)\frac{1}{g'(a)} = f'(a) \end{aligned}$$

($g'(a) > 0$ *by assumption). Therefore*

$$|h'(0)| = |f'(a)| \le 1.$$

Thus, we have shown that

$$|f'(a)| \le 1.$$

Exercise 6. *Let* G_1 *and* G_2 *be simply connected regions neither of which is the whole plane. Let* f *be a one-one analytic mapping of* G_1 *onto* G_2*. Let* $a \in G_1$ *and put* $\alpha = f(a)$*. Prove that for any one-one analytic map* h *of* G_1 *into* G_2 *with* $h(a) = \alpha$ *it follows that* $|h'(a)| \le |f'(a)|$*. Suppose* h *is not assumed to be one-one; what can be said?*

Solution. *Define the function* $F(z) = f^{-1}(h(z))$*. This is well-defined since* $f : G_1 \xrightarrow[onto]{1-1} G_2$ *and* $h : G_1 \xrightarrow{1-1} G_2$*. Clearly* $F(z)$ *is analytic since* f *and* h *are analytic,* $F(z)$ *is one-one and* $F : G_1 \to G_1$ *by construction. We have*

$$F(a) = f^{-1}(h(a)) = f^{-1}(\alpha) = a.$$

Thus by Exercise 5, we get

$$|F'(a)| \le 1.$$

We have

$$F'(z) = \left(f^{-1}(h(z))\right)' = f'(h(z))h'(z) = \frac{1}{f'(f^{-1}(h(z)))}h'(z)$$

where the last step follows by Proposition 2.20 provided $f'(f^{-1}(h(z))) \neq 0$*. So*

$$F'(a) = \frac{1}{f'(f^{-1}(h(a)))}h'(a) = \frac{h'(a)}{f'(f^{-1}(\alpha))} = \frac{h'(a)}{f'(a)}$$

which is well-defines since $f'(a) \neq 0$ *by assumption (f is one-one). Since*

$$|F'(a)| = \frac{|h'(a)|}{|f'(a)|} \le 1$$

gives

$$|h'(a)| \leq |f'(a)|.$$

If h is not one-one, then we get the same result, since I do not need the assumption that $F(z)$ has to be one-one for exercise 5.

Exercise 7. *Let G be a simply connected region and suppose that G is not the whole plane. Let $\Delta = \{\xi : |\xi| < 1\}$ and suppose that f is an analytic, one-one map of G onto Δ with $f(a) = 0$ and $f'(a) > 0$ for some point a in G. Let g be any other analytic, one-one map of G onto Δ and express g in terms of f.*

Solution. *Let $a \in G$ such that $f(a) = 0$ and $f'(a) > 0$. Since g is one-one mapping from G onto Δ, we clearly have $g(a) = \alpha$, where $\alpha \in \Delta$ (α does not have to be zero). Next, define*

$$g_1(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

Clearly, g_1 is a Möbius transformation mapping Δ to Δ and in addition g_1 is one-one and analytic. Now define

$$g_2(z) = g_1(g(z)).$$

Then, g_2 is one-one and analytic and maps G onto Δ. In addition, we have

$$g_2(a) = g_1(g(a)) = g_1(\alpha) = 0. \tag{7.10}$$

But $g_2'(a)$ need not necessarily satisfy $g_2'(a) > 0$. However, $g_2'(a) \neq 0$ $\forall z \in G$ since g_2 is one-one. Therefore, define the rotation

$$g_3(z) = e^{i\theta} z$$

for $\theta \in [0, 2\pi)$. Clearly g_3 maps Δ onto Δ. g_3 is one-one and analytic. Finally, let

$$h(z) = g_3(g_2(z)).$$

Also h is one-one, analytic and maps G onto Δ. We have

$$h(a) = g_3(g_2(a)) \underset{(7.10)}{=} g_3(0) = 0.$$

And,

$$\begin{aligned} h'(z) &= (g_3(g_2(z)))' = (g_3(g_1(g(z))))' = g_3'(g_1(g(z)))g_1'(g(z))g'(z) \\ &= e^{i\theta} \frac{(1 - \overline{\alpha} g(z)) \cdot 1 - (g(z) - \alpha) \cdot (-\overline{\alpha})}{(1 - \overline{\alpha} g(z))^2} g'(z). \end{aligned}$$

So,

$$h'(a) = e^{i\theta} \frac{(1 - \overline{\alpha}\alpha) - 0}{(1 - \overline{\alpha}\alpha)^2} g'(a) = e^{i\theta} \frac{1 - |\alpha|^2}{(1 - |\alpha|^2)^2} g'(a) = e^{i\theta} \frac{1}{1 - |\alpha|^2} g'(a).$$

Clearly, we can choose a $\theta \in [0, 2\pi)$ such that $h'(a) > 0$ (depending on the sign of $g'(a)$. Recall $g'(a) \neq 0$ since g is one-one and $\alpha \in \Delta$, so $\frac{1}{1-|\alpha|^2} > 0$). For example: If $g'(a) < 0$, then pick $\theta = \pi$, so $e^{i\theta} = -1$ or $\theta = -Arg(g'(a))$ will work, too.
By uniqueness of the map f (f satisfies the properties of the Riemann Mapping Theorem), we have

$$f(z) = h(z)$$

since h satisfies the properties of the Riemann Mapping Theorem, too. Thus,

$$f(z) = h(z) = g_3(g_2(z)) = e^{i\theta}g_2(z) = e^{i\theta}g_1(g(z)) = e^{i\theta}\frac{g(z) - \alpha}{1 - \bar{\alpha}g(z)}.$$

So

$$\begin{aligned} & 1 - \bar{\alpha}g(z) = e^{i\theta}g(z) - \alpha e^{i\theta} \\ \iff \quad & f(z) - \bar{\alpha}g(z)f(z) = e^{i\theta}g(z) - \alpha e^{i\theta} \\ \iff \quad & (e^{i\theta} + \bar{\alpha}f(z))g(z) = f(z) + \alpha e^{i\theta} \\ \iff \quad & g(z) = \frac{f(z) + \alpha e^{i\theta}}{e^{i\theta} + \bar{\alpha}f(z)}. \end{aligned}$$

Exercise 8. *Let r_1, r_2, R_1, R_2, be positive numbers such that $R_1/r_1 = R_2/r_2$; show that ann$(0; r_1, R_1)$ and ann$(0; r_2, R_2)$ are conformally equivalent.*

Solution. *The function $f : \mathbb{C} \to \mathbb{C}$; $f(z) = \frac{r_2}{r_1}z$ is analytic and it maps the annulus ann$(0; r_1, R_1)$ bijectively into ann$(0; r_2, R_2)$. The inner boundary $\{z : |z| = r_1\}$ of the domain is mapped to the inner boundary of ann$(0; r_2, R_2)$. For the outer boundary note that by assumption $R_1 = \frac{R_2 r_1}{r_2}$, so if $|z| = R_1$ then $|f(z)| = \frac{r_2 r_1}{R_1} = R_2$. The inverse function is $f^{-1}(z) = \frac{r_1}{r_2}z$, an analytic and non-constant function also. Therefore the result is proved.*

Exercise 9. *Show that there is an analytic function f defined on $G =$ann$(0; 0, 1)$ such that f' never vanishes and $f(G) = B(0; 1)$.*

Solution. *Not available.*

7.5 The Weierstrass Factorization Theorem

Exercise 1. *Show that $\prod(1 + z_n)$ converges absolutely iff $\prod(1 + |z_n|)$ converges.*

Solution. *Assume $Re(z_n) > -1$. Then the product $\prod(1 + z_n)$ converges absolutely iff $\sum_{k=1}^{\infty} z_n$ converges absolutely (by Corollary 5.6 p. 166 applied to $w_n = 1 + z_n$). $\sum_{k=1}^{\infty} z_n$ converges absolutely iff $\sum_{n=1}^{\infty} |z_n|$ converges (by definition).*
Claim: $\sum_{n=1}^{\infty} |z_n|$ converges iff $\sum_{n=1}^{\infty} \log(1 + |z_n|)$ converges.
Then $\sum_{n=1}^{\infty} \log(1+|z_n|)$ converges iff $\prod(1+|z_n|)$ converges (by Proposition 5.2 p.165 applied to $w_n = 1+|z_n|$).
Thus, ones we have proved the claim, we obtain: $\prod(1 + z_n)$ converges absolutely iff $\prod(1 + |z_n|)$ converges.
Proof of the claim: Let $x_n = |z_n|$ for convenience. Clearly $x_n \geq 0$ $\forall n \in \mathbb{N}$.
$\Rightarrow$: Assume $\sum |z_n|$ converges, that is $\sum x_n$ converges. Therefore, we have $x_n \to 0$. So given $\epsilon > 0$, we have

$$0 \leq \log(1 + |x_n|) \leq (1 + \epsilon)x_n$$

for sufficiently large n where the last inequality follows since $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$ by Exercise 2. By the comparison test, $\sum \log(1 + x_n)$ converges, that is $\sum \log(1 + |z_n|)$ converges.
$\Leftarrow$: Assume $\sum \log(1 + |z_n|)$ converges, that is $\sum \log(1 + x_n)$ converges. Therefore, we have $\log(1 + x_n) \to 0$ and therefore $x_n \to 0$. So given $\epsilon > 0$, we have

$$(1 - \epsilon)x_n \leq \log(1 + x_n)$$

for sufficiently large n where the last inequality follows since $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$ by Exercise 2. By the comparison test, $\sum x_n$ converges, that is $\sum |z_n|$ converges.
Therefore, $\sum_{n=1}^{\infty} |z_n|$ converges iff $\sum_{n=1}^{\infty} \log(1 + |z_n|)$ converges.

Exercise 2. *Prove that* $lim_{z\to 0}\frac{\log(1+z)}{z} = 1$.

Solution. *First note that,*

$$f(z) = \frac{\log(1+z)}{z}$$

has a removable singularity at $z = 0$*, since*

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} z\frac{\log(1+z)}{z} = \lim_{z\to 0}\log(1+z) = \log(1) = 0$$

(by Theorem 1.2 Chapter 5 p. 103). We know

$$\log(1+z) \underset{p.165}{=} \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}$$

if $|z| < 1$*. Therefore,*

$$f(z) = \frac{\log(1+z)}{z} = \frac{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}}{z} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^{k-1}}{k} = 1 + \sum_{k=2}^{\infty}(-1)^{k+2}\frac{z^{k-1}}{k} = 1 + \sum_{k=1}^{\infty}(-1)^{k}\frac{z^k}{k+1}$$

where the sum is an analytic function. Hence,

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\left(1 + \sum_{k=1}^{\infty}(-1)^k\frac{z^k}{k+1}\right) = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(\lim_{z\to 0} z)^k}{k+1} = 1 + \sum_{k=1}^{\infty} 0 = 1$$

and therefore

$$lim_{z\to 0}\frac{\log(1+z)}{z} = 1.$$

Exercise 3. *Let f and g be analytic functions on a region G and show that there are analytic functions f_1, g_1, and h on G such that $f(z) = h(z)f_1(z)$ and $g(z) = h(z)g_1(z)$ for all z in G; and f_1 and g_1 have no common zeros.*

Solution. *Let f and g be analytic functions on the region G. Let $\{a_j\}$ be the zeros of f with multiplicity n_j. Let $\{b_j\}$ be the zeros of g with multiplicity $\tilde{n}_j$.*
If f and g have no common zeros, let $h = 1$. Then $f_1 = f$ and $g_1 = g$. Clearly f_1, g_1 and h are analytic on G such that $f(z) = h(z)f_1(z)$ and $g(z) = h(z)g_1(z)$ for all z in G and f_1 and g_1 have no common zeros.
Otherwise let $\{z_j\}$ be the set of common zeros of f and g with multiplicity $m_j = \min(n_j, \tilde{n}_j)$. Then $\{z_j\}$ form a sequence of distinct points in G with no limit points on G (since the zeros are isolated). Then there is an analytic function h defined on G whose only zeros are at the points z_j with multiplicity m_j (by Theorem 5.15 p. 170). Let $f_1 = \frac{f}{h}$ and $g_1 = \frac{g}{h}$, then clearly f_1 and g_1 are analytic functions on G such that $f(z) = h(z)f_1(z)$ and $g(z) = h(z)g_1(z)$ for all z in G. (since f_1 and g_1 have removable singularities at the z_j's by construction; the multiplicity of z_j's in f and g is greater or equal than the multiplicity of z_j's in h). In addition, we have that f_1 and g_1 have no common zeros by construction.

Exercise 4. *(a) Let* $0 < |a| < 1$ *and* $|z| \leq r < 1$*; show that*

$$\left|\frac{a + |a|z}{(1-\bar{a}z)a}\right| \leq \frac{1+r}{1-r}$$

(b) Let $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ and $\sum(1 - |a_n|) < \infty$. Show that the infinite product

$$B(z) = \prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \bar{a}_n z}\right)$$

converges in $H(B(0;1))$ and that $|B(z)| \leq 1$. What are the zeros of B? ($B(z)$ is called a Blaschke Product.)
(c) Find a sequence $\{a_n\}$ in $B(0;1)$ such that $\sum(1-|a_n|) < \infty$ and every number $e^{i\theta}$ is a limit point of $\{a_n\}$.

Solution. *Not available.*

Exercise 5. *Discuss the convergence of the infinite product $\prod_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 0$.*

Solution. *Not available.*

Exercise 6. *Discuss the convergence of the infinite products $\prod \left[1 + \frac{i}{n}\right]$ and $\prod \left|1 + \frac{i}{n}\right|$.*

Solution. *First consider $\prod_{n=1}^{\infty} \left(1 + \frac{i}{n}\right)$. We claim that this product does not converge.*
Let $z_n := 1 + \frac{i}{n} = r_n e^{i\theta_n}$ with $r_n = \sqrt{1 + \frac{1}{n^2}}$ and $\theta = \arctan\left(\frac{1}{n}\right)$. If there is $z = re^{i\theta}$ such that $\prod_{n=1}^{\infty} z_n = z$, then $\prod_{n=1}^{\infty} r_n = r$ and $\sum_{n=1}^{\infty} \theta_n = \theta$.
Since $\frac{\arctan(x)}{x} \geq \frac{1}{2}$ for all values $x \in (0,1]$ the estimate

$$\sum_{n=1}^{\infty} \theta_n = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right) \geq \frac{1}{2} \sum_{n_1}^{\infty} \frac{1}{n}$$

is valid. The right hand side does not converge, therefore the angles do not converge and there is no $z \in \mathbb{C}$ with the desired properties (note that $r_n > 1$ for all n, thus $z \neq 0$).

The second infinite product is a product of real numbers and $z_n = |1 + \frac{i}{n}| = \sqrt{1 + \frac{1}{n^2}}$. The estimate $1 + \frac{1}{n^2} \leq \left(1 + \frac{1}{n^2}\right)^2$ implies that $1 \leq z_n \leq 1 + \frac{1}{n^2}$. Thus for any integer $N \in \mathbb{N}$

$$\prod_{n=1}^{N} 1 \leq \prod_{n=1}^{N} z_n \leq \prod_{n=1}^{N} \left(1 + \frac{1}{n^2}\right)$$

where the last term is absolutely convergent by an example in class. Therefore the sequence of the partial products $\prod_{n=1}^{N} z_n$ is an increasing, bounded sequence that converges in the real numbers. With $\log(z_n) > 0$ for all n, the comparison with the sum of logarithms (Proposition 5.4) gives absolute convergence.

Exercise 7. *Show that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.*

Solution. *Consider*

$$\begin{aligned}
P_n &= \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right) = \prod_{k=2}^{n} \left(\frac{k-1}{k}\right)\left(\frac{k+1}{k}\right) \\
&= \exp\left\{\log \prod_{k=2}^{n} \left(\frac{k-1}{k}\right)\left(\frac{k+1}{k}\right)\right\} = \exp\left\{\sum_{k=2}^{n} \log\left[\left(\frac{k-1}{k}\right)\left(\frac{k+1}{k}\right)\right]\right\} \\
&= \exp\left\{\sum_{k=2}^{n} [\log(k-1) - \log(k)]\right\} + \exp\left\{\sum_{k=2}^{n} [\log(k+1) - \log(k)]\right\} \\
&= \exp\{[\log(1) - \log(n)]\} + \exp\{[-\log(2) + \log(n+1)]\} \\
&= \exp\{[-\log(2n) + \log(n+1)]\} = \exp\left\{\log\left(\frac{n+1}{2n}\right)\right\} = \frac{n+1}{2n}.
\end{aligned}$$

So,

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Exercise 8. *For which values of z do the products* $\prod(1-z^n)$ *and* $\prod(1+z^{2n})$ *converge? Is there an open set G such that the product converges uniformly on each compact subset of G? If so, give the largest such open.*

Solution. *Consider the infinite product* $\prod_{n=1}^{\infty}(1-z^n)$*. By definition and Corollary 5.6 this converges absolutely if and only if* $\sum_{n=1}^{\infty}-z^n$ *converges absolutely. From Calculus it is known that the convergence is also uniform for* $|z| \leq r < 1$ *with* $r \in [0,1)$*.*
If z is an n^{th} *root of unity for a positive integer n, then one of the factors in the infinite product becomes* 0 *and the product converges to 0, but not absolutely according to Definition 5.5. If there is no n such that z is* n^{th} *root of unity then no factor of the infinite product equals zero and there is a subsequence* $\{(1-z^{n_k}\}_k$ *such that* $|1-z^{n_k}| > 1.5$*. We conclude that* $\prod_{n=1}^{\infty}(1+z^n)$ *does not converge.*
With the denseness of $\bigcup_{n\in\mathbb{N}}\{z \in \mathbb{C} : z \text{ is } n^{\text{th}} \text{ root of unity}\}$ *in the unit sphere it follows that the maximal set G such that the convergence is uniform on compact subsets is the unit disk.*
The behavior of $\prod_{n=1}^{\infty}(1+z^{2n})$ *is almost the same as the previous one. Substitute* $w := z^2$*, then the convergence is uniform for* $|w| \leq r < 1$*, so* $|z| \leq sqrtr < 1$*. Also if* $|z|$ *-and hence* $|w|$*- is greater than* 1*, then the infinite product diverges. Similar to the previous case one factor equals* 0 *if z is a* $2n^{th}$ *root of* -1*.*
The maximal region G such that the convergence is uniform on compact sets is again the open unit disk.

Exercise 9. *Use Theorem 5.15 to show there is an analytic function f on* $D = \{z : |z| < 1\}$ *which is not analytic on any open set G which properly contains D.*

Solution. *Not available.*

Exercise 10. *Suppose G is an open set and* $\{f_n\}$ *is a sequence in* $H(G)$ *such that* $f(z) = \prod f_n(z)$ *converges in* $H(G)$*. (a) Show that*

$$\sum_{k=1}^{\infty}\left[f_k'(z)\prod_{n\neq k} f_n(z)\right]$$

converges in $H(G)$ *and equals* $f'(z)$*. (b) Assume that f is not the identically zero function and let K be a compact subset of G such that* $f(z) \neq 0$ *for all z in K. Show that*

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\frac{f_n'(z)}{f_n(z)}$$

and the convergence is uniform over K.

Solution. *Not available.*

Exercise 11. *A subset* $\mathcal{J}$ *of* $H(G)$*, G a region, is an ideal iff: (i) f and g in* $\mathcal{J}$ *implies* $af+bg$ *is in* $\mathcal{J}$ *for all complex numbers a and b; (ii) f in* $\mathcal{J}$ *and g any function in* $H(G)$ *implies* fg *is in* $\mathcal{J}$*.* $\mathcal{J}$ *is called a proper ideal if* $\mathcal{J} \neq (0)$ *and* $\mathcal{J} \neq H(G)$*;* $\mathcal{J}$ *is a maximal ideal if* $\mathcal{J}$ *is a proper ideal and whenever* $\mathcal{L}$ *is an ideal with* $\mathcal{J} \subset \mathcal{L}$ *then either* $\mathcal{L} = \mathcal{J}$ *or* $\mathcal{L} = H(G)$*;* $\mathcal{J}$ *is a prime ideal if whenever f and* $g \in H(G)$ *and* $fg \in \mathcal{J}$ *then either* $f \in \mathcal{J}$ *or* $g \in \mathcal{J}$*. If* $f \in H(G)$ *let* $\mathcal{Z}(f)$ *be the set of zeros of f counted according to their multiplicity. So* $\mathcal{Z}((z-a)^3) = \{a,a,a\}$*. If* $\mathcal{S} \subset H(G)$ *then* $\mathcal{Z}(\mathcal{S}) = \cap\{\mathcal{Z}(f) : f \in \mathcal{S}\}$*, where the zeros are again counted according to their multiplicity. So if* $\mathcal{S} = \{(z-a)^3(z-b), (z-a)^2\}$ *then* $\mathcal{Z}(\mathcal{S}) = \{a,a\}$*.*
(a) If f and $g \in H(G)$ *then f divides g (in symbols,* $f|g$ *if there is an h in* $H(G)$ *such that* $g = fh$*. Show that* $f|g$ *iff* $\mathcal{Z} \subset \mathcal{Z}(g)$*.*
(b) If $\mathcal{S} \subset H(G)$ *and* $\mathcal{S}$ *contains a non-zero function then f is a greatest common divisor of* $\mathcal{S}$ *if: (i)* $f|g$ *for each g in* $\mathcal{S}$ *and (ii) whenever* $h|g$ *for each g in* $\mathcal{S}$*,* $h|f$*. In symbols,* $f = g.c.d.\mathcal{S}$*. Prove that* $f = g.c.d.\mathcal{S}$*. iff* $\mathcal{Z}(f) = \mathcal{Z}(\mathcal{S})$ *and show that each non-empty subset of* $H(G)$ *has a g.c.d.*
(c) If $A \subset G$ *let* $\mathcal{J}(a) = \{f \in H(G) : \mathcal{Z}(f) \supset A\}$*. Show that* $\mathcal{J}(A)$ *is a closed ideal in* $H(G)$ *and* $\mathcal{J}(A) = (0)$

iff A has a limit point in G.
(d) Let $a \in G$ and $\mathcal{J} = \mathcal{J}(\{a\})$. Show that $\mathcal{J}$ is a maximal ideal.
(e) Show that every maximal ideal in $H(G)$ is a prime ideal.
(f) Give an example of an ideal which is not a prime ideal.

Solution. *Not available.*

Exercise 12. *Find an entire function f such that $f(n + in) = 0$ for every integer n (positive, negative or zero). Give the most elementary example possible (i.e., choose the p_n to be as small as possible).*

Solution. *Not available.*

Exercise 13. *Find an entire function f such that $f(m + in) = 0$ for all possible integers m, n. Find the most elementary solution possible.*

Solution. *Not available.*

7.6 Factorization of the sine function

Exercise 1. *Show that* $\cos \pi z = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2}\right]$.

Solution. *We know by the double-angle identity of sine* $\sin(2z) = 2\sin(z)\cos(z)$ *(this is proved easily by using the definition) or* $\sin(2\pi z) = 2\sin(\pi z)\cos(\pi z)$. *Since, we know* $\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$, *we obtain*

$$\begin{aligned} & \sin(2\pi z) = 2\sin(\pi z)\cos(\pi z) \\ \Longleftrightarrow\quad & 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z) \\ \Longleftrightarrow\quad & 2\pi z \prod_{m=1}^{\infty}\left(1 - \frac{4z^2}{(2m)^2}\right)\prod_{m=1}^{\infty}\left(1 - \frac{4z^2}{(2m-1)^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z) \end{aligned}$$

where the last statement follows by splitting the product into a product of the even and odd terms (rearrangement of the terms is allowed). Hence

$$\begin{aligned} & 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n)^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z) \\ \Longleftrightarrow\quad & 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z). \end{aligned}$$

Thus,

$$\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

Exercise 2. *Find a factorization for* $\sinh z$ *and* $\cosh z$.

Solution. *We know*

$$\begin{aligned}
\sinh(z) &= \frac{e^z - e^{-z}}{2} = \frac{e^{-i(iz)} - e^{-(-i)(iz)}}{2} \\
&= \frac{-e^{i(iz)} + e^{-i(iz)}}{2} = i\frac{-e^{i(iz)} + e^{-i(iz)}}{2i} \\
&= -i\frac{e^{i(iz)} - e^{-i(iz)}}{2i} = -i\sin(iz) \\
&= -i(iz)\prod_{n=1}^{\infty}\left(1 - \frac{(iz)^2}{n^2\pi^2}\right) = z\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)
\end{aligned}$$

since by p. 175 Equation 6.2

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \Rightarrow \sin(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Therefore,

$$\sinh(z) = z\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right).$$

We have

$$\begin{aligned}
\cosh(z) &= \frac{e^z + e^{-z}}{2} = \frac{e^{-i(iz)} + e^{-(-i)(iz)}}{2} \\
&= \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \cos(iz) \\
&= \prod_{n=1}^{\infty}\left(1 - \frac{4(iz)^2}{(2n-1)^2\pi^2}\right) = \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{(2n-1)^2\pi^2}\right)
\end{aligned}$$

since by Exercise 1 p. 176

$$\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right) \Rightarrow \cos(z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2\pi^2}\right).$$

Therefore,

$$\cosh(z) = \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{(2n-1)^2\pi^2}\right).$$

Exercise 3. *Find a factorization of the function* $\cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right)$.

Solution. *Not available.*

Exercise 4. *Prove Wallis's formula:* $\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}$.

Solution. *Not available.*

7.7 The gamma function

Exercise 1. *Show that* $0 < \gamma < 1$. *(An approximation to* γ *is* .57722. *It is unknown whether* γ *is rational or irrational.)*

Solution. *Not available.*

Exercise 2. *Show that* $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ *for z not an integer. Deduce from this that* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Solution. *Assume z is not an integer, then by Gauss's Formula p. 178 we get*

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)} \lim_{m\to\infty} \frac{m!m^{1-z}}{(1-z)(2-z)\cdots(m+1-z)} \\
&= \lim_{n\to\infty} \frac{n!n!n^{z+1-z}}{z(1+z)(1-z)(2+z)(2-z)\cdots(n+z)(n-z)(n+1-z)} \\
&= \lim_{n\to\infty} \frac{(n!)^2 n}{z(1^2-z^2)(2^2-z^2)\cdots(n^2-z^2)(n+1-z)} \\
&= \lim_{n\to\infty} \frac{(n!)^2 n}{z(n!n!)\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{2^3}\right)\cdots\left(1-\frac{z^2}{n^2}\right)(n+1-z)} \\
&= \lim_{n\to\infty} \frac{1}{z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots\left(1-\frac{z^2}{n^2}\right)\frac{n+1-z}{n}} \\
&= \frac{1}{z\lim_{n\to\infty}\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots\left(1-\frac{z^2}{n^2}\right)\underbrace{\lim_{n\to\infty}\frac{n+1-z}{n}}_{=1}} \\
&= \frac{1}{z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} = \frac{\pi}{\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} \\
&= \frac{\pi}{\sin(\pi z)} = \pi\csc(\pi z)
\end{aligned}$$

where the step before the last step follows by p. 175 Equation 6.2. So

$$\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$$

for z not an integer. Now, let $z = \frac{1}{2}$, *then*

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \pi\csc\left(\frac{\pi}{2}\right)$$

which implies

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi\cdot 1} = \sqrt{\pi}.$$

Exercise 3. *Show:* $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$. *(Hint: Consider the function* $\Gamma(z)\Gamma(z+\frac{1}{2})\Gamma(2z)^{-1}$.*)*

Solution. *Following the hint define a function* f *on its domain* $G = \{z \in \mathbb{C} : z \notin (\mathbb{Z} - \mathbb{N})\}$ *by*

$$f(z) = \frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z}{2}+\frac{1}{2}\right).$$

Let x be a positive real number. It suffices to show that $f(x)$ satisfies the Bohr-Mollerup Theorem. By the logarithmic convexity of Γ is follows that

$$\log f(x) = (x-1)\log 2 - \log\sqrt{\pi} + \log\Gamma\left(\frac{x}{2}\right) + \log\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)$$

is a convex function. Also f satisfies the functional equation of the Gamma-function:

$$\begin{aligned} f(x+1) &= \frac{2^x}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\Gamma\left(\frac{x}{2}+1\right) \\ &= \frac{2^x}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\frac{x}{2}\Gamma\left(\frac{x}{2}\right) \\ &= \frac{x2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\Gamma\left(\frac{x}{2}\right) \\ &= xf(x). \end{aligned}$$

Lastly, also $f(1) = \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} = 1$; thus by Theorem 7.13, f agrees with Γ on the positive real numbers. With the Identity Theorem then f agrees with Γ on the whole domain G.

Exercise 4. *Show that $\log\Gamma(z)$ is defined for z in $\mathbb{C} - (-\infty, 0]$ and that*

$$\log\Gamma(z) = -\log z - \gamma z - \sum_{n=1}^{\infty}\left[\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right].$$

Solution. *Not available.*

Exercise 5. *Let f be analytic on the right half plane $Re\, z > 0$ and satisfy: $f(1) = 1$, $f(z+1) = zf(z)$, and $\lim_{n\to\infty}\frac{f(z+n)}{n^z f(n)} = 1$ for all z. Show that $f = \Gamma$.*

Solution. *Not available.*

Exercise 6. *Show that*

$$\Gamma(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t}t^{z-1}\,dt$$

for $z \neq 0, -1, -2, \ldots$ (not for $Re\, z > 0$ alone).

Solution. *Write $\Gamma(z) = \Phi(z) + \Psi(z)$ where*

$$\Phi(z) = \int_0^1 e^{-t}t^{z-1}\,dt$$

and

$$\Psi(z) = \int_1^{\infty} e^{-t}t^{z-1}\,dt. \tag{7.11}$$

We can write

$$\begin{aligned} \Phi(z) &= \int_0^1 e^{-t}t^{z-1}\,dt = \int_0^1 \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}t^{z-1}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 t^n t^{z-1}\,dt \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{t^{n+z}}{n+z}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(z+n)} \end{aligned}$$

Thus,

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt \tag{7.12}$$

Claim 1: $\Gamma(z)$ *given by (7.12) is the analytic continuation of (7.11), that is* $\Gamma(z)$ *given by (7.12) is defined for all* $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$.
Proof of Claim 1: We know from the book that $\Psi(z)$ *is analytic for* $Re\,(z) > 0$.
Claim 2: $\Psi(z)$ *is analytic for* $Re\,(z) \le 0$. *Thus* $\Psi(z)$ *is analytic on* $\mathbb{C}$.
Proof of Claim 2: Assume $Re\,(z) \le 0$. *Then*

$$|t^{z-1}| = t^{Re(z)-1} \underset{t\in[1,\infty), Re(z)\le 0}{\le} t^{-1}.$$

But since $e^{-\frac{1}{2}t} t^{Re\,(z)-1} \to 0$ *as* $t \to \infty$, *there exists a constant* $C > 0$ *such that* $t^{Re(z)-1} \le Ce^{\frac{1}{2}}$ *when* $t \ge 1$. *Hence, we have*

$$|e^{-t} t^{z-1}| \le |e^{-t}| \cdot |t^{z-1}| = e^{-t} t^{Re(z)-1} \le e^{-t} C e^{\frac{1}{2}t} = Ce^{-\frac{1}{2}t}$$

and therefore $Ce^{-\frac{1}{2}t}$ *is integrable on* $(1, \infty)$. *By Fubini's Theorem for any* $\{\gamma\} \subset G = \{z : Re(z) \le 0\}$,

$$\int_{\gamma} \int_1^{\infty} e^{-t} t^{z-1}\, dt\, dz = \int_1^{\infty} \int_{\gamma} e^{-t} t^{z-1}\, dz\, dt = 0$$

which implies

$$\int_1^{\infty} e^{-t} t^{z-1}\, dt \in H(G).$$

In summary,

$$\Psi(z) = \int_1^{\infty} e^{-t} t^{z-1}\, dt \in H(\mathbb{C}).$$

Thus, Claim 2 is proved.
It remains to show that

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}$$

is analytic on $\mathbb{C} - \{0, -1, -2, \ldots\}$. *Note that* $\Phi(z)$ *is uniformly and absolutely convergent as a series in any closed domain which contains none of the points* $0, -1, -2, \ldots$ *and thus provides the analytic continuation of* $\Phi(z)$. *Since*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt$$

is analytic and we know (Theorem 7.15 p. 180)

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$$

for $Re(z) > 0$, *we get*

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt$$

is the analytic continuation of (7.11) for $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$.

Exercise 7. *Show that*

$$\int_0^\infty \sin(t^2)\,dt = \int_0^\infty \cos(t^2) = \frac{1}{2}\sqrt{\frac{1}{2}\pi}.$$

Solution. *We have shown that* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ *(see Exercise 2 p. 185). Thus*

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}\,dt = 2a\int_0^\infty e^{-a^2t^2}\,dt$$

(because of Theorem 7.15 p. 180). The last step involves integrating a complex integral. Hence, we have

$$\int_0^\infty e^{-a^2t^2}\,dt = \frac{\sqrt{\pi}}{2a}. \tag{7.13}$$

Let $a = \frac{1-i}{\sqrt{2}}$, then $a^2 = \frac{1-2i+i^2}{2} = -i$ and therefore

$$\begin{aligned}
\int_0^\infty e^{-\left(\frac{1-i}{\sqrt{2}}\right)^2 t^2}\,dt &= \int_0^\infty e^{-(-i)t^2}\,dt = \int_0^\infty e^{it^2}\,dt \\
&= \int_0^\infty \cos(t^2) + i\sin(t^2)\,dt = \int_0^\infty \cos(t^2)\,dt + i\int_0^\infty \sin(t^2)\,dt \\
&\underset{(7.13)}{=} \frac{\sqrt{\pi}}{2\frac{1-i}{\sqrt{2}}} = \frac{\sqrt{\pi}}{\sqrt{2}}\frac{1+i}{(1-i)(1+i)} = \frac{1}{2}\sqrt{\frac{1}{2}}\pi(1+i) = \frac{1}{2}\sqrt{\frac{1}{2}\pi} + i\frac{1}{2}\sqrt{\frac{1}{2}\pi}.
\end{aligned}$$

Thus,

$$\int_0^\infty \cos(t^2)\,dt + i\int_0^\infty \sin(t^2)\,dt = \frac{1}{2}\sqrt{\frac{1}{2}\pi} + i\frac{1}{2}\sqrt{\frac{1}{2}\pi}$$

which implies

$$\int_0^\infty \cos(t^2)\,dt = \frac{1}{2}\sqrt{\frac{1}{2}\pi}$$

and

$$\int_0^\infty \sin(t^2) = \frac{1}{2}\sqrt{\frac{1}{2}\pi}.$$

Exercise 8. *Let $u > 0$ and $v > 0$ and express $\Gamma(u)\Gamma(v)$ as a double integral over the first quadrant of the plane. By changing to polar coordinates show that*

$$\Gamma(u)\Gamma(v) = 2\Gamma(u+v)\int_0^{\frac{\pi}{2}} (\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta.$$

The function

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

is called the beta function. By changes of variables show that

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt$$

Can this be generalized to the case when u and v are complex numbers with positive real part?

Solution. *Let $u > 0$ and $v > 0$, then by Theorem 7.15 p. 180 and three changes of variables (indicated by "CoV") we obtain*

$$\begin{aligned}
\Gamma(u)\Gamma(v) &= \int_0^\infty e^{-s}s^{u-1}\,ds\int_0^\infty e^{-t}t^{v-1}\,dt\\
&\underset{CoV:s=x^2,t=y^2}{=} 2\int_0^\infty e^{-x^2}x^{2u-2}x\,dx\cdot 2\int_0^\infty e^{-y^2}y^{2v-2}y\,dy\\
&= 4\int_0^\infty\int_0^\infty e^{-x^2-y^2}x^{2u-1}y^{2v-1}\,dx\,dy\\
&\underset{CoV:x=r\cos(\theta),y=r\sin\theta}{=} 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r^{2u-1}(\cos\theta)^{2u-1}r^{2v-1}(\sin\theta)^{2v-1}r\,dr\,d\theta\\
&= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r^{2u+2v-1}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}r\,dr\,d\theta\\
&= 4\int_0^\infty e^{-r^2}r^{2u+2v-1}\,dr\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta\\
&\underset{CoV:t=r^2}{=} 4\frac{1}{2}\int_0^\infty e^{-t}t^{u+v-1}\,dt\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta\\
&= 2\Gamma(u+v)\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta.
\end{aligned}$$

This implies

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}=2\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta.$$

Next, we show

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt.$$

We have

$$\begin{aligned}
B(u,v) &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}=2\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\,d\theta\\
&\underset{CoV:t=\cos^2\theta}{=} 2\int_0^1\left(\sqrt{t}\right)^{2u-2}\left(\sqrt{1-t}\right)^{2v-2}\,dt\\
&= \int_0^1 t^{u-1}(1-t)^{v-1}\,dt.
\end{aligned}$$

Hence,

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt.$$

Finally, we show

$$B(u,v)=\int_0^\infty\frac{t^{u-1}}{(1+t)^{u+v}}\,dt.$$

We have

$$\begin{aligned}
B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}\ dt \\
&\underset{CoV:s=\frac{t}{1-t}}{=} \int_0^\infty \left(\frac{s}{1+s}\right)^{u-1}\left(1-\frac{s}{1+s}\right)^{v-1}(1+s)^{-2}\ ds \\
&= \int_0^\infty s^{u-1}\frac{1}{(1+s)^{u-1}}\frac{1}{(1+s)^{v-1}}(1+s)^{-2}\ ds \\
&= \int_0^\infty s^{u-1}\frac{1}{(1+s)^{u+v-2}}(1+s)^{-2}\ ds \\
&= \int_0^\infty \frac{s^{u-1}}{(1+s)^{u+v}}\ ds.
\end{aligned}$$

Hence,

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt.$$

Yes, the beta function can be generalized to the case when u and v are complex numbers with positive real part. We have seen that

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt = 2\int_0^{\pi/2}(\cos\theta)^{2u-1}(\sin\theta)^{2v-1}\ d\theta$$

for $u > 0$ and $v > 0$. Thus it remains to show that

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt \in H(G\times G)$$

where $G\times G \subset \mathbb{C}\times\mathbb{C}$ with $G = \{z : Re(z) > 0\}$.
Let $Re(u) > 0$ and $Re(v) > 0$.

$$\int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt = \underbrace{\int_0^1 \frac{t^{u-1}}{(1+t)^{u+v}}\ dt}_{I} + \underbrace{\int_1^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt}_{II}.$$

I) We have

$$\begin{aligned}
\int_0^1 |t^{u-1}(1+t)^{-u-v}|\ dt &\leq \int_0^1 |t^{u-1}|\cdot|(1+t)^{-u-v}|\ dt \\
&\leq \int_0^1 t^{Re(u)-1}(1+t)^{-Re(u)-Re(v)}\ dt \\
&\leq \int_0^1 t^{Re(u)-1} < \infty
\end{aligned}$$

where the last step follows since $Re(u) > 0$ and the previous one since $Re(u) > 0$ and $Re(v) > 0$ and $t \in (0,1)$. This implies $|t^{u-1}(1+t)^{-u-v}|$ is integrable on $(0,1)$. By Fubini's Theorem for any $\{\gamma\} \subset G\times G$

$$\int_\gamma\int_0^1 \frac{t^{u-1}}{(1+t)^{u+v}}\ dt\ dz = \int_0^1\int_\gamma \frac{t^{u-1}}{(1+t)^{u+v}}\ dz\ dt = 0$$

which implies
$$\int_0^1 \frac{t^{u-1}}{(1+t)^{u+v}}\ dt \in H(G\times G).$$
II) We have
$$\begin{aligned}
\int_1^\infty |t^{u-1}(1+t)^{-u-v}|\ dt &\leq \lim_{n\to\infty}\int_1^n |t^{u-1}(1+t)^{-u-v}|\ dt\\
&\leq \lim_{n\to\infty}\int_1^n |t^{u-1}|\cdot|(1+t)^{-u-v}|\ dt\\
&= \lim_{n\to\infty}\int_1^n t^{Re(u)-1}(1+t)^{-Re(u)-Re(v)}\ dt\\
&= \lim_{n\to\infty}\int_1^n t^{Re(u)-1}\frac{1}{t^{Re(u)+Re(v)}\left(\frac{1}{t}+1\right)^{Re(u)+Re(v)}}\ dt\\
&= \lim_{n\to\infty}\int_1^n t^{-1-Re(u)}\frac{1}{\left(\frac{1}{t}+1\right)^{Re(u)+Re(v)}}\ dt\\
&\leq \lim_{n\to\infty}\int_1^n \frac{1}{t^{1+Re(v)}}\ dt < \infty
\end{aligned}$$
where the last step follows since $Re(v) > 0$ and the previous one since $Re(u)+Re(v) > 0$, $t \geq 1$ and therefore $\frac{1}{(\frac{1}{t}+1)^{Re(u)+Re(v)}} \leq 1$. This implies $|t^{u-1}(1+t)^{-u-v}|$ is integrable on $(1,\infty)$. By Fubini's Theorem for any $\{\gamma\}\subset G\times G$
$$\int_\gamma\int_1^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt\ dz = \int_1^\infty\int_\gamma \frac{t^{u-1}}{(1+t)^{u+v}}\ dz\ dt = 0$$
which implies
$$\int_1^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt \in H(G\times G).$$
In summary
$$\int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\ dt \in H(G\times G).$$
Exercise 9. *Let α_n be the volume of the ball of radius one in $\mathbb{R}^n$ ($n \geq 1$). Prove by induction and iterated integrals that*
$$\alpha_n = 2\alpha_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\ dt$$
Solution. *Define the n-dimensional ball with radius r by*
$$B_n(r) = \{x\in\mathbb{R}^n : |x|\leq r\}$$
and define the volume of $B_n(r)$ by $V_n(r)$. Clearly, $V_1(r) = 2r$, $V_2(r) = \pi r^2$, $V_3(r) = \frac{4}{3}\pi r^3$ and therefore $V_1(1) = 2$, $V_2(1)$ and $V_3 = \frac{4}{3}\pi$.
Claim: For arbitrary n we have
$$V_n(r) = r^n V_n(1)$$

which implies

$$V_n(r) = r^n V_n(1) \underset{(def)}{=} r^n \alpha_n \tag{7.14}$$

Assume the claim is true, then we write

$$B_n(1) = \{x \in \mathbb{R}^n : |x| \le 1\} = \{(x_1, x_2, \ldots, x_{n-1}, t) \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + t^2 \le 1\}.$$

For a fixed $t \in [-1, 1]$ we have the balls

$$B_{n-1}\left(\sqrt{1-t^2}\right) = \{(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : x_1^2 + x_2^2 + \ldots + x_{n-1}^2 \le 1\}$$

and by (7.14) we obtain

$$V_{n-1}\left(\sqrt{1-t^2}\right) = \left(\sqrt{1-t^2}\right)^{n-1} V_{n-1}(1) = \left(1-t^2\right)^{\frac{n-1}{2}} \alpha_{n-1}.$$

Clearly, the ball $B_n(1)$ is the union of all disjoint balls $B_{n-1}\left(\sqrt{1-t^2}\right)$ as t varies over $[-1, 1]$. Hence

$$\alpha_n = V_n(1) = \int_{-1}^{1} \left(1-t^2\right)^{\frac{n-1}{2}} \alpha_{n-1}\, dt = 2\alpha_{n-1} \int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}}\, dt$$

which proves the formula above.
Finally, we prove the claim $V_n(r) = r^n V_n(1)$. We have by using a symmetry argument

$$\begin{aligned}
V_n(r) &= \int_{-r}^{r} \int_{-\sqrt{r^2-x_1^2}}^{\sqrt{r^2-x_1^2}} \int_{-\sqrt{r^2-x_1^2-x_2^2}}^{\sqrt{r^2-x_1^2-x_2^2}} \cdots \int_{-\sqrt{r^2-\sum_{i=1}^{n-1} x_i^2}}^{\sqrt{r^2-\sum_{i=1}^{n-1} x_i^2}} 1\, dx_n\, dx_{n-1} \cdots dx_2\, dx_1 \\
&= 2^n \int_0^r \int_0^{\sqrt{r^2-x_1^2}} \int_0^{\sqrt{r^2-x_1^2-x_2^2}} \cdots \int_0^{\sqrt{r^2-\sum_{i=1}^{n-1} x_i^2}} 1\, dx_n\, dx_{n-1} \cdots dx_2\, dx_1 \\
&\underset{CoV: x_i = r y_i}{=} 2^n \int_0^r \int_0^{\sqrt{1-y_1^2}} \int_0^{\sqrt{1-y_1^2-y_2^2}} \cdots \int_0^{\sqrt{1-\sum_{i=1}^{n-1} y_i^2}} 1 \cdot \underbrace{r^n}_{Jacobian}\, dy_n\, dy_{n-1} \cdots dy_2\, dy_1 \\
&= r^n \int_{-1}^{1} \int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} \int_{-\sqrt{1-y_1^2-y_2^2}}^{\sqrt{1-y_1^2-y_2^2}} \cdots \int_{-\sqrt{1-\sum_{i=1}^{n-1} y_i^2}}^{\sqrt{1-\sum_{i=1}^{n-1} y_i^2}} 1\, dy_n\, dy_{n-1} \cdots dy_2\, dy_1 \\
&= r^n V_n(1).
\end{aligned}$$

This implies $V_n(r) = r^n V_n(1)$. (Note that this is kind of obvious, since we scale the unit ball in $\mathbb{R}^n$ by r in each dimension to obtain the ball with radius r in $\mathbb{R}^n$). We could have proved the formula by mathematical induction. Using spherical coordinates simplifies the computation if preferred.

Exercise 10. *Show that*

$$\alpha_n = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}$$

where α_n is defined in problem 9. Show that if $n = 2k$, $k \ge 1$, then $\alpha_n = \pi^k / k!$

Solution. *Not available.*

Exercise 11. *The Gaussian psi function is defined by*

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

(a) Show that Ψ *is meromorphic in* $\mathbb{C}$ *with simple poles at* $z = 0, -1, \ldots$ *and* $Res(\Psi; -n) = -1$ *for* $n \geq 0$.
(b) Show that $\Psi(1) = -\gamma$.
(c) Show that $\Psi(z+1) - \Psi(z) = \frac{1}{z}$.
(d) Show that $\Psi(z) - \Psi(1-z) = -\pi \cot \pi z$.
(e) State and prove a characterization of Ψ *analogous to the Bohr-Mollerup Theorem.*

Solution. *Not available.*

7.8 The Riemann zeta function

Exercise 1. *Let* $\xi(z) = z(z-l)\pi^{-\frac{1}{2}z}\zeta(z)\Gamma(\frac{1}{2}z)$ *and show that* ξ *is an entire function which satisfies the functional equation* $\xi(z) = \xi(1-z)$.

Solution. *The function* ξ *is analytic on* $\mathbb{C} - (\{-2k : k \in \mathbb{N}_0\} \cup \{1\})$ *as a product of analytic functions. The factors* $z, z-1$ *and* $\pi^{-\frac{1}{2}z}$ *are entire. The zeta-function has a simple pole at* $z = 1$, *hence* $\lim_{z\to 1}(z-1)\zeta(z)$ *exists in* $\mathbb{C}$ *and the function* ξ *is well-defined at* $z = 1$.
The function $\Gamma\left(\frac{1}{2}z\right)$ *has simple poles at the non-positive even integers. For* $z = 0, \lim_{z\to 0} z\Gamma\left(\frac{1}{2}z\right) \in \mathbb{C}$. *For negative even integers* $z = 2k, -k \in \mathbb{N}$ *the poles of Gamma coincide with the simple zeros of the zeta-function, hence* $\lim_{z\to 2k} \zeta(z)\Gamma\left(\frac{1}{2}z\right)$ *exists in* $\mathbb{C}$ *also. Therefore the function* ξ *is analytic at each point.*
The functional equation of ξ *is related to the functional equation of* ζ *(8.3, p. 192), evaluated at* $1-z$, *i.e.*

$$\zeta(1-z) = 2(2\pi)^{-z}\Gamma(z)\zeta(z)\sin\left(\frac{1}{2}\pi(1-z)\right). \tag{7.15}$$

Furthermore the proof uses variants of problems 1, 2 of the homework set 5, in particular

$$\Gamma\left(\frac{1}{2} - \frac{z}{2}\right)\Gamma\left(\frac{1}{2} + \frac{z}{2}\right) = \pi \csc\left(\frac{1}{2}\pi(1-z)\right) \tag{7.16}$$

$$\sqrt{\pi}\Gamma(z) = 2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1}{2} + \frac{z}{2}\right). \tag{7.17}$$

Then the following chain of equalities holds.

$$\begin{aligned}
\xi(1-z) &= (1-z)(-z)\pi^{-\frac{1}{2}+\frac{z}{2}}\zeta(1-z)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)\\
&\overset{(7.15)}{=} z(z-1)\pi^{-\frac{1}{2}+\frac{z}{2}}\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)2^{1-z}\pi^{-z}\zeta(z)\Gamma(z)\sin\left(\frac{1}{2}\pi(1-z)\right)\\
&= \xi(z)\pi^{-\frac{1}{2}}2^{1-z}\frac{\Gamma(z)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\sin\left(\frac{1}{2}\pi(1-z)\right)\\
&\overset{(7.16)}{=} \xi(z)\pi^{-\frac{1}{2}}2^{1-z}\frac{\Gamma(z)}{\Gamma\left(\frac{z}{2}\right)}\frac{\pi\csc\left(\frac{1}{2}\pi(1-z)\right)}{\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)}\sin\left(\frac{1}{2}\pi(1-z)\right)\\
&= \xi(z)\pi^{\frac{1}{2}}2^{1-z}\frac{\Gamma(z)}{\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)}\\
&\overset{(7.17)}{=} \xi(z)\pi^{\frac{1}{2}}2^{1-z}\frac{1}{\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)}\frac{2^{z-1}}{\pi^{\frac{1}{2}}}\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)\\
&= \xi(z).
\end{aligned}$$

Exercise 2. *Use Theorem 8.17 to prove that* $\sum p_n^{-1} = \infty$. *Notice that this implies that there are an infinite number of primes.*

Solution. *Euler's Theorem asserts that for real values* $x > 1$, $\zeta(x) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-x}}\right)$, *where* p_n *are the prime numbers. The left hand side is unbounded as* $x \searrow 1$ *and so is the right hand side.*
Now the argument goes as follows. Seeking contradiction suppose that $\sum_{n=1}^{\infty}\frac{1}{p_n}$ *is finite. All summands are positive, therefore the sum converges absolutely. By Proposition 5.4, p.165, also* $\sum_{n=1}^{\infty}\log\left(1-\frac{1}{p_n}\right)$ *converges absolutely. But this implies that*

$$-\sum_{n=1}^{\infty}\log\left(1-\frac{1}{p_n}\right) = \log\prod_{n=1}^{\infty}\left(\frac{1}{1-p_n}\right) = \log\zeta(1) < \infty$$

which gives the desired contradiction because the $\log(\zeta(z))$ *has a simple pole at* $z = 1$.

Exercise 3. *Prove that* $\zeta^2(z) = \sum_{n=1}^{\infty}\frac{d(n)}{n^z}$ *for Re* $z > 1$, *where* $d(n)$ *is the number of divisors of* n.

Solution. *Not available.*

Exercise 4. *Prove that* $\zeta(z)\zeta(z-1) = \sum_{n=1}^{\infty}\frac{\sigma(n)}{n^z}$, *for Re* $z > 1$, *where* $\sigma(n)$ *is the sum of the divisors of* n.

Solution. *Not available.*

Exercise 5. *Prove that* $\frac{\zeta(z-1)}{\zeta(z)} = \sum_{n=1}^{\infty}\frac{\varphi(n)}{n^z}$ *for Re* $z > 1$, *where* $\varphi(n)$ *is the number of integers less than* n *and which are relatively prime to* n.

Solution. *Not available.*

Exercise 6. *Prove that* $\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^z}$ *for Re* $z > 1$, *where* $\mu(n)$ *is defined as follows. Let* $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ *be the factorization of* n *into a product of primes* $p_1, \ldots, p_m$ *and suppose that these primes are distinct. Let* $\mu(1) = 1$*; if* $k_1 = \ldots = k_m = 1$ *then let* $\mu(n) = (-1)^m$*; otherwise let* $\mu(n) = 0$.

Solution. *Not available.*

Exercise 7. *Prove that* $\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}$ *for Re* $z > 1$, *where* $\Lambda(n) = \log p$ *if* $n = p^m$ *for some prime* p *and* $m \geq 1$*; and* $\Lambda(n) = 0$ *otherwise.*

Solution. *Not available.*

Exercise 8. *(a) Let* $\eta(z) = \zeta'(z)/\zeta(z)$ *for Re* $z > 1$ *and show that* $\lim_{z \to z_0}(z - z_0)\eta(z)$ *is always an integer for Re* $z_0 \geq 1$*. Characterize the point* z_0 *(in its relation to* ζ*) in terms of the sign of this integer.*
(b) Show that for $\epsilon > 0$

$$Re\ \eta(1 + \epsilon + it) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-1(1+\epsilon)} \cos(t \log n)$$

where $\Lambda(n)$ *is defined in Exercise 7.*
(c) Show that for all $\epsilon > 0$,

$$3Re\ \eta(1 + \epsilon) + 4Re\ \eta(1 + \epsilon + it) + Re\ \eta(1 + \epsilon + 2it) \leq 0.$$

(d) Show that $\zeta(z) \neq 0$ *if Re* $z = 1$ *(or* 0*).*

Solution. *Not available.*

Chapter 8

Runge's Theorem

8.1 Runge's Theorem

Exercise 1. *Prove Corollary 1.14 if it is only assumed that E^- meets each component of $\mathbb{C}_\infty - G$.*

Solution. *Not available.*

Exercise 3. *Let G be the open unit disk $B(0;1)$ and let $K = \{z : \frac{1}{4} \le |z| \le \frac{3}{4}\}$. Show that there is a function f analytic on some open subset G_1 containing K which cannot be approximated on K by functions in $H(G)$.*

Remarks. The next two problems are concerned with the following question. Given a compact set K contained in an open set $G_1 \subset G$, can functions in $H(G_1)$ be approximated on K by functions in $H(G)$? Exercise 2 says that for an arbitrary choice of K, G, and G_1 this is not true. Exercise 4 below gives criteria for a fixed K and G such that this can be done for any G_1. Exercise 3 is a lemma which is useful in proving Exercise 4.

Solution. *Not available.*

Exercise 3. *Let K be a compact subset of the open set G and suppose that any bounded component D of $G - K$ has $D^- \cap \partial G \neq \square$. Then every component of $\mathbb{C}_\infty - K$ contains a component of $\mathbb{C}_\infty - G$.*

Solution. *Not available.*

Exercise 4. *Let K be a compact subset of the open set G; then the following are equivalent:*
(a) If f is analytic in a neighborhood of K and $\epsilon > 0$ then there is a g in $H(G)$ with $|f(z) - g(z)| < \epsilon$ for all z in K;
(b) If D is a bounded component of $G - K$ then $D^- \cap \partial G \neq \square$;
(c) If z is any point in $G - K$ then there is a function f in $H(G)$ with

$$|f(z)| > \sup\{|f(w)| : w \text{ in } K\}.$$

Solution. *Not available.*

Exercise 5. *Can you interpret part (c) of Exercise 4 in terms of $\hat{K}$?*

Solution. *Not available.*

Exercise 6. *Let K be a compact subset of the region G and define $\hat{K}_G = \{z \in G : |f(z)| \le \|f\|_K$ for all f in $H(G)\}$.*
(a) Show that if $\mathbb{C}_\infty - G$ is connected then $\hat{K}_G = \hat{K}$.
(b) Show that $d(K, \mathbb{C} - G) = d(\hat{K}_G, \mathbb{C} - G)$.
(c) Show that $\hat{K}_G \subset$ the convex hull of $K \equiv$ the intersection of all convex subsets of $\mathbb{C}$ which contain K.
(d) If $\hat{K}_G \subset G_1 \subset G$ and G_1 is open then for every g in $H(G_1)$ and $\epsilon > 0$ there is a function f in $H(G)$ such that $|f(z) - g(z)| < \epsilon$ for all z in $\hat{K}_G$. (Hint: see Exercise 4.)
(e) $\hat{K}_G =$ the union of K and all bounded components of $G - K$ whose closure does not intersect ∂G.

Solution. *Not available.*

8.2 Simple connectedness

Exercise 1. *The set $G = \{re^{it} : -\infty < t < 0$ and $1 + e^t < r < 1 + 2e^t\}$ is called a cornucopia. Show that G is simply connected. Let $K = G^-$; is int$\hat{K}$ connected?*

Solution. *he result follows once it is established that every closed piecewise smooth curve in G is 0-homotopic. This requires the following subclaim: let $z \in G$ then the argument of z is uniquely defined. Suppose otherwise then there are real r_1, r_2, t_1, t_2 such that $z = r_1 e^{it_1} = r_2 e^{it_2}$. Suppose WLOG that $t_1 < t_2 < 0$ then there is a positive integer k such that $t_1 = 2k\pi = t_2$ and therefore $r_2 > 1+e^{t_2} = 1+e^{t_1}e^{k\pi} > 1+2e^{t_1} > r_1$ contradicting the fact that $|z| = r_1 = r_2$.*
Define

$$R : (-\infty, 0) \to G; \qquad R(t) = 1 + \frac{3}{2}e^t.$$

Let $\gamma : [0,1] \to G$ be an arbitrary closed piecewise smooth curve in G with radius function $\rho(s)$ and argument function $\tau(s)$ such that

$$\gamma(s) = \rho(s)e^{i\tau(s)},$$

and define

$$\begin{aligned}
&\Gamma : [0,1] \times [0,1] \to G,\\
&\Gamma(s,0) = \gamma(s),\\
&\Gamma(s,1) = R(\tau(s)),\\
&\Gamma(0,u) = \Gamma(1,u) \qquad \forall\, u \in [0,1].
\end{aligned}$$

Also define a function

$$\begin{aligned}
&\Theta : [0,1] \times [0,1] \to G,\\
&\Theta(s,0) = R(\tau(s)),\\
&\Theta(s,1) = R\left(\tau\left(\frac{1}{2}\right)\right),\\
&\Theta(0,u) = \Theta(1,u) \qquad \forall\, u \in [0,1].
\end{aligned}$$

The function $\Theta \circ \Gamma$ satisfies Definition IV 6.1 (p.88), the curve γ is homotopic to the point $R\left(\tau\left(\frac{1}{2}\right)\right)$ and by Definition IV 6.14 (p.93) the set G is simply connected.

Exercise 2. *If K is polynomially convex, show that the components of the interior of K are simply connected.*

Solution. *Not available.*

8.3 Mittag-Leffler's Theorem

Exercise 1. *Let G be a region and let $\{a_n\}$ and $\{b_m\}$ be two sequences of distinct points in G without limit points in G such that an $a_n \neq b_m$ for all n, m. Let $S_n(z)$ be a singular part at a_n and let p_m be a positive integer. Show that there is a meromorphic function f on G whose only poles and zeros are $\{a_n\}$ and $\{b_m\}$ respectively, the singular part at $z = a_n$ is $S_n(z)$, and $z = b_m$ is a zero of multiplicity p_m.*

Solution. *Let G be a region and let $\{b_m\}$ be a sequence of distinct points in G with no limit point in G; and let $\{p_m\}$ be a sequence of integers. By Theorem 5.15 p.170 there is an analytic function g defined on G whose only zeros are at the points b_m; furthermore, b_m is a zero of g of multiplicity p_m.*
Since $g \in H(g)$ and $\{a_n\} \in G$, g has a Taylor series in a neighborhood $B(a_n; R_n)$ of each a_n, that is

$$g_n(z) = \sum_{k=0}^{\infty} \alpha_k (z - a_n)^k \in B(a_n; R_n)$$

where $\alpha_k = \frac{1}{k!} g^{(k)}(a_n)$. Goal: Try to use this series to create a singular part $r_n(z)$ at a_n such that $r_n(z) g_n(z) = s_n(z)$ or

$$r_n(z) \sum_{k=0}^{\infty} \alpha_k (z - a_n)^k = \sum_{j=1}^{m_n} \frac{A_{jn}}{(z - a_n)^j} \iff \sum_{j=1}^{m_n} \frac{A_{jn}}{(z - a_n)^j} = \sum_{k=0}^{\infty} \alpha_k r_n(z)(z - a_n)^k.$$

Claim:

$$r_n(z) = \sum_{j=1}^{m_n} B_{jk} (z - a_n)^{-j-k} \tag{8.1}$$

works.
Proof of the claim:

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha_k r_n(z)(z - a_n)^k &= \sum_{k=0}^{\infty} \alpha_k \sum_{j=1}^{m_n} B_{jk} (z - a_n)^{-j-k} (z - a_n)^k \\ &= \sum_{k=0}^{\infty} \alpha_k \sum_{j=1}^{m_n} B_{jk} (z - a_n)^{-j} \\ &= \sum_{j=1}^{m_n} (z - a_n)^{-j} \sum_{k=0}^{\infty} \alpha_k B_{jk} = \sum_{j=1}^{m_n} \frac{A_{jn}}{(z - a_n)^j} \end{aligned}$$

where the last step follows by choosing

$$\sum_{k=0}^{\infty} \alpha_k B_{jk} = A_{jn}.$$

Since G is a region, G is open. Let $\{a_n\}$ be a sequence of distinct points without a limit point in G and such that $a_n \neq b_m$ for all n, m. Let $\{r_n(z)\}$ be the sequence of rational functions given by

$$r_n(z) = \sum_{j=1}^{m_n} \frac{B_{jk}}{(z - a_n)^{j+k}}$$

(see (8.1)). By Mittag-Leffler's Theorem, there is a meromorphic function h on G whose poles are exactly the points $\{a_n\}$ and such that the singular part of h at a_n is $r_n(z)$.
Set $f = g \cdot h$. Then by construction f is the meromorphic function on G whose only poles and zeros are $\{a_n\}$ and $\{b_m\}$ respectively, the singular part at $z = a_n$ is $S_n(z)$, and $z = b_m$ is a zero of multiplicity p_m. (Note that the zeros do not cancel the poles since by assumption $a_n \neq b_m$ $\forall$ n, m).

Exercise 2. *Let $\{a_n\}$ be a sequence of points in the plane such that $|a_n| \to \infty$, and let $\{b_n\}$ be an arbitrary sequence of complex numbers.*
(a) Show that if integers $\{k_n\}$ can be chosen such that

$$\sum_{k=n}^{\infty} \left(\frac{r}{a_n}\right)^{k_n} \frac{b_n}{a_n} \qquad (3.4)$$

converges absolutely for all $r > 0$ then

$$\sum_{k=n}^{\infty} \left(\frac{r}{a_n}\right)^{k_n} \frac{b_n}{z - a_n} \qquad (3.5)$$

converges in $M(\mathbb{C})$ to a function f with poles at each point $z = a_n$.
(b) Show that if $\limsup |b_n| < \infty$ then (3.4) converges absolutely if $k_n = n$ for all n.
(c) Show that if there is an integer k such that the series

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n^{k+1}} \qquad (3.6)$$

converges absolutely, then (3.4) converges absolutely if $k_n = k$ for all n.
(d) Suppose there is an $r > 0$ such that $|a_n - a_m| \geq r$ for all $n \neq m$. Show that $\sum |a_n|^{-3} < \infty$. In particular, if the sequence $\{b_n\}$ is bounded then the series (3.6) with $k = 2$ converges absolutely. (This is somewhat involved and the reader may prefer to prove part (f) directly since this is the only application.)
(e) Show that if the series (3.5) converges in $M(\mathbb{C})$ to a meromorphic function f then

$$f(z) = \sum_{n=1}^{\infty} \left[\frac{b_n}{z - a_n} + \frac{b_n}{a_n}\left\{1 + \left(\frac{z}{a_n}\right) + \ldots + \left(\frac{z}{a_n}\right)^{k_n - 1}\right\}\right]$$

(f) Let ω and ω' be two complex numbers such that $Im(\omega'/\omega) \neq 0$. Using the previous parts of this exercise show that the series

$$\zeta(z) = \frac{1}{z} + \sum{}' \left(\frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2}\right),$$

where the sum is over all $w = 2n\omega + 2n'\omega$ for n, $n = 0, \pm 1, \pm 2, \ldots$ but not $w = 0$, is convergent in $M(\mathbb{C})$ to a meromorphic function ζ with simple poles at the points $2n\omega + 2n'\omega'$. This function is called the Weierstrass zeta function.
(g) Let $\mathcal{P}(z) = -\zeta'(z)$; $\mathcal{P}$ is called the Weierstrass pe function. Show that

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum{}' \left(\frac{1}{(z - w)^2} - \frac{1}{w^2}\right)$$

where the sum is over the same w as in part (f). Also show that

$$\mathcal{P}(z) = \mathcal{P}(z + 2n\omega + 2n'\omega')$$

for all integers n and n'. That is, $\mathcal{P}$ is doubly periodic with periods 2ω and $2\omega'$.

Solution. *Not available.*

Exercise 3. *This exercise shows how to deduce Weierstrass's Theorem for the plane (Theorem VII. 5.12) from Mittag-Leffler's Theorem.*
(a) Deduce from Exercises 2(a) and 2(b) that for any sequence $\{a_n\}$ in $\mathbb{C}$ with $\lim a_n = \infty$ and $a_n \neq 0$ there is a sequence of integers $\{k_n\}$ such that

$$h(z) = \sum_{n=1}^{\infty} \left[\frac{1}{z - a_n} + \frac{1}{a_n} + \frac{1}{a_n}\left(\frac{z}{a_n}\right) + \ldots + \frac{1}{a_n}\left(\frac{z}{a_n}\right)^{k_n - 1} \right]$$

is a meromorphic function on $\mathbb{C}$ with simple poles at $a_1, a_2, \ldots$. The remainder of the proof consists of showing that there is a function f such that $h = f'/f$. This function f will then have the appropriate zeros.
(b) Let z be an arbitrary but fixed point in $\mathbb{C} - \{a_1, a_2, \ldots\}$. Show that if γ_1 and γ_2 are any rectifiable curves in $\mathbb{C} - \{a_1, a_2, \ldots\}$ from 0 to z and h is the function obtained in part (a), then there is an integer m such that

$$\int_{\gamma_1} h - \int_{\gamma_2} h = 2\pi i m.$$

(c) Again let h be the meromorphic function from part (a). Prove that for $z \neq a_1, a_2, \ldots$ and γ any rectifiable curve in $\mathbb{C} - \{a_1, a_2, \ldots\}$,

$$f(z) = \exp\left(\int_{\gamma} h\right)$$

defines an analytic function on $\mathbb{C} - \{a_1, a_2, \ldots\}$ with $f'/f = h$. (That is, the value of $f(z)$ is independent of the curve γ and the resulting function f is analytic.
(d) Suppose that $z \in \{a_1, a_2, \ldots\}$; show that z is a removable singularity of the function f defined in part (c). Furthermore, show that $f(z) = 0$ and that the multiplicity of this zero equals the number of times that z appears in the sequence $\{a_1, a_2, \ldots\}$.
(e) Show that

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left[\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{2}\left(\frac{z}{a_n}\right)^{k_n}\right] \qquad (3.7)$$

Remark. We could have skipped parts (b), (c), and (d) and gone directly from (a) to (e). However this would have meant that we must show that (3.7) converges in $H(\mathbb{C})$ and it could hardly be classified as a new proof. The steps outlined in parts (a) through (d) give a proof of Weierstrass's Theorem without introducing infinite products.

Solution. *Not available.*

Exercise 4. *This exercise assumes a knowledge of the terminology and results of Exercise VII. 5.11.*
(a) Define two functions f and g in $H(G)$ to be relatively prime (in symbols, $(f, g) = 1$) if the only common divisors of f and g are non-vanishing functions in $H(G)$. Show that $(f, g) = 1$ iff $\mathcal{Z}(f) \cap \mathcal{Z}(g) = \square$.
(b) If $(f, g) = 1$, show that there are functions f_1, g_1 in $H(G)$ such that $f f_1 + g g_1 = 1$ (Hint: Show that there is a meromorphic function φ on G such that $f_1 = \varphi g \in H(G)$ and $g | (1 - f f_1)$.)
(c) Let $f_1, \ldots, f_n \in H(G)$ and $g = g.c.d\,\{f_1, \ldots, f_n\}$. Show that there are functions $\varphi_1, \ldots, \varphi_n$ in $H(G)$ such that $g = \varphi_1 f_1 + \ldots + \varphi_n f_n$. (Hint: Use (b) and induction.)
(d) If $\{\mathcal{J}_\alpha\}$ is a collection of ideals in $H(G)$, show that $\mathcal{J} = \bigcap_\alpha \mathcal{J}_\alpha$ is also an ideal. If $\mathcal{S} \subset H(G)$ then let $\mathcal{J} = \cap\{\mathcal{S} : \mathcal{S}$ is an ideal of $H(G)$ and $\mathcal{S} \subset \mathcal{J}\}$. Prove that $\mathcal{J}$ is the smallest ideal in $H(G)$ that contains $\mathcal{S}$ and $\mathcal{J} = \{\varphi_1 f_1 + \ldots + \varphi_n f_n : \varphi_k \in H(G), f_k \in \mathcal{S}$ for $1 \leq k \leq n\}$. $\mathcal{J}$ is called the ideal generated by $\mathcal{S}$ and is denoted by $\mathcal{J} = (\mathcal{S})$. If $\mathcal{S}$ is finite then $(\mathcal{S})$ is called a finitely generated ideal. If $\mathcal{S} = \{f\}$ for a single function f then (f) is called a principal ideal.
(e) Show that every finitely generated ideal in $H(G)$ is a principal ideal.

(f) An ideal $\mathcal{J}$ is called a fixed ideal if $\mathcal{Z}(\mathcal{J}) \neq \square$; otherwise it is called a free ideal. Prove that if $\mathcal{J} = (\mathcal{S})$ then $\mathcal{Z}(\mathcal{J}) = \mathcal{Z}(\mathcal{S})$ and that a proper principal ideal is fixed.
(g) Let $f_n(z) = \sin(2^{-n}z)$ for all $n \geq 0$ and let $\mathcal{J} = (\{f_1, f_2, \ldots\})$. Show that $\mathcal{J}$ is a fixed ideal in $H(\mathbb{C})$ which is not a principal ideal.
(h) Let $\mathcal{J}$ be a fixed ideal and prove that there is an f in $H(G)$ with $\mathcal{Z}(f) = \mathcal{Z}(\mathcal{J})$ and $\mathcal{J} \subset (f)$. Also show that $\mathcal{J} = (f)$ if $\mathcal{J}$ is finitely generated.
(i) Let $\mathcal{M}$ be a maximal ideal that is fixed. Show that there is a point a in G such that $\mathcal{M} = ((z - a))$.
(j) Let $\{a_n\}$ be a sequence of distinct points in G with no limit point in G. Let $\mathcal{J} = \{f \in H(G) : f(a_n) = 0$ for all but a finite number of the $a_n\}$. Show that $\mathcal{J}$ is a proper free ideal in $H(G)$.
(k) If $\mathcal{J}$ is a free ideal show that for any finite subset $\mathcal{S}$ of $\mathcal{J}$, $\mathcal{Z}(\mathcal{S}) \neq \square$. Use this to show that $\mathcal{J}$ can contain no polynomials.
l) Let $\mathcal{J}$ be a proper free ideal; then $\mathcal{J}$ is a maximal ideal iff whenever $g \in H(G)$ and $\mathcal{Z}(g) \cap \mathcal{Z}(f) \neq \square$ for all f in $\mathcal{J}$ then $g \in \mathcal{J}$.

Solution. *Not available.*

Exercise 5. *Let G be a region and let $\{a_n\}$ be a sequence of distinct points in G with no limit point in G. For each integer $n \geq 1$ choose integers $k_n \geq 0$ and constants $A_n^{(k)}$, $0 \leq k \leq k_n$. Show that there is an analytic function f on G such that $f^{(k)}(a_n) = k!A_n^{(k)}$. (Hint: Let g be an analytic function on G with a zero at a_n of multiplicity $k_n + 1$. Let h be a meromorphic function on G with poles at each an of order $k_n + 1$ and with singular part $S_n(z)$. Choose the S_n so that $f = gh$ has the desired property.)*

Solution. *Not available.*

Exercise 6. *Find a meromorphic function with poles of order 2 at $1, \sqrt{2}, \sqrt{3}, \ldots$ such that the residue at each pole is 0 and $\lim (z - \sqrt{n})^2 f(z) = 1$ for all n.*

Solution. *We claim that the function*

$$f(z) = \sum_{k=1}^{\infty} \frac{3\sqrt{n}z^2 - 2z^3}{n^{3/2}(z - \sqrt{n})^2}$$

has the desired properties. f has poles of order 2 at $z = \sqrt{n}, n \in \mathbb{N}$ (and no other poles) and it is analytic for every $z \in \mathbb{C}$ that is not one of the poles. Moreover f converges in $M(\mathbb{C})$. To see this let $r > 0$ and choose $N \in \mathbb{N}$ so large that $\sqrt{n} > 2r$ for all $n \geq N$. Then for $|z| \leq r$ and $n \in \mathbb{N}$

$$\left|\frac{3\sqrt{n}z^2 - 2z^3}{n^{3/2}(z - \sqrt{n})^2}\right| \leq \frac{r^2(3\sqrt{n} + \sqrt{n})}{n^{3/2}(\sqrt{n} - r)^2} \leq \frac{16r^2}{n^2} =: M_n.$$

This is the majorant independent of z for the Weierstrass' Criterion.
Next fix $m \in \mathbb{N}$ and consider the limit in equation $\lim_{z \to \sqrt{n}} (z - \sqrt{n})^2 f(z) = 1$,

$$\lim_{z \to \sqrt{m}} (z - \sqrt{m})^2 f(z) = \lim_{z \to \sqrt{m}} (z - \sqrt{m})^2 \sum_{n=1}^{\infty} \frac{3\sqrt{n}z^2 - 2z^3}{n^{3/2}(z - \sqrt{n})^2}$$

$$\begin{aligned}
&= \lim_{z \to \sqrt{m}} \sum_{n=1}^{m-1} \frac{(z - \sqrt{m})(3\sqrt{n}z^2 - 2z^3)}{n^{3/2}(z - \sqrt{n})^2} + \frac{3m^{3/2} - 2m^{3/2}}{m^{3/2}} \\
&\quad + \lim_{z \to \sqrt{m}} \sum_{m+1}^{\infty} \frac{(z - \sqrt{n})(3\sqrt{n}z^2 - 2z^3)}{n^{3/2}(z - \sqrt{n})^2} \\
&= 0 + 1 + 0 = 1.
\end{aligned}$$

Lastly we have to check that the residuals are 0 *at the poles. By Proposition V 2.4, p.113, for a function* $g(z) := (z - \sqrt{m})^2 f(z)$*, it has to be verified that* $g'(z)|_{z=\sqrt{m}} = 0$.
First find the derivative of f *as*

$$f'(z) = \sum_{n=1}^{\infty} \frac{6\sqrt{n}z^2 - 6nz - 2z^3}{n^{3/2}(z - \sqrt{m})^3}$$

then compute the residuum at $z = \sqrt{m}$

$$\operatorname{Res}(f, \sqrt{m}) = g'(z)|_{z=\sqrt{m}} = 2(z - \sqrt{m})f(z) + (z - \sqrt{m})^2 f'(z)\Big|_{z=\sqrt{m}}$$

$$= \Bigg[\sum_{n=1}^{\infty} \frac{2(z - \sqrt{m})(3\sqrt{n}z^2 - 2z^3)}{n^{3/2}(z - \sqrt{n})^2} + \sum_{n=1}^{\infty} \frac{(z - \sqrt{m})^2(6\sqrt{n}z^2 - 6nz - 2z^3)}{n^{3/2}(z - \sqrt{m})^3}\Bigg]_{z=\sqrt{m}}$$

$$= \Bigg[\sum_{n=1}^{\infty} \Big(2(z - \sqrt{(n)})(3\sqrt{n}z^2 - 2z^3) + (z - \sqrt{m})(6\sqrt{n}z^2) - 6nz - 2z^3)\Big) \cdot \frac{z - \sqrt{m}}{n^{3/2}(z - \sqrt{n})^3}\Bigg]_{z=\sqrt{m}}$$

If $n \neq m$ *and* $z = \sqrt{m}$*, the last factor equals zero, in the case* $n = m$*,* $(z - \sqrt{m})^2$ *cancels directly and we are left with*

$$\frac{12\sqrt{m} - 2z^3 - 6mz}{n^{3/2}(z - \sqrt{m})}\Bigg|_{z=m} = \frac{-6z(z - \sqrt{m})}{n^{3/2}}\Bigg|_{z=m} = 0.$$

The residuum vanishes for all $\sqrt{m}, m \in \mathbb{N}$ *and therefore the function considered in this exercise has all the required properties.*

Chapter 9

Analytic Continuation and Riemann Surfaces

9.1 Schwarz Reflection Principle

Exercise 1. *Let γ be a simple closed rectifiable curve with the property that there is a point a such that for all z on γ the line segment $[a, z]$ intersects $\{\gamma\}$ only at z; i.e. $[a, z] \cap \{\gamma\} = \{z\}$. Define a point w to be inside γ if $[a, w] \cap \{\gamma\} = \square$ and let G be the collection of all points that are inside γ.*
(a) Show that G is a region and $G^- = G \cup \{\gamma\}$.
(b) Let $f : G^- \to \mathbb{C}$ be a continuous function such that f is analytic on G. Show that $\int_\gamma f = 0$.
(c) Show that $n(\gamma; z) = \pm 1$ if z is inside γ and $n(\gamma; z) = 0$ if $z \notin G^-$.
Remarks. It is not necessary to assume that γ has such a point a as above; each part of this exercise remains true if γ is only assumed to be a simple closed rectifiable curve. Of course, we must define what is meant by the inside of γ. This is difficult to obtain. The fact that a simple closed curve divides the plane into two pieces (an inside and an outside) is the content of the Jordan Curve Theorem. This is a very deep result of topology.

Solution. *Not available.*

Exercise 2. *Let G be a region in the plane that does not contain zero and let G^* be the set of all points z such that there is a point w in G where z and w are symmetric with respect to the circle $|\xi| = 1$. (See III. 3.17.)*
(a) Show that $G^* = \{z : (1/\overline{z}) \in G\}$.
(b) If $f : G \to \mathbb{C}$ is analytic, define $f^* : G^* \to \mathbb{C}$ by $f^*(z) = \overline{f(1/\overline{z})}$. Show that f^* is analytic.
(c) Suppose that $G = G^*$ and f is an analytic function defined on G such that $f(z)$ is real for z in G with $|z| = 1$. Show that $f = f^*$.
(d) Formulate and prove a version of the Schwarz Reflection Principle where the circle $|\xi| = 1$ replaces $\mathbb{R}$. Do the same thing for an arbitrary circle.

Solution. *Not available.*

Exercise 3. *Let G, G_+, G_-, G_0 be as in the statement of the Schwarz Reflection Principle and let $f : G_+ \cup G_0 \to \mathbb{C}_\infty$ be a continuous function such that f is meromorphic on G_+. Also suppose that for x in G_0 $f(x) \in \mathbb{R}$. Show that there is a meromorphic function $g : G \to \mathbb{C}_\infty$ such that $g(z) = f(z)$ for z in $G_+ \cup G_0$. Is it possible to allow f to assume the value ∞ on G_0?*

Solution. *Not available.*

9.2 Analytic Continuation Along a Path

Exercise 1. *The collection $\{D_0, D_1, \ldots, D_n\}$ of open disks is called a chain of disks if $D_{j-1} \cap D_j \neq \square$ for $1 \leq j \leq n$. If $\{(f_j, D_j) : 0 \leq j \leq n\}$ is a collection of function elements such that $\{D_0, D_1, \ldots, D_n\}$ is a chain of disks and $f_{j-1}(z) = f_j(z)$ for z in $D_{j-1} \cap D_j$, $1 \leq j \leq n$; then $\{(f_j, D_j) : 0 \leq j \leq n\}$ is called an analytic continuation along a chain of disks. We say that (f_n, D_n) is obtained by an analytic continuation of (f_0, D_0) along a chain of disks.*
(a) Let $\{(f_j, D_j) : 0 \leq j \leq n\}$ be an analytic continuation along a chain of disks and let a and b be the centers of the disks D_0 and D_n respectively. Show that there is a path γ from a to b and an analytic continuation $\{(g_t, B_t)\}$ along γ such that $\{\gamma\} \subset \bigcup_{j=0}^{n} D_j$, $[f_0]_a = [g_0]_a$ and $[f_n]_b = [g_1]_b$.
(b) Conversely, let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along a path $\gamma : [0, 1] \to \mathbb{C}$ and let $a = \gamma(0)$, $b = \gamma(1)$. Show that there is an analytic continuation along a chain of disks $\{(g_j, B_j) : 0 \leq j \leq n\}$ such that $\{\gamma\} \subset \bigcup_{j=0}^{n} B_j$, $[f_0]_a = [g_0]_a$ and $[f_1]_b = [g_n]_b$.

Solution. *Not available.*

Exercise 2. *Let $D_0 = B(1; 1)$ and let f_0 be the restriction of the principal branch of $\sqrt{z}$ to D_0. Let $\gamma(t) = \exp(2\pi it)$ and $\sigma(t) = \exp(4\pi it)$ for $0 \leq t \leq 1$.*
(a) Find an analytic continuation $\{(f_t, D_t) : 0 \leq t \leq 1\}$ of (f_0, D_0) along γ and show that $[f_1]_1 = [-f_0]_1$.
(b) Find an analytic continuation $\{(g_t, B_t) : 0 \leq t \leq 1\}$ of (f_0, D_0) along σ and show that $[g_1]_1 = [g_0]_1$.

Solution. *Not available.*

Exercise 3. *Let f be an entire function, $D_0 = B(0; 1)$, and let γ be a path from 0 to b. Show that if $\{(f_t, D_t) : 0 \leq t \leq 1\}$ is a continuation of (f, D_0) along γ then $f_1(z) = f(z)$ for all z in D_1 (This exercise is rather easy; it is actually an exercise in the use of the terminology.)*

Solution. *Not available.*

Exercise 4. *Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. Show that $\{(f_t', D_t) : 0 \leq t \leq 1\}$ is also a continuation along γ.*

Solution. *Not available.*

Exercise 5. *Suppose $\gamma : [0, 1] \to \mathbb{C}$ is a closed path with $\gamma(0) = \gamma(1) = a$ and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ such that $[f_1]_a = [f_0']_a$ and $f_0 \neq 0$. What can be said about (f_0, D_0)?*

Solution. *Not available.*

Exercise 6. *Let $D_0 = B(1; 1)$ and let f_0 be the restriction to D_0 of the principal branch of the logarithm. For an integer n let $\gamma(t) = \exp(2\pi int)$, $0 \leq t \leq 1$. Find a continuation $\{(f_t, D_t) : 0 \leq t \leq 1\}$ along γ of (f_0, D_0) and show that $[f_1]_1 = [f_0 + 2\pi in]$.*

Solution. *Not available.*

Exercise 7. *Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. Suppose G is a region such that $f_t(D_t) \subset G$ for all t, and suppose there is an analytic function $h : G \to \mathbb{C}$ such that $h(f_0(z)) = z$ for all z in D_0. Show that $h(f_t(z)) = z$ for all z in D_t and for all t. Hint: Show that $T = \{t : h(f_t(z)) = z$ for all z in $D_t\}$ is both open and closed in $[0, 1]$.*

Solution. *Not available.*

Exercise 8. *Let $\gamma : [0,1] \to \mathbb{C}$ be a path with $\gamma(0) = 1$ and $\gamma(t) \neq 0$ for any t. Suppose that $\{(f_t, D_t) : 0 \leq t \leq 1\}$ is an analytic continuation of $f_0(z) = \log z$. Show that each f_t is a branch of the logarithm.*

Solution. *Not available.*

9.3 Monodromy Theorem

Exercise 1. *Prove that the set T defined in the proof of Lemma 3.2 is closed.*

Solution. *Not available.*

Exercise 2. *Let (f, D) be a function element and let $a \in D$. If $\gamma : [0,1] \to \mathbb{C}$ is a path with $\gamma(0) = a$ and $\gamma(1) = b$ and $\{(f_t, D_t) : 0 \leq t \leq 1\}$ is an analytic continuation of (f, D) along γ, let $R(t)$ be the radius of convergence of the power series expansion of f_t at $z = \gamma(t)$.*
(a) Show that $R(t)$ is independent of the choice of continuation. That is, if a second continuation $\{(g_t, B_t)\}$ along γ is given with $[g_0]_a = [f]_a$ and $r(t)$ is the radius of convergence of the power series expansion of g_t about $z = \gamma(t)$ then $r(t) = R(t)$ for all t.
(b) Suppose that $D = B(1;1)$, f is the restriction of the principal branch of the logarithm to D, and $\gamma(t) = 1 + at$ for $0 \leq t \leq 1$ and $a > 0$. Find $R(t)$.
(c) Let (f, D) be as in part (b), let $0 < a < 1$ and let $\gamma(t) = (1 - at)\exp(2\pi it)$ for $0 \leq t \leq 1$. Find $R(t)$.
(d) For each of the functions $R(t)$ obtained in parts (b) and (c), find $\min\{R(t) : 0 \leq t \leq 1\}$ as a function of a and examine the behavior of this function as $a \to \infty$ or $a \to 0$.

Solution. *Not available.*

Exercise 3. *Let $\Gamma : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function such that $\Gamma(0,u) = a$, $\Gamma(1,u) = b$ for all u. Let $\gamma_u(t) = \Gamma(t,u)$ and suppose that $\{(f_{t,u}, D_{t,u}) : 0 \leq t \leq 1\}$ is an analytic continuation along γ_u such that $[f_{0,u}]_a = [f_{0,v}]_a$ for all u and v in $[0,1]$. Let $R(t,u)$ be the radius of convergence of the power series expansion of $f_{t,u}$ about $z = \Gamma(t,u)$. Show that either $R(t,u) \equiv \infty$ or $R : [0,1] \times [0,1] \to (0,\infty)$ is a continuous function.*

Solution. *Not available.*

Exercise 4. *Use Exercise 3 to give a second proof of the Monodromy Theorem.*

Solution. *Not available.*

9.4 Topological Spaces and Neighborhood Systems

Exercise 1. *Prove the propositions which were stated in this section without proof.*

Solution. *Not available.*

Exercise 2. *Let $(X, \mathcal{T})$ and $(\Omega, \mathcal{S})$ be topological spaces and let $Y \subset X$. Show that if $f : X \to \Omega$ is a continuous function then the restriction of f to Y is a continuous function of $(Y, \mathcal{T}_Y)$ into $(\Omega, \mathcal{S})$.*

Solution. *Not available.*

Exercise 3. *Let X and Ω be sets and let $\{\mathcal{N}_x : x \in X\}$ and $\{\mathcal{M}_\omega : \omega \in \Omega\}$ be neighborhood systems and let $\mathcal{T}$ and $\mathcal{S}$ be the induced topologies on X and Ω respectively.*
(a) Show that a function $f : X \to \Omega$ is continuous iff when $x \in X$ and $\omega = f(x)$, for each Δ in $\mathcal{M}_\omega$ there is a U in $\mathcal{N}_x$ such that $f(U) \subset \Delta$.
(b) Let $X = \Omega = \mathbb{C}$ and let $\mathcal{N}_z = \mathcal{M}_z = \{B(z;\epsilon) : \epsilon > 0\}$ for each z in $\mathbb{C}$. Interpret part (a) of this exercise for this particular situation.

Solution. *Not available.*

Exercise 4. *Adopt the notation of Exercise 3. Show that a function $f : X \to \Omega$ is open iff for each x in X and U in $\mathcal{N}_x$ there is a set Δ in $\mathcal{M}_\omega$ (where $\omega = f(x)$) such that $\Delta \subset f(U)$.*

Solution. *Not available.*

Exercise 5. *Adopt the notation of Exercise 3. Let $Y \subset X$ and define $\mathcal{U}_Y = \{Y \cap U : U \in \mathcal{N}_Y\}$ for each y in Y. Show that $\{\mathcal{U}_Y : y \in Y\}$ is a neighborhood system for Y and the topology it induces on Y is $\mathcal{T}_Y$.*

Solution. *Not available.*

Exercise 6. *Adopt the notation of Exercise 3. For each point (x, ω) in $X \times \Omega$ let*

$$\mathcal{U}_{(x,\omega)} = \{U \times \Delta : U \in \mathcal{N}_x, \Delta \in \mathcal{M}_\omega\}$$

(a) Show that $\{\mathcal{U}_{(x,\omega)} : (x, \omega) \in X \times \Omega\}$ is a neighborhood system on $X \times \Omega$ and let $\mathcal{P}$ be the induced topology on $X \times \Omega$.
(b) If $U \in \mathcal{T}$ and $\Delta \in \mathcal{S}$, call the set $U \times \Delta$ an open rectangle. Prove that a set is in $\mathcal{P}$ iff it is the union of open rectangles.
(c) Define $p_1 : X \times \Omega \to X$ and $p_2 : X \times \Omega \to X$ by $p_1(x, \omega) = x$ and $p_2(x, \omega) = \omega$. Show that p_1 and p_2 are open continuous maps. Furthermore if $(Z, \mathcal{R})$ is a topological space show that a function $f : (Z, \mathcal{R}) \to (X \times \Omega, \mathcal{P})$ is continuous iff $p_1 \circ f : Z \to X$ and $p_2 \circ f : Z \to \Omega$ are continuous.

Solution. *Not available.*

9.5 The Sheaf of Germs of Analytic Functions on an Open Set

Exercise 1. *Define $F : \mathcal{S}(\mathbb{C}) \to \mathbb{C}$ by $F(z, [f]_z) = f(z)$ and show that F is continuous.*

Solution. *Not available.*

Exercise 2. *Let $\mathcal{F}$ be the complete analytic function obtained from the principal branch of the logarithm and let $G = \mathbb{C} - \{0\}$. If D is an open subset of G and $f : D \to \mathbb{C}$ is a branch of the logarithm show that $[f]_a \in \mathcal{F}$ for all a in D. Conversely, if (f, D) is a function element such that $[f]_a \in \mathcal{F}$ for some a in D, show that $f : D \to \mathbb{C}$ is a branch of the logarithm. (Hint: Use Exercise 2.8.)*

Solution. *Not available.*

Exercise 3. *Let $G = \mathbb{C} - \{0\}$, let $\mathcal{F}$ be the complete analytic function obtained from the principal branch of the logarithm, and let $(\mathcal{R}, \rho)$ be the Riemann surface of $\mathcal{F}$ (so that G is the base of $\mathcal{R}$). Show that $\mathcal{R}$ is homeomorphic to the graph $\Gamma = \{(z, e^z) : z \in G\}$ considered as a subset of $\mathbb{C} \times \mathbb{C}$. (Use the map $h : \mathcal{R} \to \Gamma$ defined by $h(z, [f]_z) = (f(z), z)$ and use Exercise 2.) State and prove an analogous result for branches of $z^{1/n}$.*

Solution. *Not available.*

Exercise 4. *Consider the sheaf $\mathcal{S}(\mathbb{C})$, let $B = \{z : |z-1| < 1\}$, let l be the principal branch of the logarithm defined on B, and let $l_1(z) = l(z) + 2\pi i$ for all z in B. (a) Let $D = \{z : |z| < 1\}$ and show that $(\frac{1}{2}, [l]_{\frac{1}{2}})$ and $(\frac{1}{2}, [l_1]_{\frac{1}{2}})$ belong to the same component of $\rho^{-1}(D)$. (b) Find two disjoint open subsets of $\mathcal{S}(\mathbb{C})$ each of which contains one of the points $(\frac{1}{2}, [l]_{\frac{1}{2}})$ and $(\frac{1}{2}, [l_1]_{\frac{1}{2}})$.*

Solution. *Not available.*

9.6 Analytic Manifolds

Exercise 1. *Show that an analytic manifold is locally compact. That is, prove that if $a \in X$ and U is an open neighborhood of a then there is an open neighborhood V of a such that $V^- \subset U$ and V^- is compact.*

Solution. *Not available.*

Exercise 2. *Which of the following are analytic manifolds? What is its analytic structure if it is a manifold? (a) A cone in $\mathbb{R}^3$. (b) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1$ or $x_1^2 + x_2^2 > 1$ and $x_3 = 0\}$.*

Solution. *Not available.*

Exercise 3. *The following is a generalization of Proposition 6.3(b). Let (X, Φ) be an analytic manifold, let Ω be a topological space, and suppose there is a continuous function h of X onto Ω that is locally one-one (that is, if $x \in X$ there is an open set U such that $x \in U$ and h is one-one on U). If $(U, \varphi) \in \Phi$ and h is one-one on U let $\Delta = h(U)$ and let $\psi : \Delta \to \mathbb{C}$ be defined by $\psi(\omega) = \varphi \circ (h/U)^{-1}(\omega)$. Let Ψ be the collection of all such pairs (Δ, ψ). Prove that (Ω, Ψ) is an analytic manifold and h is an analytic function from X to Ω.*

Solution. *Not available.*

Exercise 4. *Let $T = \{z : |z| = 1\} \times \{z : |z| = 1\}$; then T is a torus. (This torus is homeomorphic to the usual hollow doughnut in $\mathbb{R}^3$.) If ω and ω' are complex numbers such that Im $(\omega/\omega') \neq 0$ then ω and ω', considered as elements of the vector space $\mathbb{C}$ over $\mathbb{R}$, are linearly independent. So each z in $\mathbb{C}$ can be uniquely represented as $z = t\omega + t'\omega'$; t, t' in $\mathbb{R}$. Define $h : \mathbb{C} \to T$ by $h(t\omega + t\omega') = (e^{2\pi it}, e^{2\pi it'})$. Show that h induces an analytic structure on T. (Use Exercise 3.) (b) If ω, ω' and ζ, ζ' are two pairs of complex numbers such that Im $(\omega/\omega') \neq 0$ and Im $(\zeta/\zeta') \neq 0$, define $\sigma(s\zeta + s'\zeta') = (e^{2\pi is}, e^{2\pi is'})$ and $\tau(t\omega + t'\omega') = (e^{2\pi it}, e^{2\pi it'})$. Let $G = \{t\omega + t'\omega' : 0 < t < 1, 0 < t' < 1\}$ and $\Omega = \{s\zeta + s'\zeta' : 0 < s < 1, 0 < s' < 1\}$; show that both σ and τ are one-one on G and Ω respectively. (Both G and Ω are the interiors of parallelograms.) If Φ_τ and Φ_σ are the analytic structures induced on T by τ and σ respectively, and if the identity map of (T, Φ_τ) into (T, Φ_σ) is analytic then show that the function $f : G \to \Omega$ defined by $f = \sigma^{-1} \circ \tau$ is analytic. (To say that the identity map of (T, Φ_τ) into (T, Φ_σ) is analytic is to say that Φ_τ and Φ_σ are equivalent structures.) (c) Let $\omega = 1$, $\omega' = i$, $\zeta = 1$, $\zeta = \alpha$ where Im $\alpha \neq 0$; define σ, τ, G, Ω and f as in part (b). Show that Φ_τ and Φ_σ are equivalent analytic structures if and only if $\alpha = i$. (Hint: Use the Cauchy-Riemann equations.) (d) Can you generalize part (c)? Conjecture a generalization?*

Solution. *Not available.*

Exercise 5. *(a) Let f be a meromorphic function defined on $\mathbb{C}$ and suppose f has two independent periods ω and ω'. That is, $f(z) = f(z + n\omega + n'\omega')$ for all z in $\mathbb{C}$ and all integers n and n', and Im $(\omega/\omega') \neq 0$. Using the notation of Exercise 4(a) show that there is an analytic function $F : T \to \mathbb{C}_\infty$ such that $f = F \circ h$. (For an example of a meromorphic function with two independent periods see Exercise VIII. 4.2(g).)*
(b) Prove that there is no non-constant entire function with two independent periods.

Solution. *Not available.*

Exercise 6. *Show that an analytic surface is arcwise connected.*

Solution. *Not available.*

Exercise 7. *Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an analytic function.*
(a) Show that either $f \equiv \infty$ or $f^{-1}(\infty)$ is a finite set.
(b) If $f \not\equiv \infty$, let $a_1, \dots, a_n$ be the points in $\mathbb{C}$ where f takes on the value ∞. Show that there are polynomials $p_0, p_1, \dots, p_n$ such that

$$f(z) = p_0(z) + \sum_{k=1}^{n} p_k\left(\frac{1}{z - a_k}\right)$$

for z in $\mathbb{C}$.
(c) If f is one-one, show that either $f(z) = az + b$ (some a, b in $\mathbb{C}$) or $f(z) = \frac{a}{z-c} + b$ (some a, b, c in $\mathbb{C}$).

Solution. *Not available.*

Exercise 8. *Furnish the details of the discussion of the surface for $\sqrt{z}$ at the end of this section.*

Solution. *Not available.*

Exercise 9. *Let $G = \left\{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}\right\}$ and define $f : G \to \mathbb{C}$ by $f(z) = \sin z$. Give a discussion for f similar to the discussion of $\sqrt{z}$ at the end of this section.*

Solution. *Not available.*

9.7 Covering spaces

Exercise 1. *Suppose that (X, ρ) is a covering space of Ω and (Ω, π) is a covering space of Y; prove that $(X, \pi \circ \rho)$ is also a covering space of Y.*

Solution. *Not available.*

Exercise 2. *Let (X, ρ) and (Y, σ) be covering spaces of Ω and Λ respectively. Define $\rho \times \sigma : X \times Y \to \Omega \times \Lambda$ by $(\rho \times \sigma)(x, y) = (\rho(x), \sigma(y))$ and show that $(X \times Y, \rho \times \sigma)$ is a covering space of $\Omega \times \Lambda$.*

Solution. *Not available.*

Exercise 3. *Let (Ω, ψ) be an analytic manifold and let (X, ρ) be a covering space of Ω. Show that there is an analytic structure Φ on X such that ρ is an analytic function from (X, Φ) to (Ω, ψ).*

Solution. *Not available.*

Exercise 4. *Let (X, ρ) be a covering space of Ω and let $\omega \in \Omega$. Show that each component of $\rho^{-1}(\omega)$ consists of a single point and $\rho^{-1}(\omega))$ has no limit points in X.*

Solution. *Not available.*

Exercise 5. *Let Ω be a pathwise connected space and let (X, ρ) be a covering space of Ω. If ω_1 and ω_2 are points in Ω, show that $\rho^{-1}(\omega_1)$ and $\rho^{-1}(\omega_2)$ have the same cardinality. (Hint: Let γ be a path in Ω from ω_1 to ω_2; if $x_1 \in \rho^{-1}(\omega_1)$ and $\tilde{\gamma}$ is the lifting of γ with initial point $x_1 = \tilde{\gamma}(0)$, let $f(x_1) = \tilde{\gamma}(1)$. Show that f is a one-one map of $\rho^{-1}(\omega_1)$ onto $\rho^{-1}(\omega_2)$.)*

Solution. *Not available.*

Exercise 6. *In this exercise all spaces are regions in the plane.*
(a) Let (G, f) be a covering of Ω and suppose that f is analytic; show that if Ω is simply connected then f is one-one. (Hint: If $f(z_1) = f(z_2)$ let γ be a path in G from z_1 to z_2 and consider a certain analytic continuation along $f \circ \gamma$; apply the Monodromy Theorem.)
(b) Suppose that (G_1, f_1) and (G_2, f_2) are coverings of the region Ω such that both f_1 and f_2 are analytic. Show that if G_1 is simply connected then there is an analytic function $f : G_1 \to G_2$ such that (G_1, f) is a covering of G_2 and $f_1 = f_2 \circ f$. That is the diagram is commutative.
(c) Let (G_1, f_1), (G_2, f_2) and Ω be as in part (b) and, in addition, assume that both G_1 and G_2 are simply connected. Show that there is a one-one analytic function mapping G_1 onto G_2.

Solution. *Not available.*

Exercise 7. *Let G and Ω be regions in the plane and suppose that $f : G \to \Omega$ is an analytic function such that (G, f) is a covering space of Ω. Show that for every region Ω_1 contained in Ω which is simply connected there is an analytic function $g : \Omega_1 \to G$ such that $f(g_1(\omega)) = \omega$ for all ω in Ω_1.*

Solution. *Not available.*

Exercise 8. *What is a simply connected covering space of the figure eight?*

Solution. *Not available.*

Exercise 9. *Give two nonhomeomorphic covering spaces of the figure eight that are not simply connected.*

Solution. *Not available.*

Exercise 10. *Prove that the closed curve in Exercise IV.6.8 is not homotopic to zero in the doubly punctured plane.*

Solution. *Not available.*

Chapter 10

Harmonic Functions

10.1 Basic properties of harmonic functions

Exercise 1. *Show that if u is harmonic then so are $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$.*

Solution. *Let u be harmonic, that is*

$$u_{xx} + u_{yy} = 0. \tag{10.1}$$

Since $u : G \to \mathbb{R}$ is harmonic, u is infinitely times differentiable by Proposition 1.3 p. 252. Thus, the third partial derivatives of u exist and are continuous (therefore $u_{xyy} = u_{yyx}$ and $u_{yxx} = u_{xxy}$). Now, we show that u_x is harmonic, that is

$$(u_x)_{xx} + (u_x)_{yy} = 0.$$

$$(u_x)_{xx} + (u_x)_{yy} = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx} = (u_{xx} + u_{yy})_x \underset{(10.1)}{=} 0_x = 0.$$

Thus, u_x is harmonic. Finally, we show that u_y is harmonic, that is

$$(u_y)_{xx} + (u_y)_{yy} = 0.$$

$$(u_y)_{xx} + (u_y)_{yy} = u_{yxx} + u_{yyy} = u_{xxy} + u_{yyy} = (u_{xx} + u_{yy})_y \underset{(10.1)}{=} 0_y = 0.$$

Thus, u_y is harmonic.

Exercise 2. *If u is harmonic, show that $f = u_x - iu_y$ is analytic.*

Solution. *Since, u is harmonic, we have*

$$u_{xx} + u_{yy} = 0. \tag{10.2}$$

In addition, u is infinitely times differentiable by Proposition 1.3 p. 252. Thus, all the partials are continuous (therefore $u_{xy} = u_{yx}$).
To show: $f = u_x - iu_y$ is analytic which is equivalent to showing
a) $Re(f) = u_x$ and $Im(f) = -u_y$ are harmonic and
b) u_x and $-u_y$ have to satisfy the Cauchy Riemann equations (by Theorem III 2.29).

a) By the previous exercise, we have seen that u_x and u_y are harmonic. Therefore, $-u_y$ is harmonic, too. Thus, part a) is proved.
b) We have to show that the Cauchy-Riemann equations are satisfied, that is

$$(u_x)_x = -(u_y)_y \quad \text{and} \quad (u_x)_y = -(-u_y)_x.$$

The first is true by (10.2) since

$$(u_x)_x = -(u_y)_y \iff u_{xx} + u_{yy} = 0.$$

The second is true since $u_{xy} = u_{yx}$. Therefore

$$(u_x)_y = -(-u_y)_x \iff u_{xy} = u_{yx} \iff u_{xy} = u_{yx}.$$

In summary, $f = u_x - iu_y$ is analytic provided u is harmonic.

Exercise 3. *Let $p(x, y) = \sum_{k,l=0}^{n} a_{kl} x^k y^l$ for all x, y in $\mathbb{R}$.*
Show that p is harmonic iff:
(a) $k(k-1)a_{k,l-2} + l(l-1)a_{k-2,l} = 0$ for $2 \leq k, l \leq n$;
(b) $a_{n-1,l} = a_{n,l} = 0$ for $2 \leq l \leq n$;
(c) $a_{k,n-1} = a_{k,n} = 0$ for $2 \leq k \leq n$.

Solution. *Not available.*

Exercise 4. *Prove that a harmonic function is an open map. (Hint: Use the fact that the connected subsets of $\mathbb{R}$ are intervals.)*

Solution. *Not available.*

Exercise 5. *If f is analytic on G and $f(z) \neq 0$ for any z show that $u = \log|f|$ is harmonic on G.*

Solution. *Not available.*

Exercise 6. *Let u be harmonic in G and suppose $\bar{B}(a; R) \subset G$. Show that*

$$u(a) = \frac{1}{\pi R^2} \iint_{\bar{B}(a;R)} u(x, y)\, dx\, dy.$$

Solution. *Let D be a disk such that $\bar{B}(a; R) \subset D \subset G$. There exist an $f \in H(D)$ such that $Re(f) = u$ on D. From Cauchy's Integral Formula, we have*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - a}\, dw = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

where $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. So

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

Multiply by r, we get

$$f(a)r = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) r\, d\theta$$

and then integrate with respect to r yields

$$\begin{aligned}
&\int_0^R f(a) r\, dr = \int_0^R \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta}) r\, d\theta\, dr \\
\iff\quad & f(a)\frac{r^2}{2} = \frac{1}{2\pi}\int_0^R\int_0^{2\pi} f(a+re^{i\theta}) r\, d\theta\, dr \\
\iff\quad & f(a) = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R f(a+re^{i\theta}) r\, dr\, d\theta \\
\iff\quad & f(a) = \frac{1}{\pi R^2}\iint_{\bar{B}(a;R)} f(x,y)\, dx\, dy.
\end{aligned}$$

where the last step follows by changing from polar coordinates to rectangular coordinates. Now

$$u(a) = Re(f(a)) = \frac{1}{\pi R^2}\iint_{\bar{B}(a;R)} Re(f(x,y))\, dx\, dy = \frac{1}{\pi R^2}\iint_{\bar{B}(a;R)} u(x,y)\, dx\, dy.$$

Thus

$$u(a) = \frac{1}{\pi R^2}\iint_{\bar{B}(a;R)} u(x,y)\, dx\, dy.$$

Exercise 7. *For* $|z| < 1$ *let*

$$u(z) = Im\left[\left(\frac{1+z}{1-z}\right)^2\right].$$

Show that u is harmonic and $\lim_{r\to 1^-} u(re^{i\theta}) = 0$ *for all* θ*. Does this violate Theorem 1.7? Why?*

Solution. *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$*. Let* $f(z) = \left(\frac{1+z}{1-z}\right)^2$*. Clearly,* $f \in H(D)$ *since* $z = 1$ *is not in D. By Theorem III 2.29 we obtain that* $v = Re(f)$ *and* $u = Im(f)$ *are harmonic. Therefore*

$$u(z) = Im\left[\left(\frac{1+z}{1-z}\right)^2\right]$$

is harmonic. Next, we will show

$$\lim_{r\to 1^-} u(re^{i\theta}) = 0$$

for all θ. We have

$$\begin{aligned}
u(re^{i\theta}) &= Im\left[\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^2\right] \\
&= Im\left[\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^2\left(\frac{1-re^{i\theta}}{1-re^{i\theta}}\right)^2\right] \\
&= Im\left[\left(\frac{(1+re^{i\theta})(1-re^{i\theta})}{(1-re^{i\theta})(1-re^{i\theta})}\right)^2\right] \\
&= Im\left[\frac{(1+re^{i\theta}-re^{-i\theta}-r^2)^2}{(1-re^{i\theta}-re^{-i\theta}+r^2)^2}\right] \\
&= Im\left[\frac{(1+2ri\sin\theta-r^2)^2}{(1-2r\cos\theta+r^2)^2}\right] \\
&= Im\left[\frac{((1-r^2)+2ri\sin\theta)^2}{(1-2r\cos\theta+r^2)^2}\right] \\
&= Im\left[\frac{(1-r^2)^2+4ri\sin\theta(1-r^2)-4r^2\sin^2\theta}{(1-2r\cos\theta+r^2)^2}\right] \\
&= \frac{4r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}.
\end{aligned}$$

Note that the denominator is zero iff $r = 1$ and $\theta = 2\pi k$, $k \in \mathbb{Z} \iff z = 1$.
If we assume, that $r = 1$ and $\theta \neq 2\pi k$, $k \in \mathbb{Z}$, then

$$\lim_{r\to 1^-} u(re^{i\theta}) = u(e^{i\theta}) = \frac{4\cdot 1\cdot(1-1^2)\sin\theta}{(1-2\cdot 1\cdot\cos\theta+1^2)^2} = \frac{0}{2^2(1-\cos\theta)^2} = 0.$$

If $\theta = 2\pi k$, $k \in \mathbb{Z}$, then

$$\lim_{r\to 1^-} u(re^{2\pi k}) = \lim_{r\to 1^-}\frac{4r(1-r^2)\sin(2\pi k)}{(1-2r\cos(2\pi k)+r^2)^2} = \lim_{r\to 1^-}\frac{0}{(1-r)^4} = 0.$$

Thus,

$$\lim_{r\to 1^-} u(re^{i\theta}) = 0.$$

This does not violate Theorem 1.7. The reason is the following: If we pick $u = u$ and $v = 0$, then the condition

$$\limsup_{z\to a} u(z) \leq 0$$

does not hold $\forall a \in \partial D$.
Claim: $\limsup_{z\to 1} u(z) > 0$.
Proof of the claim:

$$u(z) = u(x,y) = Im\left[\left(\frac{1+x+iy}{1-x-iy}\right)^2\right] = \frac{4y(1-x^2-y^2)}{(1-2x+x^2+y^2)^2}.$$

If we fix $x = 1$, then

$$u(1,y) = \frac{4y(-y^2)}{y^4} = \frac{-4y^3}{y^4} = -\frac{4}{y} \quad \forall y \neq 0.$$

But $-\frac{4}{y} \to \infty$ as $y \to 0^-$.

Exercise 8. *Let $u : G \to \mathbb{R}$ be a function with continuous second partial derivatives and define $U(r,\theta) = u(r\cos\theta, r\sin\theta)$.*
(a) Show that

$$r^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = r^2\frac{\partial^2 U}{\partial r^2} + r\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \theta^2} = r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial \theta^2}.$$

So if $0 \notin G$ then u is harmonic iff

$$r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial \theta^2} = 0.$$

(b) Let u have the property that it depends only on $|z|$ and not $\arg z$. That is, $u(z) = \varphi(|z|)$. Show that u is harmonic iff $u(z) = a\log|z| + b$ for some constants a and b.

Solution. *Not available.*

Exercise 9. *Let $u : G \to \mathbb{R}$ be harmonic and let $A = \{z \in G : u_x(z) = u_y(z) = 0\}$; that is, A is the set of zeros of the gradient of u. Can A have a limit point in G?*

Solution. *Let u be harmonic and not constant. By Exercise 1 p. 255, we know that u_x and u_y are harmonic. By exercise 2 p. 255, we know that $f = u_x - iu_y$ is analytic. Define $B = \{z \in G : f(z) = 0\} = \{z \in G : u_x(z) - iu_y(z) = 0\}$. Clearly*

$$A = \{z \in G : u_x(z) = u_y(z) = 0\} = B$$

since $u_x(z) - iu_y(z) = 0$ iff $u_x(z) = u_y(z) = 0$ ($u_x(z)$ and $u_y(z)$ are real). Since f is analytic, the zeros are isolated and therefore B cannot have a limit point in G (Corollary 3.10 p. 79). This implies that A cannot have a limit point in G.
If u is constant, then $A = B = G$ and thus A can have a limit point in G. (In this case $f \equiv 0$ and by Theorem 3.7 p. 78 A has a limit point in G).

Exercise 10. *State and prove a Schwarz Reflection Principle for harmonic functions.*

Solution. *Not available.*

Exercise 11. *Deduce the Maximum Principle for analytic functions from Theorem 1.6.*

Solution. *Not available.*

10.2 Harmonic functions on a disk

Exercise 1. *Let $D = \{z : |z| < 1\}$ and suppose that $f : D^- \to \mathbb{C}$ is a continuous function such that both $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic. Show that*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)\, dt$$

for all $re^{i\theta}$ in D. Using Definition 2.1 show that f is analytic on D iff

$$\int_{-\pi}^{\pi} f(e^{it})e^{int}\, dt = 0$$

for all $n \geq 1$.

Solution. *Let $D = \{z : |z| < 1\}$ and suppose $f = u + iv : D \to \mathbb{C}$ is a continuous function such that u and v are harmonic. Clearly u and v are continuous functions mapping D to $\mathbb{R}$. By Corollary 2.9 we have*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})\,dt \quad \text{for } 0 \le r < 1 \text{ and all } \theta \tag{10.3}$$

and

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)v(e^{it})\,dt \quad \text{for } 0 \le r < 1 \text{ and all } \theta. \tag{10.4}$$

Hence,

$$\begin{aligned} f(re^{i\theta}) &= u(re^{i\theta}) + iv(re^{i\theta}) \underset{(10.3),(10.4)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})\,dt + i\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)v(e^{it})\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\left[u(e^{it}) + iv(r^{i\theta})\right]\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})\,dt \end{aligned}$$

for $0 \le r < 1$ and all θ. Thus, we have proved

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)\,dt$$

for all $re^{i\theta}$ in D.
Next, we will prove

$$f \text{ is analytic } \iff \int_{-\pi}^{\pi} f(e^{i\theta})e^{int}\,dt = 0$$

for all $n \ge 1$. $\Rightarrow$: Let f be analytic on D and assume $n \ge 1$ (n integer). Define $g(z) = f(z)z^{n-1}$. Clearly g is analytic on D. Let $\gamma = re^{it}$, $-\pi \le t \le \pi$, $0 < r < 1$, then by Cauchy's Theorem

$$\begin{aligned} & \int_\gamma g(z)\,dz = 0 \\ \iff\ & \int_\gamma f(z)z^{n-1}\,dz = 0 \\ \iff\ & \int_{-\pi}^{\pi} f(re^{it})ire^{it}e^{it(n-1)}r^{n-1}\,dt = 0 \\ \iff\ & i\int_{-\pi}^{\pi} f(re^{it})e^{int}r^n\,dt = 0 \\ \iff\ & \int_{-\pi}^{\pi} f(re^{it})e^{int}r^n\,dt = 0 \end{aligned}$$

which implies

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi} f(re^{it})e^{int}r^n\,dt = 0 \underset{\text{by Ex 3a p. 262}}{\Rightarrow} \int_{-\pi}^{\pi} f(e^{it})e^{int}\,dt = 0 \quad \forall n \ge 1.$$

$\Leftarrow$: *Assume* $\int_{-\pi}^{\pi} f(re^{it})e^{int}r^n \, dt = 0$ *for all* $n \geq 1$. *We have for all* $z \in D$

$$\begin{aligned}
f(z) &= f(re^{i\theta}) \underset{\text{Part 1}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it}) \sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-t)}\, dt \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\, dt \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{-1} r^{|n|}e^{in\theta}\underbrace{\int_{-\pi}^{\pi} f(e^{it})e^{-int}\, dt}_{=0 \text{ by assumption}} + \frac{1}{2\pi}\sum_{n=0}^{\infty} r^{n}e^{in\theta}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\, dt \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty} z^n \int_{-\pi}^{\pi} f(e^{it})e^{-int}\, dt.
\end{aligned}$$

Now, for any closed rectifiable curve γ *in* D *we get*

$$\int_\gamma f(z)\, dz = \int_\gamma \frac{1}{2\pi}\sum_{n=0}^{\infty} z^n \int_{-\pi}^{\pi} f(e^{it})e^{-int}\, dt\, dz = \frac{1}{2\pi}\sum_{n=0}^{\infty}\underbrace{\int_\gamma z^n\, dz}_{=0}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\, dt$$

since z^n *is analytic for* $n \geq 0$.
Thus f *is analytic on* D.

Exercise 2. *In the statement of Theorem 2.4 suppose that* f *is piecewise continuous on* ∂D. *Is the conclusion of the theorem still valid? If not, what parts of the conclusion remain true?*

Solution. *Not available.*

Exercise 3. *Let* $D = \{z : |z| < 1\}$, $T = \partial D = \{z : |z| = 1\}$
(a) Show that if $g : D^- \to \mathbb{C}$ *is a continuous function and* $g_r : T \to \mathbb{C}$ *is defined by* $g_r(z) = g(rz)$ *then* $g_r(z) \to g(z)$ *uniformly for* z *in* T *as* $r \to 1^-$.
(b) If $f : T \to \mathbb{C}$ *is a continuous function define* $\tilde{f} : D^- \to \mathbb{C}$ *by* $\tilde{f}(z) = f(z)$ *for* z *in* T *and*

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)\, dt$$

(So Re $\tilde{f}$ *and Im* $\tilde{f}$ *are harmonic in D). Define* $\tilde{f}_r : T \to \mathbb{C}$ *by* $\tilde{f}_r(z) = \tilde{f}(rz)$. *Show that for each* $r < 1$ *there is a sequence* $\{p_n, (z, \overline{z})\}$ *of polynomials in* z *and* $\overline{z}$ *such that* $p_n(z, \overline{z}) \to \tilde{f}_r(z)$ *uniformly for* z *in* T. *(Hint: Use Definition 2.1.)*
(c) ***Weierstrass approximation theorem for T****. If* $f : T \to \mathbb{C}$ *is a continuous function then there is a sequence* $\{p_n(z, \overline{z})\}$ *of polynomials in* z *and* $\overline{z}$ *such that* $p_n(z, \overline{z}) \to f(z)$ *uniformly for* z *in* T.
(d) Suppose $g : [0, 1] \to \mathbb{C}$ *is a continuous function such that* $g(0) = g(1)$. *Use part (c) to show that there is a sequence* $\{p_n\}$ *of polynomials such that* $p_n(t) \to g(t)$ *uniformly for* t *in* $[0, 1]$.
(e) ***Weierstrass approximation theorem for [0,1].*** *If* $g : [0, 1] \to \mathbb{C}$ *is a continuous function then there is a sequence* $\{p_n\}$ *of polynomials such that* $p_n(t) \to g(t)$ *uniformly for* t *in* $[0, 1]$. *(Hint: Apply part (d) to the function* $g(t) + (1 - t)g(1) + tg(0)$*.)*
(f) Show that if the function g *in part (e) is real valued then the polynomials can be chosen with real coefficients.*

Solution. *Not available.*

Exercise 4. *Let G be a simply connected region and let Γ be its closure in $\mathbb{C}_\infty$; $\partial_\infty G = \Gamma - G$. Suppose there is a homeomorphism φ of Γ onto $D^-(D = \{z : |z| < 1\})$ such that φ is analytic on G.*
(a) Show that $\varphi(G) = D$ and $\varphi(\partial_\infty G) = \partial D$.
(b) Show that if $f : \partial_\infty G \to \mathbb{R}$ is a continuous function then there is a continuous function $u : \Gamma \to \mathbb{R}$ such that $u(z) = f(z)$ for z in $\partial_\infty G$ and u is harmonic in G.
(c) Suppose that the function f in part (b) is not assumed to be continuous at ∞. Show that there is a continuous function $u : G^- \to \mathbb{R}$ such that $u(z) = f(z)$ for z in ∂G and u is harmonic in G (see Exercise 2).

Solution. *Not available.*

Exercise 5. *Let G be an open set, $a \in G$, and $G_0 = G - \{a\}$. Suppose that u is a harmonic function on G_0 such that $\lim_{z\to a} u(z)$ exists and is equal to A. Show that if $U : G \to \mathbb{R}$ is defined by $U(z) = u(z)$ for $z \neq a$ and $U(a) = A$ then U is harmonic on G.*

Solution. *Not available.*

Exercise 6. *Let $f : \{z : Re\, z = 0\} \to \mathbb{R}$ be a bounded continuous function and define $u : \{z : Re\, z > 0\} \to \mathbb{R}$ by*

$$u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x f(it)}{x^2 + (y-t)^2}\, dt.$$

Show that u is a bounded harmonic function on the right half plane such that for c in $\mathbb{R}$, $f(ic) = \lim_{z\to ic} u(z)$.

Solution. *Not available.*

Exercise 7. *Let $D = \{z : |z| < l\}$ and suppose $f : \partial D \to \mathbb{R}$ is continuous except for a jump discontinuity at $z = 1$. Define $u : D \to \mathbb{R}$ by (2.5). Show that u is harmonic. Let v be a harmonic conjugate of u. What can you say about the behavior of $v(r)$ as $r \to 1^-$? What about $v(re^{i\theta})$ as $r \to 1^-$ and $\theta \to 0$?*

Solution. *Not available.*

10.3 Subharmonic and superharmonic functions

Exercise 1. *Which of the following functions are subharmonic? superharmonic? harmonic? neither subharmonic nor superharmonic? (a) $\varphi(x,y) = x^2 + y^2$; (b) $\varphi(x,y) = x^2 - y^2$; (c) $\varphi(x,y) = x^2 + y$; (d) $\varphi(x,y) = x^2 - y$; (e) $\varphi(x,y) = x + y^2$; (f) $\varphi(x,y) = x - y^2$.*

Solution. *Note that $\int_{-\pi}^{\pi} \sin\theta\, d\theta = 0$, $\int_{-\pi}^{\pi} \cos\theta\, d\theta = 0$, $\int_{-\pi}^{\pi} \sin^2\theta\, d\theta = \pi$ and $\int_{-\pi}^{\pi} \cos^2\theta\, d\theta = \pi$. For all $a = (\alpha, \beta) \in \mathbb{C}$ and any $r > 0$ we have*
a)

$$\begin{aligned}
\varphi(a + re^{i\theta}) &= \varphi(\alpha + r\cos\theta, \beta + r\sin\theta) = (\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 \\
&= \alpha^2 + 2\alpha r\cos\theta + r^2\cos^2\theta + \beta^2 + 2\beta r\sin\theta + r^2\sin^2\theta \\
&= \alpha^2 + \beta^2 + 2\alpha r\cos\theta + 2\beta r\sin\theta + r^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(a + re^{i\theta})\, d\theta &= \alpha^2 + \beta^2 + \frac{\alpha r}{\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\, d\theta}_{=0} + \frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\, d\theta}_{=0} + r^2 \\
&= \alpha^2 + \beta^2 + r^2 \geq \alpha^2 + \beta^2 = \varphi(a).
\end{aligned}$$

Hence, $\varphi \in$ Subhar(G).
b)

$$\begin{aligned}
\varphi(a + re^{i\theta}) &= \varphi(\alpha + r\cos\theta, \beta + r\sin\theta) = (\alpha + r\cos\theta)^2 - (\beta + r\sin\theta)^2 \\
&= \alpha^2 + 2\alpha r\cos\theta + r^2\cos^2\theta - \beta^2 - 2\beta r\sin\theta - r^2\sin^2\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a + re^{i\theta})\,d\theta &= \alpha^2 - \beta^2 + \frac{\alpha r}{\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0} + \frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos^2\theta\,d\theta}_{=\pi} - \frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0} \\
&- \frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin^2\theta\,d\theta}_{=\pi} = \alpha^2 - \beta^2 + \frac{r^2}{2} - \frac{r^2}{2} = \alpha^2 - \beta^2 = \varphi(a).
\end{aligned}$$

Hence, $\varphi \in$ Har(G), therefore also $\varphi \in$ Subhar(G) and $\varphi \in$ Superhar(G).
c)

$$\begin{aligned}
\varphi(a + re^{i\theta}) &= \varphi(\alpha + r\cos\theta, \beta + r\sin\theta) = (\alpha + r\cos\theta)^2 + (\beta + r\sin\theta) \\
&= \alpha^2 + 2\alpha r\cos\theta + r^2\cos^2\theta + \beta + r\sin\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a + re^{i\theta})\,d\theta &= \alpha^2 + \beta + \frac{\alpha r}{\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0} + \frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos^2\theta\,d\theta}_{=\pi} + \frac{r}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0} \\
&= \alpha^2 + \beta + \frac{r^2}{2} \geq \alpha^2 + \beta = \varphi(a).
\end{aligned}$$

Hence, $\varphi \in$ Subhar(G).
d)

$$\begin{aligned}
\varphi(a + re^{i\theta}) &= \varphi(\alpha + r\cos\theta, \beta + r\sin\theta) = (\alpha + r\cos\theta)^2 - (\beta + r\sin\theta) \\
&= \alpha^2 + 2\alpha r\cos\theta + r^2\cos^2\theta - \beta - r\sin\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a + re^{i\theta})\,d\theta &= \alpha^2 - \beta + \frac{\alpha r}{\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0} + \frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos^2\theta\,d\theta}_{=\pi} - \frac{r}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0} \\
&= \alpha^2 - \beta + \frac{r^2}{2} \geq \alpha^2 - \beta = \varphi(a).
\end{aligned}$$

Hence, $\varphi \in$ Subhar(G).
e)

$$\begin{aligned}
\varphi(a + re^{i\theta}) &= \varphi(\alpha + r\cos\theta, \beta + r\sin\theta) = (\alpha + r\cos\theta) + (\beta + r\sin\theta)^2 \\
&= \alpha + r\cos\theta + \beta^2 + 2\beta r\sin\theta + r^2\sin^2\theta.
\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a+re^{i\theta})\,d\theta &= \alpha+\beta^2+\frac{r}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0}+\frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0}+\frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin^2\theta\,d\theta}_{=\pi}\\ &= \alpha+\beta^2+\frac{r^2}{2}\geq\alpha+\beta^2=\varphi(a).\end{aligned}$$

Hence, $\varphi\in$ Subhar(G).
f)

$$\begin{aligned}\varphi(a+re^{i\theta}) &= \varphi(\alpha+r\cos\theta,\beta+r\sin\theta)=(\alpha+r\cos\theta)-(\beta+r\sin\theta)^2\\ &= \alpha+r\cos\theta-\beta^2-2\beta r\sin\theta-r^2\sin^2\theta.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a+re^{i\theta})\,d\theta &= \alpha-\beta^2+\frac{r}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0}-\frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0}-\frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin^2\theta\,d\theta}_{=\pi}\\ &= \alpha-\beta^2-\frac{r^2}{2}\leq\alpha-\beta^2=\varphi(a).\end{aligned}$$

Hence, $\varphi\in$ Superhar(G).

Exercise 2. *Let Subhar(G) and Superhar(G) denote, respectively, the sets of subharmonic and superharmonic functions on G.*
(a) Show that Subhar(G) and Superhar(G) are closed subsets of $\mathbb{C}(G;\mathbb{R})$.
(b) Does a version of Harnack's Theorem hold for subharmonic and superharmonic functions?

Solution. *Not available.*

Exercise 3. *(This exercise is difficult.) If G is a region and if $f:\partial_\infty G\to\mathbb{R}$ is a continuous function let u_f be the Perron Function associated with f. This defines a map $T:(\partial_\infty G;\mathbb{R})\to$ Har(G) by $T(f)=u_f$. Prove:*
(a) T is linear (i.e., $T(a_1f_1+a_2f_2)=a_1T(f_1)+a_2T(f_2)$).
(b) T is positive (i.e., if $f(a)\geq 0$ for all a in $\partial_\infty G$ then $T(f)(z)\geq 0$ for all z in G).
(c) T is continuous. Moreover, if $\{f_n\}$ is a sequence in $C(\partial_\infty G;\mathbb{R})$ such that $f_n\to f$ uniformly then $T(f_n)\to T(f)$ uniformly on G.
(d) If the Dirichlet Problem can be solved for G then T is one-one. Is the converse true?

Solution. *Not available.*

Exercise 4. *In the hypothesis of Theorem 3.11, suppose only that f is a bounded function on $\partial_\infty G$; prove that the conclusion remains valid. (This is useful if G is an unbounded region and g is a bounded continuous function on ∂G. If we define $f:\partial_\infty G\to\mathbb{R}$ by $f(z)=g(z)$ for z in ∂G and $f(\infty)=0$ then the conclusion of Theorem 3.11 remains valid. Of course there is no reason to expect that the harmonic function will have predictable behavior near ∞ — we could have assigned any value to $f(\infty)$. However, the behavior near points of ∂G can be studied with hope of success.)*

Solution. *Not available.*

Exercise 5. *Show that the requirement that G_1 is bounded in Corollary 3.5 is necessary.*

Solution. *Not available.*

Exercise 6. *If $f : G \to \Omega$ is analytic and $\varphi : \Omega \to \mathbb{R}$ is subharmonic, show that $\varphi \circ f$ is subharmonic if f is one-one. What happens if $f'(z) \neq 0$ for all z in G?*

Solution. *Clearly f is continuous, since $f : G \to \Omega$ is analytic and φ is continuous, since $\varphi : \Omega \to \mathbb{R}$ is subharmonic. Define $u = \varphi \circ f$, then $u : G \to \mathbb{R}$ is continuous. If we can show that for every bounded region G_1 such that $\overline{G_1} \subset G$ and for every continuous function $u_1 : \overline{G_1} \to \mathbb{R}$ that is harmonic in G_1 and satisfies $u(z) \leq u_1(z)$ for z in ∂G_1, we have*

$$u(z) \leq u_1$$

for z in G_1, then u is subharmonic (by Corollary 3.5 p. 265).
Hence, let G_1 be a bounded region G_1 such that $\overline{G_1} \subset G$ and let $u_1 : \overline{G_1} \to \mathbb{R}$ that is harmonic in G_1 and satisfies $u(z) \leq u_1(z)$ for z on ∂G_1. We have to show $u(z) \leq u_1(z)\ \forall z \in G_1$.
Since f is one-one, we know that f^{-1} exists and is analytic. So define

$$\varphi_1 = u_1 \circ f^{-1} : \overline{\Omega} \to \mathbb{R}$$

with $f^{-1} : \overline{\Omega}_1 \to \overline{G}_1$. Clearly φ_1 is harmonic (harmonic composite analytic is harmonic). Since $\varphi : \Omega \to \mathbb{R}$ is subharmonic, we can use Theorem 3.4 to obtain: For every region Ω_1 contained in Ω and every harmonic function $\tilde{\varphi}$ on Ω_1, $\varphi - \tilde{\varphi}$ satisfies the Maximum Principle on Ω_1. In fact $\varphi - \varphi_1$ satisfies the Maximum Principle on Ω_1. By assumption we have

$$u(z) \leq u_1(z) \quad \forall z \in G_1$$

which is the same as

$$u(z) = \varphi(f(z)) \leq u_1(z) = \varphi_1(f(z)) \quad \forall z \in \partial G_1$$

since f maps G_1 onto Ω_1, and thus ∂G_1 onto $\partial \Omega_1$ (we choose this by assumption). Thus,

$$\varphi(f(z)) - \varphi_1(f(z)) \leq 0 \quad \forall z \in \partial G_1.$$

Since $\varphi - \varphi_1$ satisfies the Maximum Principle on Ω_1, this implies

$$\varphi(f(z)) - \varphi_1(f(z)) \leq 0 \quad \forall z \in G_1,$$

that is

$$u(z) - u_1(z) \leq 0 \quad \forall z \in G_1$$

and we are done. Therefore, $u = \varphi \circ f$ is subharmonic on G.

If $f'(z) \neq 0$ for all z in G, then the previous result holds locally for some neighborhood in G. (f is locally one-one and onto and f^{-1} is analytic and hence $u = \varphi \circ f$ is locally subharmonic). Here is the claim: Let $f : G \to \mathbb{C}$ be analytic, let $z_0 \in A$, and let $f'(z_0) \neq 0$. Then there is a neighborhood U of z_0 and a neighborhood V of $w_0 = f(z_0)$ such that $f : U \to V$ is one-one and onto and $f^{-1} : V \to U$ is analytic.
Proof of the claim: $f(z) - w_0$ has a simple zero at z_0 since $f'(z_0) \neq 0$. We can use Theorem 7.4 p. 98 to find $\epsilon > 0$ and $\delta > 0$ such that each w with $|w - w_0| < \delta$ has exactly one pre-image z with $|z - z_0| < \epsilon$. Let $V = \{w \mid |w - w_0| < \delta\}$ and let U be the inverse image of V under the map f restricted to $\{z \mid |z - z_0| < \epsilon\}$. By Theorem 7.4 p. 98, f maps U one-one onto V. Since f is continuous, U is a neighborhood of z_0. By the Open Mapping Theorem (Theorem 7.5 p. 99), $f = (f^{-1})^{-1}$ is an open map, so f^{-1} is continuous from V to U. To finally show that it is analytic, we can use Proposition 3.7 p. 125. We have

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|z - z_0| = \epsilon} \frac{z f'(w)}{f(z) - w}\, dz.$$

Note, if we assume that φ is twice differentiable, then the result holds globally.

10.4 The Dirichlet Problem

Exercise 1. *Let $G = B(0;1)$ and find a barrier for G at each point of the boundary.*

Solution. *Not available.*

Exercise 2. *Let $G = \mathbb{C} - (\infty, 0]$ and construct a barrier for each point of $\partial_\infty G$.*

Solution. *Not available.*

Exercise 3. *Let G be a region and a a point in $\partial_\infty G$ such that there is a harmonic function $u : G \to \mathbb{R}$ with $\lim_{z\to a} u(z) = 0$ and $\liminf_{z\to w} u(z) > 0$ for all w in $\partial_\infty G$, $w \neq a$. Show that there is a barrier for G at a.*

Solution. *Not available.*

Exercise 4. *This exercise asks for an easier proof of a special case of Theorem 4.9. Let G be a bounded region and let $a \in \partial G$ such that there is a point b with $[a,b] \cap G^- = \{a\}$. Show that G has a barrier at a. (Hint: Consider the transformation $(z-a)(z-b)^1$.)*

Solution. *Not available.*

10.5 Green's Function

Exercise 1. *(a) Let G be a simply connected region, let $a \in G$, and let $f : G \to D = \{z : |z| < 1\}$ be a one-one analytic function such that $f(G) = D$ and $f(a) = 0$. Show that the Green's Function on G with singularity at a is $g_a(z) = -\log|f(z)|$.*
(b) Find the Green's Functions for each of the following regions: (i) $G = \mathbb{C} - (\infty, 0]$; (ii) $G = \{z : Re\, z > 0\}$; (iii) $G = \{z : 0 < Im\, z < 2\pi\}$.

Solution. *Solution to part a):*
Let G be simply connected, let $a \in G$, and let $f : G \to D = \{z : |z| < 1\}$ be a one-one analytic function such that $f(G) = D$ and $f(a) = 0$.
To show: $g_a(z) = -\log|f(z)|$ is a Green's Function on G with singularity at a. Clearly, $g_a(z)$ has a singularity if $|f(z)| = 0$. This happens if $f(z) = 0$. By assumption $f(z) = 0$ iff $z = a$. (Note that there is no other point $z \in G$ such that $f(z) = 0$ since f is assumed to be one-one). This implies $g_a(z)$ has a singularity at a. It remains to verify a), b) and c) of Definition 5.1.
a) g_a is harmonic in $G - \{a\}$. This follows directly by Exercise 5 p. 255 in $G - \{a\}$.
b) $G(z) = g_a(z) + \log|z-a|$ is harmonic in a disk about a. Let the disk be $B(a;r)$. Clearly $g_a(z) = -\log|f(z)|$ is harmonic on $B(a;r) - \{a\}$. By Exercise 5 p. 255, we can argue again that $\log|z-a|$ is harmonic on $B(a;r) - \{a\}$ (choose $f(z) = z - a$ and $f(z) = 0$ iff $z = a$). Hence $G(z) = g_a(z) + \log|z-a| = -\log|f(z)| + \log|z-a|$ is harmonic on $B(a;r) - \{a\}$ as a sum of two harmonic functions. But

$$G(z) = -\log|f(z)| + \log|z-a| = -\log\left(\frac{|f(z)|}{|z-a|}\right) = -\log\left(\frac{|z-a|\cdot|\tilde{f}(z)|}{|z-a|}\right) = -\log|\tilde{f}(z)|$$

where $f(z) = (z-a)\tilde{f}(z)$ by assumption and $\tilde{f}(z)$ is analytic on G with no zero. So $G(z)$ has a removable singularity at a and therefore $G(z)$ is harmonic on $B(a;r)$.
c) $\lim_{z\to w} g_a(z) = 0$ for each w in $\partial_\infty G$. We have $|f(z)| = 1$ $\forall z \in \partial_\infty G$ by assumption ($f(\partial_\infty G) = \partial D$). Hence

$$\lim_{z\to w} g_a(z) = \lim_{w\to a} -\log|f(w)| = -\log 1 = 0$$

for each a in $\partial_\infty G$. This implies $g_a(z) = -\log|f(z)|$ is a Green's Function on G with singularity at a.
Solution to part b) ii):
Clearly, G is a simply connected region (not the whole plane). If we can find a map, say $h : G \to D$, then the Green's Function, say g_a, is given by

$$g_a(z) = -\log|h(z)|$$

by Part 1 a). Note that h has to satisfy the following assumptions to use Part 1 a):
a) $h : G \to D$ one-one and analytic
b) $h(a) = 0$
c) $h(G) = D$.
The existence and uniqueness of this function h with the assumptions a), b) and c) is guaranteed by the Riemann Mapping Theorem p. 160 Theorem 4.2. It remains to find h:
Claim: $h : G \to D$ given by

$$h(z) = f(g(z))$$

where

$$g(z) = \frac{z-1}{z+1} \text{ and } f(z) = \frac{z - g(a)}{1 - \overline{g(a)}z}$$

works. Thus, the Green's Function for the region $G = \{z : Re(z) > 0\}$ is given by $g_a(z) = -\log|h(z)|$ where $h(z)$ is given by the Claim.
Proof of the Claim:
We will invoke the Orientation Principle (p. 53) to find $g : G \to D$ where g is one-one and analytic. Then

$$g(z) = \frac{z-1}{z+1}$$

(see p. 53 in the book for the derivation).
From p. 162 we have seen that the Möbius Transformation (which is one-one and analytic)

$$f(z) = \frac{z - g(a)}{1 - \overline{g(a)}z}$$

maps D onto D such that $f(g(a)) = 0$. Hence $h(z) = f(g(z))$ maps G onto D. In addition h is one-one, analytic and $h(a) = 0$ and therefore satisfies a), b) and c).

Exercise 2. *Let g_a be the Green's Function on a region G with singularity at $z = a$. Prove that if ψ is a positive superharmonic function on $G - \{a\}$ with $\liminf_{z\to a}[\psi(z) + \log|z-a|] > -\infty$, then $g_a(z) \le \psi(z)$ for $z \ne a$.*

Solution. *Not available.*

Exercise 3. *This exercise gives a proof of the Riemann Mapping Theorem where it is assumed that if G is a simply connected region, $G \ne \mathbb{C}$, then: (i) $\mathbb{C}_\infty - G$ is connected, (ii) every harmonic function on G has a harmonic conjugate, (iii) if $a \notin G$ then a branch of $\log(z-a)$ can be defined.*
(a) Let G be a bounded simply connected region and let $a \in G$; prove that there is a Green's Function g_a on G with singularity at a. Let $u(z) = g_a(z) + \log|z-a|$ and let v be the harmonic conjugate of u. If $\varphi = u + iv$ let $f(z) = e^{i\alpha}(z-a)e^{-\varphi(z)}$ for a real number α. (So f is analytic in G.) Prove that $|f(z)| = \exp(-g_a(z))$ and that $\lim_{z\to w}|f(z)| = 1$ for each w in ∂G (Compare this with Exercise 1). Prove that for $0 < r < 1$, $C_r = \{z : |f(z)| = r\}$ consists of a finite number of simple closed curves in G (see Exercise VI.1.3). Let G_r be a component of $\{z : |f(z)| < r\}$ and apply Rouche's Theorem to get that $f(z) = 0$ and $f(z) - w_0 = 0$,

$|w_0| < r$, have the same number of solutions in G_r. Prove that f is one-one on G_r. From here conclude that $f(G) = D = \{z : |z| < 1\}$ and $f'(a) > 0$, for a suitable choice of α.
(b) Let G be a simply connected region with $G \neq \mathbb{C}$, but assume that G is unbounded and $0, \infty \in \partial_\infty G$. Let l be a branch of $\log z$ on G, $a \in G$, and $\alpha = l(a)$. Show that l is one-one on G and $l(z) \neq \alpha + 2\pi i$ for any z in G. Prove that $\varphi(z) = [l(z) - \alpha - 2\pi i]^{-1}$ is a conformal map of G onto a bounded simply connected region in the plane. (Show that l omits all values in a neighborhood of $\alpha + 2\pi i$.)
(c) Combine parts (a) and (b) to prove the Riemann Mapping Theorem.

Solution. *Not available.*

Exercise 4. *(a) Let G be a region such that $\partial G = \gamma$ is a simple continuously differentiable closed curve. If $f : \partial G \to \mathbb{R}$ is continuous and $g(z, a) = g_a(z)$ is the Green's Function on G with singularity at a, show that*

$$h(a) = \int_\gamma f(z) \frac{\partial g}{\partial n}(z, a)\, ds \qquad (5.5)$$

is a formula for the solution of the Dirichlet Problem with boundary values f; where $\frac{\partial g}{\partial n}$ is the derivative of g in the direction of the outward normal to γ and ds indicates that the integral is with respect to arc length. (Note: these concepts are not discussed in this book but the formula is sufficiently interesting so as to merit presentation.) (Hint: Apply Green's formula

$$\iint_\Omega [u\Delta v - v\Delta u]\, dx\, dy = \int_{\partial\Omega} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right] ds$$

with $\Omega = G - \{z : |z - a| \leq \delta\}$, $\delta < d(a, \{\gamma\})$, $u = h$, $v = g_a(z) = g(z, a)$.)
(b) Show that if $G = \{z : |z| < 1\}$ then (5.5) reduces to equation (2.5).

Solution. *Not available.*

Chapter 11

Entire Functions

11.1 Jensen's Formula

Exercise 1. *In the hypothesis of Jensen's Formula, do not suppose that $f(0) \neq 0$. Show that if f has a zero at $z = 0$ of multiplicity m then*

$$\log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r = -\sum_{k=1}^{n}\log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

Solution. *We have $\frac{r(z-a_k)}{r^2-\bar{a}_k z}$ maps $B(0;r)$ onto itself and takes the boundary to the boundary. Let*

$$F(z) = f(z)\frac{r^m}{z^m}\prod_{k=1}^{n}\frac{r^2-\bar{a}_k z}{r(z-a_k)}.$$

Then $F \in H(G)$ and F has no zeros in $B(0;r)$ and $|F(z)| = |f(z)|$ if $|z| = r$. Thus, by a known result, we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|F(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

Also

$$\begin{aligned}
g(z) \quad &\equiv \quad \frac{f(z)}{z^m} = \frac{\sum_{i=0}^{\infty}\frac{f^{(i)}(0)z^i}{i!}}{z^m} \\
&= \quad \frac{\sum_{i=m}^{\infty}\frac{f^{(i)}(0)z^i}{i!}}{z^m} \\
&= \quad \sum_{i=m}^{\infty}\frac{f^{(i)}(0)z^{i-m}}{i!} = \frac{f^{(m)}(0)}{m!}z^0 + \frac{f^{(m+1)}(0)}{(m+1)!}z^1 + \ldots
\end{aligned}$$

(f has a zero at $z = 0$ of multiplicity m) which implies $g(0) = \frac{f^{(m)}(0)}{m!}$. Thus,

$$F(0) = g(0)r^m\prod_{k=1}^{n}\left(-\frac{r}{a_k}\right).$$

Therefore

$$|F(0)| = |g(0)|r^m \prod_{k=1}^{n} \frac{r}{|a_k|}$$

implies

$$\log|F(0)| = \log|g(0)| + m\log r + \sum_{k=1}^{n} \log\frac{r}{|a_k|} = \log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r + \sum_{k=1}^{n}\log\frac{r}{|a_k|}.$$

Hence,

$$\log|F(0)| = \log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r + \sum_{k=1}^{n}\log\frac{r}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta$$

which is equivalent to

$$\log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r = -\sum_{k=1}^{n}\log\frac{r}{|a_k|} + \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta$$

which proves the assertion.

Exercise 2. *Let f be an entire function, $M(r) = \sup\{|f(re^{i\theta})| : 0 \le \theta \le 2\pi\}$, $n(r)$ = the number of zeros of f in $B(0;r)$ counted according to multiplicity. Suppose that $f(0) = 1$ and show that $n(r)\log 2 \le \log M(2r)$.*

Solution. *Since f is an entire function, it is analytic on $\bar{B}(0;2r)$. Suppose $a_1,\dots,a_n$ are the zeros of f in $B(0;r)$ repeated according to multiplicity and $b_1,\dots,b_m$ are the zeros of f in $B(0;2r) - B(0;r)$. That is, $a_1,\dots,a_n,b_1,\dots,b_m$ are the zeros of f in $B(0;2r)$ repeated according to multiplicity. Since $f(0) = 1 \ne 0$, we have by Jensen's formula*

$$0 = \log 1 = \log|f(0)| = -\sum_{k=1}^{n}\log\left(\frac{2r}{|a_k|}\right) - \sum_{k=1}^{m}\log\left(\frac{2r}{|b_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log|f(2re^{i\theta})|\,d\theta$$

which implies

$$\begin{aligned}
\sum_{k=1}^{n}\log\left(\frac{2r}{|a_k|}\right) &= -\underbrace{\sum_{k=1}^{m}\log\left(\frac{2r}{|b_k|}\right)}_{\ge 0 \text{ since } r \le |b_k| < 2r} + \frac{1}{2\pi}\int_0^{2\pi}\log|f(2re^{i\theta})|\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\log|f(2re^{i\theta})|\,d\theta \\
&\underset{\text{Def of } M(r)}{\le} \frac{1}{2\pi}\log|M(2r)|\int_0^{2\pi} d\theta = \log|M(2r)|.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{n}\log\left(\frac{2r}{|a_k|}\right) \le \log|M(2r)|.$$

But

$$\begin{aligned}
\sum_{k=1}^{n}\log\left(\frac{2r}{|a_k|}\right) &= \sum_{k=1}^{n(r)}\log\left(\frac{2r}{|a_k|}\right) = \log\prod_{k=1}^{n(r)}\left(\frac{2r}{|a_k|}\right) \\
&= \log\left(2^n\prod_{k=1}^{n(r)}\frac{r}{|a_k|}\right) = \log\left(2^{n(r)}\right) + \log\prod_{k=1}^{n(r)}\frac{2r}{|a_k|} \\
&\ge n(r)\log(2)
\end{aligned}$$

since $0 < |a_k| < r$ *implies* $\frac{r}{|a_k|} > 1\ \forall k$ *which means* $\prod_{k=1}^{n(r)} \frac{r}{|a_k|} > 1$ *so the logarithm is greater than zero. Thus,*

$$n(r)\log(2) \leq \log|M(2r)|$$

which proves the assertion.

Exercise 3. *In Jensen's Formula do not suppose that* f *is analytic in a region containing* $\bar{B}(0;r)$ *but only that* f *is meromorphic with no pole at* $z = 0$. *Evaluate*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Solution. *Not available.*

Exercise 4. *(a) Using the notation of Exercise 2, prove that*

$$\int_0^r \frac{n(t)}{t}\, dt = \sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right)$$

where $a_1, \ldots, a_n$ *are the zeros of* f *in* $B(0;r)$.
(b) Let f *be meromorphic without a pole at* $z = 0$ *and let* $n(r)$ *be the number of zeros of* f *in* $B(0;r)$ *minus the number of poles (each counted according to multiplicity). Evaluate*

$$\int_0^r \frac{n(t)}{t}\, dt.$$

Solution. *Not available.*

Exercise 5. *Let* $D = B(0;1)$ *and suppose that* $f : D \to \mathbb{C}$ *is an analytic function which is bounded.*
(a) If $\{a_n\}$ *are the non-zero zeros of* f *in* D *counted according to multiplicity, prove that* $\sum(1 - |a_n|) < \infty$. *(Hint: Use Proposition VII. 5.4).*
(b) If f *has a zero at* $z = 0$ *of multiplicity* $m \geq 0$, *prove that* $f(z) = z^m B(z)\exp(-g(z))$ *where* B *is a Blaschke Product (Exercise VII. 5.4) and* g *is an analytic function on* D *with Re* $g(z) \geq -\log M$ *(*$M = \sup\{|f(z)| : |z| < 1\}$*).*

Solution. *Not available.*

11.2 The genus and order of an entire function

Exercise 1. *Let* $f(z) = \sum c_n z^n$ *be an entire function of finite genus* μ*; prove that*

$$\lim_{n\to\infty} c_n (n!)^{1/(\mu+1)} = 0.$$

(Hint: Use Cauchy's Estimate.)

Solution. *Define* $M(r) = \max\{f(re^{i\theta}) : 0 \leq \theta \leq 2\pi\} = \max_{|z|=r}|f(z)|$. *For sufficiently large* $r = |z|$, *we have*

$$M(r) < e^{\alpha r^{\mu+1}} \tag{11.1}$$

by Theorem 2.6 p. 283. By Cauchy's estimate p. 73, we have

$$|c_n| \leq \frac{M(r)}{r^n}$$

where $c_n = \frac{f^{(n)}(0)}{n!}$. *Hence,*

$$|c_n| \le \frac{M(r)}{r^n} \underset{(11.1)}{<} e^{\alpha r^{\mu+1}}. \tag{11.2}$$

Define, $r = n^{\frac{1}{\mu+1}}\lambda$, *where* λ *is a positive constant, and choose* $\lambda > e$ *for convenience. Then (11.2) can be written in the form*

$$|c_n| < \frac{e^{\alpha n \lambda^{\mu+1}}}{(n^n)^{\frac{1}{\mu+1}}\,\lambda^n} \iff (n^n)^{\frac{1}{\mu+1}}\,|c_n| < \frac{e^{\alpha n \lambda^{\mu+1}}}{\lambda^n} = \left(\frac{e^{\alpha\lambda^{\mu+1}}}{\lambda}\right)^n.$$

Now, if we choose α *such that* $\alpha\lambda^{\mu+1} = 1$, *the last inequality becomes*

$$(n^n)^{\frac{1}{\mu+1}}\,|c_n| < \left(\frac{e}{\lambda}\right)^n < \epsilon$$

for any $\epsilon > 0$, *provided n is taken large enough,* $n > N$ *say. Since* $n! < n^n$, *for* $n \ge 2$, *we have*

$$(n!)^{\frac{1}{\mu+1}}|c_n| < \epsilon$$

for $n > N$ *or*

$$\lim_{n\to\infty} c_n (n!)^{1/(\mu+1)} = 0.$$

Exercise 2. *Let* f_1 *and* f_2 *be entire functions of finite orders* λ_1 *and* λ_2 *respectively. Show that* $f = f_1 + f_2$ *has finite order* λ *and* $\lambda \le \max(\lambda_1, \lambda_2)$. *Show that* $\lambda = \max(\lambda_1, \lambda_2)$ *if* $\lambda_1 \ne \lambda_2$ *and give an example which shows that* $\lambda < \max(\lambda_1, \lambda_2)$ *with* $f \ne 0$.

Solution. *Not available.*

Exercise 3. *Suppose* f *is an entire function and* A, B, a *are positive constants such that there is a* r_0 *with* $|f(z)| \le \exp(A|z|^a + B)$ *for* $|z| > r_0$. *Show that* f *is of finite order* $\le a$.

Solution. *Not available.*

Exercise 4. *Prove that if* f *is an entire function of order* λ *then* f' *also has order* λ.

Solution. *Let* f *be an entire function of order* λ, *then* f *has power series*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

By Exercise 5 e) p. 286 we have

$$\alpha = \liminf_{n\to\infty} \frac{-\log|c_n|}{n\log n} = \frac{1}{\lambda}.$$

Differentiating f *yields*

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} = \sum_{n=0}^{\infty} \underbrace{(n+1)c_{n+1}}_{d_n} z^n.$$

Now, let λ' be the order of f', then

$$\begin{aligned}
\frac{1}{\lambda'} &= \liminf_{n\to\infty} \frac{-\log|d_n|}{n\log n} = \liminf_{n\to\infty} \frac{-\log|(n+1)c_{n+1}|}{n\log n} \\
&= \liminf_{n\to\infty} \frac{-\log(n+1) - \log|c_{n+1}|}{n\log n} \\
&= \liminf_{n\to\infty} \left[\frac{-\log(n+1)}{n\log n} - \frac{\log|c_{n+1}|}{n\log n}\cdot\frac{(n+1)\log(n+1)}{(n+1)\log(n+1)}\right] \\
&= \liminf_{n\to\infty} \left[\underbrace{\frac{-\log(n+1)}{n\log n}}_{\to 0} + \underbrace{\frac{-\log|c_{n+1}|}{(n+1)\log(n+1)}}_{\to\frac{1}{\lambda}}\cdot\underbrace{\frac{n+1}{n}}_{\to 1}\cdot\underbrace{\frac{\log(n+1)}{\log(n)}}_{\to 1}\right] \\
&= \frac{1}{\lambda}.
\end{aligned}$$

Hence, $\lambda' = \lambda$ and therefore f' also has order λ.

Exercise 5. *Let $f(z) = \sum c_n z^n$ be an entire function and define the number α by*

$$\alpha = \liminf_{n\to\infty} \frac{-\log|c_n|}{n\log n}$$

(a) Show that if f has finite order then $\alpha > 0$. (Hint: If the order of f is λ and $\beta > \lambda$ show that $|c_n| \leq r^{-n}\exp(r^\beta)$ for sufficiently large r, and find the maximum value of this expression.)
(b) Suppose that $0 < \alpha < \infty$ and show that for any $\epsilon > 0$, $\epsilon < \alpha$, there is an integer p such that $|c_n|^{1/n} < n^{-(\alpha-\epsilon)}$ for $n > p$. Conclude that for $|z| = r > 1$ there is a constant A such that

$$|f(z)| < Ar^p + \sum_{n=1}^{\infty}\left(\frac{r}{n^{\alpha-\epsilon}}\right)^n$$

(c) Let p be as in part (b) and let N be the largest integer $\leq (2r)^{1/(\alpha-\epsilon)}$. Take r sufficiently large so that $N > p$ and show that

$$\sum_{n=N+1}^{\infty}\left(\frac{r}{n^{\alpha-\epsilon}}\right)^n < 1 \text{ and } \sum_{n=p+1}^{N}\left(\frac{r}{n^{\alpha-\epsilon}}\right)^n < B\exp\left((2r)^{1/(\alpha-\epsilon)}\right)$$

where B is a constant which does not depend on r.
(d) Use parts (b) and (c) to show that if $0 < \alpha < \infty$ then f has finite order λ and $\lambda \leq \alpha^{-1}$.
(e) Prove that f is of finite order iff $\alpha > 0$, and if f has order λ then $\lambda = \alpha^{-1}$.

Solution. *Not available.*

Exercise 6. *Find the order of each of the following functions: (a) $\sin z$; (b) $\cos z$; (c) $\cosh\sqrt{z}$; (d) $\sum_{n=1}^{\infty} n^{-an}z^n$ where $a > 0$. (Hint: For part (d) use Exercise 5.)*

Solution. *c) The order is $\lambda = \frac{1}{2}$. Note that $\sinh^2 z = \cosh^2 z - 1$ and that $\cos x$ is bounded for real values x, so*

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \leq \cosh^2 y \leq \cosh^2 |z|.$$

Thus

$$|\cos z| \le \cosh |z| = \frac{1}{2}(e^{|z|} + e^{-|z|}) \le \frac{1}{2}e^{|z|} \le e^{|z|},$$

and therefore

$$|\cos\sqrt{z}| \le \exp(|z|^{1/2}).$$

This implies that the order $\lambda \le \frac{1}{2}$. *Next we want to show that* $\lambda = \frac{1}{2}$. *Seeking contradiction assume that there is* $\epsilon > 0$ *and* $r > 0$ *such that*

$$|\cos\sqrt{z}| \le \exp(|z|^{\frac{1}{2}-\epsilon}) \tag{11.3}$$

for all $|z| > r$.
Let $z = -n^2$ *for an* $n \in \mathbb{N}$ *with* $n > \max\{\sqrt{r}, 4, 2^{1/(2\epsilon)}\}$ *then in particular (i)* $|z| > r$, *(ii)* $-\ln 2 + n > \frac{n}{2}$, *and (iii)* $\frac{1}{2} > \frac{1}{n^{2\epsilon}}$ *and*

$$|f(z)| = |\cos\sqrt{z}| = \frac{1}{2}|e^n + e^{-n}| > \frac{1}{2}e^n = e^{-\log 2 + n} \underset{(ii)}{>} e^{\frac{n}{2}} \underset{(iii)}{>} \exp(n^{-2\epsilon}n) = \exp(n^{2(\frac{1}{2}-\epsilon)}) = \exp(|z|^{\frac{1}{2}-\epsilon}).$$

This shows that no $\epsilon > 0$ *can be found that satisfies equation (11.3). This implies* $\lambda = \frac{1}{2}$.

Exercise 7. *Let* f_1 *and* f_2 *be entire functions of finite order* λ_1, λ_2; *show that* $f = f_1 f_2$ *has finite order* $\lambda \le \max(\lambda_1, \lambda_2)$.

Solution. *Define* $M(r) = \max_{|z|=r} |f(z)|$, $M_1(r) = \max_{|z|=r} |f_1(z)|$ *and* $M_2(r) = \max_{|z|=r} |f_2(z)|$. *Since* f_1 *and* f_2 *are entire functions of finite order* λ_1 *and* λ_2, *respectively, we have for* $\epsilon > 0$,

$$M_1(r) < e^{r^{\lambda_1+\epsilon/2}} \tag{11.4}$$

and

$$M_2(r) < e^{r^{\lambda_2+\epsilon/2}} \tag{11.5}$$

for sufficiently large $r = |z|$ *by Proposition 2.14. Hence*

$$\begin{aligned} M(r) &= \max_{|z|=r} |f(z)| = \max_{|z|=r} |f_1(z)\cdot f_2(z)| \\ &\le \max_{|z|=r} |f_1(z)| \cdot \max_{|z|=r} |f_2(z)| \\ &= M_1(r)M_2(r) \underset{\text{by (11.4),(11.5)}}{<} e^{r^{\lambda_1+\epsilon/2}} e^{r^{\lambda_2+\epsilon/2}} \\ &= e^{r^{\lambda_1+\epsilon/2}+r^{\lambda_2+\epsilon/2}} \underset{\lambda\le\max(\lambda_1,\lambda_2)}{\le} e^{2r^{\lambda+\epsilon/2}} < e^{r^{\lambda+\epsilon}} \end{aligned}$$

provided r *is sufficiently large. Hence, the order of* $f(z) = f_1(z)\cdot f_2(z)$ *has finite order* $\lambda \le \max(\lambda_1, \lambda_2)$.

Exercise 8. *Let* $\{a_n\}$ *be a sequence of non-zero complex numbers. Let* $\rho = \inf\{a : \sum |a_n|^{-a} < \infty\}$; *the number called the exponent of convergence of* $\{a_n\}$.
(a) If f *is an entire function of rank* p *then the exponent of convergence* ρ *of the non-zero zeros of* f *satisfies:* $p \le \rho \le p + 1$.
(b) If ρ' = *the exponent of convergence of* $\{a_n\}$ *then for every* $\epsilon > 0$, $\sum |a_n|^{-(\rho+\epsilon)} < \infty$ *and* $\sum |a_n|^{-(\rho-\epsilon)} = \infty$.
(c) Let f *be an entire function of order* λ *and let* $\{a_1, a_2, \ldots\}$ *be the non-zero zeros of* f *counted according to multiplicity. If* ρ *is the exponent of convergence of* $\{a_n\}$ *prove that* $\rho \le \lambda$. *(Hint: See the proof of (3.5) in the next section.)*

(d) Let $P(z) = \prod_{n=1}^{\infty} E_p(z/a_n)$ be a canonical product of rank p, and let ρ be the exponent of convergence of $\{a_n\}$. Prove that the order of P is ρ. (Hint: If λ is the order of P, $\rho \leq \lambda$; assume that $|a_l| \leq |a_2| \leq \ldots$ and fix z, $|z| > 1$. Choose N such that $|a_n| \leq 2|z|$ if $n \leq N$ and $|a_n| > 2|z|$ if $n \leq N + 1$. Treating the cases $\rho < p + 1$ and $\rho = p + 1$ separately, use (2.7) to show that for some $\epsilon > 0$

$$\sum_{n=N+1}^{\infty} \log \left| E_p\left(\frac{z}{a_n}\right)\right| < A|z|^{\rho+\epsilon}.$$

Prove that for $|z| \geq \frac{1}{2}$, $\log|E_p(z)| < B|z|^p$ where B is a constant independent of z. Use this to prove that

$$\sum_{n=1}^{N} \log \left| E_p\left(\frac{z}{a_n}\right)\right| < C|z|^{\rho+\epsilon}$$

for some constant C independent of z.)

Solution. *Not available.*

Exercise 9. *Find the order of the following entire functions:*
(a)

$$f(z) = \prod_{n=1}^{\infty}(1 - a^n z), \qquad 0 < |a| < 1$$

b)

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n!}\right)$$

c)

$$f(z) = [\Gamma(z)]^{-1}.$$

Solution. *b) Considering $g(z) := \prod_{m=1}^{\infty}\left(1 - \frac{z}{n!}\right)$ one notes that $g(z)$ is already in form of the canonical product with the entire function in the exponent being constantly zero and $p = 0$. Hence also the genus $\mu = 0$. The simple zeros of $g(z)$ are at $z = n!$ and from Exercise 8a) it follows that $\lambda \leq 1$. In fact we will show that $\lambda = 0$. It suffices to show that the exponent of convergence $\rho = 0$ and to employ Exercise 8d). Let $m \leq \mathbb{N}$, let $n > 4m$ then*

$$n! \geq n \cdot \underbrace{(n-1)\cdot 2}_{\geq n} \cdot \underbrace{(n-2)\cdot 3}_{\geq n} \ldots \underbrace{(n-(2k-1))\cdot 2m}_{\geq n}.$$

Each product of consecutive numbers is of the form $(n - (k - 1))k$, for $1 \leq k \leq 2m$. By choice of n, $\frac{k}{n} \leq \frac{1}{2}$ and $n - k + 1 > 2m$ which justifies the individual estimates by n. In total this yields $n! > n^{2m}$ for n large, or equivalently

$$\sum_{n=4m}^{\infty} (n!)^{-\frac{1}{m}} \leq \sum_{n=4m}^{\infty} n^{-2}$$

and the sum on the left converges for every positive integer m. We conclude that $\rho = \inf\{m :$ the left sum converges$\} = 0$. By Part d) of Exercise 8 we conclude that the order of $g(z)$ is zero.

11.3 Hadamard Factorization Theorem

Exercise 1. *Let f be analytic in a region G and suppose that f is not identically zero. Let $G_0 = G - \{z : f(z) = 0\}$ and define $h : G_0 \to \mathbb{R}$ by $h(z) = \log|f(z)|$. Show that $\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} = \frac{f'}{f}$ on G_0.*

Solution. *Let f be analytic in a region G and suppose that f is not identically zero. Let $G_0 = G - \{z : f(z) = 0\}$, then $h : G_0 \to \mathbb{R}$ given by $h(z) = \log|f(z)|$ is well defined as well as $\frac{f'}{f}$ is well defined on G_0.*
Let $f = u(x, y) + iv(x, y) = u + iv$. Since f is analytic, the Cauchy-Riemann (C-R) equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. We have by p. 41 Equation 2.22 and 2.23

$$f' = u_x + iv_x \quad \text{and} \quad f' = -iu_y + v_y$$

and thus

$$2f' = u_x + iv_x - iu_y + v_y$$

implies

$$f' = \frac{1}{2}\left(u_x + iv_x - iu_y + v_y\right) \underset{\text{(C-R)}}{=} \frac{1}{2}\left(u_x - iu_y - iu_y + u_x\right) = \frac{1}{2}\left(2u_x - 2iu_y\right) = u_x - iu_y.$$

Therefore,

$$\frac{f'}{f} = \frac{u_x - iu_y}{u + iv}. \tag{11.6}$$

Next, we calculate $\frac{\partial h}{\partial x} = h_x$ and $\frac{\partial h}{\partial y} = h_y$ where $h(z) = \log|f(z)| = \frac{1}{2}\log(u^2 + v^2)$. Using the chain rule, we get

$$h_x = h_u u_x + h_v v_x = \frac{u}{u^2 + v^2} u_x + \frac{v}{u^2 + v^2} v_x$$

and

$$h_y = h_u u_y + h_v v_y = \frac{u}{u^2 + v^2} u_y + \frac{v}{u^2 + v^2} v_y$$

and hence

$$\begin{aligned}
\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} &= h_x - ih_y \\
&= \frac{u}{u^2 + v^2} u_x + \frac{v}{u^2 + v^2} v_x - i\frac{u}{u^2 + v^2} u_y - i\frac{v}{u^2 + v^2} v_y \\
&= \frac{u}{u^2 + v^2}(u_x - iu_y) + \frac{v}{u^2 + v^2}(v_x - iv_y) \\
&\underset{\text{(C-R)}}{=} \frac{u}{u^2 + v^2}(u_x - iu_y) + \frac{v}{u^2 + v^2}(-u_y - iu_x) \\
&= \frac{u}{u^2 + v^2}(u_x - iu_y) - \frac{iv}{u^2 + v^2}(u_x - iu_y) \\
&= \frac{u - iv}{u^2 + v^2}(u_x - iu_y) \\
&= \frac{u - iv}{(u - iv)(u + iv)}(u_x - iu_y) \\
&= \frac{u_x - iu_y}{u + iv}.
\end{aligned}$$

Compare this with (11.6) and we obtain the assertion

$$\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} = \frac{f'}{f}.$$

Exercise 2. *Refer to Exercise 2.7 and show that if $\lambda_1 \neq \lambda_2$ then $\lambda = \max(\lambda_1; \lambda_2)$.*

Solution. *Not available.*

Exercise 3. *(a) Let f and g be entire functions of finite order λ and suppose that $f(a_n) = g(a_n)$ for a sequence $\{a_n\}$ such that $\sum |a_n|^{-(\lambda+1)} = \infty$. Show that $f = g$.*
(b) Use Exercise 2.8 to show that if f, g and $\{a_n\}$ are as in part (a) with $\sum |a_n|^{-(\lambda+\epsilon)} = \infty$ for some $\epsilon > 0$ then $f = g$.
(c) Find all entire functions f of finite order such that $f(\log n) = n$.
(d) Give an example of an entire function with zeros $\{\log 2, \log 3, \ldots\}$ and no other zeros.

Solution. *a) Since f and g are entire functions of finite order λ, also $F = f - g$ is an entire function of finite order λ (see Exercise 2 p. 286). Suppose that $f(a_n) = g(a_n)$ for a sequence $\{a_n\}$ such that $\sum |a_n|^{-(\lambda+1)} = \infty$. Assume $F \not\equiv 0$ ($f \not\equiv g$), we will derive a contradiction. Since $F \not\equiv 0$, the sequence $\{a_n\}$ are the zeros of F, because*

$$F(a_n) = f(a_n) - g(a_n) = 0$$

since $f(a_n) = g(a_n)$ by assumption. By Hadamard's Factorization Theorem (p. 289), F has finite genus $\mu \leq \lambda$ since $F \in H(\mathbb{C})$ with finite order λ. Since by definition $\mu = \max(p, q)$ where p is the rank of F, we certainly have

$$p \leq \lambda.$$

By the definition of the rank (p. 281 Definition 2.1), we have

$$\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty,$$

where a_n's are the zeros of F. Therefore by the comparison test, we also have

$$\sum_{n=1}^{\infty} |a_n|^{-(\lambda+1)} \underset{p\leq\lambda}{\leq} \sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$$

so $\sum_{n=1}^{\infty} |a_n|^{-(\lambda+1)} < \infty$ contradicting the assumption $\sum_{n=1}^{\infty} |a_n|^{-(\lambda+1)} = \infty$. Hence $F \equiv 0$, and therefore we obtain $f = g$.
b) Since f and g are entire functions of finite order λ, we also have $F = f - g$ is an entire function of finite order λ (see Exercise 2 p. 286). Suppose that $f(a_n) = g(a_n)$ for a sequence $\{a_n\}$ such that $\sum |a_n|^{-(\lambda+\epsilon)} = \infty$. Assume $F \not\equiv 0$ ($f \not\equiv g$), we will derive a contradiction. Since $F \not\equiv 0$, the sequence $\{a_n\}$ are the zeros of F, because

$$F(a_n) = f(a_n) - g(a_n) = 0$$

since $f(a_n) = g(a_n)$ by assumption. Let ρ be the exponent of convergence of $\{a_n\}$, then

$$\rho \leq \lambda$$

by Exercise 8 c) p. 286-287. Hence, according to Exercise 8 b) p. 286-287, we have

$$\sum_{n=1}^{\infty} |a_n|^{-(\rho+\epsilon)} < \infty.$$

Therefore by the comparison test, we also have

$$\sum_{n=1}^{\infty} |a_n|^{-(\lambda+\epsilon)} \underset{\rho\leq\lambda}{\leq} \sum_{n=1}^{\infty} |a_n|^{-(\rho+\epsilon)} < \infty.$$

Hence $\sum_{n=1}^{\infty} |a_n|^{-(\lambda+\epsilon)} < \infty$ *contradicting the fact* $\sum_{n=1}^{\infty} |a_n|^{-(\lambda+\epsilon)} = \infty$. *Thus,* $F \equiv 0$ *and therefore we get the assertion* $f = g$.

Chapter 12

The Range of an Analytic Function

12.1 Bloch's Theorem

Exercise 1. *Examine the proof of Bloch's Theorem to prove that* $L \geq 1/24$.

Solution. *Mimic the proof of Block's Theorem up to the application of Schwarz's Lemma, i.e. set* $K(r) = \max\{|f'(z)||z| = r\}, h(r) = (1-r)K(r), r_0 = \sup\{r : h(r) = 1\}$. *Choose a with* $|a| = r_0, |f'(a)| = K(r_0) = \frac{1}{1-r_0}$. *Also choose* $\rho_0 = \frac{1}{2}(1-r_0)$ *and infer that for all* $z \in B(a, \rho)$

$$|f'(z)| \leq \frac{1}{\rho_0}, \text{ and } |f'(z) - f'(a)| < \frac{3|z-a|}{2\rho_0^2}.$$

For $z \in B(0, \rho_0)$ *define* $g(z) = f(z+a) - f(a)$. *By convexity of* $B(a, \rho_0)$ *the line segment* $\gamma = [a, a+z]$ *is completely contained in* $B(a, \rho_0)$. *It follows that*

$$|g(z)| = |\int \gamma f'(w)\,\mathrm{d}w| \leq \frac{1}{\rho_0}|z| \leq 1 =: M.$$

The function g is not necessarily injective on $B(0, \rho_0)$ *but the problem only asks for an estimate on the Landau constant for which injectivity is not required. In this setting Lemma 1.2 gives*

$$g(B(0, \rho_0)) \supset B\left(0, \frac{\rho^2 |g'(0)|^2}{6M}\right) = B\left(0, \frac{1}{4 \cdot 6}\right) = B\left(0, \frac{1}{24}\right).$$

This statement rewritten for f yields

$$f(B(a, \rho_0)) \supset B\left(f(a), \frac{1}{24}\right).$$

Exercise 2. *Suppose that in the statement of Bloch's Theorem it is only assumed that f is analytic on D. What conclusion can be drawn? (Hint: Consider the functions* $f_s(z) = s^{-1} f(sz)$, $0 < s < 1$.*) Do the same for Proposition 1.10.*

Solution. *Not available.*

12.2 The Little Picard Theorem

Exercise 1. *Show that if f is a meromorphic function on $\mathbb{C}$ such that $\mathbb{C}_\infty - f(\mathbb{C})$ has at least three points then f is a constant. (Hint: What if $\infty \neq f(\mathbb{C})$?)*

Solution. *First consider an entire function f that misses three points in $\mathbb{C}_\infty$, one on which is ∞. Then there are two distinct numbers $a, b \in \mathbb{C}$ that are not in the image of f. By Little Picard's Theorem the function f must be a constant.*
Next consider a meromorphic function on $\mathbb{C}$ that is not entire. Hence $\infty \in f(\mathbb{C})$ and by assumption there are distinct $a, b, c \in \mathbb{C} - f(\mathbb{C})$. Define a function

$$g(z) := \frac{1}{f(z) - a}, \qquad z \in \mathbb{C}$$

then g is entire because $f(z) - a \neq 0$ for all $z \in \mathbb{C}$ and $\frac{1}{b-a}, \frac{1}{c-a} \in \mathbb{C} - g(\mathbb{C})$. Again using Little Picard's Theorem conclude that g is a constant function. It cannot be the zero-function, because f was assumed to be meromorphic, hence $f(z) \neq \infty$ for at least some $z \in \mathbb{C}$. Thus g is a nonzero constant function and $f(z) = g(z)^{-1} + a$ is also a constant function and the result is established.

Exercise 2. *For each integer $n \geq 1$ determine all meromorphic functions f and g on $\mathbb{C}$ with a pole at ∞ such that $f^n + g^n = 1$.*

Solution. *Not available.*

12.3 Schottky's Theorem

No exercises are assigned in this section.

12.4 The Great Picard Theorem

Exercise 1. *Let f be analytic in $G = B(0; R) - \{0\}$ and discuss all possible values of the integral*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a}\, dz$$

where γ is the circle $|z| = r < R$ and a is any complex number. If it is assumed that this integral takes on certain values for certain numbers a, does this imply anything about the nature of the singularity at $z = 0$?

Solution. *Not available.*

Exercise 2. *Show that if f is a one-one entire function then $f(z) = az + b$ for some constants a and b, $a \neq 0$.*

Solution. *Not available.*

Exercise 3. *Prove that the closure of the set $\mathcal{F}$ in Theorem 4.1 equals $\mathcal{F}$ together with the constant functions ∞, 0, and 1.*

Solution. *Not available.*