# Forced Burgers' Equation and Physics Informed Neural Network

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# Burgers' Equation

- **Mathematical form:**[1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad x \in [0, L], \quad t \in [0, T]$$

- **Forced Burgers' Equation:** with a force term

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + f(x, t)$$

- $f(x) = \sin(x)$ and $x \in [0, 2\pi]$

$$u(0, x) = \sin x$$
$$u(t, 0) = u(t, 2\pi) = 0$$

[1] J M Burgers, 1948

# Steady State Solution

- For **steady state solution**[2]

$$\frac{\partial u}{\partial t} = 0 \quad when \quad t \to \infty$$

- Now the equation is

$$u\frac{du}{dx} = \nu\frac{d^2u}{dx^2} + \sin x$$

- Method of matched asymptotic expansion(MMAE)

$$u = 2\,\mathrm{sgn}(x-\pi)(\sin\frac{x}{2} - 1) + 2\tanh\frac{\pi - x}{\nu}$$

- This solution is valid for small $\nu$.

[2]Kreiss and Kreiss,1986; Banerjee, D, 2019

# Steady State Solution

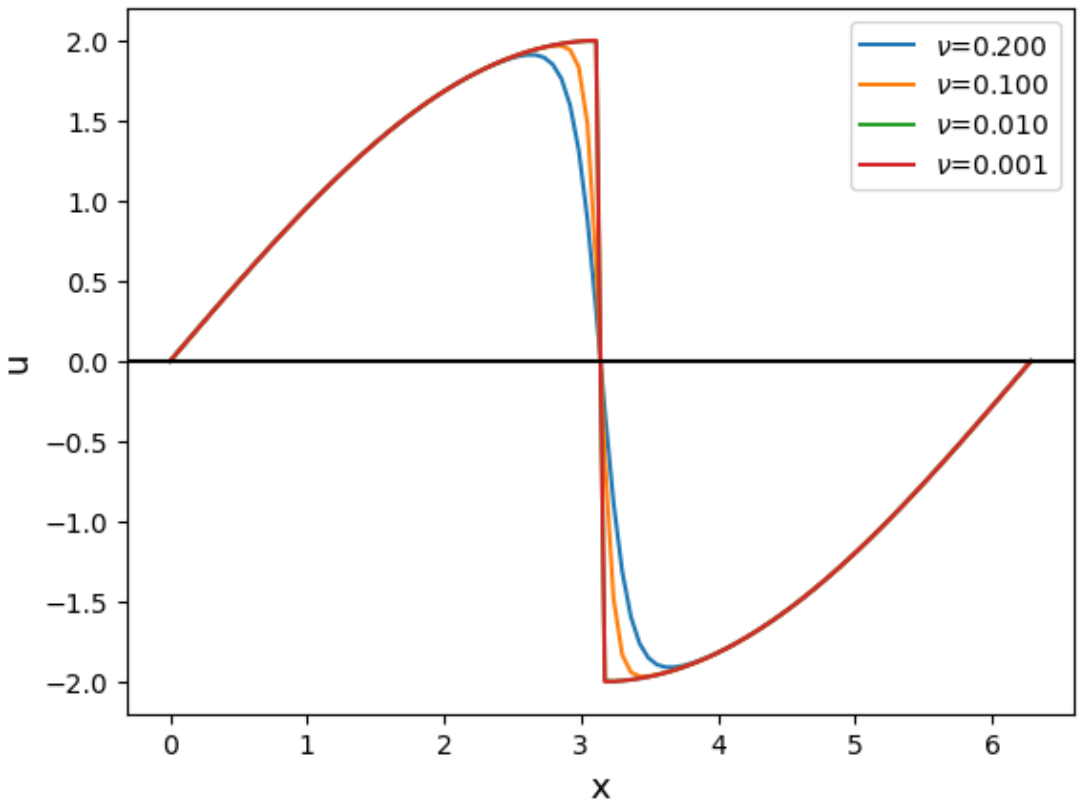
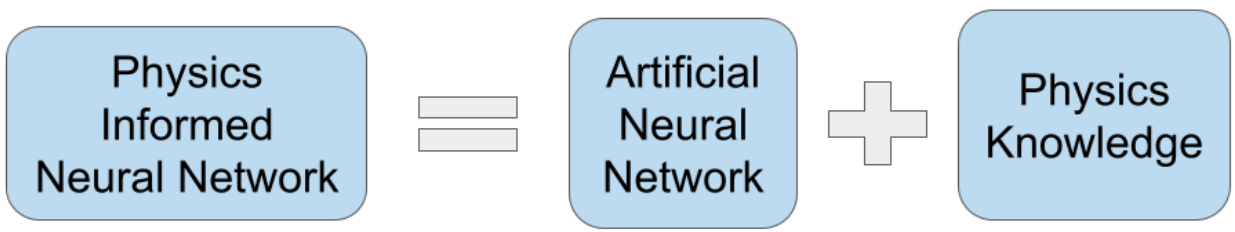



**Figure:** Steady velocity fields for different viscosity

# Physics Informed Neural Network

- **Physics Informed Neural Network(PINN)**[3] is a technique which uses Neural Network as a solution of a Partial Differential Equation.
- PINN approximates PDE solutions by minimizing a loss function that reflects the PDE, Boundary condition, Initial Condition, constraints etc.




[3] Rassi et al,2017

# PINN Building Blocks

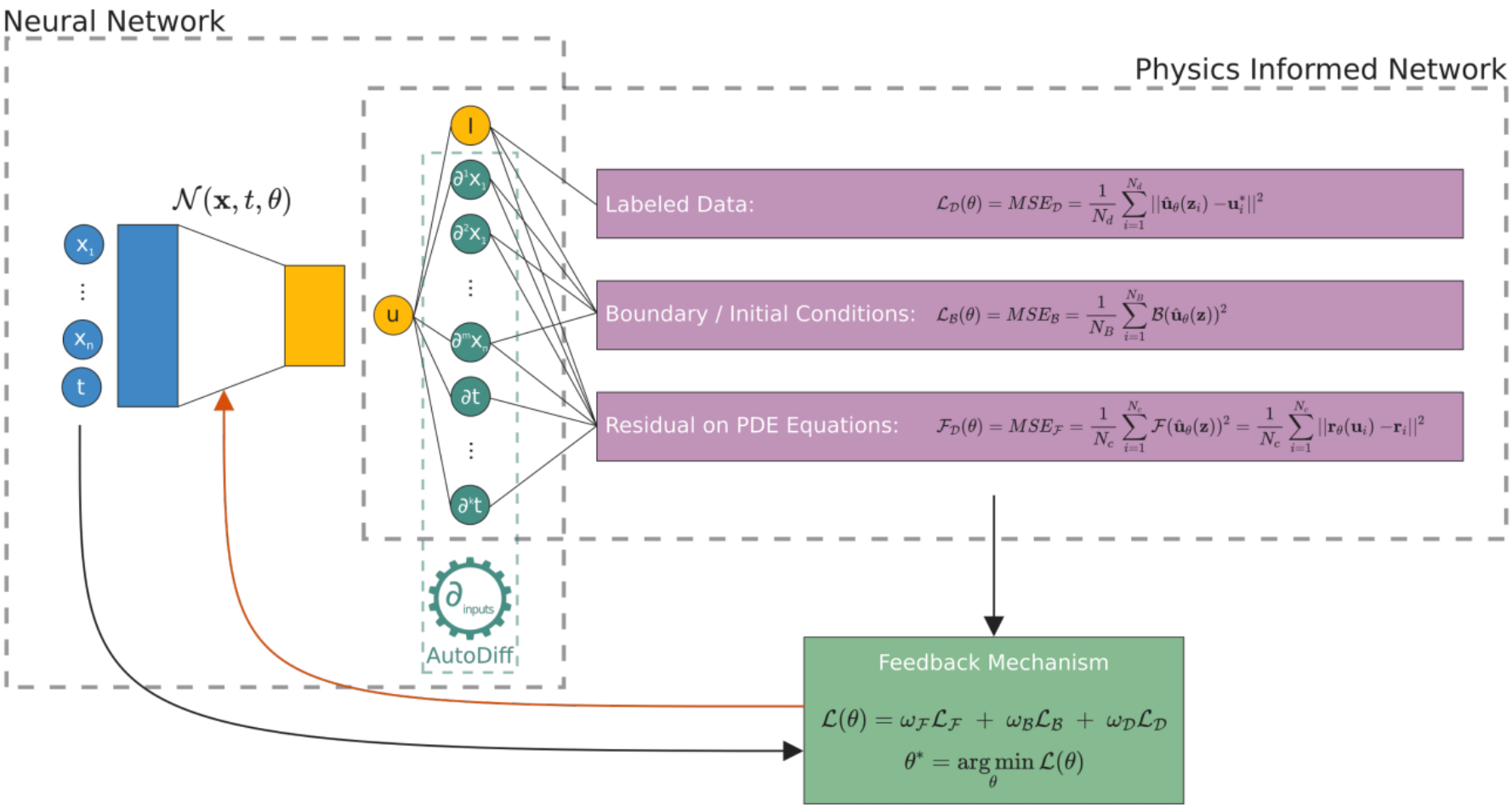
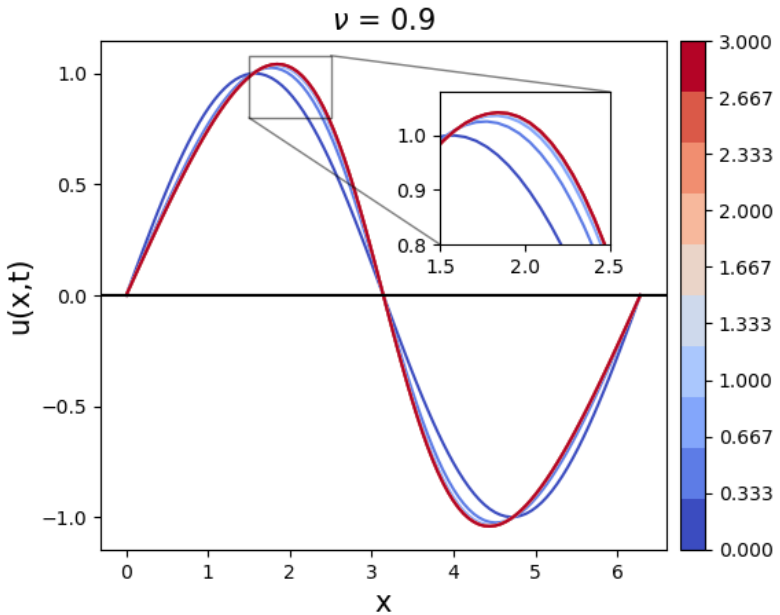
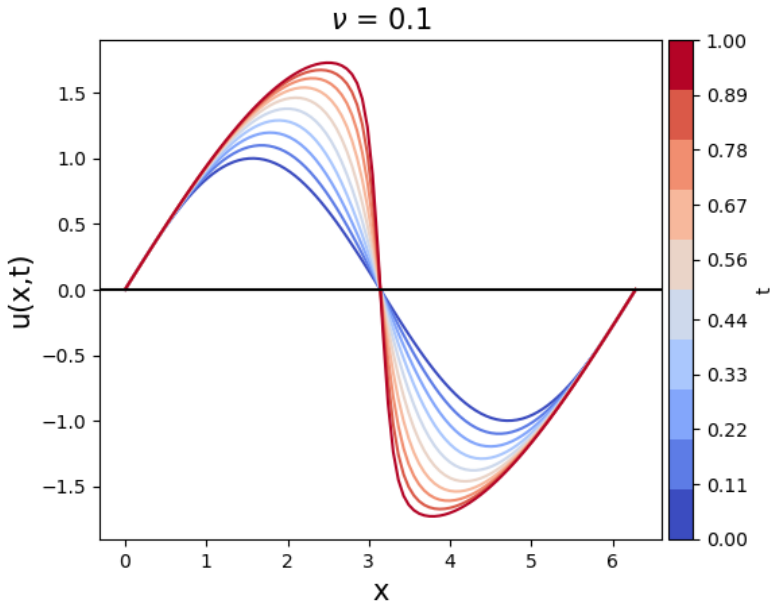


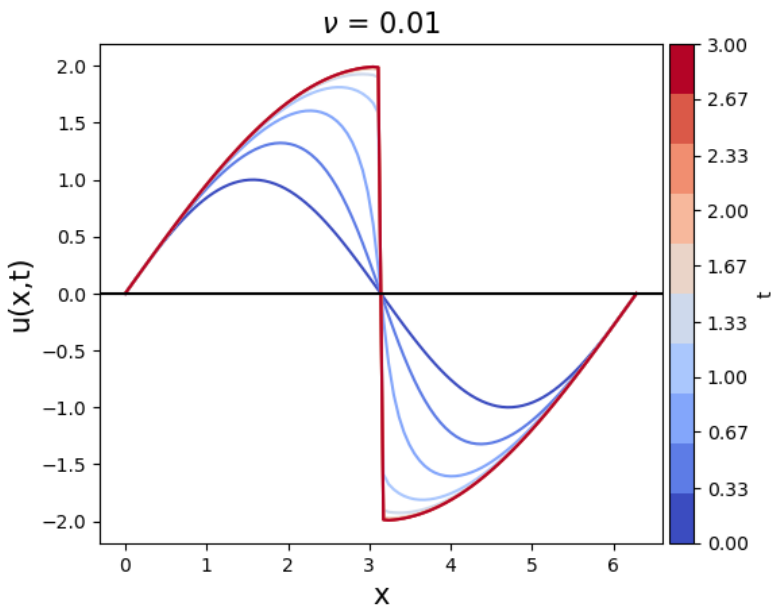

**Figure:** Building Block of PINN [4]

[4] S Cuomo et al, 2022

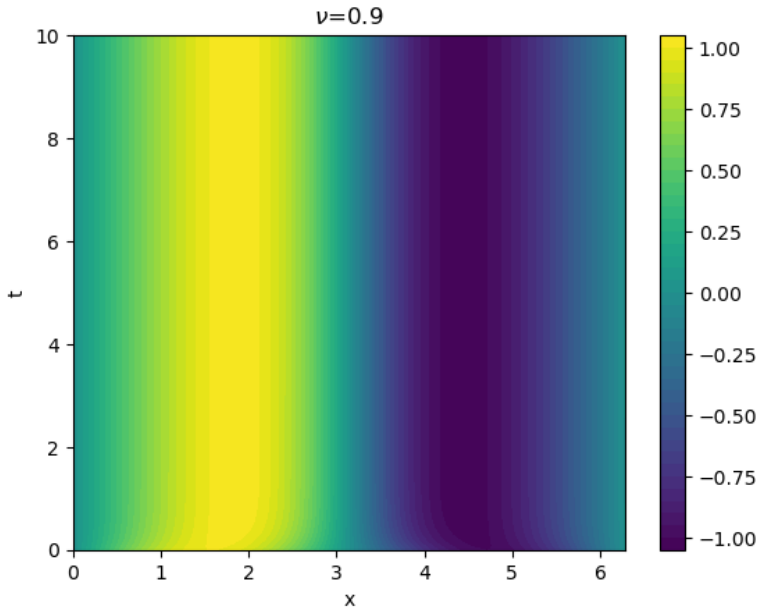
# Results



# Results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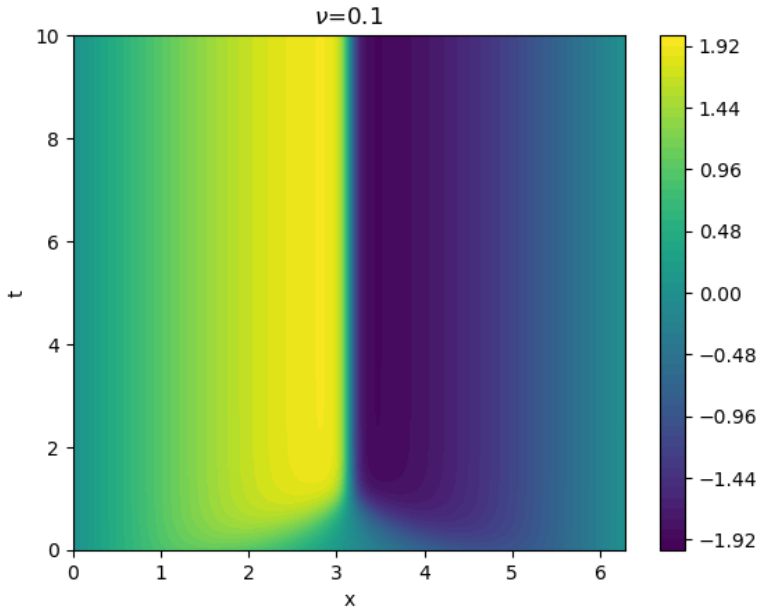
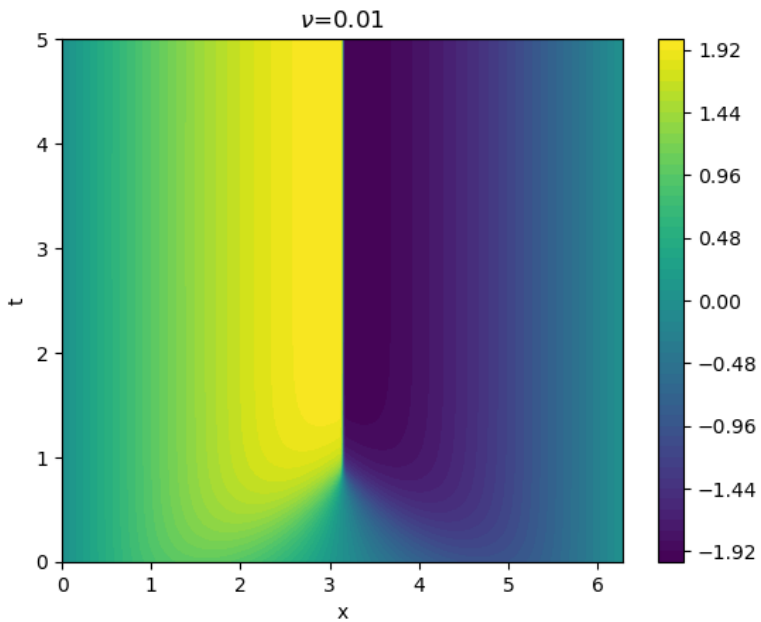
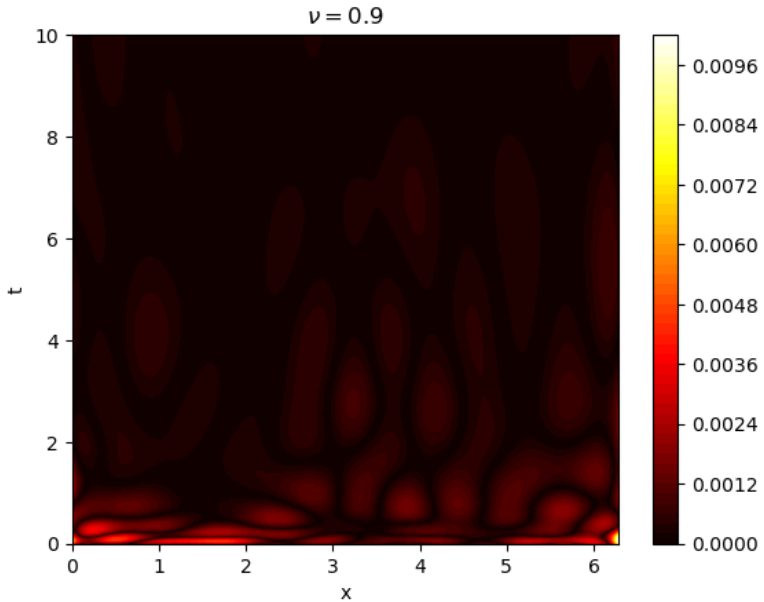
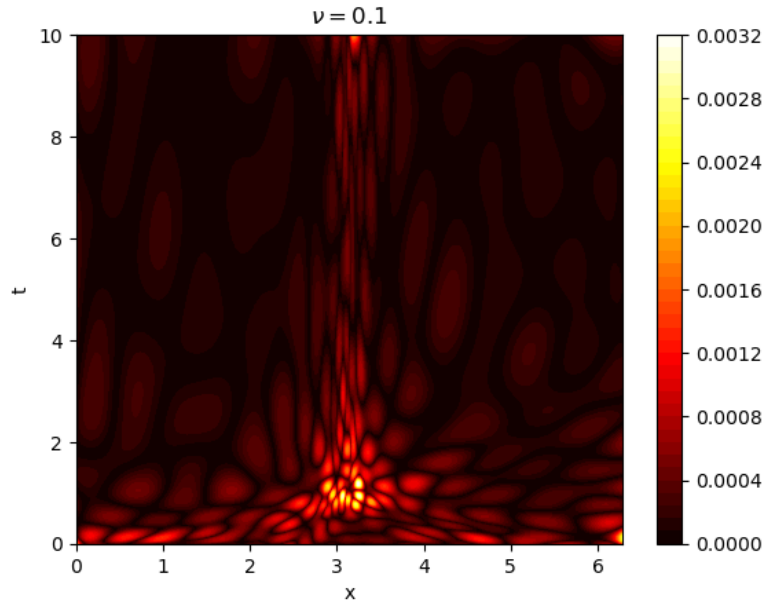
t
x
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

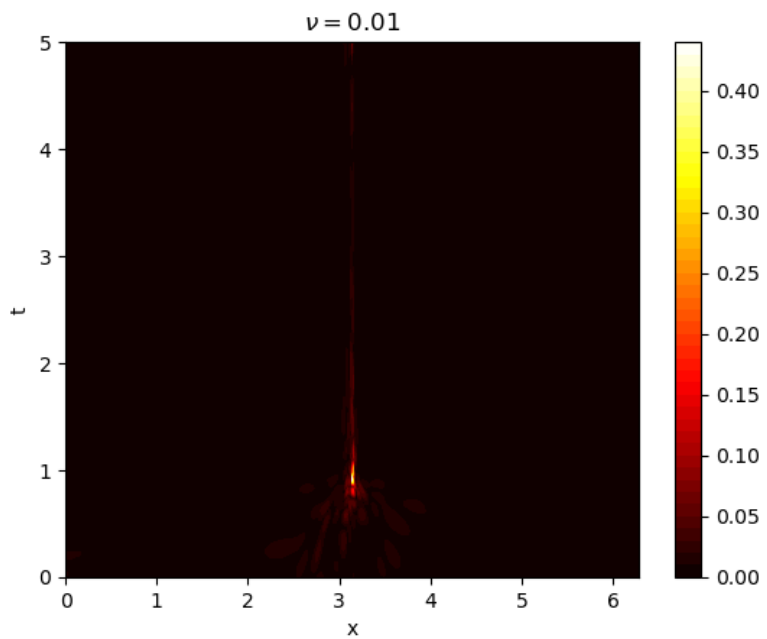

ν=0.1
t
x
1.92
1.44
0.96
0.48
0.00
−0.48
−0.96
−1.44
−1.92


ν=0.01
t
x
1.92
1.44
0.96
0.48
0.00
−0.48
−0.96
−1.44
−1.92

# Results

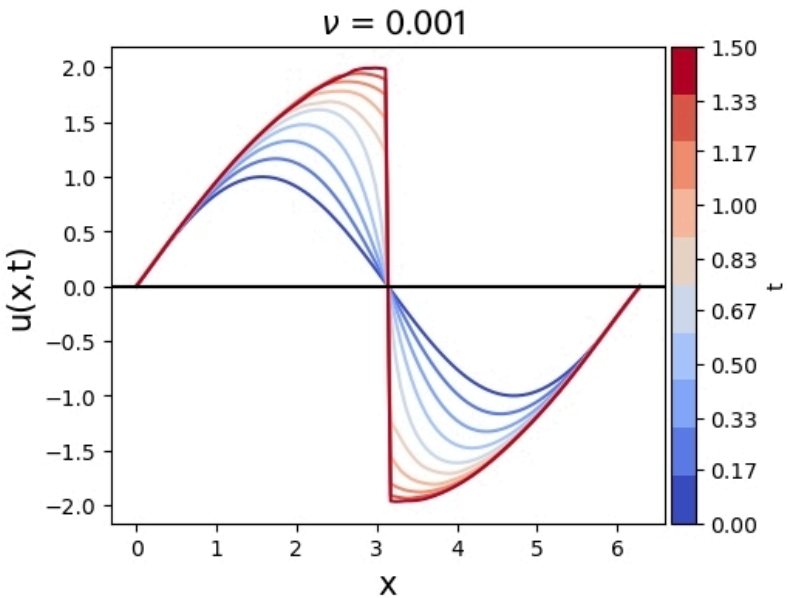

$\nu = 0.9$
t
x
0 1 2 3 4 5 6
0 2 4 6 8 10
0.0096
0.0084
0.0072
0.0060
0.0048
0.0036
0.0024
0.0012
0.0000

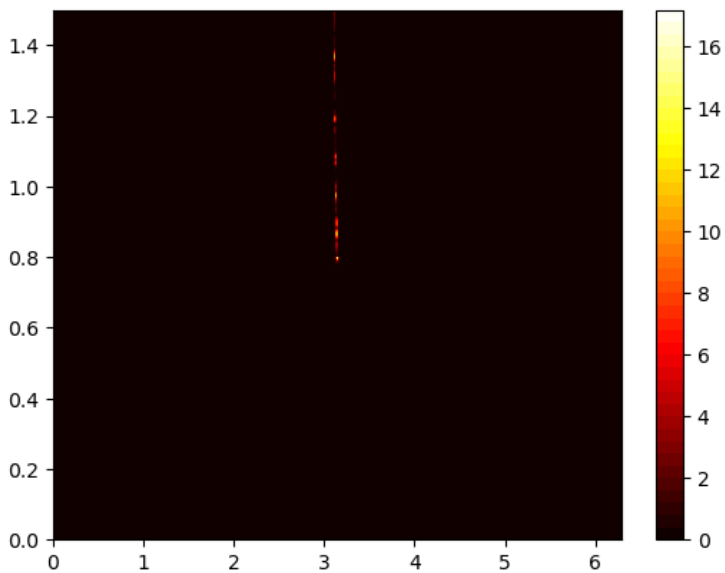

$\nu = 0.1$
t
x
0 1 2 3 4 5 6
0 2 4 6 8 10
0.0032
0.0028
0.0024
0.0020
0.0016
0.0012
0.0008
0.0004
0.0000


$\nu = 0.01$
t
x
0 1 2 3 4 5 6
0 1 2 3 4 5
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00

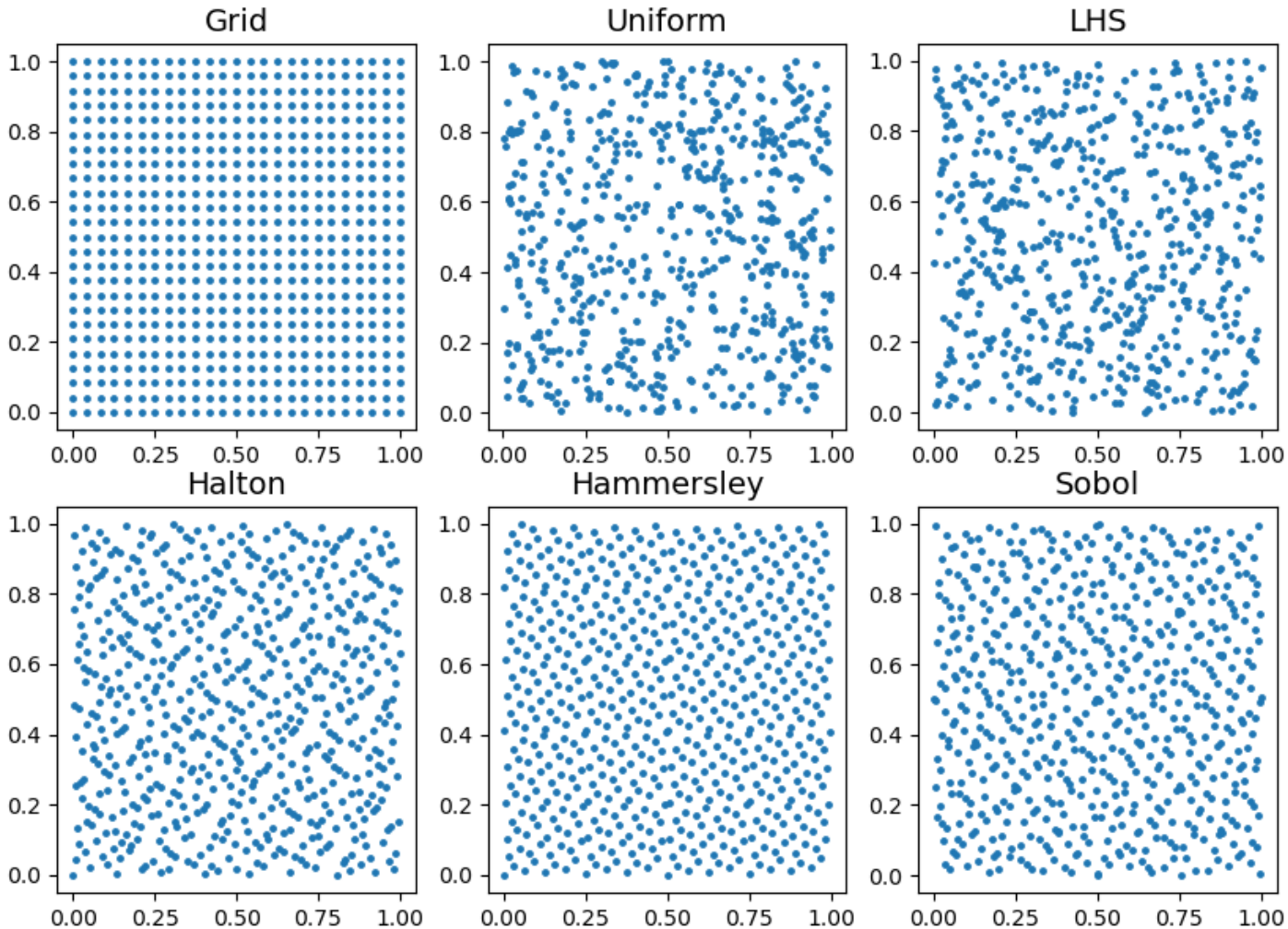
# Problems

- Over-fitting
- Loss of information
- Non-linearity



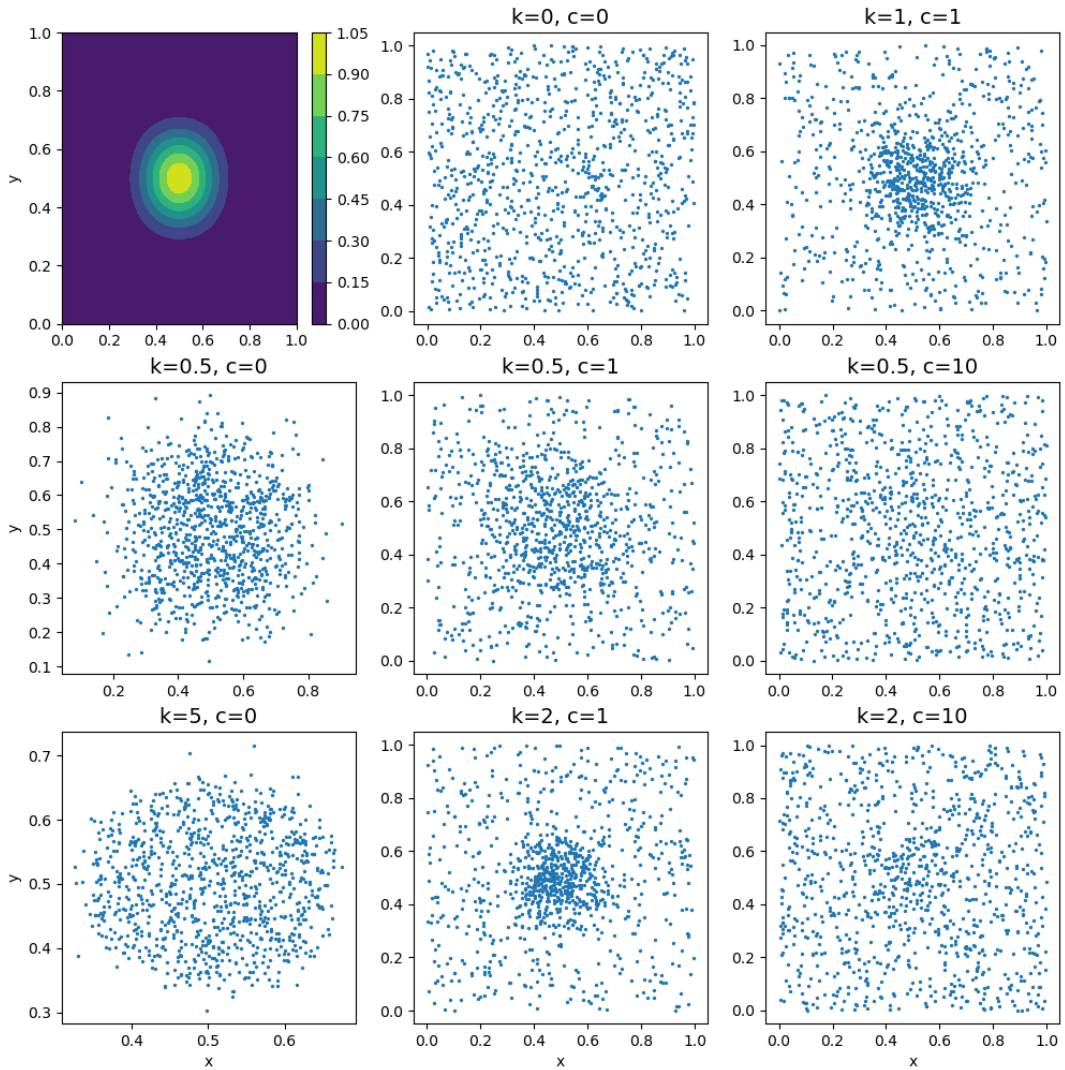
# Fixed Residual Point Sampling



# Resampling

- Residual points are resampled after some interval in any of the above sampling methods.

# Residual Based Adaptive Distribution(RAD)



# RAD

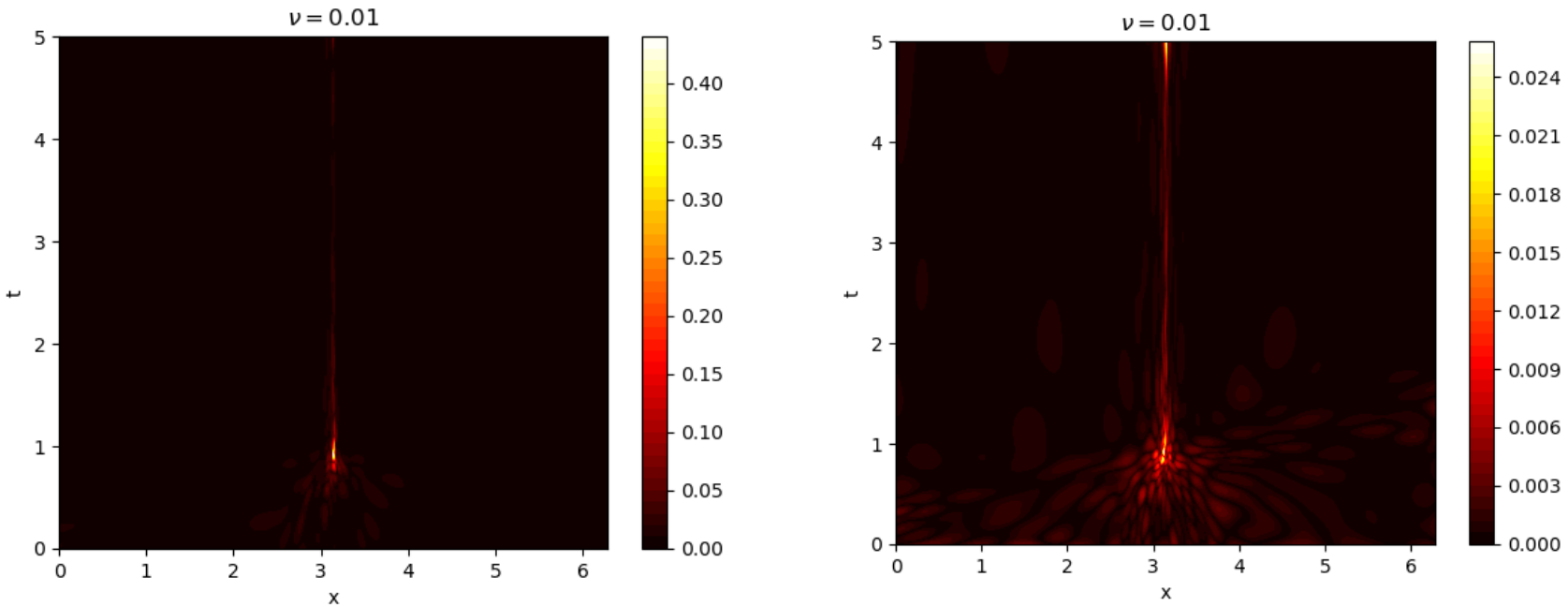
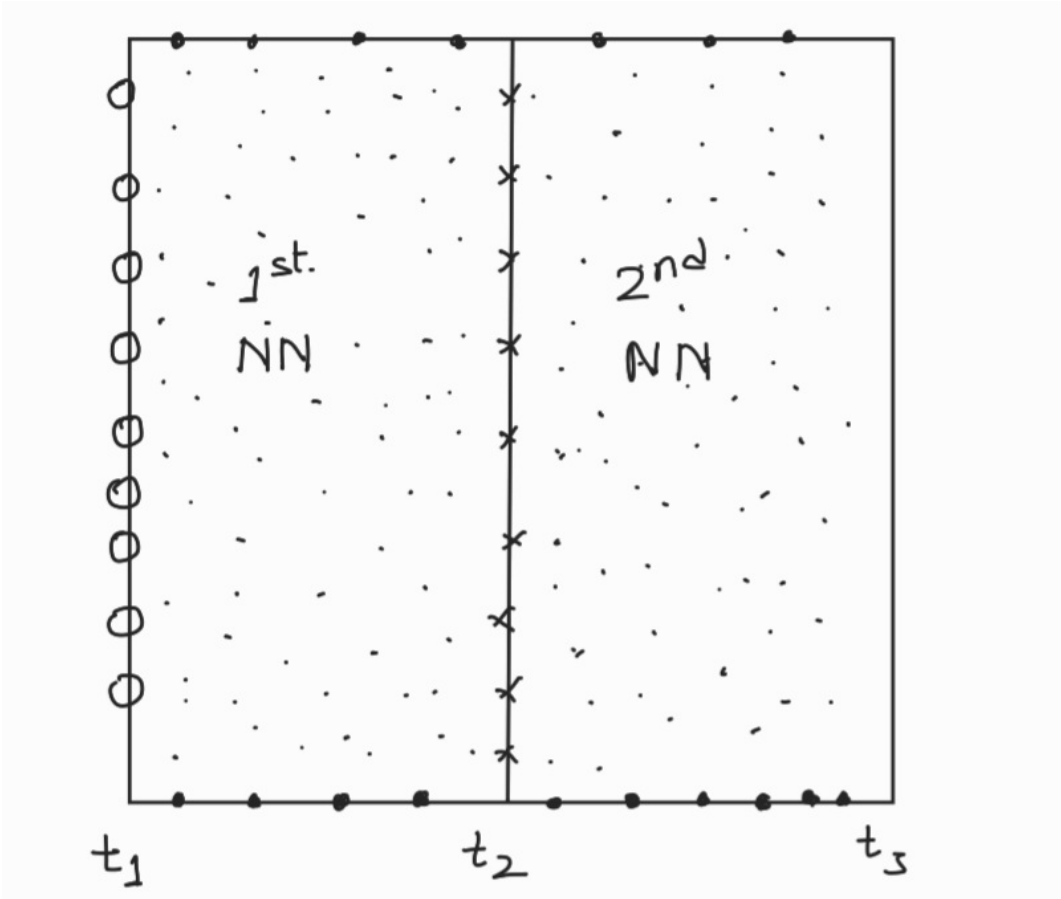
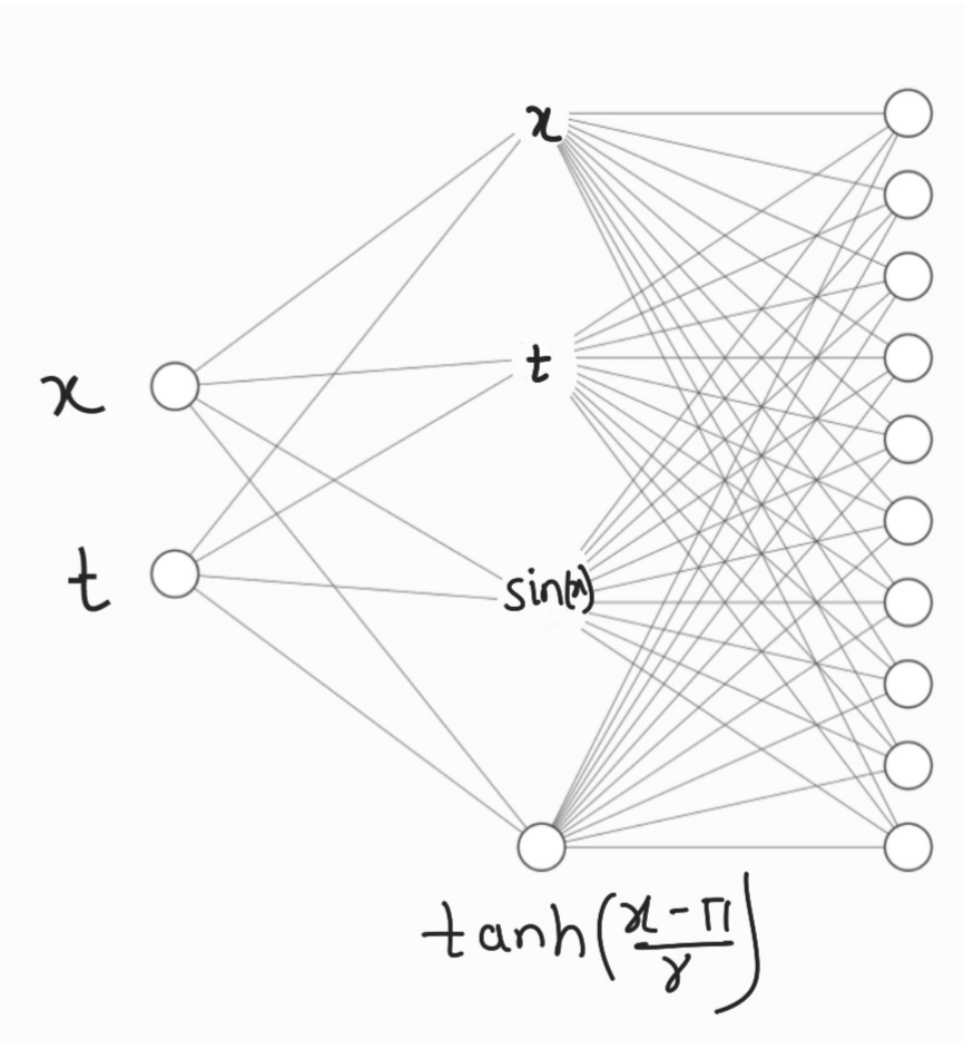


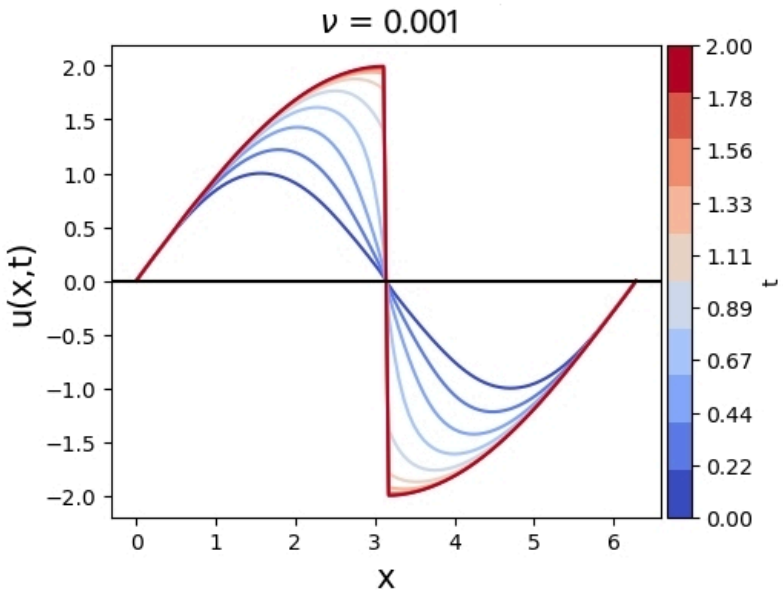

**Figure:** Velocity field for $\nu = 0.01$ after applying RAD

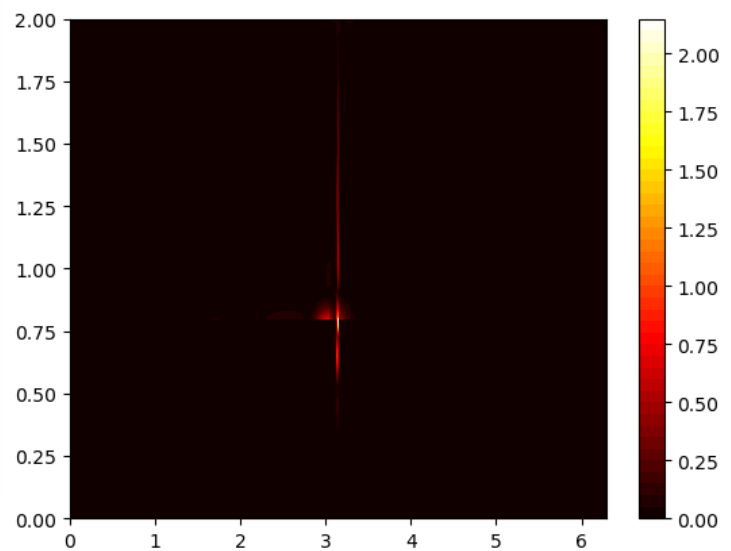
# Time Domain Decomposition



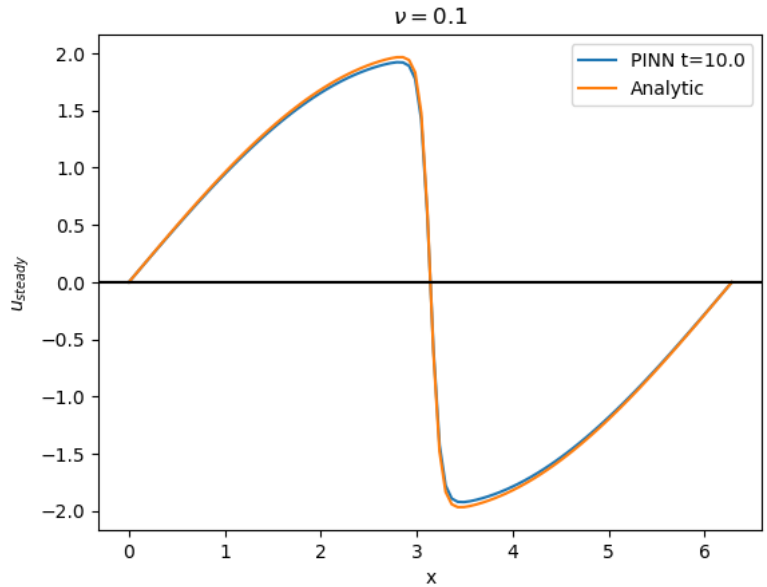
# Feature Transform

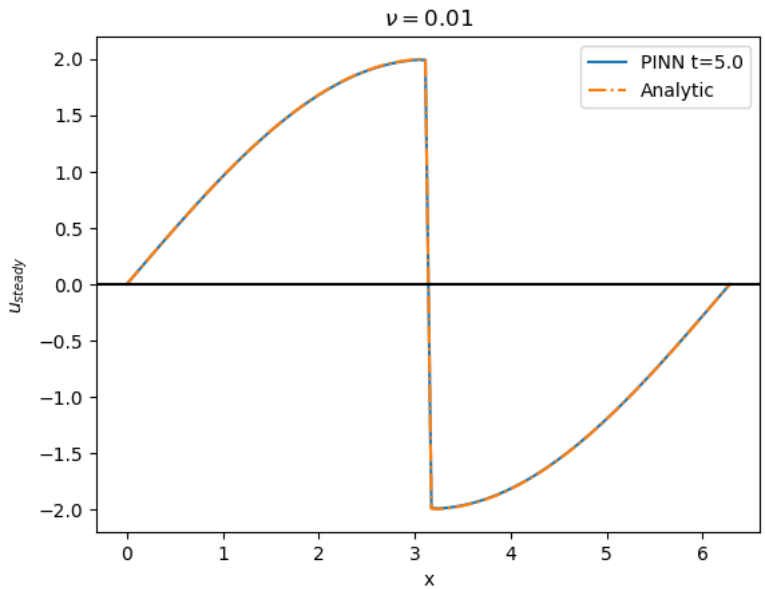


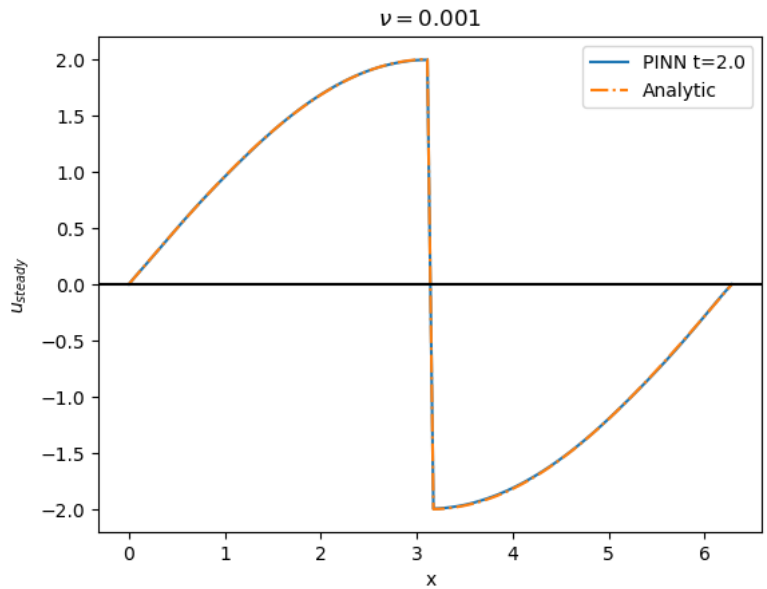



# Compare to Analytical Result



# Conclusion

- PINNs combines **strength of Neural Network** and **physics-based modeling** to solve physical problems accurately and efficiently.
- Still active area of research and need more development in some part.

# References

1. Scientific Machine Learning through Physics-Informed Neural Networks: Where we are and What's next.-Salvatore Cuomo1 , Vincenzo Schiano Di Cola, Fabio Giampaolo , Gianluigi Rozza , Maziar Raissi and Francesco Piccialli
2. Lu, Lu and Meng, Xuhui and Mao, Zhiping and Karniadakis, George Em, DeepXDE: A deep learning library for solving differential equations
3. P. Mehta, M. Bukov, C.-H. Wang et al. / Physics Reports 810 (2019) 1–124)
4. Physics Informed Deep Learning (Part I): Data-driven Solutions of Nonlinear Partial Differential Equations-Maziar Raissi, Paris Perdikaris and George Em Karniadakis

# Artificial Neural Network

- The primary component of an ANN is **'stylized neurons'**.
- A neuron consists of a **linear transformation** followed by a **non-linear activation function**. There are different types of Non -linear activation functions like Perceptrons, Sigmoid, Tanh, ReLU, ELU, etc.
- ANN consists of such neurons stacked in layers. A **Deep Neural Network(DNN)** has more than two hidden layers.

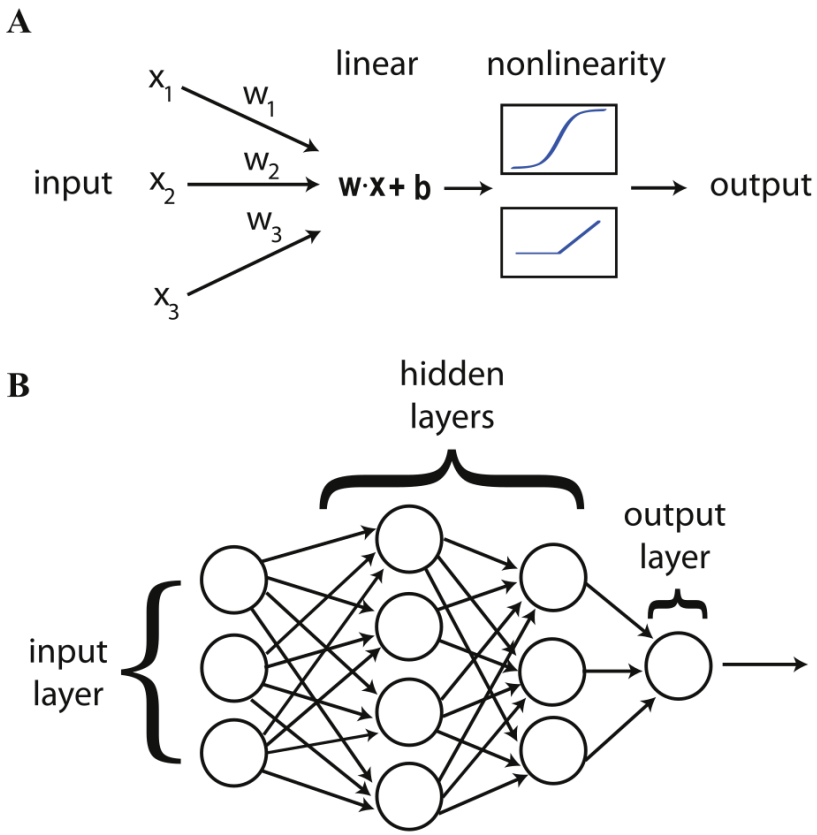



**Figure:** structure of Neural Network (Source: P. Mehta, M. Bukov, C.-H. Wang et al. / Physics Reports 810 (2019) 1–124)

# Artificial Neural Network

- **Universal Approximation Theorem:** A neural network with single hidden layer can approximate any continuous, multi-input/multi-output function with arbitrary accuracy.

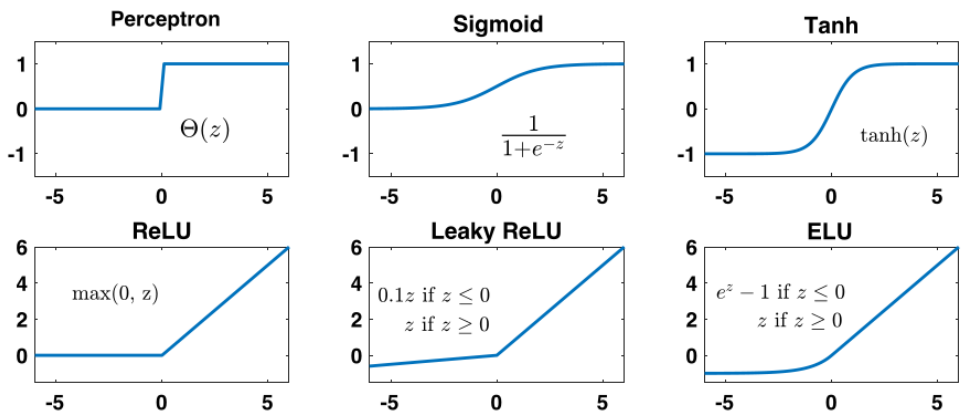



**Figure:** Some Non-linear Activation Functions (Source: P. Mehta, M. Bukov, C.-H. Wang et al. / Physics Reports 810 (2019) 1–124)

# DeepXDE

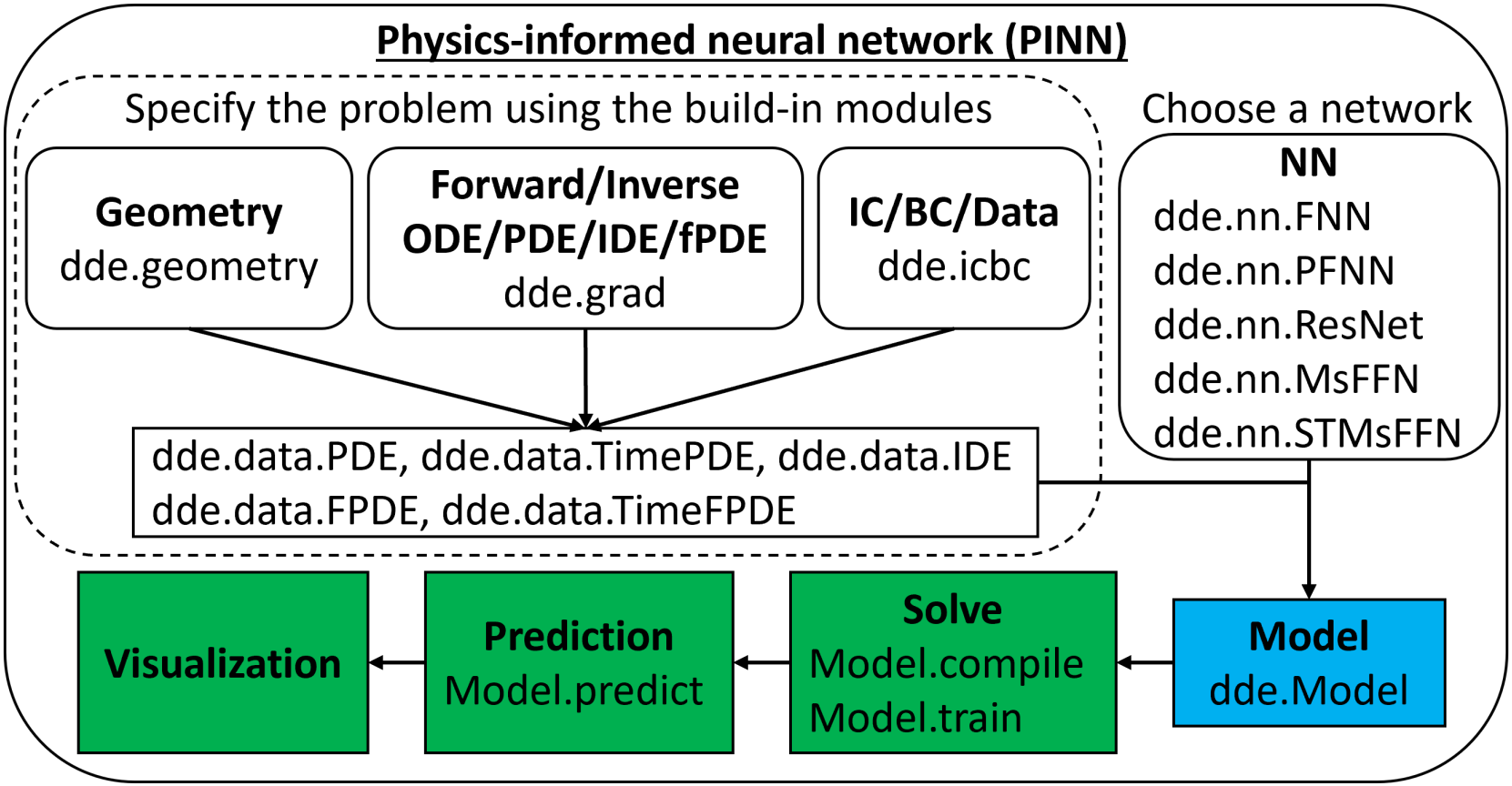



**Figure:** Implementation of PINN by DeepXDE (source:Lu, Lu and Meng, Xuhui and Mao, Zhiping and Karniadakis, George Em, DeepXDE: A deep learning library for solving differential equations)

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Thank You

भारतीय प्रौद्योगिकी संस्थान खड़गपुर

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