Asymmetric Oscillator with position-dependent mass

Diganta Samanta(21PH40020)
Sayantan Ghosh(21PH40047)
Sujay Rangdar(21PH40053)
Indian Institute of Technology Kharagpur

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Introduction

- A position-dependent mass (PDM) system is a physical system where the mass of an object or particle varies as a function of its position. In classical mechanics, the mass of an object is considered a constant property, but in quantum mechanics, PDM is used to describe systems that cannot be explained by a constant mass.
- We have studied its classical approach from the paper da Costa (2023): 012102.
- It has wide applications in semiconductors, nonlinear optics, many-body theory, etc.

Deformed Classical Oscillator with PDM

- Lagrangian of a PDM system,

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}m(x)(\dot{x}^2 - \omega_0^2 x^2) \tag{1}$$

Where m(x) is position-dependent mass and $\frac{1}{2}m(x)\omega_0^2 x^2$ is quadratic potential.

- Corresponding Hamiltonian is

$$\begin{aligned}\mathcal{H}(x,p) &= \dot{x}p - \mathcal{L} \\ &= \frac{1}{2}m(x)\dot{x}^2 + \frac{1}{2}m(x)\omega_0^2 x^2\end{aligned} \tag{2}$$

- The equation of motion becomes

$$m(x)(\ddot{x} + \omega_0^2 x) + \frac{1}{2}m\prime(x)(\dot{x}^2 + \omega_0^2 x^2) = 0 \tag{3}$$

- The mass function in the problem of a harmonic oscilltor with PDM introduced by Costa Filho et al. has the form

$$m(x) = \frac{m_0}{(1+\gamma x)^2}, \quad (x > -1/\gamma and \gamma > 0), \tag{4}$$

- So the Lagrangian becomes

$$\mathcal{L}(x, \dot{x}) = \frac{m_0}{2}\left[\frac{\dot{x}^2 - \omega_0^2 x^2}{(1+\gamma x)^2}\right] \tag{5}$$

- and the corresponding Hamiltonian is

$$\mathcal{H}(x, p) = \frac{(1+\gamma x)^2 p^2}{2m_0} + \frac{m_0 \omega_0^2 x^2}{2(1+\gamma x)^2} \tag{6}$$

The potential term $V(x) = \frac{m_0 \omega_0^2 x^2}{2(1+\gamma x)^2}$ is semi-confined since $lim_{x \to -\frac{1}{\gamma}} V(x) = +\infty$ and $lim_{x \to +\infty} V(x) = W_\gamma$ with well depth $W_\gamma = m_0 \omega_0^2 / 2\gamma^2$ depending on the deformation parameter γ.

- The motion equation is

$$\ddot{x} - \frac{\gamma \dot{x}^2}{1+\gamma x} + \frac{\omega_0^2 x}{1+\gamma x} = 0 \tag{7}$$

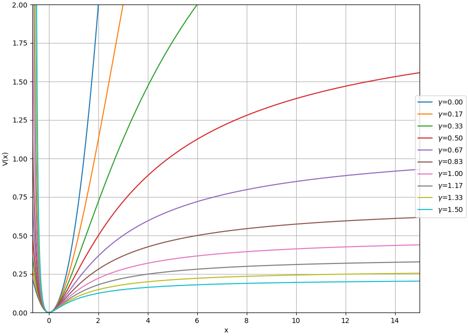

Figure: Potential for different γ

Linearize solution

If we write the ODE in the coupled form

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \frac{\gamma y^2}{1+\gamma x} - \frac{\omega_0^2 x}{1+\gamma x}\end{aligned} \tag{8}$$

The fixed point for this system is (0,0). The linearized form is

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\gamma^2 y^2+\omega_0^2}{(1+\gamma x)^2} & \frac{2\gamma y}{(1+\gamma x)^2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \tag{9}$$

at fixed point (0,0)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \tag{10}$$

So the eigenvalues of this linearized system are $\pm i\omega_0$. In linearized form, it is a harmonic oscillator.

Analytical Solution

Let the energy is $E = \frac{1}{2}m_0\omega_0^2 A_0^2$. If we substitute p by $m(x)\dot{x}$ in (6), the equation has the form

$$\frac{(1+\gamma x)^2}{2m_0}\frac{m_0^2\dot{x}^2}{(1+\gamma x)^4} + \frac{m_0\omega_0^2 x^2}{2(1+\gamma x)^2} = \frac{1}{2}m_0\omega_0^2 A_0^2 \qquad (11)$$

Taking $\Omega_\gamma = \omega_0\sqrt{1-\gamma^2 A_0^2}$, $\Lambda_\gamma = i\Omega_\gamma$ and $A_\gamma = \frac{A_0}{1-\gamma^2 A_0^2}$ we get

$$\frac{(x-\gamma A_0 A_\gamma)^2}{A_\gamma^2} + \frac{\dot{x}^2}{\Omega_\gamma^2 A_\gamma^2} = 1 \quad (0 \leq \gamma A_0 < 1), \qquad (12a)$$

$$\frac{\dot{x}^2}{\omega_0^2 A_0^2} = \frac{2x}{A_0} + 1 \quad (\gamma A_0 = 1), \qquad (12b)$$

$$\frac{(x-\gamma A_0 A_\gamma)^2}{A_\gamma^2} - \frac{\dot{x}^2}{\Lambda_\gamma^2 A_\gamma^2} = 1 \quad (\gamma A_0 > 1) \qquad (12c)$$

- The analytical solution is

$$x(t) = \begin{cases} A_\gamma(cos[\Omega_\gamma(t - t_0)] + \gamma A_0) & x \leq 0 \\ \frac{A_0}{2}[\omega_0^2(t - t_0)^2 - 1] \\ A_\gamma(-cosh[\Lambda_\gamma(t - t_0)] + \gamma A_0), & x \geq 0 \end{cases} \tag{13}$$

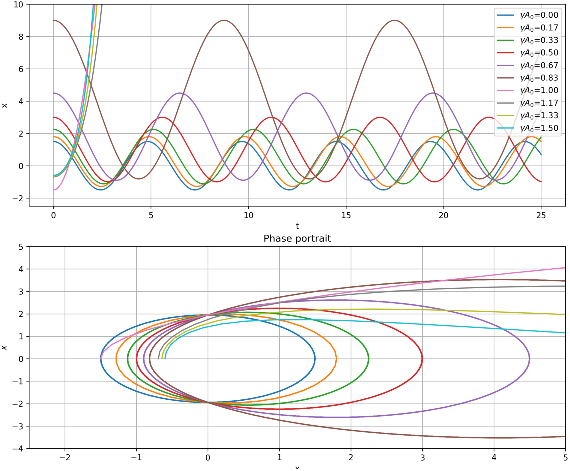

Figure: Phase potrait and x vs t graph for different deformation parameter.[Analytic]

Numerical solution

For numerical solution we use RK4 method

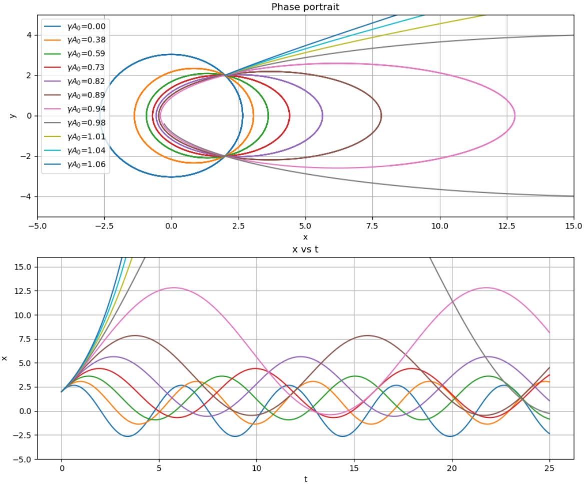

Figure: Phase potrait and x vs t graph for different deformation parameter[Numerical]

Final Remarks

In this short term, we have learned

- how a classical system is treated using the Hamiltonian and Lagrangian approaches.
- being position dependency on mass, how it is behaving like Harmonic Oscillator.
- how a nonlinear differential equation is treated and how RK4 method is behaving for this.

References

Bruno G. da Costa, Ignacio S. Gomez, and Biswanath Rath.
Exact solution and coherent states of an asymmetric oscillator with position-dependent mass.
Journal of Mathematical Physics, 64(1):012102, jan 2023.

Thank You

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INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR