Physics Informed Neural Networks for solving Partial Differential Equations

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Contents

Introduction

- **Machine Learning(ML)** is a sub-field of Artificial Intelligence.
- ML has a goal to develop algorithms that can learn from data automatically.
- **Artificial Neural Network(ANN), Neural Network(NN), or Neural Net** is a model of Machine Learning which is inspired by **neurons in the human brain**.
- **PINN** is an application of ANN for solving Problems in physics.

Artificial Neural Network

- The primary component of an ANN is **'stylized neurons'**.
- A neuron consists of a **linear transformation** followed by a **non-linear activation function**. There are different types of Non -linear activation functions like Perceptrons, Sigmoid, Tanh, ReLU, ELU, etc.
- ANN consists of such neurons stacked in layers. A deep Neural Network(DNN) has more than two hidden layers.

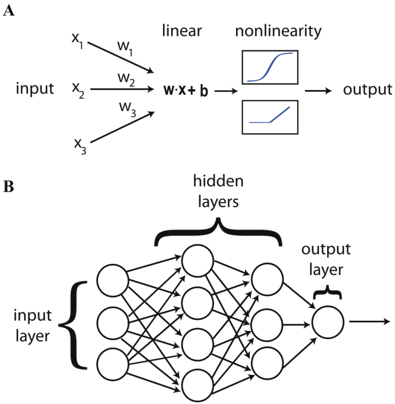


Figure: structure of Neural Network (Source: P. Mehta, M. Bukov, C.-H. Wang et al. / Physics Reports 810 (2019) 1–124)

Artificial Neural Network

- **Universal Approximation Theorem:** A neural network with single hidden layer can approximate any continuous, multi-input/multi-output function with arbitrary accuracy.

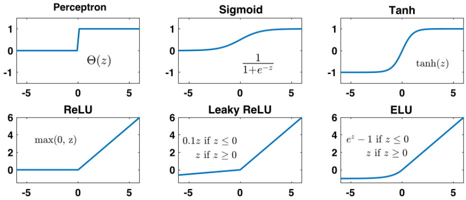


Figure: Some None Linear Activation Functions (Source: P. Mehta, M. Bukov, C.-H. Wang et al. / Physics Reports 810 (2019) 1–124)

Physics Informed Neural Network

- **Physics Informed Neural Network(PINN)** is a technique which uses Neural Network as a solution of a Partial Differential Equation.
- PINN approximates PDE solutions by minimizing a loss function that reflects the PDE, Boundary condition, Initial Condition, constraints etc.
- PINN has four parts
  1. Neural Network Architecture
  2. Auto Differentiation
  3. Loss Function
  4. Optimizer

PINN Building Blocks

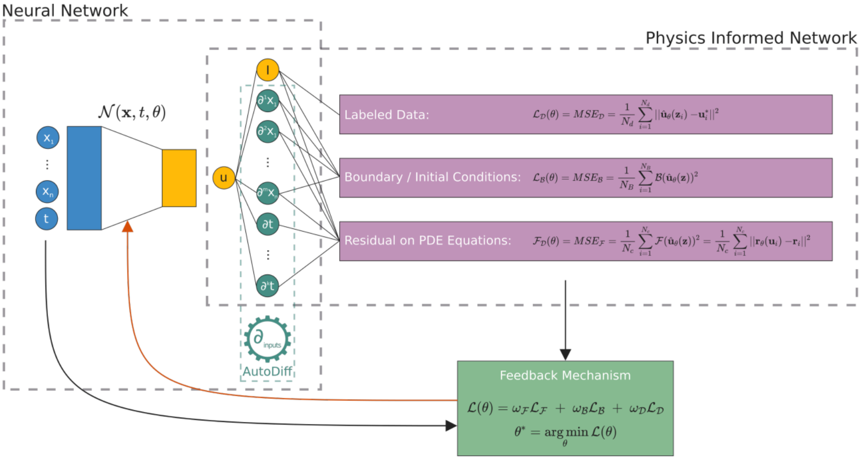


Figure: Building Block of PINN (source:Scientific Machine Learning through Physics-Informed Neural Networks,S Cuomo, V S di Cola, F Giampaolo, G Rozza, M Raissi, F Piccialli arXiv:2201.05624)

Solving Burgers' Equation

- I have solved **Burgers' Equation**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad x \in [-1, 1], \quad t \in [0, 1]$$

- Dirichlet boundary conditions and initial conditions are

$$u(-1, t) = u(1, t) = 0, \quad u(x, 0) = -\sin \pi x.$$

- I have solved for four values of viscosity $(\nu = 0.1, 0.01, 0.001, 0.0001)$

Results for Burgers' Equation

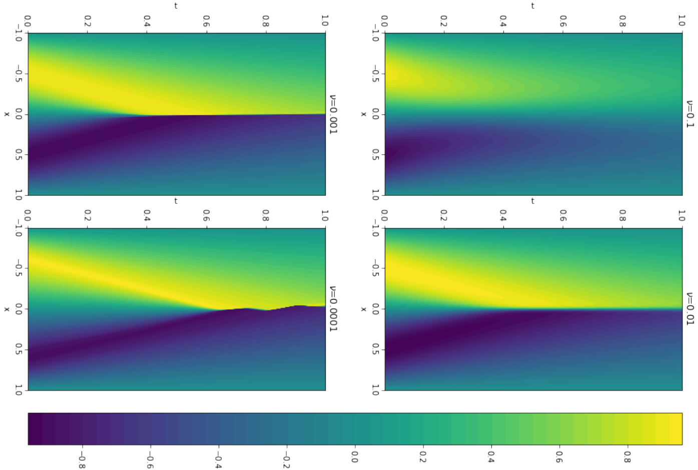


Figure: Solution plots of Burgers' Equation

Solving Heat Equation

- I have solved **1d Heat Equation**

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad x \in [0,1], \quad t \in [0,1]$$

- Dirichlet boundary conditions:

$$u(0,t) = u(1,t) = 0$$

- I have solved for two initial conditions
  - Sinusoidal initial condition(taking $\alpha = 0.4$)

$$u(x,0) = \sin \pi x$$

  - Dirac Delta initial condition(taking $\alpha = 0.008$)

$$u(x,0) = \delta(x - \frac{L}{2})$$

    Where length of the rod = L = 1. I have generated Dirac function by using Gaussian distribution with variance very less than one.

Results for Heat Equation

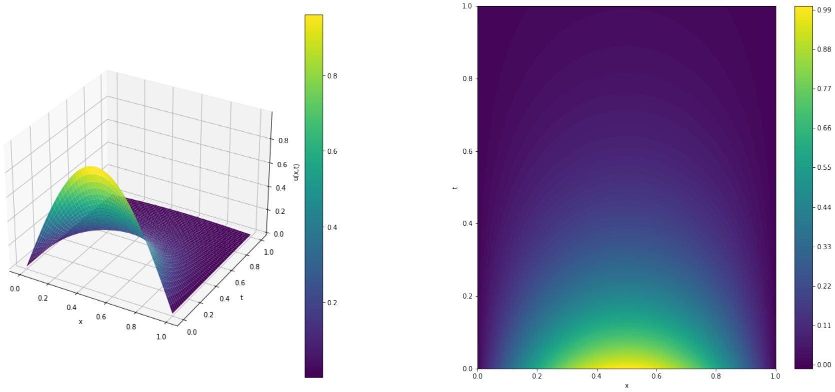
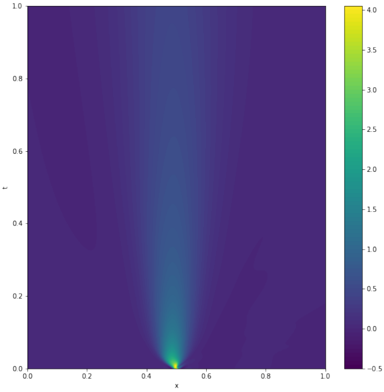


Figure: Plots of solution of Heat equation for periodic initial function

Results for Heat Equation



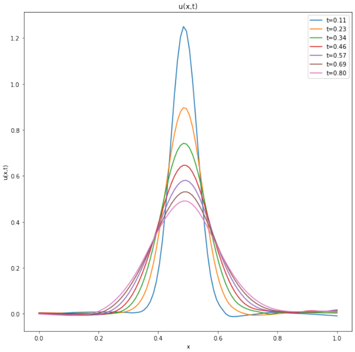


Figure: Plots of solution of Heat equation for Dirac delta initial function

Advantages of PINN

- A trained PINN can predict the values on the simulation grid of **different resolutions** without being retrained.
- A trained PINN can predict the values on the simulation grid of different resolutions without being retrained. PINN uses **automatic differentiation** to compute required derivatives which is superior to numerical or symbolical differentiation.
- Besides using experimental data, PINN uses the underlying physics during the network training. So
  - If the **size of the data is small**, it can give an accurate solution.
  - It can predict a precise value if the **boundary conditions are not provided**, and there is sufficient data.

References

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2. Lu, Lu and Meng, Xuhui and Mao, Zhiping and Karniadakis, George Em, DeepXDE: A deep learning library for solving differential equations
3. P. Mehta, M. Bukov, C.-H. Wang et al. / Physics Reports 810 (2019) 1–124)
4. Physics Informed Deep Learning (Part I): Data-driven Solutions of Nonlinear Partial Differential Equations-Maziar Raissi, Paris Perdikaris and George Em Karniadakis

DeepXDE

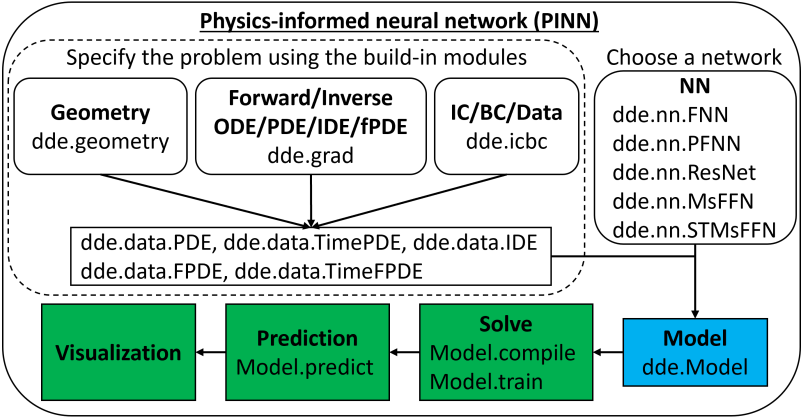

Figure: Implementation of PINN by DeepXDE (source:Lu, Lu and Meng, Xuhui and Mao, Zhiping and Karniadakis, George Em, DeepXDE: A deep learning library for solving differential equations)

Training data for Burgers' equation

ν	Optimizer	Training time (second)	final train loss
0.1	L-BFGS	128.997	2.05×10^{-06}
0.01	L-BFGS	214.848	4.50×10^{-06}
0.001	adam(lr=0.001) + L-BFGS	1417.553	2.05×10^{-06}
0.0001	adam(lr=0.001) + L-BFGS	1166.165	2.89×10^{-02}

Table: Training times and final losses for different viscosity

Training data for heat equation

Initial function	Optimizer	Training time (second)	final train loss
Periodic	adam(lr=0.001) + L-BFGS	145.807	6.72×10^{-07}
Dirac delta	adam(lr=0.001) + L-BFGS	145.332	3.60×10^{-02}

Table: Training times and final losses for different initial function

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