Module 2: Stacking Layers

The Multi-Layer Perceptron and Universal Approximation (1969-1986)

Neural Networks for Finance

BSc Lecture Series

November 26, 2025

Where We Left Off

Module 1 Summary
We learned that a single perceptron:

- Takes weighted inputs
- Applies a threshold
- Outputs a binary decision
- Can only draw **linear** boundaries

The Perceptron Equation:

$$y = f\left(\sum_{i=1}^{n} w_i x_i + b\right)$$

The Problem
The perceptron cannot solve XOR or any non-linearly separable problem.

The AI Winter:

- Minsky-Papert (1969) critique
- Funding dried up
- "Neural networks don't work"

Today's Question:
What if we stack multiple perceptrons together?

The perceptron: powerful but limited

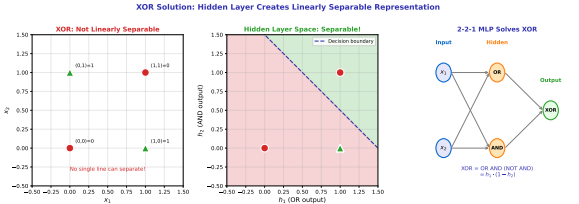
The XOR Problem Revisited

Why One Line Isn't Enough

x_1	x_2	XOR
0	0	0
0	1	1
1	0	1
1	1	0

The Geometry:

- Opposite corners have same label
- No single line can separate them
- We need *multiple* boundaries



xor_solution_ml

Some patterns require more than a single line

The Finance Parallel

Single Analyst (Perceptron)
One junior analyst screening stocks:

- Looks at a few metrics
- Applies simple rules
- Makes direct decisions
- Limited perspective

Limitation:
"Buy if P/E < 15 AND momentum > 0"
This is a single linear rule.

Investment Team (MLP)
A hierarchical team:

- Junior analysts find patterns
- Senior analysts synthesize
- CIO makes final call
- Complex reasoning emerges

Capability:
"Consider value metrics, momentum signals, AND market regime together"
Multiple non-linear patterns.

Key Insight: Hierarchical processing enables complex pattern recognition.

A single analyst sees simple patterns. A team sees complex ones.

Module 2 Roadmap

What We'll Cover

1. **Historical Context**
   - AI Winter survival
   - Backprop rediscovery (1986)
2. **MLP Architecture**
   - Intuition: The firm analogy
   - Math: Matrix notation
3. **Activation Functions**
   - Why non-linearity matters
   - Sigmoid, Tanh, ReLU
4. **Universal Approximation**
   - The fundamental theorem
   - Implications and limits
5. **Loss Functions**
   - MSE for regression
   - Cross-entropy for classification

Learning Objectives:

- Understand MLP architecture
- Master matrix notation
- Know when to use which activation
- Appreciate universal approximation

From single perceptron to universal function approximation

The AI Winter (1969-1982)

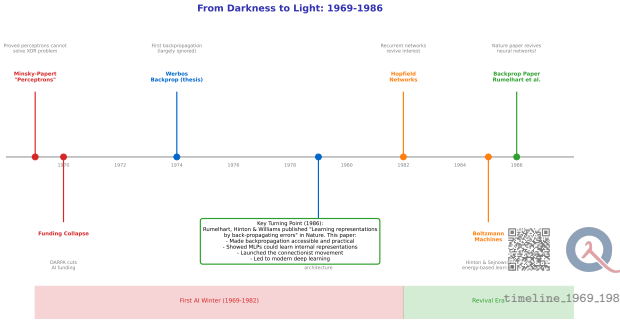
After Minsky-Papert

The neural network winter:

- Government funding cut
- Researchers moved to other fields
- “Connectionism is dead”
- Symbolic AI dominated

The Mood:

- Perceptrons can’t solve XOR
- Multi-layer networks exist but...
- No efficient training algorithm
- Why bother?



After Minsky-Papert, neural network research nearly died

Underground Progress

Paul Werbos (1974)
PhD thesis at Harvard:

- Derived backpropagation
- For general non-linear systems
- Applied to neural networks
- Largely ignored

Why Ignored?

- Published in economics, not CS
- AI winter was at its coldest
- No computational power to test
- No community to spread ideas

Parallel Discoveries

1970s:

- Linnainmaa: automatic differentiation
- Control theory: similar ideas

1980s:

- Parker (1982): rediscovery
- LeCun (1985): independent work
- Rumelhart/Hinton/Williams (1986): fame

Lesson: Good ideas can be discovered multiple times before they "take off."

The key ideas existed but were ignored

1982: Hopfield Networks

John Hopfield
A physicist (not AI researcher) revived interest:

- Connected neural networks to physics
- Energy-based formulation
- Published in PNAS (prestigious)
- Showed neural nets could store memories

The Impact:

- Legitimized neural network research
- Attracted physicists to the field
- New mathematical tools
- Funding started returning

Why Physics Helped

Physics Connection:

- Neurons $\leftrightarrow$ spins in magnets
- Learning $\leftrightarrow$ energy minimization
- Networks $\leftrightarrow$ statistical mechanics

Finance Parallel:
Physicists would later apply similar ideas to:

- Option pricing
- Market dynamics
- Risk modeling
- Quantitative finance

John Hopfield: Physicist rediscovers neural networks

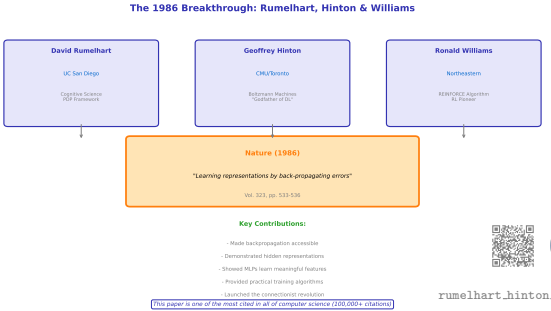
1986: The Backpropagation Paper

The Paper That Changed Everything
Rumelhart, Hinton, Williams in Nature (1986):
"Learning representations by back-propagating errors"

Key Contributions:

- Clear algorithm presentation
- Demonstrated on real problems
- Published in high-impact journal
- Well-communicated to broad audience

The Result:
Neural network renaissance begins.



Nature paper: "Learning representations by back-propagating errors"

What Made 1986 Different?

Werbos (1974)

+ Correct algorithm
+ General framework
- Wrong field (economics)
- No demonstrations
- No community
- No computers

Rumelhart et al. (1986)

+ Correct algorithm
+ Clear presentation
+ Compelling demos
+ High-profile venue (Nature)
+ Growing community
+ Computers available

Lesson for Researchers:
Being right isn't enough. You need:

- The right timing
- The right communication
- The right audience
- The right technology

The right idea at the right time with the right people

Discussion Question

"Backpropagation was discovered multiple times (1974, 1982, 1986). Why do some discoveries get ignored while others take off? What role did timing play?"

Consider:

- Publication venue matters
- Community readiness
- Computational infrastructure
- Demonstration quality
- Today: transformers (2017) exploded
- LSTMs existed since 1997
- What changed?

Think-Pair-Share: 3 minutes

The Neural Network Renaissance

After 1986
Neural networks were back:

- Funding returned
- New conferences (NIPS, now NeurIPS)
- "Connectionism" movement
- Real applications emerged

Key Milestones:

- 1989: LeNet for digit recognition
- 1990s: Speech recognition
- 1990s: Financial applications begin

But Challenges Remained

Not everything worked:

- Deep networks hard to train
- Vanishing gradients
- Limited compute power
- Another "winter" in 2000s

True Revolution: 2012
AlexNet on ImageNet marked the deep learning era. (Module 4)

But first, we need to understand the architecture...

Neural networks are back - and this time they can learn

The Investment Firm Analogy

Hierarchical Decision Making

Level 1: Junior Analysts (Hidden Layer 1)

- Look at raw data
- Find basic patterns
- "This looks like a value stock"
- "This has momentum"

Level 2: Senior Analysts (Hidden Layer 2)

- Combine junior reports
- Higher-level synthesis
- "Value + momentum = quality"

Level 3: CIO (Output Layer)

- Final buy/sell decision
- Combines all analyses
- Single decision point

Key Properties:

1. Information flows upward
2. Each level adds abstraction
3. Later layers see patterns in patterns
4. Final layer integrates everything

This is an MLP!

Hierarchical decision making

Input Layer: The Data Gatherers

The Input Layer
What it does:

- Receives raw data
- One neuron per feature
- No computation
- Just passes data forward

In Finance:

- P/E ratio
- Momentum (returns)
- Volume
- Volatility
- Sector indicators
- Market cap

Notation

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where:

- n = number of features
- x_i = value of feature i

Example (n=4):

$$\mathbf{x} = \begin{pmatrix} 15 \\ 0.08 \\ 1.2M \\ 0.25 \end{pmatrix} = \begin{pmatrix} \text{P/E} \\ \text{Return} \\ \text{Volume} \\ \text{Vol} \end{pmatrix}$$

The input layer receives raw information

Hidden Layers: The Pattern Finders

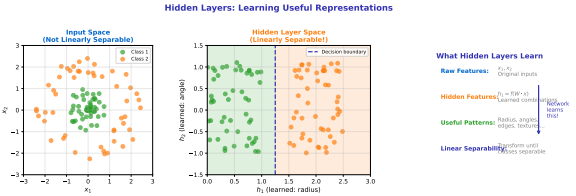
What Hidden Layers Do

They discover intermediate patterns:

- Not explicitly programmed
- Emerge from training
- Often uninterpretable
- But highly useful

Each Hidden Neuron:

- Receives weighted inputs
- Applies activation function
- Outputs a single number
- "Detects" a specific pattern



hidden_layer_representation

"They see things in the data you didn't explicitly ask for"

Finance Example: What Hidden Layers Find

Hypothetical Hidden Neurons

Hidden Neuron 1: "Value Detector"

- Positive weight on low P/E
- Positive weight on high book value
- Activates for value stocks

Hidden Neuron 2: "Momentum Detector"

- Positive weight on recent returns
- Positive weight on volume
- Activates for trending stocks

Hidden Neuron 3: "Risk Detector"

- Positive weight on volatility
- Positive weight on debt
- Activates for risky stocks

The Output Layer

Combines hidden neuron outputs:

$$\text{Buy} = f(w_1 \cdot \text{Value} + w_2 \cdot \text{Momentum} - w_3 \cdot \text{Risk})$$

Key Insight:

We never told the network what "value" or "momentum" means. It *discovered* these concepts from data.

Caveat:

Real hidden neurons may not be this interpretable. They might detect patterns we can't name.

Hidden neurons learn abstract concepts

Output Layer: The Final Decision

The Output Layer
Takes hidden representations and produces:

- Classification: probability of class
- Regression: continuous prediction
- Multiple outputs possible

For Binary Classification:
Single output neuron with sigmoid:

$$\hat{y} = \sigma(w^T h + b)$$

Output $\in (0, 1)$ interpreted as probability.

For Regression:
Single output neuron with no activation (or linear):

$$\hat{y} = w^T h + b$$

Output is predicted value.

Finance Examples

Buy/Sell Classification:

- Output: $P(\text{Buy})$
- If > 0.5: recommend Buy
- If < 0.5: recommend Sell

Return Prediction:

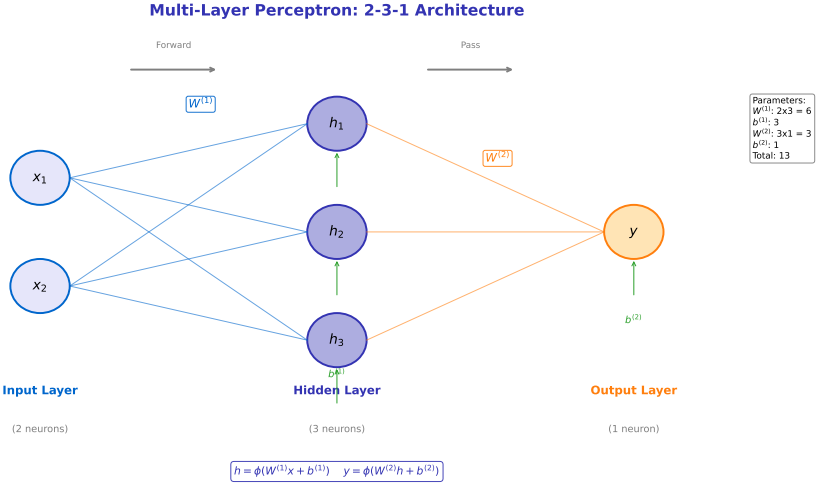
- Output: predicted return
- Could be next-day, next-month
- Continuous value

Multi-Class (Sector):

- Multiple output neurons
- Softmax activation
- Each output = probability of sector

The output layer synthesizes everything into a decision

The Full MLP Architecture



Why Are They Called "Hidden"?

We Don't Observe Them Directly
Observable:

- Input layer: the features we provide
- Output layer: the prediction we get

Hidden:

- Internal representations
- Not directly specified
- Learned automatically
- "Hidden" from us

We Don't Tell Them What to Learn

Traditional ML:
"Here are features: P/E, momentum, volume"
We engineer the features.

Deep Learning Philosophy:
"Here is raw data. Find useful patterns."
Network discovers features.

Trade-off:
More automatic, but less interpretable.

Hidden layers discover features automatically

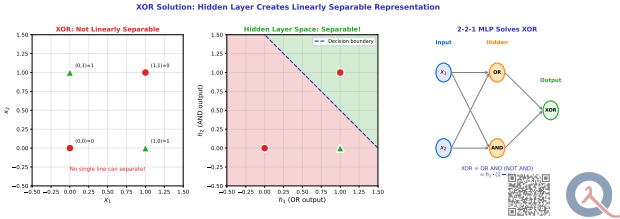
How MLPs Solve XOR

The Two-Hidden-Neuron Solution

Hidden Neuron 1:
Learns: "Is it in the upper-right region?"
$h_1 = \sigma(w_{11}x_1 + w_{12}x_2 + b_1)$

Hidden Neuron 2:
Learns: "Is it in the lower-left region?"
$h_2 = \sigma(w_{21}x_1 + w_{22}x_2 + b_2)$

Output Neuron:
Combines: "If h_1 XOR h_2, output 1"
Each hidden neuron draws *one* line. Together, they create a non-linear boundary.



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Multiple decision boundaries working together

Discussion Question

"If hidden layers find features automatically, why do we still need feature engineering in finance?"

Consider:

Arguments for Feature Engineering:

- Domain knowledge helps
- Less data needed
- More interpretable
- Faster training

Arguments Against:

- Human biases
- Miss non-obvious patterns
- Deep learning works on raw data
- ImageNet revolution

Reality: In finance, hybrid approaches often work best.

Think-Pair-Share: 3 minutes

Universal Approximation: The Big Promise

A Remarkable Theorem

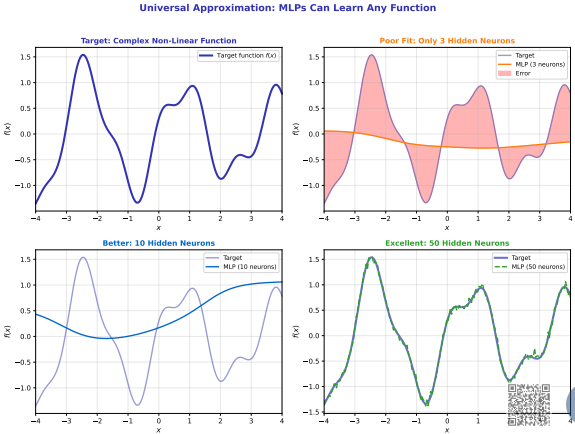
With just *one* hidden layer and enough neurons, an MLP can approximate **any** continuous function to arbitrary accuracy.

Implications:

- MLPs are universal function approximators
- No pattern is too complex (in theory)
- The architecture is not the bottleneck

Caveats:

- "Enough neurons" may be exponential
- Finding the right weights is hard
- Theory vs practice gap



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Universal Approximation Theorem (Cybenko, 1989): A feedforward network with a single hidden layer containing a finite number of neurons can approximate any continuous function on compact subsets of $\mathbb{R}^n$

MLPs can learn ANY pattern (in theory)

Now Let's Formalize

What You Already Know
From the intuition section:

- Layers process sequentially
- Each layer transforms its input
- Hidden layers find patterns
- Output layer makes predictions

What's Next

- Matrix notation for efficiency
- Precise forward pass equations
- Parameter counting
- Worked numerical examples

Why Matrix Notation?

Without Matrices:
Write $n \times m$ separate equations for each weight.

With Matrices:

$$\mathbf{h} = f(\mathbf{W}\mathbf{x} + \mathbf{b})$$

One equation captures everything.

Benefits:

- Compact notation
- Efficient computation (GPUs)
- Easier to implement
- Clearer understanding

You understand the intuition. Let's write it precisely.

Matrix Notation: Why Matrices?

Single Neuron (Scalar)

$$h = f(w_1x_1 + w_2x_2 + w_3x_3 + b)$$

As Dot Product:

$$h = f(\mathbf{w}^T\mathbf{x} + b)$$

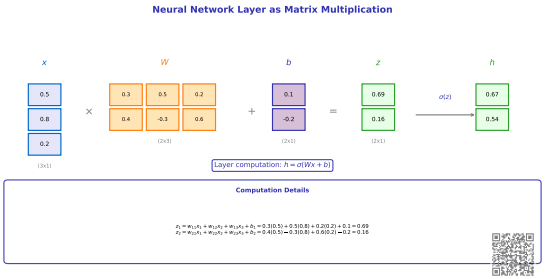
where $\mathbf{w}, \mathbf{x} \in \mathbb{R}^3$

Multiple Neurons (Matrix):

$$\mathbf{h} = f(\mathbf{W}\mathbf{x} + \mathbf{b})$$

where $\mathbf{W} \in \mathbb{R}^{m \times n}$

Each *row* of $\mathbf{W}$ is the weights for one hidden neuron.



matrix_multiplication_visua

Matrices make neural network math elegant

The Weight Matrix

Weight Matrix $\mathbf{W}^{(l)}$
For layer l:

$$\mathbf{W}^{(l)} \in \mathbb{R}^{n_l \times n_{l-1}}$$

where:

- n_l = neurons in layer l
- n_{l-1} = neurons in layer $l-1$

Entry $W_{ij}^{(l)}$:
Weight from neuron j in layer $l-1$ to neuron i in layer l.

Bias Vector $\mathbf{b}^{(l)}$

$$\mathbf{b}^{(l)} \in \mathbb{R}^{n_l}$$

One bias per neuron in layer l.

Example: 4-3 Layer
Input: 4 neurons, Hidden: 3 neurons

$$\mathbf{W}^{(1)} = \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \end{pmatrix}$$

Size: 3×4 (12 weights)
$\mathbf{b}^{(1)} \in \mathbb{R}^3$ (3 biases)

Each layer has its own weight matrix

Forward Pass: Layer by Layer

One Layer Computation

$$\mathbf{z}^{(l)} = \mathbf{W}^{(l)}\mathbf{a}^{(l-1)} + \mathbf{b}^{(l)}$$

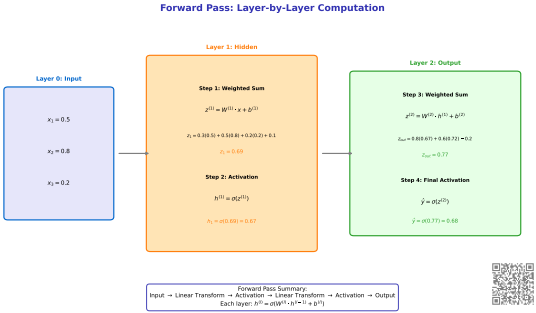
$$\mathbf{a}^{(l)} = f(\mathbf{z}^{(l)})$$

where:

- $\mathbf{z}^{(l)}$: pre-activation (weighted sum)
- $\mathbf{a}^{(l)}$: activation (after f)
- $\mathbf{a}^{(0)} = \mathbf{x}$: input

The Steps:

1. Matrix multiply: $\mathbf{W}^{(l)}\mathbf{a}^{(l-1)}$
2. Add bias: $+\mathbf{b}^{(l)}$
3. Apply activation: $f(\cdot)$



layer_by_layer_computatio

Computing outputs one layer at a time

The Complete Forward Pass

For an L-Layer Network

Input:

$$\mathbf{a}^{(0)} = \mathbf{x}$$

Hidden Layers ($l = 1, \ldots, L-1$):

$$\mathbf{z}^{(l)} = \mathbf{W}^{(l)}\mathbf{a}^{(l-1)} + \mathbf{b}^{(l)}$$

$$\mathbf{a}^{(l)} = f(\mathbf{z}^{(l)})$$

Output Layer:

$$\mathbf{z}^{(L)} = \mathbf{W}^{(L)}\mathbf{a}^{(L-1)} + \mathbf{b}^{(L)}$$

$$\hat{\mathbf{y}} = g(\mathbf{z}^{(L)})$$

where g may differ from f.

Example: 2-Layer Network

Layer 1 (hidden):

$$\mathbf{z}^{(1)} = \mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}$$

$$\mathbf{a}^{(1)} = \text{ReLU}(\mathbf{z}^{(1)})$$

Layer 2 (output):

$$\mathbf{z}^{(2)} = \mathbf{W}^{(2)}\mathbf{a}^{(1)} + \mathbf{b}^{(2)}$$

$$\hat{y} = \sigma(\mathbf{z}^{(2)})$$

Compact Form:

$$\hat{y} = \sigma(\mathbf{W}^{(2)}\text{ReLU}(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}) + \mathbf{b}^{(2)})$$

Chaining layer computations together

Dimensions Matter

Dimension Checking
For $\mathbf{z} = \mathbf{W}\mathbf{x} + \mathbf{b}$:

$\mathbf{W}$:	$(n_{\text{out}} \times n_{\text{in}})$
$\mathbf{x}$:	$(n_{\text{in}} \times 1)$
$\mathbf{W}\mathbf{x}$:	$(n_{\text{out}} \times 1)$
$\mathbf{b}$:	$(n_{\text{out}} \times 1)$
$\mathbf{z}$:	$(n_{\text{out}} \times 1)$

Rule:
Inner dimensions must match.
$(m \times \mathbf{n}) \times (\mathbf{n} \times p) = (m \times p)$

Example: 4-3-1 Network

Layer 1:

- $\mathbf{W}^{(1)}$: 3×4
- $\mathbf{x}$: 4×1
- $\mathbf{z}^{(1)}$: 3×1

Layer 2:

- $\mathbf{W}^{(2)}$: 1×3
- $\mathbf{a}^{(1)}$: 3×1
- $\mathbf{z}^{(2)}$: 1×1 (scalar)

Common Error: Transposed matrices. Always check dimensions!

Matrix dimensions must be compatible

Worked Example: 2-3-1 Network

Network Setup

Input: $\mathbf{x} = \begin{pmatrix} 0.5 \\ 0.8 \end{pmatrix}$

Layer 1 weights:

$$\mathbf{W}^{(1)} = \begin{pmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \\ 0.5 & 0.2 \end{pmatrix}$$

$$\mathbf{b}^{(1)} = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.0 \end{pmatrix}$$

Layer 2 weights:

$$\mathbf{W}^{(2)} = \begin{pmatrix} 0.6 & 0.3 & 0.4 \end{pmatrix}$$

$$b^{(2)} = -0.2$$

Forward Pass

Layer 1:

$$\mathbf{z}^{(1)} = \begin{pmatrix} 0.2(0.5) + 0.4(0.8) + 0.1 \\ 0.3(0.5) + 0.1(0.8) - 0.1 \\ 0.5(0.5) + 0.2(0.8) + 0.0 \end{pmatrix} = \begin{pmatrix} 0.52 \\ 0.13 \\ 0.41 \end{pmatrix}$$

$$\mathbf{a}^{(1)} = \text{ReLU}(\mathbf{z}^{(1)}) = \begin{pmatrix} 0.52 \\ 0.13 \\ 0.41 \end{pmatrix}$$

Layer 2:

$$z^{(2)} = 0.6(0.52) + 0.3(0.13) + 0.4(0.41) - 0.2 = 0.315$$

$$\hat{y} = \sigma(0.315) = 0.578$$

Output: 57.8% probability of class 1

Following the numbers through the network

Counting Parameters

Parameters per Layer
For layer l with n_{l-1} inputs and n_l outputs:

Weights: $n_l \times n_{l-1}$
Biases: n_l
Total: $n_l \times n_{l-1} + n_l = n_l(n_{l-1} + 1)$

Network Total:

$$\text{Params} = \sum_{l=1}^{L} n_l(n_{l-1} + 1)$$

Example: 4-10-5-1 Network

Layer 1 ($4 \to 10$):

$$10 \times 4 + 10 = 50$$

Layer 2 ($10 \to 5$):

$$5 \times 10 + 5 = 55$$

Layer 3 ($5 \to 1$):

$$1 \times 5 + 1 = 6$$

Total: 111 parameters

For 100 training samples: < 2 samples per parameter.
Risk of overfitting!

How many weights does your network have?

Discussion Question

"A 4-10-5-1 network has how many parameters? Calculate and discuss: is this a lot or a little for stock prediction?"

Answer: 111 parameters

Consider:

Stock Data Context:

- Daily data: ∼252 days/year
- 10 years = 2,520 samples
- 111 params: 23 samples/param
- Seems okay...

But Also Consider:

- Financial regimes change
- Not all data equally relevant
- Need train/val/test split
- Model complexity vs data size

Exercise: 3 minutes

Finance Example: Multi-Factor Stock Prediction

A Realistic Setup

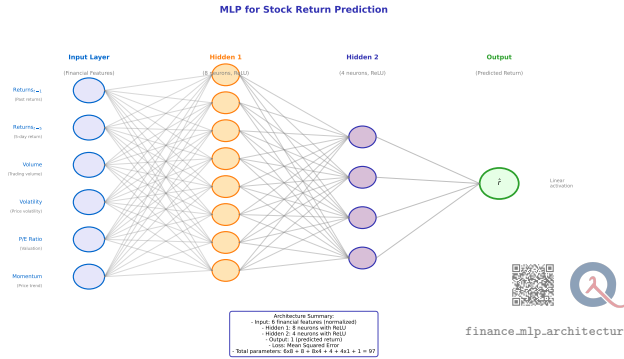
Input Features (10):

- P/E, P/B, EV/EBITDA (value)
- 1m, 3m, 6m returns (momentum)
- 20d volatility (risk)
- Volume ratio (liquidity)
- Sector one-hot (2 features)

Architecture:

- Hidden 1: 20 neurons (ReLU)
- Hidden 2: 10 neurons (ReLU)
- Output: 1 neuron (sigmoid)

Total: 441 parameters



Multiple factors combined through hidden layers

Why Non-Linearity?

The Core Problem
Without activation functions:

$$\mathbf{a}^{(1)} = \mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}$$
$$\hat{\mathbf{y}} = \mathbf{W}^{(2)}\mathbf{a}^{(1)} + \mathbf{b}^{(2)}$$

Substituting:

$$\hat{\mathbf{y}} = \mathbf{W}^{(2)}(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}) + \mathbf{b}^{(2)}$$
$$= (\mathbf{W}^{(2)}\mathbf{W}^{(1)})\mathbf{x} + (\mathbf{W}^{(2)}\mathbf{b}^{(1)} + \mathbf{b}^{(2)})$$
$$= \mathbf{W}'\mathbf{x} + \mathbf{b}'$$

Result: A single linear transformation!

The Solution
Non-linear activation functions:

$$\mathbf{a}^{(l)} = f(\mathbf{z}^{(l)})$$

where f is non-linear.

Why This Works:

- Non-linearity breaks the collapse
- Composition of non-linear functions
- Can approximate any function

Key Insight:
Non-linearity is what makes deep networks "deep" in a meaningful sense.

Non-linearity is essential for learning complex patterns

Linear Networks Collapse

Mathematical Proof
For any number of linear layers:

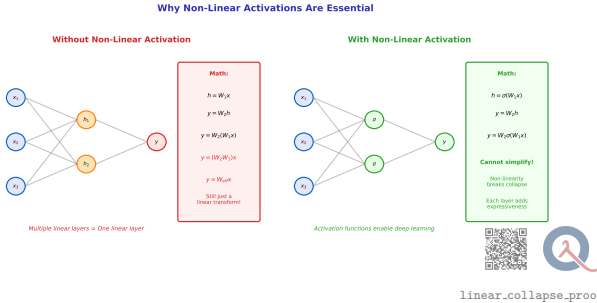
$$\mathbf{y} = \mathbf{W}^{(L)}\mathbf{W}^{(L-1)} \dots \mathbf{W}^{(1)}\mathbf{x}$$

Since matrix multiplication is associative:

$$= (\mathbf{W}^{(L)}\mathbf{W}^{(L-1)} \dots \mathbf{W}^{(1)})\mathbf{x}$$

$$= \mathbf{W}^{\text{eff}}\mathbf{x}$$

Conclusion:
100 linear layers = 1 linear layer.
No benefit from depth without non-linearity.



Stacked linear layers = single linear layer

The Sigmoid Function

Definition

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Properties:

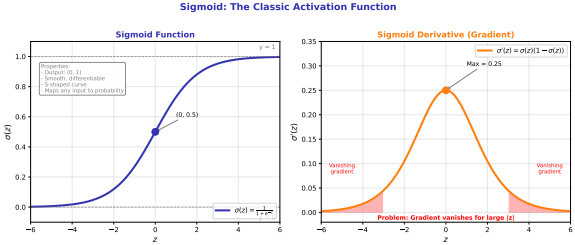
- Range: $(0, 1)$
- Smooth and differentiable
- $\sigma(0) = 0.5$
- Symmetric: $\sigma(-z) = 1 - \sigma(z)$

Derivative:

$$\sigma'(z) = \sigma(z)(1 - \sigma(z))$$

Use Cases:

- Binary classification (output)
- Probability interpretation
- Historical (hidden layers)



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The classic activation: squashes to probability

Sigmoid: Properties and Problems

Advantages

+ Bounded output $(0, 1)$
+ Smooth gradient
+ Probability interpretation
+ Historically important

Disadvantages

- **Vanishing gradients**
  For $|z| > 4$: $\sigma'(z) \approx 0$
  Gradients become tiny
  Deep networks can't learn
- Not zero-centered
  All positive outputs
  Zig-zag weight updates
- Computationally expensive
  Requires exp function

The Vanishing Gradient Problem

When z is very positive or negative:

z	$\sigma'(z)$
0	0.25
2	0.10
4	0.018
6	0.0025

Gradients shrink exponentially through layers!

Result: Early layers learn very slowly in deep networks. This limited deep learning until ReLU.

Smooth and bounded, but gradients can vanish

The Tanh Function

Definition

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = 2\sigma(2z) - 1$$

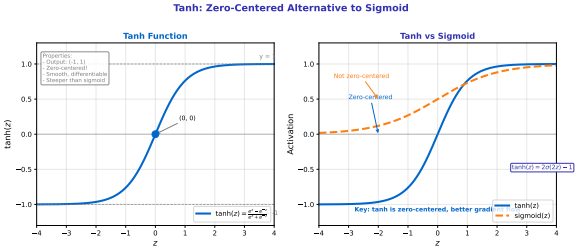
Properties:

- Range: $(-1, 1)$
- Zero-centered
- $\tanh(0) = 0$
- Odd function: $\tanh(-z) = -\tanh(z)$

Derivative:

$$\tanh'(z) = 1 - \tanh^2(z)$$

Advantage over Sigmoid:
Zero-centered outputs lead to more stable gradient updates.



tanh_functio

Zero-centered: range (-1, 1)

ReLU: Rectified Linear Unit

Definition

$$\text{ReLU}(z) = \max(0, z) = \begin{cases} z & z > 0 \\ 0 & z \leq 0 \end{cases}$$

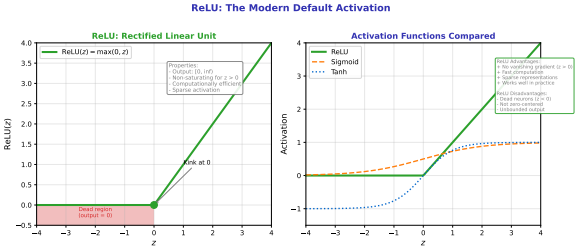
Properties:

- Range: $[0, \infty)$
- Not bounded above
- Not differentiable at $z = 0$
- Piecewise linear

Derivative:

$$\text{ReLU}'(z) = \begin{cases} 1 & z > 0 \\ 0 & z \leq 0 \end{cases}$$

The Modern Default for hidden layers.



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Simple but powerful: the modern default

Why ReLU Works So Well

Advantages

+ **No vanishing gradient**
  Gradient is 1 for $z > 0$
  Signal propagates through layers
+ **Computationally cheap**
  Just comparison and assignment
  No exponentials
  6x faster than sigmoid
+ **Sparse activation**
  Many neurons output 0
  Efficient representation
+ **Biological plausibility**
  Neurons can be "off"

Disadvantages

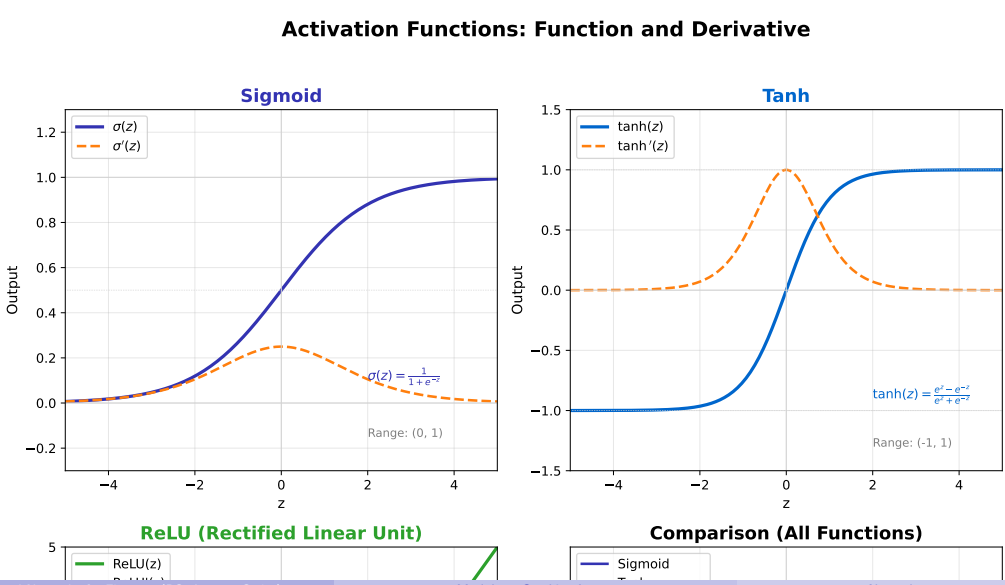
- **"Dying ReLU" problem**
  If $z < 0$ always: gradient $= 0$
  Neuron never updates
  Can "die" permanently
- Not zero-centered
- Unbounded (can explode)

Variants:

- Leaky ReLU: $\max(0.01z, z)$
- ELU: z if $z > 0$, $\alpha(e^z - 1)$ otherwise
- GELU: used in transformers

Cheap to compute, gradients don't vanish (for positive inputs)

Activation Functions: Comparison



Discussion Question

"Which activation function would you use for: (a) predicting stock returns, (b) buy/sell classification? Why?"

Consider:

(a) Stock Returns (Regression)

- Output: continuous value
- Can be positive or negative
- Hidden: ReLU or tanh
- Output: **Linear (none)**
- Returns are unbounded

(b) Buy/Sell (Classification)

- Output: probability $\in (0, 1)$
- Two mutually exclusive classes
- Hidden: ReLU
- Output: **Sigmoid**
- Or softmax for multi-class

Think-Pair-Share: 3 minutes

Choosing the Right Activation

Hidden Layer Guidelines

Default: ReLU

- Works well in most cases
- Fast and stable

If dying ReLU: Leaky ReLU

- Small negative slope
- Prevents dead neurons

For RNNs: Tanh

- Bounded outputs help stability
- Zero-centered

Output Layer Guidelines

Task	Activation
Binary class	Sigmoid
Multi-class	Softmax
Regression	Linear
Bounded regression	Sigmoid/tanh
Positive only	ReLU

Finance Examples:

- Return prediction: Linear
- Direction prediction: Sigmoid
- Sector classification: Softmax
- Volatility: ReLU or Softplus

Output layer choice depends on your problem type

The Fundamental Question

How Powerful Are Neural Networks?
We've seen that MLPs can:

- Solve XOR (non-linear patterns)
- Combine features hierarchically
- Learn from data

But a Deeper Question:
Are there functions that MLPs fundamentally *cannot* represent?

Or can they approximate *anything*?

Why This Matters

If MLPs are limited:

- Need to check if problem is solvable
- Architecture constraints matter
- Some patterns impossible

If MLPs are universal:

- Architecture is not the bottleneck
- Challenges are elsewhere (data, training)
- Theoretical guarantee of capability

Spoiler: MLPs are universal approximators!

Just how powerful are neural networks?

Universal Approximation Theorem

The Theorem (Informal)
A feedforward network with:

- One hidden layer
- Sufficient hidden neurons
- Non-linear activation (e.g., sigmoid)

can approximate any continuous function on a compact domain to arbitrary accuracy.

Key Contributors:

- Cybenko (1989): sigmoid
- Hornik (1991): general activations
- Further extensions since

Formal Statement
Let $f : [0,1]^n \to \mathbb{R}$ be continuous.
For any $\epsilon > 0$, there exists an MLP $\hat{f}$ with:

$$|\hat{f}(\mathbf{x}) - f(\mathbf{x})| < \epsilon$$

for all $\mathbf{x} \in [0,1]^n$.

In Plain English:
No matter how complex the pattern, an MLP with enough hidden neurons can match it as closely as you want.

With enough hidden neurons, you can approximate any continuous function

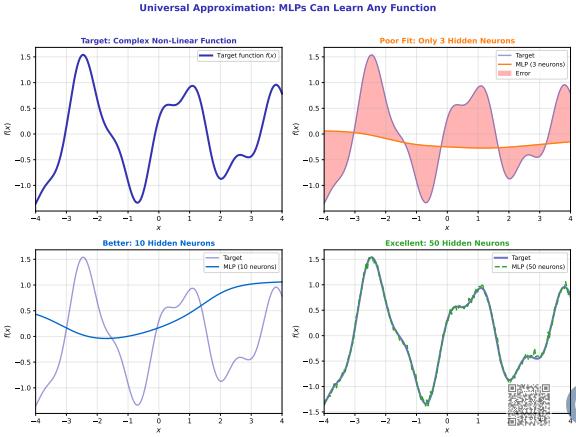
What Universal Approximation Means

The Good News

- No function is “too complex”
- MLPs are theoretically complete
- Architecture is not the limit
- One hidden layer is enough (in theory)

Visual Intuition:
Each hidden neuron contributes a “bump” or “step.”
With enough bumps, you can approximate any shape.

Think of it like approximating a curve with many small line segments.



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Universal Approximation Theorem (Cybenko, 1989): A feedforward network with a single hidden layer containing a finite number of neurons can approximate any continuous function on compact subsets of $\mathbb{R}^n$

More neurons = better approximation

What It Doesn't Mean

Common Misconceptions

"Any network can learn anything"

No. Need enough neurons

- May need exponentially many

"Training will find the solution"

No. Theorem is about existence

- Says nothing about finding weights
- Optimization may fail

"One layer is always enough"

cally no.

- Deep networks often more efficient
- Fewer parameters for same accuracy

The Gap: Existence vs Construction

The theorem says:
"A good approximation exists."
It does NOT say:

- How many neurons you need
- How to find the right weights
- How much data is required
- How long training takes
- Whether it will generalize

Analogy:
"There exists a needle in this haystack" doesn't help you find it.

Existence of a solution does not mean we can find it

Theory vs Practice

Theoretical Guarantees
Universal approximation says:

- Given infinite neurons: perfect fit
- Given infinite data: find the function
- Given infinite compute: optimize

Practical Reality
We have:

- Finite neurons: limited capacity
- Finite data: must generalize
- Finite compute: approximate solutions

What Matters More in Practice

1. **Data quality and quantity**
   - More important than architecture
2. **Regularization**
   - Prevent overfitting
3. **Optimization**
   - Finding good weights
4. **Generalization**
   - Performance on new data

Module 3 will address these practical challenges.

Universal approximation is necessary but not sufficient

Implications for Finance

The Optimistic View
If markets have patterns, MLPs can learn them:

- Non-linear relationships? Possible.
- Complex interactions? Possible.
- Hidden factors? Possible.

Theoretical Capability:
"An MLP could, in principle, capture any market pattern."

The Realistic View

Challenges Remain:

- Signal-to-noise ratio is low
- Markets are non-stationary
- Past patterns may not repeat
- Data is limited (especially for crashes)
- Overfitting is easy

The EMH Counterargument:
If markets are efficient, there's nothing systematic to learn.

Module 4 will explore this tension.

In theory, yes. In practice, many challenges remain.

Why Loss Functions?

Learning Requires an Objective
To train a neural network, we need:

1. A way to measure errors
2. A number that decreases as we improve
3. A signal for weight updates

The Loss Function:
$\mathcal{L}(\hat{\mathbf{y}}, \mathbf{y})$ measures how wrong our predictions are.

Goal of Training:
Find weights that minimize $\mathcal{L}$.

Finance Analogy

Profit & Loss (P&L):

- Measures trading performance
- Negative P&L = bad trades
- Optimize to maximize P&L

Loss Function:

- Measures prediction errors
- High loss = bad predictions
- Optimize to minimize loss

Note: "Loss" is the opposite of "profit" – we minimize loss!

To learn, we must measure mistakes

Mean Squared Error (MSE)

Definition

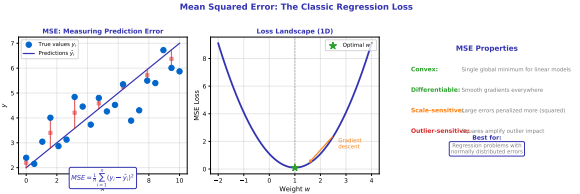
$$\mathcal{L}_{\text{MSE}} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Properties:

- Always non-negative
- Zero only if perfect predictions
- Penalizes large errors heavily
- Differentiable everywhere

Use Case:

- Regression problems
- Predicting continuous values
- Stock returns, prices, etc.



mse_visualizatio

The standard loss for predicting continuous values

Cross-Entropy Loss

Binary Cross-Entropy

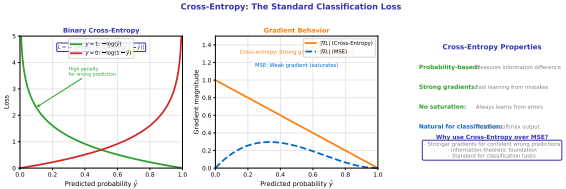
$$\mathcal{L}_{\text{BCE}} = -\frac{1}{n}\sum_{i=1}^{n}[y_i \log(\hat{y}_i) + (1-y_i)\log(1-\hat{y}_i)]$$

Properties:

- For probability outputs
- Heavily penalizes confident wrong answers
- Connected to information theory

Use Case:

- Classification problems
- Buy/sell decisions
- Any yes/no prediction



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The standard loss for classification

The Loss Landscape

Loss as a Function of Weights

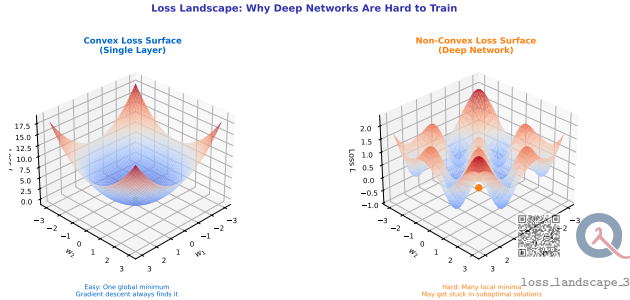
$$\mathcal{L}(\mathbf{W}, \mathbf{b})$$

For every choice of weights, there's a loss value.

The Landscape:

- High regions: bad weights
- Low regions: good weights
- Global minimum: best weights
- Local minima: traps

Training =
Finding the lowest point in this landscape.



Training = finding the lowest point in this landscape

Finance: Choosing Your Loss

Task-Specific Loss Functions

Task	Loss
Return prediction	MSE
Direction prediction	Cross-entropy
Volatility forecast	MSE
Multi-class sector	Categorical CE

Beyond Standard Losses:

- Sharpe ratio optimization
- Asymmetric losses (penalize losses more than gains)
- Custom finance metrics

Important Consideration

MSE vs Business Metric:
A model with low MSE may still lose money!

Example:

- Predict returns with 5% MSE
- But wrong on big moves
- Transaction costs eat profits
- Risk-adjusted return is poor

Lesson:
Statistical accuracy $\neq$ Trading profitability
Module 4 explores this gap.

Different problems, different loss functions

Module 2: Key Takeaways

What We Learned

1. **Historical Context**
   - AI Winter (1969-1982)
   - Backprop renaissance (1986)
   - Right idea + right time
2. **MLP Architecture**
   - Hidden layers find patterns
   - Matrix notation for computation
   - Parameter counting
3. **Activation Functions**
   - Non-linearity is essential
   - ReLU for hidden, task-specific for output
4. **Universal Approximation**
   - MLPs can learn any function
   - But existence $\neq$ construction
5. **Loss Functions**
   - MSE for regression
   - Cross-entropy for classification
   - Loss landscape visualization

The Big Picture:
We now have powerful architectures. But how do they *learn*?

From single perceptron to universal function approximator

Preview: Module 3

"We have the architecture. But how does it LEARN?"

The Missing Piece

We know:

- How to compute forward pass
- What loss functions measure
- That good weights exist

We don't know:

- How to find good weights
- How errors update weights
- How to avoid overfitting

Coming in Module 3:

- Gradient descent (intuition)
- Backpropagation (the magic)
- Training dynamics
- Overfitting and regularization
- Practical training tips

The Key: Backpropagation – the algorithm that made deep learning possible.

Mathematical details: See Appendix B (Backpropagation Derivation)

Next: The magic of backpropagation