Hyperbolic Learning in Action

Deep Learning in Hyperbolic Space

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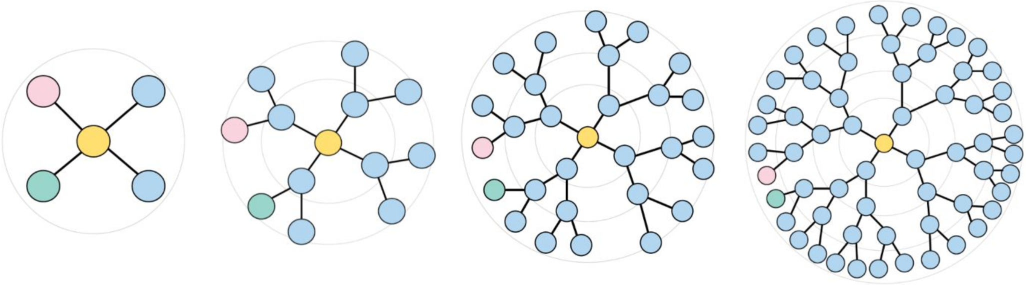
Embedding Hierarchies

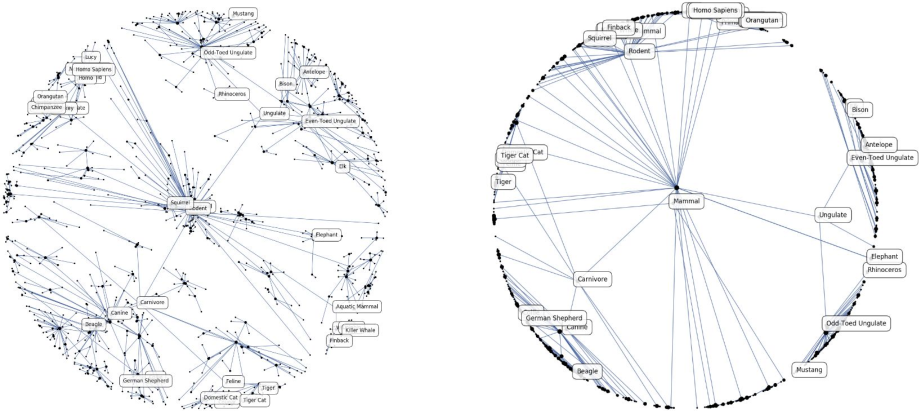
The trouble with embedding hierarchies

Hierarchies grow exponenitally in depth, Euclidean spaces grows linearly with norm.



The exponential growth of distances in hyperbolic space is a direct fit with hierarchies.

Poincaré embeddings



(a) Intermediate embedding after 20 epochs

(b) Embedding after convergence

Nickel and Kiela, *Poincaré Embeddings for Learning Hierarchical Representations*

Optimizing Poincaré embeddings

Nodes:
$\mathcal{S} = \{x_i\}_{i=1}^{n}$

Parent-child relations:
$\mathcal{D} = \{(u, v)\}$

Non-Parent-child relations:
$\mathcal{N}(u) = \{v' \,|\, (u, v') \notin \mathcal{D}\} \cup \{v\}$

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Hyperbolic representation of nodes: $\Theta = \{\theta_i\}_{i=1}^{n}$

$$\Theta' \leftarrow \operatorname{argmin} \mathcal{L}(\Theta) \quad \text{s.t.} \forall \theta_i \in \Theta : \|\theta_i\| < 1 \tag{1}$$

Pull parent-child nodes, push others.

$$\mathcal{L}(\Theta) = \sum_{(u,v) \in \mathcal{D}} \log \frac{e^{-d(\mathbf{u},\mathbf{v})}}{\sum_{\mathbf{v}' \in \mathcal{N}(u)} e^{-d(\mathbf{u},\mathbf{v}')}} \tag{2}$$

$$d(\mathbf{u}, \mathbf{v}) = \operatorname{arcosh}\left(1 + 2\frac{\|\mathbf{u} - \mathbf{v}\|^2}{\left(1 - \|\mathbf{u}\|^2\right)\left(1 - \|\mathbf{v}\|^2\right)}\right) \tag{3}$$

Optimizing Poincaré embeddings

$$\theta_{t+1} = \Re_{\theta_t}\left(-\eta_t \nabla_R \mathcal{L}\left(\theta_t\right)\right) \tag{4}$$

Optimize node embeddings with Riemmanian gradient descent.

$$\theta_{t+1} \leftarrow \mathsf{proj}\left(\theta_t - \eta_t \frac{\left(1 - \|\theta_t\|^2\right)^2}{4} \nabla_E\right) \tag{5}$$

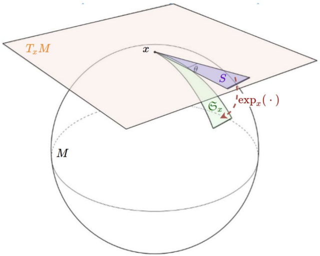
Riemmanian gradient descent = Standard gradient + scaling + projection.

Poincaré embedidngs

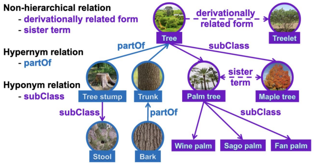
			Dimensionality					
			5	10	20	50	100	200
WordNet Reconstruction	**Euclidean**	Rank	3542.3	2286.9	1685.9	1281.7	1187.3	1157.3
		MAP	0.024	0.059	0.087	0.140	0.162	0.168
	Translational	Rank	205.9	179.4	95.3	92.8	92.7	91.0
		MAP	0.517	0.503	0.563	0.566	0.562	0.565
	Poincaré	Rank	4.9	4.02	3.84	3.98	3.9	**3.83**
		MAP	0.823	0.851	0.855	0.86	0.857	**0.87**
WordNet Link Pred.	**Euclidean**	Rank	3311.1	2199.5	952.3	351.4	190.7	81.5
		MAP	0.024	0.059	0.176	0.286	0.428	0.490
	Translational	Rank	65.7	56.6	52.1	47.2	43.2	40.4
		MAP	0.545	0.554	0.554	0.56	0.562	0.559
	Poincaré	Rank	5.7	**4.3**	4.9	4.6	4.6	4.6
		MAP	0.825	0.852	0.861	**0.863**	0.856	0.855

		Dimensionality							
		Reconstruction				Link Prediction			
		10	20	50	100	10	20	50	100
AstroPh N=18,772; E=198,110	**Euclidean**	0.376	0.788	0.969	0.989	0.508	0.815	0.946	0.960
	Poincaré	0.703	0.897	0.982	0.990	0.671	0.860	0.977	0.988
CondMat N=23,133; E=93,497	**Euclidean**	0.356	0.860	0.991	0.998	0.308	0.617	0.725	0.736
	Poincaré	0.799	0.963	0.996	0.998	0.539	0.718	0.756	0.758
GrQc N=5,242; E=14,496	**Euclidean**	0.522	0.931	0.994	0.998	0.438	0.584	0.673	0.683
	Poincaré	0.990	0.999	0.999	0.999	0.660	0.691	0.695	0.697
HepPh N=12,008; E=118,521	**Euclidean**	0.434	0.742	0.937	0.966	0.642	0.749	0.779	0.783
	Poincaré	0.811	0.960	0.994	0.997	0.683	0.743	0.770	0.774

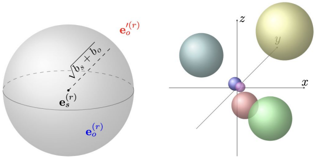
Improving Poincaré embedidngs



Hyperbolic entailment cones.

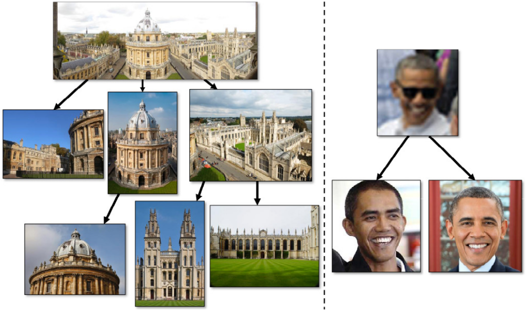


Multi-relational Poincaré embeddings.



Modelling heterogeneous hierarchies.

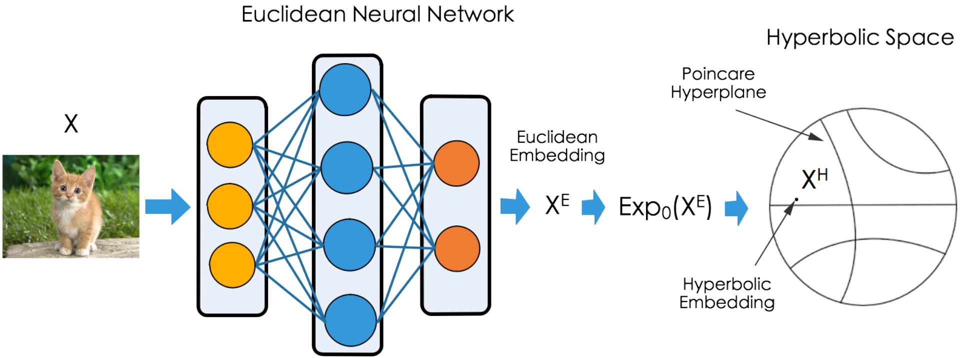
Hyperbolic Image Embeddings



Encoder	**Dataset**			
	CIFAR10	CIFAR100	CUB	*Mini*ImageNet
Inception v3 [49]	0.25	0.23	0.23	0.21
ResNet34 [14]	0.26	0.25	0.25	0.21
VGG19 [42]	0.23	0.22	0.23	0.17

Khrulkov et al., *Hyperbolic Image Embeddings*

Convolutional Networks with Hyperbolic Embeddings

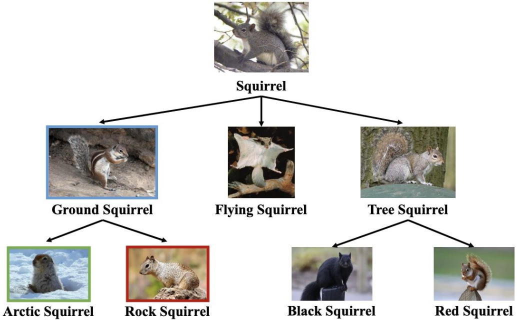


Guo et al., "Clipped Hyperbolic Classifiers Are Super-Hyperbolic Classifiers"

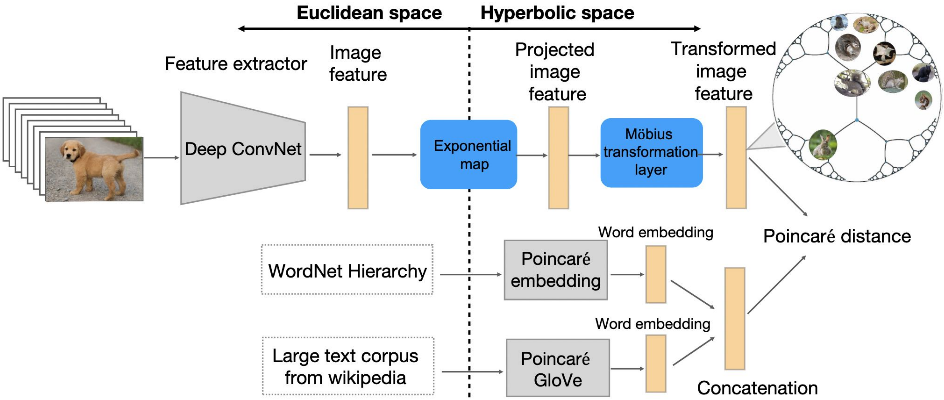
Zero-Shot Generalization

Hyperbolic Zero-Shot Visual Embedding Learning

If labels are hierarchical, does a hyperbolic embedding eneable zero-shot generalization?



Hyperbolic Zero-Shot Visual Embedding Learning



Liu et al., "Hyperbolic Visual Embedding Learning for Zero-Shot Recognition"

Hyperbolic Zero-Shot Visual Embedding Learning

If prior knowledge is a hierarchy then use hyperbolic geometry for zero-shot learning.

Data Set	Model	Hierarchical precision@k(%)				
		1	2	5	10	20
2-hops & Their Parents	DeViSE	3.2	5.3	9.5	15.6	21.2
	DeViSE*	4.5	7.0	9.9	15.6	22.0
	ConSE	4.2	6.8	12.3	18.5	25.1
	GCNZ	9.2	15.6	27.5	36.8	44.5
	Ours	**16.6**	**24.3**	**43.8**	**58.6**	**70.3**
3-hops & Their Parents	DeViSE	1.3	2.1	3.3	4.9	7.3
	DeViSE*	1.7	2.6	4.4	6.6	9.3
	ConSE	1.9	2.6	4.4	7.2	9.7
	GCNZ	2.7	4.6	8.2	12.5	15.1
	Ours	**7.9**	**12.5**	**21.4**	**28.7**	**37.5**
All	DeViSE	0.9	1.5	2.9	4.4	6.5
	DeViSE*	1.0	1.6	2.9	4.4	6.5
	ConSE	1.5	2.4	4.2	6.5	9.7
	GCNZ	2.2	3.8	7.2	10.5	13.9
	Ours	**5.1**	**6.9**	**12.9**	**16.5**	**19.3**

DeViSE: teddy, orangutan, valley, langur, cliff

GCNZ: phalanger, **red squirrel**, kangaroo, lemur, tree wallaby

Ours: **red squirrel**, **tree squirrel***, squirrel, kangaroo, phalanger

DeViSE: rugby ball, soccer ball, golf ball, basketball, cricket

GCNZ: **volleyball**, basketball, golf ball, punching bag, rugby ball

Ours: **volleyball**, **ball***, basketball, rugby ball, soccer ball

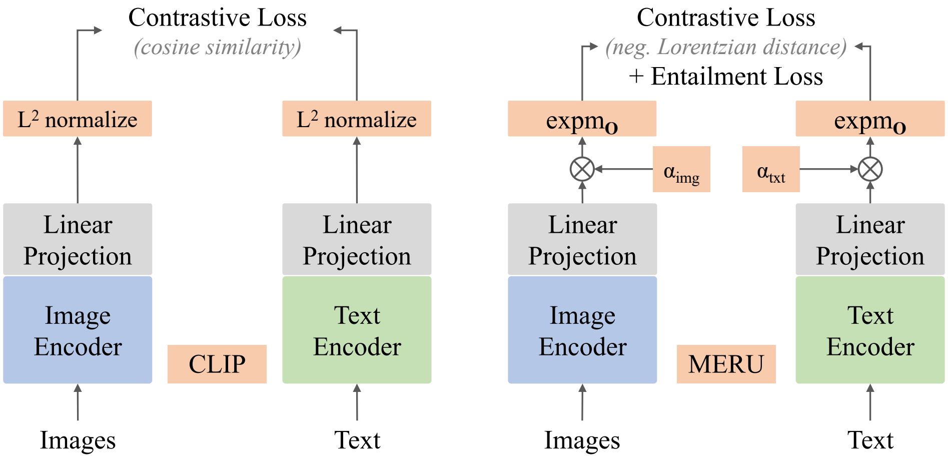
DeViSE: bullet train, freight car, school bus, police van, minibus

GCNZ: mail train, express, **passenger train**, cargo ship, shuttle bus

Ours: **passenger train**, **railroad train***, bus, school bus, trolleybus

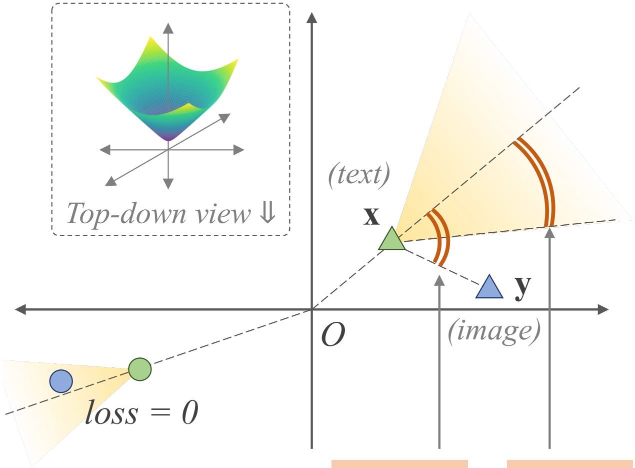
Liu et al., "Hyperbolic Visual Embedding Learning for Zero-Shot Recognition"

Hyperbolic Image-Text Representations



Desai et al., "Hyperbolic Image-text Representations"

Entailment Cones



$$loss = \boxed{\text{ext}(\mathbf{x}, \mathbf{y})} - \boxed{\text{aper}(\mathbf{x})}$$

Desai et al., "Hyperbolic Image-text Representations"

MERU

MERU	CLIP
avocado toast	*avocado toast*
healthy breakfast	*delicious*
delicious	↓
homemade	↓
fresh	↓
[ROOT]	[ROOT]

MERU	CLIP
brooklyn bridge	*photo of brooklyn bridge, new york*
new york city	*new york city*
city	*new york*
outdoors	↓
day	↓
[ROOT]	[ROOT]

MERU	CLIP
taj mahal	*taj mahal through an arch*
monument	*travel*
architecture	*inspiration*
travel	↓
day	↓
[ROOT]	[ROOT]

MERU	CLIP
sydney opera house	*sydney opera house*
opera house	*opera house*
holiday	*gift*
day	*beauty*
[ROOT]	[ROOT]

Desai et al., "Hyperbolic Image-text Representations"

MERU

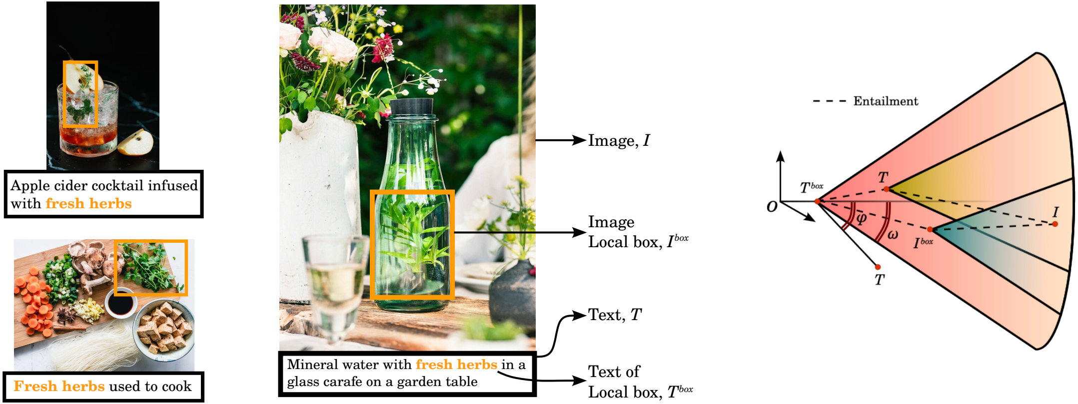
Table 2. **Zero-shot image classification.** We train MERU and CLIP models with varying parameter counts and transfer them *zero-shot* to 20 image classification datasets. Best performance in every column is highlighted in green. Hyperbolic representations from MERU match or outperform CLIP on 13 out of the first 16 datasets. On the last four datasets (gray columns), both MERU and CLIP have *near-random* performance, as concepts in these datasets are not adequately covered in the training data.

		ImageNet	Food-101	CIFAR-10	CIFAR-100	CUB	SUN397	Cars	Aircraft	DTD	Pets	Caltech-101	Flowers	STL-10	EuroSAT	RESISC45	Country211	MNIST	CLEVR	PCAM	SST2
ViT S/16	CLIP	34.3	74.5	**60.1**	24.4	**33.8**	27.5	**11.3**	**1.4**	15.0	**73.7**	63.9	47.0	88.2	18.6	31.4	**5.2**	10.0	19.4	50.2	50.1
	MERU	**34.4**	**75.6**	52.0	**24.7**	33.7	**28.0**	11.1	1.3	**16.2**	72.3	**64.1**	**49.2**	**91.1**	**30.4**	**32.0**	4.8	7.5	14.5	51.0	50.0
ViT B/16	CLIP	**37.9**	**78.9**	65.5	**33.4**	33.3	29.8	**14.4**	1.4	17.0	77.9	**68.5**	50.9	92.2	25.6	31.0	**5.8**	10.4	14.3	54.1	51.5
	MERU	37.5	78.8	**67.7**	32.7	**34.8**	**30.9**	14.0	**1.7**	**17.2**	**79.3**	**68.5**	**52.1**	**92.5**	**30.2**	**34.5**	5.6	13.0	13.5	49.8	49.9
ViT L/16	CLIP	38.4	80.3	**72.0**	**36.4**	36.3	32.0	**18.0**	1.1	16.5	78.8	**68.3**	48.6	**93.7**	26.7	35.4	6.1	14.8	13.6	51.2	51.1
	MERU	**38.8**	**80.6**	68.7	35.5	**37.2**	**33.0**	16.6	**2.2**	**17.2**	**80.0**	67.5	**52.1**	**93.7**	**28.1**	**36.5**	**6.2**	11.8	13.1	52.7	49.3

Desai et al., "Hyperbolic Image-text Representations"

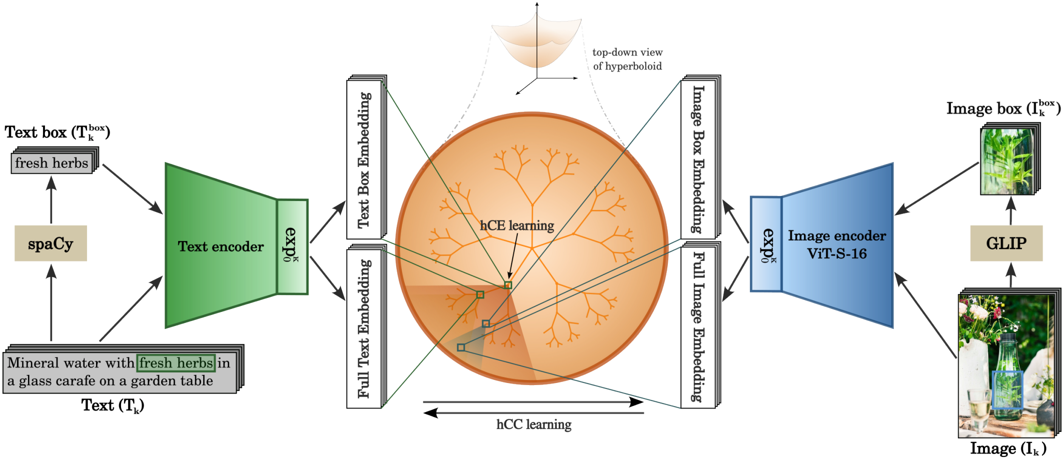
Compositional Entailment Learning

An image is not only described by a sentence but is itself a composition of multiple object boxes, each with their own textual description



Pal et al., "Compositional Entailment Learning for Hyperbolic Vision-Language Models"

Compositional Entailment Learning



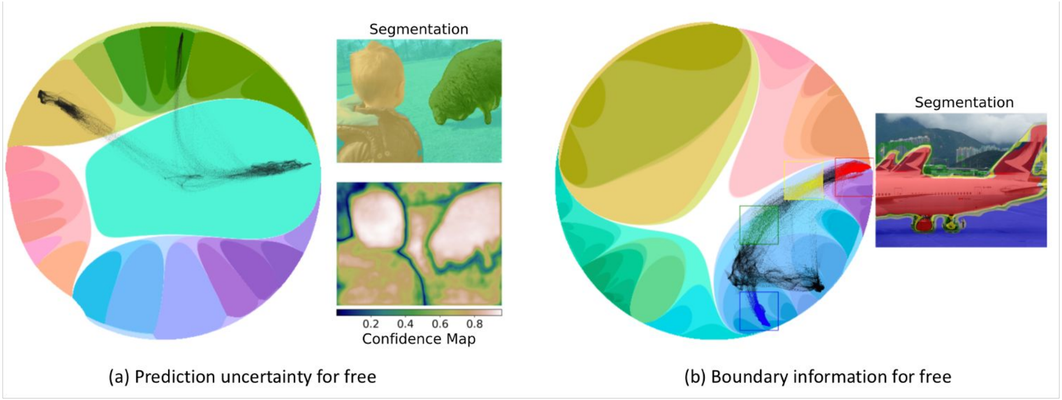
Pal et al., "Compositional Entailment Learning for Hyperbolic Vision-Language Models"

Compositional Entailment Learning

Pal et al., "Compositional Entailment Learning for Hyperbolic Vision-Language Models"

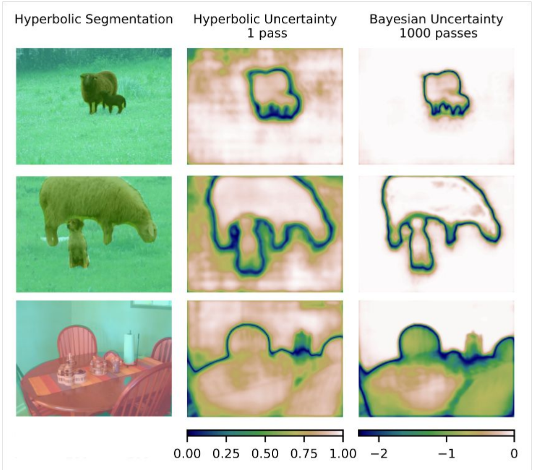
Robustness and Uncertainty

Hyperbolic Segmentation



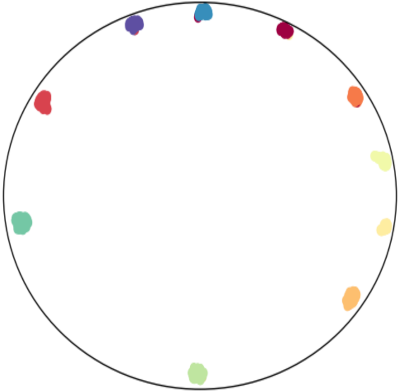
GhadimiAtigh et al., *Hyperbolic Image Segmentation*

Uncertainty and boundary information for free



GhadimiAtigh et al., *Hyperbolic Image Segmentation*

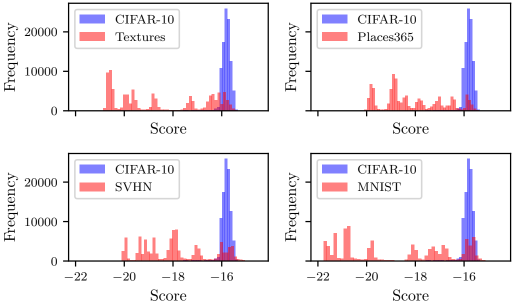
How to exploit robustness properties?

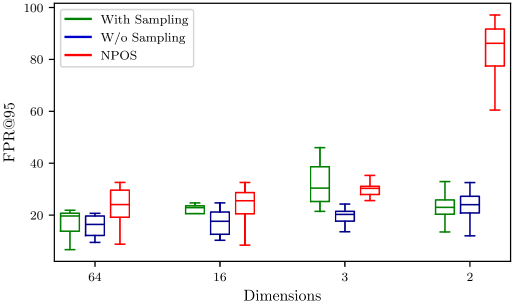


- Learn in-distribution representations promoting low variations and high separation.
- Hyperbolic geometry offers *more space* than Euclidean!

Gonzalez-Jimenez et al., *Hyperbolic Metric Learning for Visual Outlier Detection*

Hyperbolic Outlier Detection



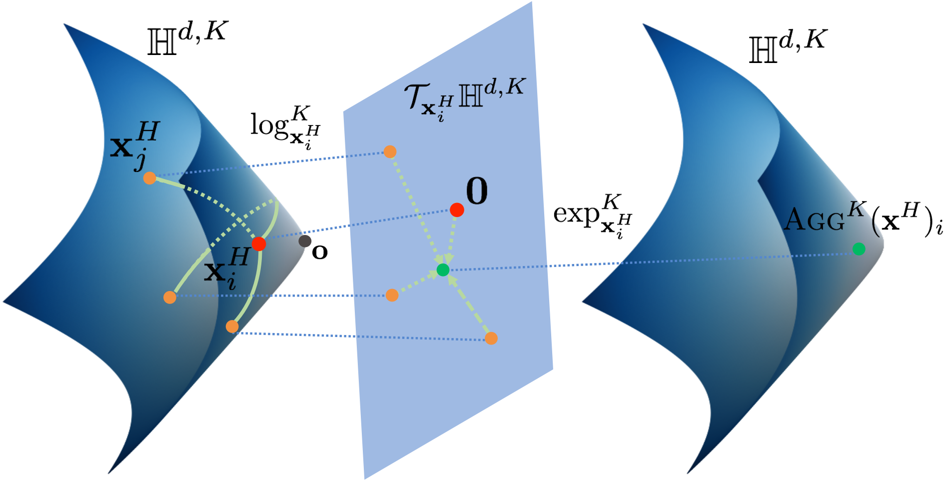


Gonzalez-Jimenez et al., *Hyperbolic Metric Learning for Visual Outlier Detection*

Fully Hyperbolic Networks

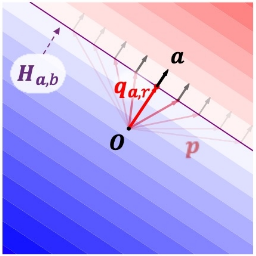
Extend Hyperbolic Space for the entire network

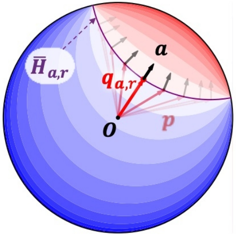
Mapping back and forth between hyperbolic and Euclidean manifolds.



Move Everything to Hyperbolic Space

The current methods depend on the tangent space for several operations and the frequent back and forth mapping is both expensive and prone to a loss of data.

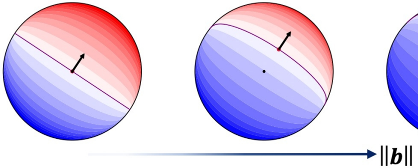




Hyperbolic Network ++

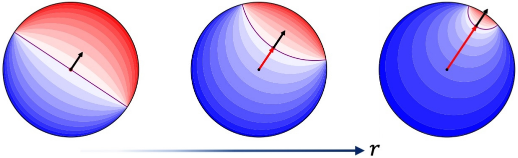
Poincaré Ball

$$y = \exp_0^c\left(W \log_0^c(x)\right) \oplus_c b \qquad (6)$$



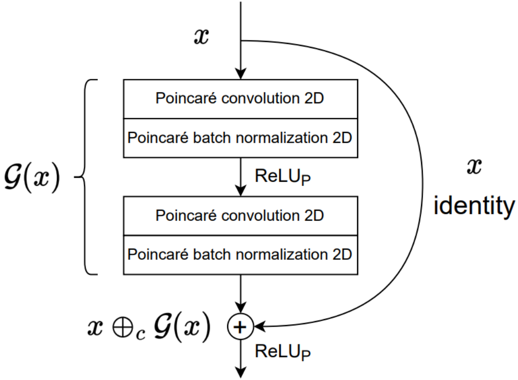
Hyperbolic Neural Networks++

$$y = \mathcal{F}^c(x; Z, r) = \mathrm{w}\left(1 + \sqrt{1 + c\|w\|^2}\right)^{-1} \qquad (7)$$



Shimizu, Mukuta, and Harada, *Hyperbolic Neural Networks++*

Poincare ResNet

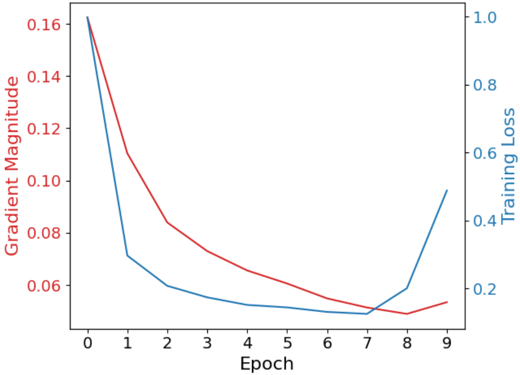


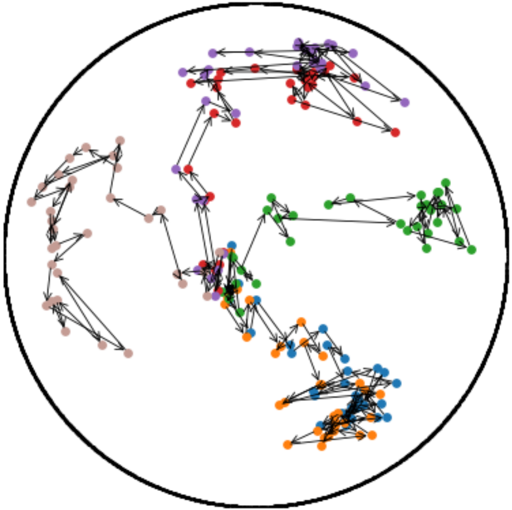
- Extend Linear Layer from Hyperbolic Neural Network++ for Convolutions.
- Poincare midpoint batch normalization for faster and equally effective alternative to Frechet Mean.
- Poincare Resnets are (i) more robust to out-of-distribution samples, (ii) more robust to adversarial attacks and (iii) complementary to Euclidean networks.

Spengler, Berkhout, and Mettes, *Poincaré ResNet*

Weaknesses

Gradients Vanishing





Guo et al., "Clipped Hyperbolic Classifiers Are Super-Hyperbolic Classifiers"

Numerical Unstability

- It will sometimes lead to catastrophic NaN problems, encountering unrepresentable values in floating point arithmetic.
- Under the 64 bit arithmetic system, the Poincare ball has a relatively larger capacity than the Lorentz model for correctly representing points.
- Lorentz model is superior to the Poincare ball from the perspective of optimization.

Mishne et al., "The Numerical Stability of Hyperbolic Representation Learning"

Prepare for Practical Session

Download codebase from:
github.com/Digital-Dermatology/hyperbolic-learning-tutorial-code

References I

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PoincareResNet

	Manifold	CIFAR-10						CIFAR-100					
		FPR95 ↓		AUROC ↑		AUPR ↑		FPR95 ↓		AUROC ↑		AUPR ↑	
		R20	R32	R20	R32	R20	R32	R20	R32	R20	R32	R20	R32
Places-365	Euclidean	64.2	72.3	**84.7**	82.0	**96.2**	95.6	89.5	93.9	62.5	57.9	89.3	87.9
	w/ HNN++	**63.8**	72.7	79.6	77.7	94.5	94.2	93.2	86.3	63.3	66.6	89.8	91.1
	Poincaré	70.2	**70.7**	82.3	**82.6**	95.7	**95.9**	**82.8**	**83.8**	**71.5**	**71.1**	**92.3**	**92.2**
SVHN	Euclidean	97.3	94.7	68.8	73.4	92.8	94.1	99.5	98.8	43.7	54.6	83.7	88.2
	w/ HNN++	73.1	79.1	**85.5**	82.2	**96.9**	96.1	92.1	88.6	66.4	68.9	91.1	92.0
	Poincaré	**66.0**	**69.3**	85.0	**83.6**	96.6	**96.3**	**76.9**	**83.0**	**76.8**	**72.6**	**94.1**	**92.9**
Textures	Euclidean	87.3	88.0	73.6	77.3	93.2	94.7	98.1	96.0	33.5	42.9	75.9	79.4
	w/ HNN++	**63.8**	**56.6**	79.6	**85.8**	94.5	**96.6**	85.9	**77.5**	58.9	65.7	86.8	89.0
	Poincaré	68.2	66.2	**82.1**	82.3	**95.5**	95.6	**83.9**	84.2	**67.7**	**68.8**	**91.0**	**91.5**

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Hyperbolic Learning in Action

Tools for Hyperbolic Learning

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Thursday 13th February, 2025

Geomstats

- An extensive library for differential geometry, supporting a wide range of manifolds and operations.
- Offers thorough documentation and tutorials, which help users get started quickly.
- Well-maintained and actively developed with frequent updates.
- Lacks in-depth focus on hyperbolic space, leading to missing important features and models.

Geomstats

Geoopt

- Started as *just for fun* paper implementation and grow to a Python Package. [1]
- Support various hyperbolic models and operations.
- The most widely used library in literature for manifold-based optimization.
- Lacks active maintenance, with outdated implementations for key operations such as sinh, cosh, etc.
- Performance issues and steep learning curve for beginners.

[1] https://www.youtube.com/watch?v=6VZ0Gk4QMME

HypLL

- A recent library with a strong focus on hyperbolic space.
- Provides support for hyperbolic layers and operations, designed like PyTorch (`hypll.nn`, `hypll.optim`).
- User-friendly for creating fully hyperbolic networks.
- Only support Poincaré Ball, other models will be implemented.

Hypll Overview

```
from hypll.tensors import TangentTensor
from hypll.optim import RiemannianAdam
from hypll.manifolds.poincare_ball import Curvature, PoincareBall
from models import hyperbolic_model
...

manifold = PoincareBall(c=Curvature(value=0.1, requires_grad=True))
model = hyperbolic_model(manifold=manifold)

optimizer = RiemannianAdam(model.parameters(), lr=0.001)
criterion = nn.CrossEntropyLoss()

for epoch in range(100):
    running_loss = 0.0
    for i, data in enumerate(trainloader, 0):
        inputs, labels = data[0].to(device), data[1].to(device)

        tangents = TangentTensor(data=inputs, man_dim=1, manifold=manifold)
        manifold_inputs = manifold.expmap(tangents)

        optimizer.zero_grad()
        outputs = model(manifold_inputs)
        loss = criterion(outputs.tensor, labels)
        loss.backward()
        optimizer.step()
        ...
```

Library	Advantages	Disadvantages
Geomstats	Extensive support for differential geometry Well maintained and documented	Not focus for hyperbolic learning Missing operations and features
Geoopt	Support many models (Lorentz, Hyperboloid, Klein, etc.) Rich in hyperbolic operations Widely used in hyperbolic papers	Slow performance (outdated code) Not maintained Difficult for beginners
Hypll	Follows PyTorch style Support hyperbolic layers for Fully Hyperbolic Networks User-friendly	Only Poincare model is supported

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Conclusions & Learning

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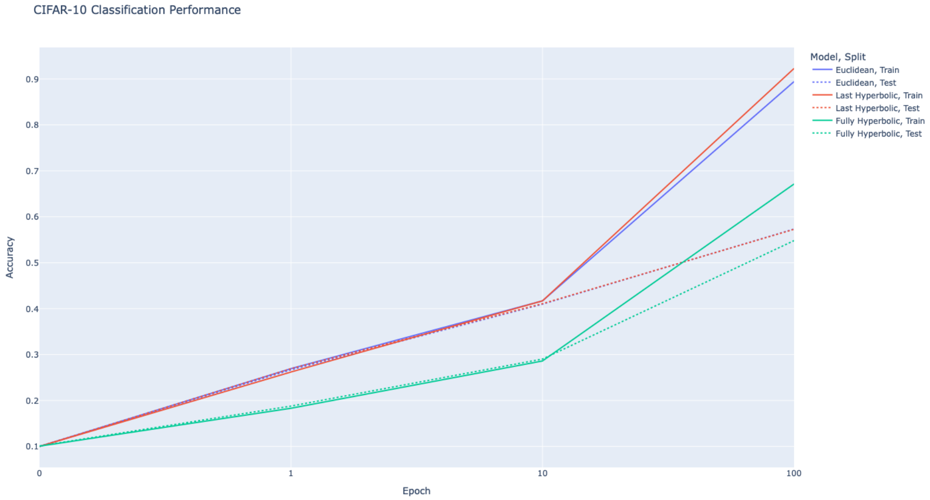
Discussion

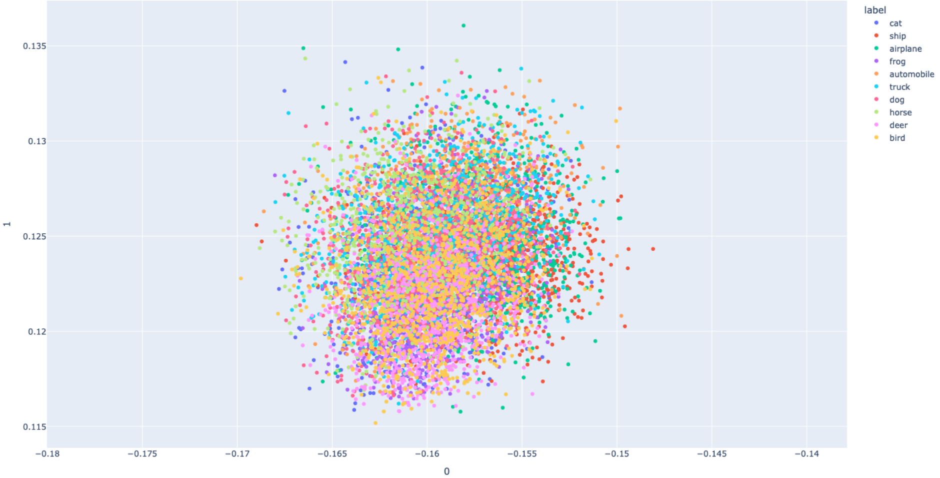
Learnings from the practical session

Tell us what you have discovered or learned from the practical session!

Performance comparison

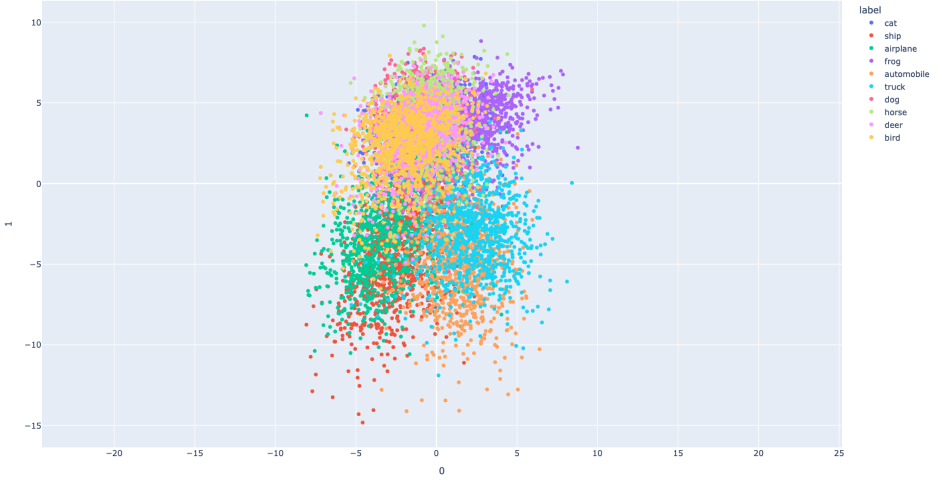


Euclidean, epoch 0



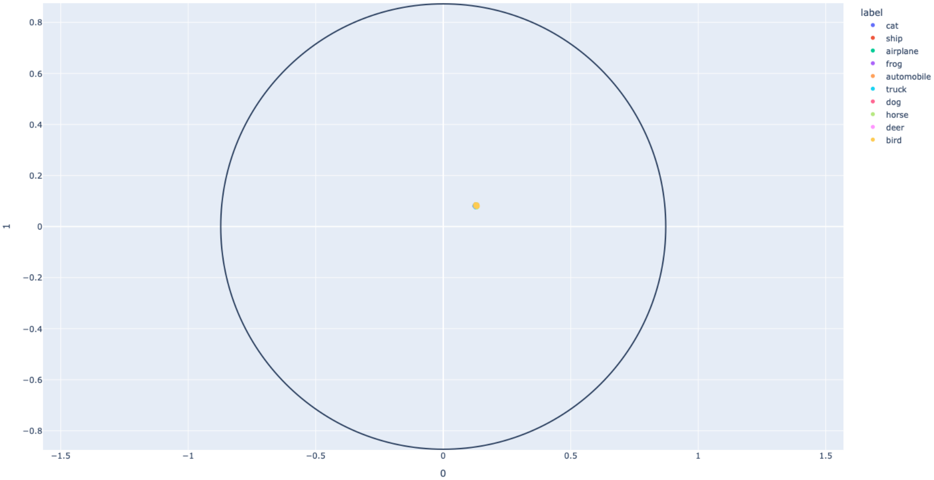
Euclidean, epoch 1

Euclidean, epoch 10

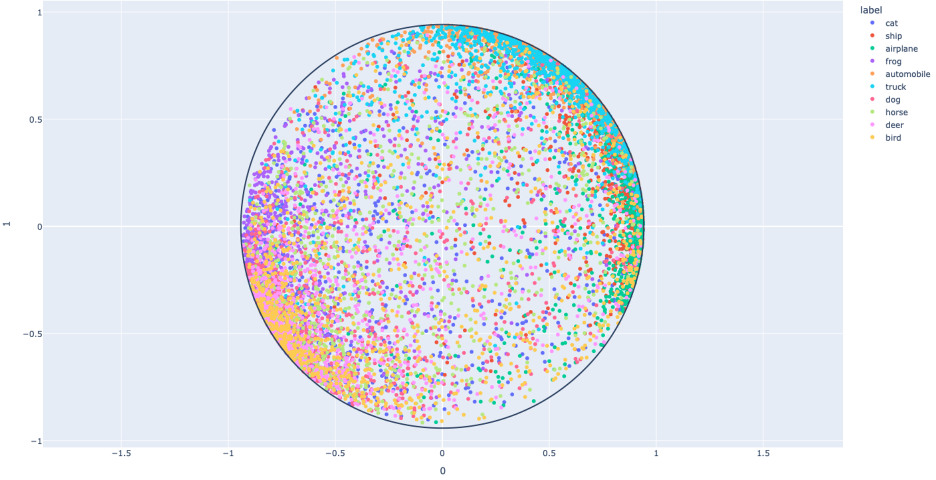


Euclidean, epoch 100

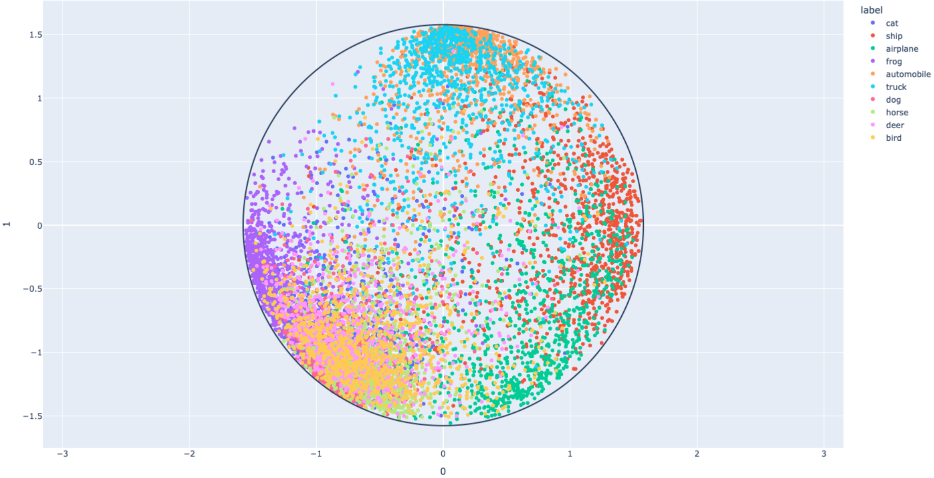
Last hyperbolic, epoch 0



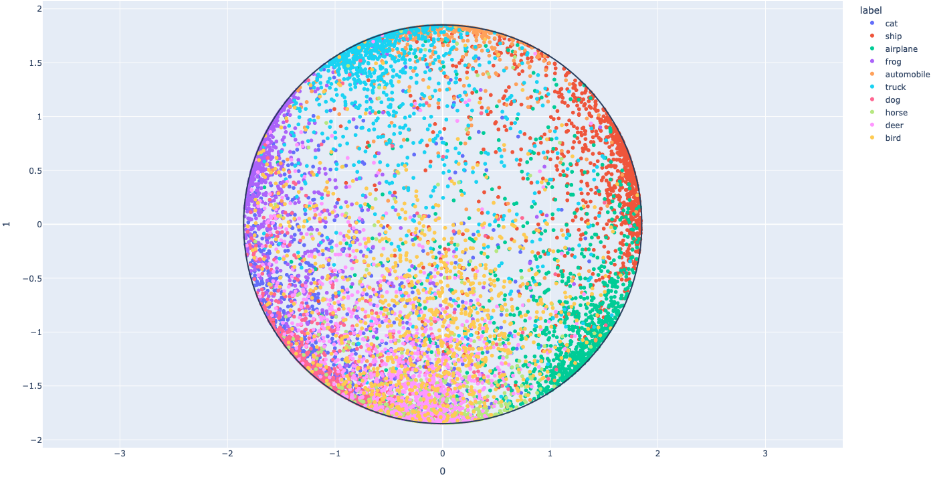
Last hyperbolic, epoch 1



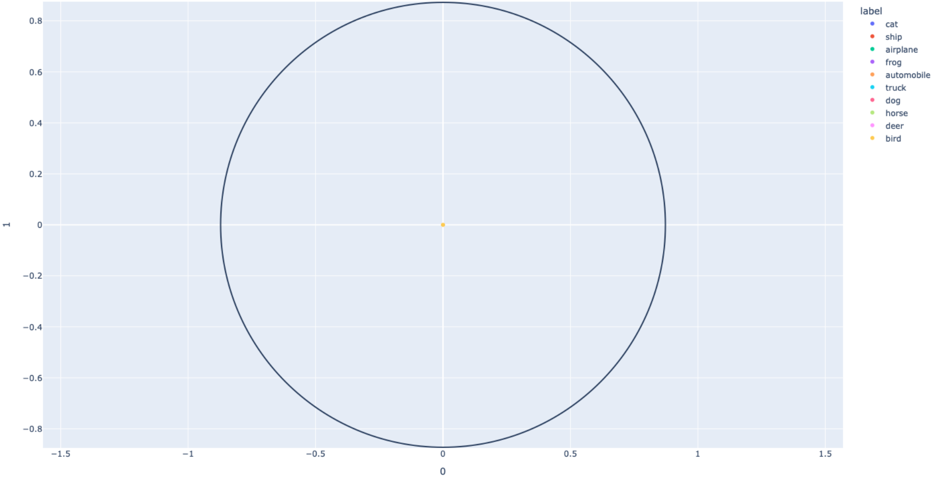
Last hyperbolic, epoch 10



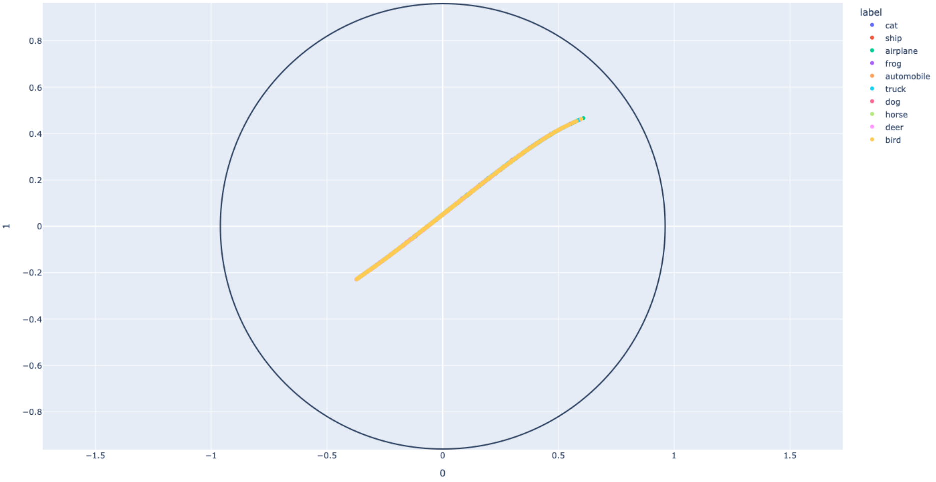
Last hyperbolic, epoch 100



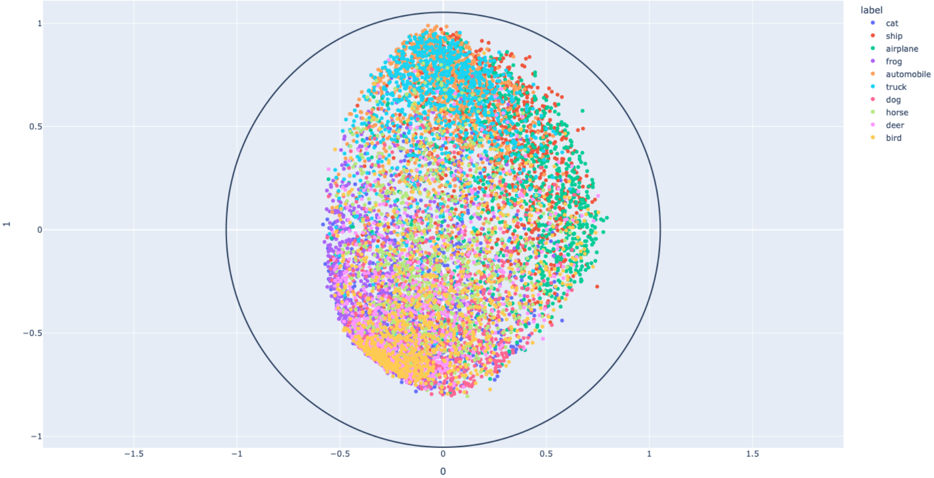
Fully hyperbolic, epoch 0



Fully hyperbolic, epoch 1



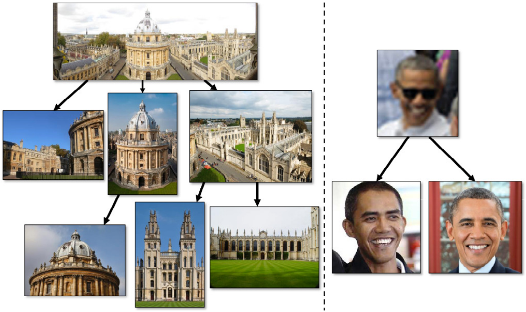
Fully hyperbolic, epoch 10



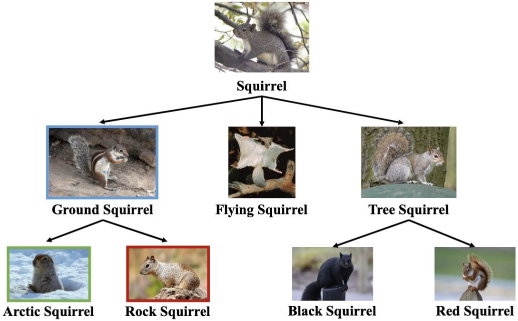
Fully hyperbolic, epoch 100

Recap

Why should we care about Hyperbolic Learning

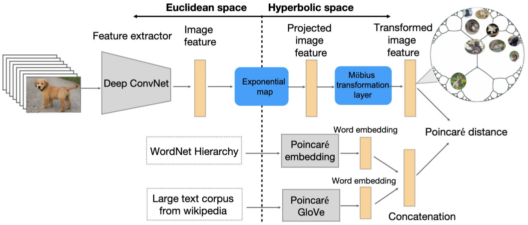


Visual Hierarchies

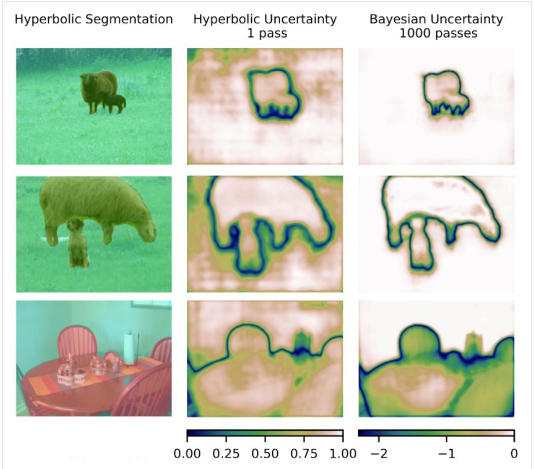


Semantic Hierarchies

Why should we care about Hyperbolic Learning



Zero-Shot Learning



Robustness

Future Potential

- Fully hyperbolic CNNs, Transformers, etc.
- Stable learning on any and all hyperbolic models.
- Fast forward and backward computation.
- Adjust curvature to data and problem.
- What model is suitable for data and problem?
- Large-scale hyperbolic learning.

Thank you

https://forms.gle/KdhQPt6e9NwKUkfGA