Solution of an Asymmetric Oscillator with position-dependent mass

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Abstract

We take the problem of the deformed oscillator with position-dependent mass in classical formalisms by introducing the effect of the mass function in both kinetic and potential energies.

Keywords— Position-dependent mass, Asymmetric Oscillator, Non-linear system

1 Introduction

The position dependent mass system of asymmetric harmonic oscillator is a complex physical system that has been the subject of significant interest in recent years. This system involves a harmonic oscillator with a non-uniform mass distribution, meaning that the mass of the oscillator varies as a function of its position. As a result, the dynamics of the system are more complicated than those of a standard harmonic oscillator, and the system exhibits a number of interesting and potentially useful properties. In this thesis, we aim to explore the behavior of the position dependent mass system of asymmetric harmonic oscillator, using both analytical and numerical methods. Specifically, we will investigate the effects of the non-uniform mass distribution on the system classically.[1]

2 Deformed Classical Oscillator with Position-dependent Mass

One dimensional classical system with position-dependent mass(PDM) is characterized by Lagrangian

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m(x)(\dot{x}^2 - \omega_0^2 x^2) \tag{1}$$

Where m(x) is position-dependent mass and $\frac{1}{2}m(x)\omega_0^2x^2$ is quadratic potential. Corresponding Hamiltonian is

$$\mathcal{H}(x, p) = \dot{x}p - \mathcal{L}$$

Where p is conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial x} = m(x)\dot{x} \tag{2}$$

So the Hamiltonian is

$$\mathcal{H}(x, p) = \frac{1}{2}m(x)\dot{x}^2 + \frac{1}{2}m(x)\omega_0^2x^2 \tag{3}$$

If we consider the Euler-Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x}$$

The equation of motion becomes

$$m(x)(\ddot{x} + \omega_0^2 x) + \frac{1}{2}m\prime(x)(\dot{x}^2 + \omega_0^2x^2) = 0 \tag{4}$$

The mass function in the problem of a harmonic oscilltor with PDM introduced by Costa Filho et al. has the form

$$m(x) = \frac{m_0}{(1 + \gamma x)^2}, \quad (x > -1/\gamma and \gamma > 0), \tag{5}$$

So the Lagrangian becomes

$$\mathcal{L}(x, \dot{x}) = \frac{m_0}{2}\left[\frac{\dot{x}^2 - \omega_0^2x^2}{(1 + \gamma x)^2}\right] \tag{6}$$

and the corresponding Hamiltonian is

$$\mathcal{H}(x, p) = \frac{(1 + \gamma x)^2p^2}{2m_0} + \frac{m_0\omega_0^2x^2}{2(1 + \gamma x)^2} \tag{7}$$

The potential term $V(x) = \frac{m_0\omega_0^2x^2}{2(1+\gamma x)^2}$ is semi-confined since $lim_{x\to -\frac{1}{\gamma}}V(x) = +\infty$ and $lim_{x\to +\infty}V(x) = W_\gamma$ with well depth $W_\gamma = m_0\omega_0^2/2\gamma^2$ depending on the deformation parameter γ. The motion equation is

$$\ddot{x} - \frac{\gamma\dot{x}^2}{1 + \gamma x} + \frac{\omega_0^2x}{1 + \gamma x} = 0 \tag{8}$$

3 Linearize solution

If we write the ODE in the coupled form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{\gamma y^2}{1+\gamma x} - \frac{\omega_0^2 x}{1+\gamma x} \end{aligned} \tag{9}$$

The fixed point for this system is (0,0). The linearized form is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\gamma^2 y^2 + \omega_0^2}{(1+\gamma x)^2} & \frac{2\gamma y}{(1+\gamma x)^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{10}$$

at fixed point (0,0)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{11}$$

So the eigenvalues of this linearized system are $\pm i\omega_0$. In linearized form, it is a harmonic oscillator.

4 Analytical Solution

It is hard to get an analytical solution from (8) by straightforward technique. So we start from the Hamiltonian (7) by taking energy which is conserved. Let the energy is $E = \frac{1}{2} m_0 \omega_0^2 A_0^2$. If we substitute p by $m(x)\dot{x}$ in (7), the equation has the form

$$\frac{(1+\gamma x)^2}{2m_0} \frac{m_0^2 \dot{x}^2}{(1+\gamma x)^4} + \frac{m_0 \omega_0^2 x^2}{2(1+\gamma x)^2} = \frac{1}{2} m_0 \omega_0^2 A_0^2 \tag{12}$$

After simplification, it becomes

$$\begin{aligned} &\dot{x}^2 + \omega_0^2 x^2 = \omega_0^2 A_0^2 (1+\gamma x)^2 \\ \text{or, } &\frac{\dot{x}^2}{\omega_0^2(1 - A_0^2\gamma^2)} + (x - \frac{A_0^2 \gamma}{1 - A_0^2 \gamma^2})^2 = \frac{A_0^2}{(1 - A_0^2\gamma^2)^2} \end{aligned} \tag{13}$$

Taking $\Omega_\gamma = \omega_0 \sqrt{1 - \gamma^2 A_0^2}$, $\Lambda_\gamma = i\Omega_\gamma$ and $A_\gamma = \frac{A_0}{1-\gamma^2 A_0^2}$ we get

$$\frac{(x - \gamma A_0 A_\gamma)^2}{A_\gamma^2} + \frac{\dot{x}^2}{\Omega_\gamma^2 A_\gamma^2} = 1 \quad (0 \le \gamma A_0 < 1), \tag{14a}$$

$$\frac{\dot{x}^2}{\omega_0^2 A_0^2} = \frac{2x}{A_0} + 1 \quad (\gamma A_0 = 1), \tag{14b}$$

$$\frac{(x - \gamma A_0 A_\gamma)^2}{A_\gamma^2} - \frac{\dot{x}^2}{\Lambda_\gamma^2 A_\gamma^2} = 1 \quad (\gamma A_0 > 1) \tag{14c}$$

From these relations, we can get phase portraits and also analytical solutions. The phase space portrait $(x, \dot{x})$ presents classical bound trajectories for $0 \le \gamma A_0 < 1$ and half-infinite trajectories for $\gamma A_0 \ge 1$. The equations of the path are conic sections: ellipse($0 \le \gamma A_0 < 1$), parabola ($\gamma A_0 = 1$), and hyperbola ($\gamma A_0 > 1$). The solution is

$$x(t) = \begin{cases} A_\gamma(cos[\Omega_\gamma(t - t_0)] + \gamma A_0) & x \le 0 \\ \frac{A_0}{2}[\omega_0^2(t - t_0)^2 - 1] \\ A_\gamma(-cosh[\Lambda_\gamma(t - t_0)] + \gamma A_0), & x \ge 0 \end{cases} \tag{15}$$

5 Numerical Solution

In this section, we deal with this problem numerically. We take eq (9) and solve it by RK4 method for different deformation parameters γA_0. we take x = $\dot{x}$ = 2 at t=0 as initial conditions.
We can see that $\gamma A_0 = \sqrt{E/W_\gamma}$. In our solution ω_0^2= 1.3.
The figure shows that the value of γA_0 decides whether the system is in the bound state or not. If γA_0 is less than 1, then the system is in the bound state. Otherwise, the system is in a scattering state.

6 Final Remarks

In this short term, we have learned

- how a classical system is treated using the Hamiltonian and Lagrangian approaches.
- being position dependency on mass, how it is behaving like Harmonic Oscillator.
- how a nonlinear differential equation is treated and how RK4 method is behaving for this.

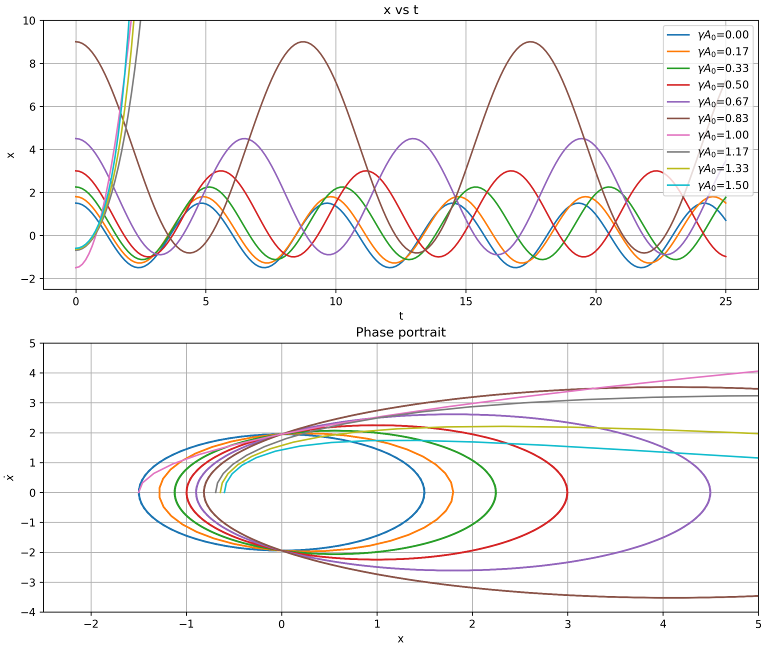

Figure 1: Phase potrait and x vs t graph for different deformation parameter.[Analytic]

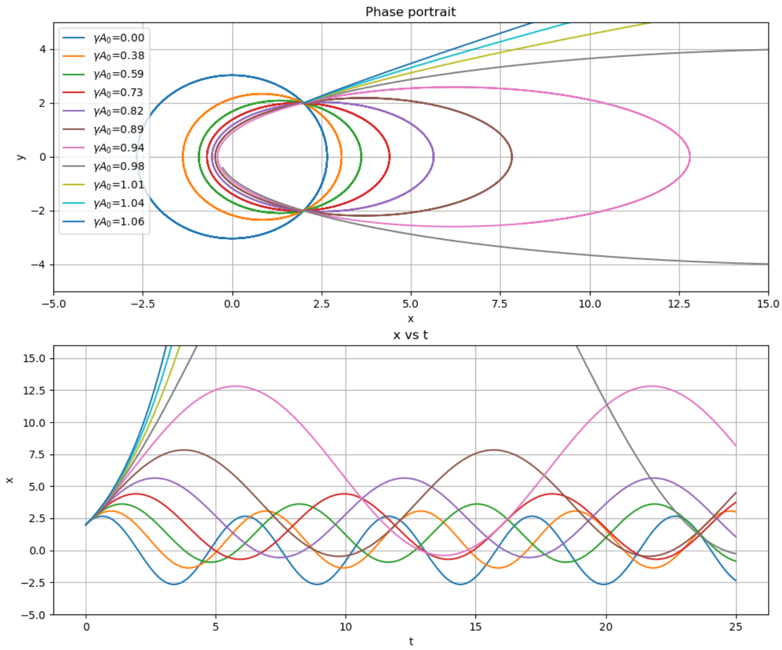

Figure 2: Phase potrait and x vs t graph for different deformation parameter[Numerical]

References

[1] Bruno G. da Costa, Ignacio S. Gomez, and Biswanath Rath. "Exact solution and coherent states of an asymmetric oscillator with position-dependent mass". In: *Journal of Mathematical Physics* 64.1 (Jan. 2023), p. 012102. DOI: 10.1063/5.0094564. URL: https://doi.org/10.1063%2F5.0094564.