Fixed-Notional Share Buyback Strategies

Mathematical Analysis and Critical Review

Strategy Analysis

November 2025

Problem Statement: Fixed-Notional Share Buyback

Objective: Execute a fixed USD amount over a trading period while minimizing execution cost.

Price Model: Geometric Brownian Motion (GBM)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Key Parameters:

- Total USD: \$1 billion
- Target Duration: 100 days (flexible: 75–125)
- Volatility (σ): 25% annualized
- Drift (μ): 0% (baseline)

Performance Metric:

$$\text{Performance (bps)} = -\frac{\text{VWAP} - \text{Benchmark}}{\text{Benchmark}} \times 10000$$

Positive = outperformance (bought cheaper than benchmark)

Strategy 1: Uniform Execution (Dollar-Cost Averaging)

Algorithm: Execute $\frac{\text{Total USD}}{\text{Target Duration}}$ each day for exactly `target_duration` days.

Mathematical Properties:

- Daily execution: $\text{USD}_t = \frac{\text{Total}}{\text{T}}$ (constant)
- Shares acquired: $\text{Shares}_t = \frac{\text{USD}_t}{P_t}$
- Total shares: $\sum_t \frac{\text{USD}_t}{P_t} = \text{USD} \cdot \sum_t \frac{1}{P_t}$

Key Result:

$$\text{VWAP} = \frac{\text{Total USD}}{\sum_t \text{Shares}_t} = \frac{T}{\sum_t \frac{1}{P_t}} = \text{Harmonic Mean}(P_1, \ldots, P_T)$$

$$\text{Benchmark} = \frac{1}{T} \sum_t P_t = \text{Arithmetic Mean}(P_1, \ldots, P_T)$$

Fundamental Inequality: H.M. $\leq$ A.M. (always)
$\Rightarrow$ **Strategy 1 has built-in positive expected performance!**

Strategy 2: Adaptive Execution

Algorithm: Adjust daily execution based on price vs. benchmark.

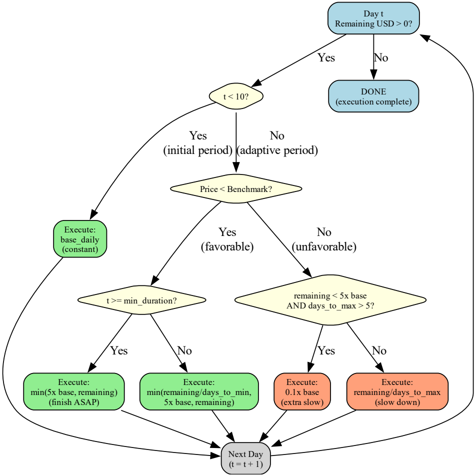
Three Phases:

1. **Initial Period (days 0–9):** Execute `base_daily` (constant)
2. **Favorable (Price $<$ Benchmark):**
   - Past `min_duration`: Execute up to $5\times$ base (finish ASAP)
   - Before `min_duration`: Speed up to finish by min_duration
3. **Unfavorable (Price $\geq$ Benchmark):**
   - Slow down to finish by `max_duration`
   - Extra-slow mode: If remaining $< 5\times$ base AND days_to_max > 5, use $0.1\times$ base

Benchmark Definition:

$$B(t) = \frac{1}{t+1}\sum_{i=0}^{t} P_i \quad \text{(Expanding Window Mean)}$$

Strategy 2: Decision Flowchart



Strategy 3: Discounted Benchmark

Modification: Same as Strategy 2, but compare price to discounted benchmark:

$$B'(t) = B(t) \times \left(1 - \frac{\text{discount\_bps}}{10000}\right)$$

Example: 100 bps discount $\Rightarrow$ compare price to 99% of benchmark

Effect:

- Price appears MORE favorable relative to $B'(t)$
- "Favorable" path triggered more often
- Average execution duration decreases
- Strategy becomes more aggressive

Implementation Note: Final performance also calculated against discounted benchmark (verified in code).

CRITIQUE: Benchmark Period Inconsistency

Problem: Decision benchmark $\neq$ Performance benchmark

Decision Benchmark	Expanding mean at day t: $\frac{1}{t+1}\sum_{i=0}^{t} P_i$
Performance Benchmark	Mean over *actual* execution period

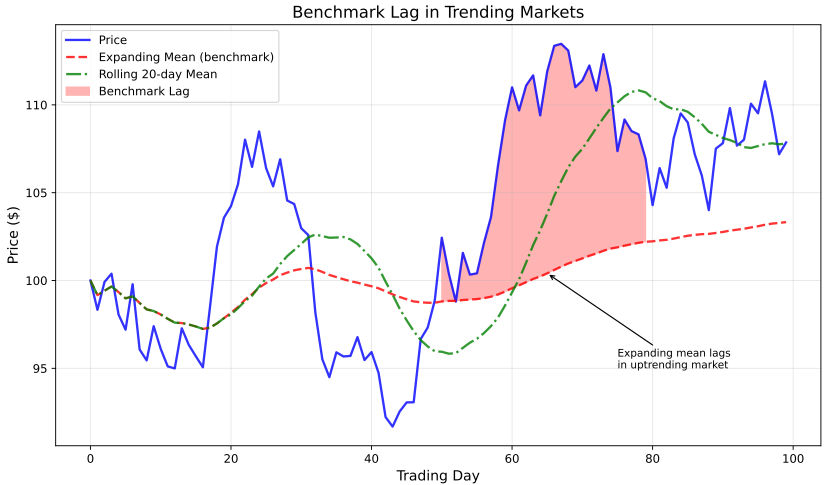
The Issue:

- If strategy finishes at day 75: Benchmark $= \frac{1}{75}\sum_{i=0}^{74} P_i$
- If strategy finishes at day 125: Benchmark $= \frac{1}{125}\sum_{i=0}^{124} P_i$
- These are **different benchmarks** for the same underlying path!

Self-Fulfilling Prophecy:

- When price $<$ benchmark, strategy finishes early
- Early finish $\Rightarrow$ benchmark calculated over fewer (favorable) days
- Artificially inflates apparent performance

CRITIQUE: Backward-Looking Benchmark



Problem: Expanding mean LAGS price in trending markets

- **Uptrend:** Price > Benchmark continuously ⇒ delays execution
- **Downtrend:** Price < Benchmark ⇒ accelerates into falling prices

CRITIQUE: Implementation Issues

1. **Arbitrary Multipliers:**
   - $5\times$ speedup has no theoretical justification
   - $0.1\times$ slowdown ("extra-slow") lacks economic basis
   - Should be derived from market impact models
2. **Binary Decision Logic:**
   - Sharp threshold at Price = Benchmark
   - Creates discontinuous execution behavior
   - Small price changes cause large execution swings
3. **No Transaction Costs:**
   - Assumes zero market impact
   - $5\times$ acceleration unrealistic for large buybacks
   - Real-world: execution speed affects price
4. **No Drift Consideration:**
   - Strategy ignores expected price trend
   - In positive drift markets, delay is costly

Suggested Improvements

1. **Fixed Benchmark Period:**
   - Always use `target_duration` days for benchmark
   - Eliminates self-fulfilling performance measurement
2. **Rolling Window Benchmark:**
   - Replace expanding mean with 20-day rolling mean
   - More responsive to recent price movements
   - Reduces lag in trending markets
3. **Smooth Transition Function:**
   - Replace binary threshold with sigmoid function
   - Example: multiplier $= f\left(\frac{P_t - B_t}{B_t}\right)$
4. **Market Impact Model:**
   - Execution amount should depend on liquidity
   - Almgren-Chriss framework for optimal execution

Key Takeaways

1. **Strategy 1 (Uniform):**
   - Simple and mathematically sound
   - VWAP = Harmonic Mean $\leq$ Arithmetic Mean = Benchmark
   - Built-in positive expected performance (dollar-cost averaging)
2. **Strategy 2/3 (Adaptive):**
   - Attempts "buy low" logic
   - Critical flaw: variable benchmark period
   - Backward-looking benchmark creates perverse incentives
3. **Main Issue:**
   - Variable benchmark period creates inconsistent performance measurement
   - Simulation results may be misleading
4. **Recommendation:**
   - Fix benchmark definition before trusting simulation results
   - Consider rolling window and smooth transition functions