07. Sigmoid Saturation

Neural Networks - From Brain to Business

Learning Goal

Understand the vanishing gradient problem caused by sigmoid saturation.

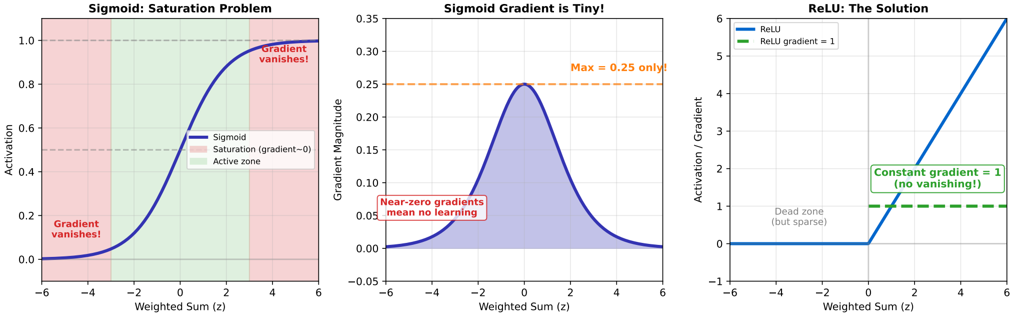
Key Concept (1/2)

The sigmoid function **saturates** at extreme values. When z is very large (e.g., z ¿ 3) or very small (z ¡ -3), the sigmoid output is nearly 1 or 0, and barely changes regardless of input changes.

This creates the **vanishing gradient problem**. During backpropagation, gradients flow backward through the network. In saturated regions, gradients become extremely small (near zero), so weights barely update. The network stops learning.

The **maximum gradient** of sigmoid is only 0.25 (at $z = 0$). In deep networks, multiplying many small gradients results in effectively zero gradient for early layers - they learn very slowly or not at all.
ReLU solves this by having a constant gradient of 1 for positive inputs. However, ReLU has its own issue: neurons can "die" if z is always negative (gradient $= 0$).

Visualization



Key Formula

Sigmoid and its derivative:

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\sigma'(z) = \sigma(z)(1 - \sigma(z))$$

Maximum derivative:

$$\max(\sigma'(z)) = \sigma(0)(1 - \sigma(0)) = 0.5 \times 0.5 = 0.25$$

ReLU and its derivative:

$$\mathrm{ReLU}(z) = \max(0, z)$$

$$\mathrm{ReLU}'(z) = \begin{cases} 1 & z > 0 \\ 0 & z \leq 0 \end{cases}$$

Imagine pushing a ball on different surfaces: - **Sigmoid at extremes**: Like pushing a ball on a flat plateau. No matter how hard you push, it barely moves. (Small gradient = little learning) - **Sigmoid at center**: Gentle slope, ball rolls smoothly. (Gradient = 0.25) - **ReLU positive region**: Constant slope, ball rolls consistently. (Gradient = 1) - **ReLU negative region**: Another flat plateau, ball doesn't move.

For efficient learning, we want consistent, non-vanishing gradients throughout.

Practice Problem 1

Problem 1
Calculate the sigmoid derivative at $z = 5$. How does it compare to the maximum derivative?

Solution

First, calculate sigmoid:

$$\sigma(5) = \frac{1}{1 + e^{-5}} = \frac{1}{1 + 0.0067} = 0.9933$$

Then, calculate derivative:

$$\sigma'(5) = \sigma(5)(1 - \sigma(5)) = 0.9933 \times 0.0067 = 0.0067$$

Comparison to maximum:

$$\frac{0.0067}{0.25} = 0.027 = 2.7\%$$

At $z = 5$, the gradient is only **2.7% of maximum**. Learning is extremely slow in this saturated region.

Practice Problem 2

Problem 2

A network has 5 layers, each with sigmoid activation. If each layer's gradient is 0.2, what is the total gradient that reaches the first layer?

Solution

Gradients multiply through layers (chain rule):

$$\begin{aligned} \text{Total gradient} &= 0.2 \times 0.2 \times 0.2 \times 0.2 \times 0.2 = 0.2^5 \\ &= 0.00032 \end{aligned}$$

The first layer receives a gradient of only **0.00032** - practically zero!
With learning rate 0.1:

$$\Delta w = 0.1 \times 0.00032 = 0.000032$$

Weight updates are negligible. The first layer cannot learn effectively.
This is the vanishing gradient problem.

Key Takeaways

- Sigmoid saturates at extreme values (gradient near 0)
- Maximum sigmoid gradient is only 0.25
- Vanishing gradients prevent learning in deep networks
- ReLU has constant gradient $= 1$ for positive inputs
- Modern networks use ReLU (or variants) for hidden layers