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Descartes versus Fermat

Analytic Geometry and Optics

Pulling together work he had quietly labored over for decades, the French philosopher and mathematician René Descartes finally published in June 1637 the book that was to make him famous. His *Discourse on Method* is considered by modern historians a major landmark in several areas, including philosophy, the history of science, and, especially, mathematical thought. David E. Smith and Marcia L. Latham, who did the definitive translation of the mathematical portion of the book, compare it to Newton's *Principia* and argue that it contributed to the great renaissance of mathematics in the 17th century.[1]

Thanks largely to this work, Descartes is commonly given credit for unifying algebra and geometry and even for the creation of analytic geometry; indeed, the Cartesian coordinate system is named after him. That's what we remember today. This being mathematics, one might expect that with Descartes' introduction of this new material, the resulting discussion was precise, well-defined, and devoid of emotional fireworks.

That, however, was not to be. What should have been a straightforward mathematical discussion turned into a mind-boggling combination of Greek tragedy and modern spy novel, with a plotline that included the Catholic Church, several of the top mathematicians of the time, and the vastly differing personalities of Descartes and his main opponent, Pierre de Fermat. The result: a drawn-out battle with a clear winner and loser, but with the ironic outcome that the winner learned little or nothing from the battle, while the loser was inspired by the battle to come up with a major principle in science and to provide important grounding for the development of the calculus.

The mathematics historian Michael Sean Mahoney writes, "Few scientific debates in history reveal so much of the personalities of the participants, or the extent to which personal factors can influence rational discourse."[2]

Descartes

Born in 1596, Descartes received his education in the early years of the 17th century. It was an exciting time, the age of Galileo, Kepler, Harvey, Gilbert, and Francis Bacon; of Shakespeare and Montaigne. It saw the beginnings of what later came to be called the Age of Reason and spawned people like Newton and Leibniz, Milton and Moliere. Yet the standard education was still largely based on the classical curriculum.

So it was for Descartes. His first five years of schooling were devoted almost entirely to Latin, Greek, and classical literature. At the age of 10, he entered La Flèche, a prestigious, highly disciplined Jesuit school, where he spent the next eight years. Among the main subjects were the works of Aristotle, but mainly as viewed through the glasses of the Jesuit fathers. It was basically via these secondhand readings that Descartes had his initial grounding in philosophy.

He was an apt pupil, but when he left—at age 18, in the year

1614–he appears to have come away with "the discovery at every turn of my own ignorance"[3] and utter disdain for the philosophies then being taught. "Philosophy," he wrote later in the *Discourse*, "affords the means of discoursing with an appearance of truth on all matters, and commands the admiration of the more simple."[4] In 1616, he earned a law degree at the University of Poitiers, to which he subsequently paid little attention.

Certainty and Method

Though contemptuous of the existing philosophies, his interest in the subject had been piqued. He began to wonder: how do we know what we know? How can we be sure that what we know, or think we know, is correct? How can it be that so many have studied so much, and yet so much of what we hear and learn is so wrong or so uncertain?

He was convinced, for example, that Copernicus was correct–that although the sun *appears* to be circling the earth, the earth actually circles the sun. In that case, however, can we depend on our senses for learning anything in the world around us?

Then he learned that Copernicus's book *De Revolutionibus*, in which Copernicus had put forth his heliocentric theory more than half a century earlier, had been censored and suspended by the Catholic Church until it would be corrected, or at least formulated so that the idea was put forth as a hypothetical one. For although Copernicus's idea was a good one, it went against church teachings, and there was no real, physical proof of its correctness. If there had been concrete proof of the concept, the church could not object. Descartes began to develop a passion for certainty, which was later to become central to his entire corpus of work.

Several ideas began to gel. Of all the subjects he had studied, mathematics seemed to provide the one real road to certainty. For, he believed, mathematics depends totally on rational thinking; it protects against errors introduced by the senses or even by measurement and experiment.

In 1618, he met and began working with Isaac Beeckman, a teacher and educational administrator who, like Descartes, had a deep interest in the association between mathematics and the physical world. Under Beeckman's guidance and encouragement, he began to focus intensely on mathematical and mechanical problems. He spent some time in the army and appears to have done some mathematical work on military architecture while there. As a kind of gentleman-soldier, he had free time to spend on his studies; still, he was not happy in the army. He complained of being idle and of being in the company of uneducated people. He left the service early in 1619.

He was highly respectful of the mathematics of the ancient Greek mathematicians, such as Pappus and Diophantus. He also, however, suspected that they deliberately held back in their presentations; that is, that they showed solutions to certain problems but kept their methods secret, in much the same way as the algebraists of Cardano's day had done.

In 1619, he wrote, "When I attended to the matter more closely, I came to see that the exclusive concern of mathematics is with questions of order or measure, and that it does not matter whether the measure in question involves numbers, shapes, stars, sounds, or any other object whatsoever. This made me realize that there must be a general science that explains everything that can be raised concerning order and measure irrespective of subject matter, and that this science should be called *mathesis universalis*–a venerable term with a well-established meaning–for it covers everything that entitles these other sciences to be called branches of mathematics. How superior it is to these subordinate sciences both in usefulness and simplicity is clear from the fact that it covers all they deal with. . . . Up to now, I have devoted all my energies to this *mathesis universalis* so that I might be able to tackle the more advanced sciences in due course."[5]

By 1619, then, he already knew that he was destined to create a philosophical system that would utilize deductive procedures as rigorous as Aristotle's, but based on his own thinking and development. He had, as he put it, discovered the foundations of a marvelous science (*mirabilis scientiae fundamenta*), on which could be based a complete philosophical system that would provide a road to learning and study, paved with certainty and clarity.

Descartes spent the next two decades developing these ideas and expanded his coverage to include nothing less than the whole world. By 1628, in fact, he had begun working on *Le Monde (The World)*, his broad mechanical explanation for the way much of the world works. In 1633, however, when he had the manuscript ready, he heard that Galileo had been hauled up before the Inquisition for espousing Copernicus's heliocentric theory. Some of Descartes' ideas in his own book, he feared, would displease the church as well. Good Catholic that he was—apparently his training at La Flèche had had some effect—he chose not to publish rather than rock the ecclesiastical boat. He felt, too, that he would rather not weaken his statements. If he was to successfully refute ancient authors such as Aristotle, his work had to be presented with at least as much certainty as their systems.

Right along, though, he was developing both his philosophy and his explanations for how things work in the world. His vortex theory postulated a matter-filled universe in which the motion of any body can be caused only by contact with another. It had enormous influence—for a while. Its advantage was that it provided a mechanical explanation for many heretofore puzzling phenomena, as well as for explanations that previously had depended on spirits and ghosts.

In the interim, he had produced the makings of several books, but, for various reasons, not one was published. Until the *Discourse*, in fact, he had no publications. His fear of offending the church was one reason. Another was that the scientific journal still lay in the future. In France, at least, its place was taken by Father Marin Mersenne, a scholarly priest whose cell in Paris had become a meeting room for some of the foremost French mathematicians of the day, including Blaise Pascal, Pierre Gassendi, Gilles Personne de Roberval, and Jean Beaugrand. Beaugrand would play a major role in the initiation of the coming conflict with Fermat.

Mersenne corresponded with other eminent mathematicians and was instrumental in facilitating the interchange of mathematical developments throughout Europe. He was often referred to as France's "walking scientific journal," and it was by his efforts that Galileo's work became known outside his own country.

Mersenne and Descartes became friendly in 1622, and Mersenne began to spread the word that a new and promising philosopher/

mathematician was developing. By 1626, thanks largely to the communications efforts of Mersenne, Descartes' reputation had grown substantially, even though he had not yet published a single word.

Discourse on Method

Descartes' *Discourse on Method* (1637) was actually a pastiche of works he had composed at various stages of his studies, though it also included some new material. Its full title was *Discourse on the Method of Rightly Conducting One's Reason and Seeking the Truth in the Sciences*. The introductory part, often referred to as the Discourse, contained the basic idea and the philosophical rationale for the entire book.

In this opening section, he laid out four laws that served as the guiding principles of his approach. "The first of these," he wrote, "was to accept nothing as true which I did not clearly recognise to be so: that is to say, carefully to avoid precipitation and prejudice in judgments, and to accept in them nothing more than what was presented to my mind so clearly and distinctly that I could have no occasion to doubt it."[6] His philosophy, then, was one of systematic doubt. Of one thing he could be certain, however, and so it was that his famous statement was born: "I think, therefore I am." One of his biographers, Stephen Gaukroger, explains, "Descartes begins by showing that, provided one's doubt is sufficiently radical, there is nothing that cannot be doubted, except that one is doubting, and this requires that there be something which exists that is doing the doubting."[7]

Of course, doubt was not enough. I noted earlier that he felt he could use mathematics as a foundation on which to build. As he put it: "I was especially delighted with the Mathematics, on account of the certitude and evidence of their reasonings: but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them."[8]

His book was not a full description of the world, but it did suggest that all physical phenomena can be explained mechanically, which turned out to be a very potent concept. Following the relatively brief introduction were three essays showing some examples of how his method could be used toward this end. Two would lie at the center of his dispute with Fermat.

The first, "Dioptrics," deals with the nature and the properties of light. Descartes saw light not as motion but as a pressure or a "tendency to motion" that was transmitted instantaneously (or very close to it) through a kind of elastic medium. This came logically out of his vortex theory: he believed that we experience the light ray in very much the same manner as an impression of movement or resistance that would travel instantly from the point of action to a blind man's hand through his stick.

Therefore, he felt, light will travel instantaneously, or nearly so, through optical media, and also that its speed will actually be greater in a denser medium such as water than in air.

He also thought of both reflection and refraction in terms of material collisions; he assumed that in reflection the light is thrown back like an elastic ball from an elastic surface, and that similar reasoning holds for refraction, except that in this case the ball breaks through the surface.

By this reasoning he came up with his law of refraction, which stated that the ratio between the sine of the angle of incidence of a light ray and the sine of its angle of refraction is a constant:

$$\sin i/\sin r = n.$$ [9]

Descartes had worked the law out mathematically, an impressive accomplishment.[10]

The second essay, "Meteors," was perhaps the first real attempt at a scientific work on the weather. It included a description of how rainbows are produced, which he based on his law of refraction.

Descartes' "Geometry"

In the third essay, "Geometry," Descartes had put together what turned out to be his main legacy in mathematics. He presented, and solved, one of the most difficult problems bequeathed to the mathematical world by the ancients. It had been thought up by the Greek geometer Apollonius in the 3rd century B.C. His contemporary Euclid and then Pappus some six centuries later did more work on it. Yet in spite of much effort by them and many later mathematicians, no one before Descartes had been able to solve it completely—that is, to provide a general solution.

Using his method, Descartes had attacked the problem some years earlier and had solved it in a matter of weeks. The problem, as stated by Descartes, was: "Having three, four or more lines given in position, it is first required to find a point from which as many other lines may be drawn, each making a given angle with one of the given lines. . . . Then, since there is always an infinite number of different points satisfying these requirements, it is also required to discover and trace the curve containing all such points."[11]

J. L. Coolidge restates the problem: "If from a point in a plane, line segments be drawn to meet four given lines of that plane at preassigned angles, and if the product of the first and third segments bear a constant ratio to the product of the second and fourth, then the locus of the point in question is a conic."[12]

Descartes' main contribution was to treat the problem algebraically and generally. The example he gave involves four lines, but his method could be generalized to *n* lines, and it could be reduced to one in which all we need to know are the lengths of certain lines. These lines are the coordinate axes, the lengths of which provide the abcissas and ordinates of needed points.

His association of equations and curves is one essential feature. Also, he located his points and curves on a single coordinate system, a step that had not been taken before. It was not, however, a rectangular coordinate system as we know it today. He used only a single unmoving axis with a moving ordinate, which was not necessarily vertical, but this was a major step nonetheless.

In essence, he had found a way to apply the algebra of Cardano and those who came after him to the geometry of the ancient mathematicians. As Descartes put it in "Geometry," "Here I beg you to observe in passing that the considerations that forced ancient writers to use arithmetical terms in geometry, thus making it impossible for them to proceed beyond a certain point where they could see clearly the relations between the two subjects, caused much obscurity and embarrassment, in their attempts at explanation."[13]

Using one of his own rules, he recognized the necessity of getting rid of the many numbers and "incomprehensible [geometric] figures" that overwhelm the procedure, as it was done before his work. Recall that equations in Cardano's time were still laid out in verbal terms. Toward the end of the 16th century, François Viète, a French lawyer attached to the court of Henry IV, had made some important advances in algebraic notation and in the general improvement of the theory of equations. Viète (1540–1603) was among the first to represent numbers by letters and to introduce at least the beginnings of a general symbolism, but his algebra still differed from ours in an important way. He still saw problems in a geometric sense; he saw the product of two line segments, for example, xx, as an area. As a result, there was even a question as to whether equations of degrees higher than three made any sense at all.

Descartes began his "Geometry" thus: "Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain lines is sufficient for its construction."[14] He saw the product of two lines, say a and b, not only as a rectangle but also as a line. Similarly, terms like x^2 and x^3 could be seen as line segments and not as a square and a cube. Result: he was able to restate a geometric problem in algebraic terms and solve it algebraically.

He also made a significant contribution to the theory of equations. He wrote, "If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction–to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways."[15] (That is, to solve the resulting simultaneous equations.)

Thus, if two curves were considered in the same system of coordinates, their points of intersection could be gotten by solving the equations of the two curves and finding the roots common to their two equations.

So now we can express relations in terms of only two variables. For example, as Emily R. Grosholz states it (as in the Pappus problem), "Distances between the fixed lines and a point *C* on the locus can be expressed in the form $ax + by + c$, and the condition which determines the locus can be expressed as an equation in two unknown quantities. For three or four fixed lines, this equation would be a quadratic equation, for five or six lines, a cubic, and so forth, the introduction of every two lines making the equation one degree higher."[16]

Descartes also brought the symbolism very close to what we use today, whereby lowercase letters at the end of the alphabet represent unknown factors, and letters at the beginning are used for constants and known terms.

Finally—and largely as a result of his "Geometry"—he is commonly credited with the creation of analytic geometry—that is, a geometry in which a point is a set of numbers located in what is now known as a (Cartesian) coordinate system, and a geometric construction can be thought of as a collection of points and described by equations or formulas. It took time for this to happen, however. In fact, the subject was not even given the name *analytic geometry* until the 19th century.

In a letter to Mersenne in 1637, he said, modestly, "I do not enjoy speaking in praise of myself, but since few people can understand my geometry, and since you wish me to give you an opinion of it, I think it would be well to say that it is all I could hope for, and that in *La Dioptrique* and *Les Meteores*, I have only tried to persuade people that my method is better than the ordinary one. I have proved this in my geometry, for in the beginning I have solved a question which, according to Pappus, could not be solved by any of the ancient geometers."[17]

As I stated earlier, he was wrong here. The ancients had solved the problem, but only for one or two specific cases. What he did was produce a general solution, which they had not done. All in all, Descartes felt that he had produced a solid, useful—and unique!—document. As far as he was concerned, all this material was new to the world. He fully believed, for example, that he had developed, for the first time, a specific method for finding the truth–that is, knowledge that is both solid and certain. Though the method would work for various kinds of knowledge, he was particularly interested in finding the "truth" in the sciences.

Dismay—and Conflict

We can only imagine, then, the shock, dismay, chagrin–and even anger–he must have felt upon seeing criticisms of his book by some of the major mathematicians of the day. Among these comments were some by a barely known lawyer and amateur mathematician named Pierre de Fermat. In order to better understand what happened when he received Fermat's comments, though, we must back up a bit.

An independent spirit like Descartes was not one to mince his words. In 1636, a contemporary of his, Jean Beaugrand, had published a book called *Geostatics*, and Descartes had issued a brutal criticism of it. Was Descartes' action at least in part payback for some earlier criticisms of Descartes by Beaugrand? Perhaps. In any case, the publication of Descartes' *Discourse* gave Beaugrand an opportunity to avenge himself. In the spring of 1637, he managed to obtain an advanced copy of Descartes' "Dioptrics" and began a vicious campaign against it. He circulated the manuscript among his own colleagues, including Fermat, apparently in hopes that it would receive some serious criticism even before it was published.

Fermat, in all innocence, and having no idea that the manuscript had been obtained unethically, issued what he thought was a purely scientific critique. He had several objections. As a firm believer in the importance of experiment, he objected to Descartes' reliance on mathematics to study physical phenomena. Specifically, he objected to Descartes' investigating the "inclination to motion" by his mathematical examination of the motion of a tangible physical object (the ball against the elastic sheet).

Another problem, he said, was in Descartes' demonstration and proof of his law of refraction; Fermat argued that it was in fact no proof at all. He stated that Descartes' result was implicit in the assumptions, that "of all the ways of resolving the determination [that is, the tendency] to motion, the author has selected only the one that leads to his conclusion; he has thereby accommodated his means to his end, and we know as little about the subject as we did before."[18]

Fermat had written his comments in the form of a letter to Mersenne. He had started it with a suggestion that we mathematicians can often "find what we are seeking by groping about in the shadows." He then amplified this with his own objections, as noted previously, and he closed with an offer. "We must seek the truth in common," he wrote, and he, Fermat, would be happy to help Descartes in his search.[19] It's not hard to imagine Descartes' reaction when he read this.

What Fermat apparently did not know was that Descartes, upon hearing of Galileo's run-in with the Inquisition, had pulled back his publication of *Le Monde*, in which he had spelled out his physical theories far more explicitly than he could in the *Discourse*. Therefore, as Mahoney explains, "Descartes' *Dioptrics* appeared without the cosmological treatise on which it was based. The short account of the nature of light that opens the *Dioptrics* could not replace the more extended and more carefully argued theory of *The World*, or *On Light*, which Descartes had withdrawn from publication. . . . Moreover, the crucial steps in his derivations of the laws of reflection and refraction depended on the laws of motion presented in the suppressed treatise. Without the precise context of *The World*, the appearance of those laws in the *Dioptrics* seemed arbitrary at best; they were not set out properly until the publication of the [Descartes'] *Principles of Philosophy* in 1644. . . . Only against that background can one understand Fermat's critique, for it focused precisely on the points that required the fuller context."[20]

At first, then, Descartes was not overly concerned. Fermat, he figured, had simply not understood what he was getting at. Nor did Descartes have any idea of what kind of competition he was facing. By the end of 1637, however, two exchanges took place that changed things considerably. Fermat had seen Descartes' "Geometry" and had expressed surprise at the lack of any work by Descartes on maxima and minima, which he felt was so important that it should have been included in a work on mathematics. Fermat had thereby sent to Mersenne his own work on this area, which included methods of finding maxima, minima, tangents to curves, and, very important in the conflict, his own work on analytic geometry. Descartes saw this work just prior to publication of his *Discourse*. Though Fermat had come at the Pappus problem from another approach, and even though this was not his latest work, his methods and procedures were uncomfortably close to Descartes' own.

By this time, other comments and criticisms were coming in. To all of these, Descartes responded mainly with anger and contempt. William R. Shea has described his response nicely.[21] I summarize it here: The French mathematicians who criticized his "Geometry" were dismissed as "two or three flies";[22] Roberval was described as "less than a rational animal";[23] Pierre Petit as "a little dog";[24] and Hobbes as "extremely contemptible."[25] Jean de Beaugrand's letters were only good to be used as "toilet paper."[26]

Descartes' reaction to the comments by Fermat was similar. In a letter to Mersenne, he compared Fermat to Ennius, an earlier Roman poet, and himself to Virgil. Virgil had been quoted in Donatus's *Life of Virgil* as feeling that he was gathering gold out of Ennius's shit.[27]

In general, as the mathematics historian J. F. Scott puts it, "Descartes firmly believed that he had nothing to learn from his contemporaries in any branch of mathematical knowledge, and in particular he leaves his readers in no doubt that he did not rate the achievements of Fermat very highly. In a letter to Mersenne he declared that none of his critics . . . had been able to achieve anything of which the ancient geometers were ignorant." Referring to these critics, he specifically mentioned "M. vostre Conseiller De Maximis et Minimis," meaning, of course, Fermat.[28]

Fermat, the Hesitant Amateur

Born in 1601, five years after Descartes, Fermat was the son of a prosperous leather merchant who also served as second consul of his town. His mother, too, had a high social standing. After a solid secondary education, he received his law degree from the University of Toulouse in 1631. Trained as a classical scholar, he was fluent in Latin and Greek and became interested in "restoring" the lost works of ancient scholars. Among them were the mathematical works of two great Greek mathematicians, Apollonius and Pappus. Still, to this point there was no indication that he would turn out to be one of the great mathematicians of his time.

Named a judge in Toulouse in 1638, he went on to become the king's councilor in 1648. Though he spent most of his life in Toulouse, he had lived some years in Bordeaux. It was during his time in Bordeaux, at the age of about 20, that he became fascinated by the work of Viète.

By the mid-1630s, he got to know Mersenne and was invited to correspond with the Paris group. By the spring of 1636, he had already been working on the ideas that would eventually cause so much heartache for Descartes.

Diffident in his writing, he tried to have it both ways. He wanted recognition, yet did not want to open himself up to criticism. Roberval offered to edit and publish some of Fermat's work, but Fermat refused categorically. Still, as he was by now in steady correspondence with his fellow mathematicians, he was becoming better known. Among these mathematicians was Beaugrand, who in fact prided himself in having "discovered" Fermat.

It was not surprising, then, that Beaugrand had sent a copy of "Dioptrics" to Fermat and asked for his comments. These would, naturally, be funneled through Mersenne. When Mersenne received them, he saw trouble brewing and dithered for several months before transmitting them to Descartes, in spite of Descartes' words in the *Discourse* that asked for comments. Finally, however, Mersenne bit the bullet and sent them, with the results we have already seen.

Attack and Response

By the end of the year, then, Descartes was dealing with a lot. He even began to suspect that he was the subject of a concerted plot to destroy his brainchild. Referring to some of his opponents, such as Pascal and Roberval, he wrote to Mersenne, "I beg you to see if they have not erased the words: *E jusques a* [*E until a*] and replaced them with *B pris en* [*B includes en*]. Because this is the way they cite me in their writing, in order to corrupt the sense of what I said."[29] Fermat's response had been even more troubling.

Descartes, who did not take well to criticism in any case, was faced with a devastating set of events. His life's work had been criticized, to his mind severely. Mahoney writes that Descartes' mathematics, as put forth in both "Dioptrics" and "Geometry," "enjoyed his most jealous protection, for the new method of mathematics had been the source of the larger philosophical method of the *Discourse.* To attack it, to correct it, or to find something not already in it, was to impugn Descartes' whole program. To claim to have achieved similar results independently of, or earlier than, Descartes was to question the uniqueness of [Descartes'] mission."[30] Yet by the end of the year, Fermat had done exactly that. Furthermore, in Descartes' eyes, these objections were coming from a disciple of his hated rival Jean Beaugrand.

Descartes was not one to calmly accept this state of affairs. A contentious man under any circumstances, he felt that this was a situation that needed correction.

Descartes took a closer look at Fermat's work and in January of 1638 began to retaliate. Whether he had in mind a specific mission–namely, to destroy Fermat's growing reputation–or was merely reacting to the criticisms is hard to say. He advanced some specific objections to Fermat's mathematics–for example, to his work on determining the tangent to a cycloid–but then, in a letter to Mersenne, he charged Fermat with deficiencies both as a mathematician and as a thinker. Fermat's methods were defective, he said, and thereby had little value. He went further, suggesting that Fermat was indebted to him for much of what he had developed.

This was particularly unfair. Although Fermat's analytical geometry came to its final form around 1635, it is well known that he had developed various aspects of it much earlier. In addition, by 1635 he had already applied his method to the locus problem that served as the starting point for both Fermat's and Descartes' work in this area. The general consensus is that Fermat was totally unaware of Descartes' work at that time.

Others by this time were entering the lists: Roberval and Pascal had sided with Fermat; Claude Mydorge and Girard Desargues took Descartes' side.

Though Fermat is now recognized as one of the great mathematicians of his era, his tendency to skimp on detail in his writings probably made it easier for Descartes' charges to stick. Even early on, before his trouble with Descartes, some of Fermat's peers were irritated by what looked to them like a supercilious attitude. He would, for example, throw out challenges that he said he had solved, but he gave no details. Had he really solved them? The activity in our own day, centering around Fermat's last theorem, is a case in point. Fermat had written in the margin of a book that he had solved this enormously difficult problem, but he added that he had no room to spell out any details. Only now, almost three centuries later, can anyone claim to have finally come up with a final proof.[31]

Anger Builds

Fermat was to take what appeared at first to be a much less contentious stance than Descartes. In December 1637, he wrote to Mersenne, "First of all I assure you that it is not due to envy or rivalry that I continue this little dispute but only to find the truth; for which I think M. Descartes will not believe me of ill will, the most so since I am well aware of his outstanding ability. . . . Before I enter into the discussion, I will add that I do not wish my letters to be more widely shared than with those with whom an intimate conversation is possible; this I entrust to you."[32]

In February, he wrote again, "I understand from your letter that my reply to M. Descartes was not appreciated, in fact that he decided to comment on my method of maxima and minima and on tangents, in which nevertheless he will find Mssr. Pascal and Roberval of the opposite opinion. Of these two things [Descartes' objections], the first [re refraction] does not surprise me, for matters of physics can always raise doubts and lead to disagreement. But I am astonished by the latter [Descartes' denigration of his methods] since it is a truth of geometry and I maintain that my methods are as certain as the first proposition in the [Euclid] *Elements*. Perhaps being presented plain and without proof they were not understood or else they seemed too easy for M. Descartes, who has made so many paths and followed such a difficult road to tangents in his *Geometrie*." [This is actually a snide remark, as will become clear later.]

"I will send you nothing more for M. Descartes since he puts such strict rules on an innocent exchange. I will be content to tell you I have found no-one here who does not agree with me that his 'Dioptrique' is not proven."[33]

And again: "I await, if you please, the reply M. Descartes made to the difficulties I showed you with his *Dioptrique* and his remarks on my 'Treatise on maxima and minima and on tangents.' If there is some rancor, as he seems to fear there is . . . , that should in no way keep you from showing them to me, for I assure you they will have no effect on my spirit, which is far away from vanity, so that M. Descartes cannot rate me so low that I would not rate myself lower. It is not that my accommodating nature obliges me to retract a truth that I already know, but I want to let you know my mood. Oblige me, please, by not hesitating to send me his writings, about which I promise you in advance to make no reply."

In the same letter, later on: "Whenever you wish my little war with M. Descartes to end, I will not be grieved, and if you arrange for me the honor of his acquaintance, I would be greatly obliged to you."[34]

In Descartes' "Geometry," he had also presented a general method for finding the normal to a curve at any point, and he was very proud of it. Unfortunately for him, Fermat's method was far more direct and closer to modern treatments. Except for simple algebraic curves, what could take pages of complex computation by Descartes could be accomplished far more expeditiously by Fermat.

Descartes began to see some of this as he gave it some thought. By then, he was also less sure of a "conspiracy" than he had been earlier. In mid-June 1638, he wrote to Mersenne, using the flamboyant terms common in those days: "I beg him [Fermat] most humbly to excuse me and to consider that I did not know him. Rather, his *De maximis* came to me in the form of a written challenge on the part of him who had already tried to refute my *Dioptrics* even before it was published, as if to smother it before its birth, having had a copy of it that had not been sent to France for that purpose. Hence it seems to me that I could not have replied to him in words any softer than I used without evincing some sort of laxity or weakness."[35]

That was what he wrote to Mersenne. Yet he had too much at stake, and his bitterness was now fed by his searing hatred of Fermat's friend and defender Gilles Personne de Roberval, so he did not really ease up. Among his more famous comments was one that he made to a colleague, Frans van Schooten, who related it later in a letter to Huygens in 1658. M. Fermat, he said, is a "Gascon." The word can be translated in several ways: it could refer to the region Fermat was from, but it is more likely to mean troublemaker or, most likely, braggart. "I am not [a Gascon]," Descartes continued. "It is true that he [Fermat] has found many pretty, special things, and that he is a man of great mind. But, as for me, I have always endeavored to examine things quite generally, in order to be able to deduce rules that also have application elsewhere."[36]

Among his other charges was that Fermat's method of finding maxima and minima and his rule of tangents were not the result of strict a priori deduction. More important, he argued that Fermat's reputation was built largely on a couple of lucky guesses. This, applied to one of the great mathematicians of the day, was particularly galling to Fermat and his followers. Unhappily, Descartes' reputation gave strength to the rumor, and by the early 1640s, Fermat was seen by some of his peers as having operated by trial and error, rather than by careful and logical analysis.

Things May Not Be What They Seem

To this point, we have to agree with E. T. Bell, who wrote of their mathematical disagreements: "It seems but natural that the somewhat touchy Descartes should have rowed with the imperturbable 'Gascon' Fermat. The soldier was frequently irritable and acid in his controversy over Fermat's method of tangents; the equable jurist was always unaffectedly courteous."[37]

This in fact seems to be a common reading of Fermat's character. In W. W. Rouse Ball's classic *A Short Account of the History of Mathematics* (1908), for example, we find: "The dispute was chiefly due to the obscurity of Descartes, but the tact and courtesy of Fermat brought it to a friendly conclusion."[38] In addition, Mahoney describes Fermat as "gentle, retiring, even shy. . . . There is much to suggest that he simply did not like controversy and that he shied away from it whenever possible."[39]

In a recent article, however, Klaus Barner, a German professor of mathematics, takes Mahoney to task, as follows: "A stereotype that goes back to Mahoney . . . and has been adopted by more recent authors, is that Fermat was a mediocre conseiller and judge who tried to avoid all social, political and religious conflicts. Nothing is further from the truth. Fermat . . . was an outstanding practitioner who . . . stood up for justice and humanity without shrinking from confrontations with the mighty."[40] Mahoney, and perhaps others, had been misled by an incomplete reading of a 1663 report, and that report in turn had relied in part on a wicked untruth perpetrated in Fermat's day by one of his enemies.[41]

It seems likely, then, that Fermat did not shrink from conflict with Descartes, but, as we'll see, his weapons and methods may have been rather more subtle than Descartes' had been.

Continuing Provocation

After the interchanges of the 1630s, the public discord between Fermat and Descartes more or less died down. In fact, it lay quietly for almost two decades. During that time, however, Descartes' attacks on Fermat's reputation were having their desired effect, and Fermat's contributions were increasingly ignored.

Interestingly, while Descartes' reputation continued to grow, it was more in the area of philosophy than in mathematics, for his *Geometry*, by now in book form, was not an easy work to deal with. Ironically, there was some of Fermat's (mathematical) reticence in it. For example, it did not provide full proofs of his work–in order, he wrote, to give others the pleasure of discovering the proofs for themselves. Yet Descartes' colleague, the Leiden mathematician Franz van Schooten, saw the gold in the book. He translated the work into Latin and added considerable explanatory commentary. The revised book came out in four subsequent editions over the years 1649–1695 and had a significant influence on a new generation of mathematicians.

At the same time, however, Descartes' acid pen alienated some of the major mathematicians of his day, including Roberval and Pascal, and he found himself defending himself against charges, for example, from Beaugrand and the British mathematician John Wallis, that he had plagiarized Viète and/or the British mathematician Thomas Harriot (1560–1621). Subsequent studies suggest this was not the case,[42] but the charges had some effect at the time. Descartes was even accused of having used Fermat's work. The most likely scenario, however, is that both men worked independently.

Too, as Mahoney puts it: "A bit more homework on his part prior to publication might have toned down his claim to unprecedented novelty and originality."[43] This might have muted the criticisms.

During the two-decade hiatus in their battle, Descartes continued his work in philosophy and metaphysics and published several well-received works. At the same time, the revised editions of his *Geometry* helped cement his reputation in the mathematical world. He died in 1650, honored and celebrated.

Retaliation

Fermat had not forgotten his hurt, however, and toward the end of the 1650s, he finally had an opportunity to retaliate. Claude Clerselier, an ardent supporter of Descartes, was in the process of preparing an edition of Descartes' letters and asked Fermat for copies of letters he wished to include in his collection. He had copies of the two key letters that Fermat had written to Mersenne for transmittal to Descartes (May and December 1637), but Clerselier had reason to believe there were others. He asked Fermat for copies of such letters. Fermat either misunderstood the request or felt he now had an opportunity to put in a good word for himself. In March of 1658, a little more than 20 years after the original controversy, he composed a long letter to Clerselier, in which he repeated his earlier criticisms and added others.

Mahoney writes, "The restatement was not entirely accurate, as Clerselier, in possession of the original letters, knew full well. Hence, it appeared to him that, by adducing new arguments against

Descartes' derivations, Fermat was seeking to reopen the dispute. He [Clerselier] felt his suspicions confirmed when, in response to his and Jacques Rohault's defense of Descartes, Fermat continued his attack all the more stoutly. The result was a series of letters between Clerselier and Fermat that continued over the next four years."[44] In all, eight letters were exchanged.

And what letters they were. We are especially interested in Fermat's, and I will give only a sample from one of his, for they are difficult to parse. All the letters were written with, as Mahoney calls it, "Baroque politesse," which "barely masks the anger and indignation in which they were written."[45]

J. D. Nicholson, who has translated and studied these letters, adds, "To understand Fermat's letters, you must understand the term *politesse*, a term which in French means right-thinking attention to manners, but in English is likely to mean the use of such manners for less than noble purposes." While Descartes' letters were likely to be direct and unadorned in their criticism of Fermat, the politesse of Fermat was so subtle that the portion of Fermat's letter that follows could easily be taken as praise if not looked at carefully.

For, says Nicholson, while Fermat is saying one thing, he often means something quite different. I will first give the polite translation,[46] and then, with Nicholson's help, I will give just a few of the possible alternate meanings. Too, the Baroque mode of expression was as prolix and excessive in writing as it was in architecture and furniture design. So take a deep breath before you venture into these two paragraphs. Fermat wrote:

> The conclusions that can be taken from the fundamental proposition of M. Descartes' *Dioptrique* are so beautiful and ought naturally to produce such lovely results throughout every part of the study of refraction that one would wish–not only for the glory of our deceased friend, but more for the augmentation and embellishment of the sciences–that this proposition were genuine and legitimately demonstrated, and all the more as it is from these [conclusions] that one is able to say that *multa sunt falsa probabiliora veris* (often, falsehoods are more acceptable than truth). . . .

> I begin from there, Monsieur, in order to let you know that I would be delighted if the differences that I have formerly had on this subject with M. Descartes were ended to his advantage. I would have been satisfied in all ways: the glory of a friend whom I have infinitely esteemed and who has passed with good reason for one of the great men of his time; the establishment of a physical truth of the greatest importance; and the easy execution of these marvelous effects. All this seems incomparably more valuable than winning my case; likewise, I would count for nothing the [Latin] phrase "He will win the fight with me," of which the friends of M. Descartes are always able to reasonably comfort his adversaries. I put myself therefore, Monsieur, in the posture of a man who wants to be vanquished. I say it loudly: I bow at last to your superior powers.[47]

In the first paragraph, did Fermat really mean the "glory of our deceased friend" or did he actually have in mind Descartes' vanity or pride? Descartes himself defined glory as a type of "love that one has for one's self."

In the second paragraph, Fermat wrote, "I would be delighted if the differences that I have formerly had with M. Descartes were ended to his advantage." Rather than "I would be delighted," however, he could also have meant "I would feel raped" or "ravaged." Did Fermat really mean "one of the great men of his time," or was he implying "one of the fat pretentious ones"?

There are also several interesting literary references. Consider, in the first paragraph, "Often, falsehoods are more certain than truth." Fermat could simply be saying that Descartes' false scientific idea was more acceptable than Fermat's own correct one, but it could also be a reference to a famous court case argued by Cicero in 81 B.C. A group of men has stolen the inheritance due a rightful heir. To keep the inheritance, they drag the heir into court on a trumped-up charge of murdering his father. Fermat, in other words, was "tried" by many of Descartes' friends in the court of public opinion and perhaps is the heir denied his due.

In the second paragraph, Fermat uses the Latin phrase "He will win the fight with me." Here he's referring to the famous classical conflict between the heroic Ajax and the smooth-talking Ulysses. In the end, Ulysses' silver tongue wins him the spoils, and Ajax plunges his sword into his own chest. It's not hard to guess who is Ulysses and who Ajax.

Another reference is seen in Fermat's line about bowing. The words "I bow at last to your superior powers" is Charles E. Bennett's translation of the first line of the Latin poet Horace's Epode 17, "A Mock Recantation."[48] In this poem, the protagonist is sentenced to spread the fame of another for all eternity. Fermat apparently is not hopeful that his letters will have the desired effect, but this doesn't mean that he isn't going to try to get the credit he believes he deserves.

There is more like this before Fermat actually gets to the corrections he wishes to score with, but it is enough to give the flavor of the letters. Also, note the construction. It's not easy to tease out the real meaning, but the key words are "one would wish . . . that this proposition were genuine and legitimately demonstrated." His contention, in spite of all the fine words, is that Descartes' proposition was not legitimately, or satisfactorily, proved.

Finally, concentrating his attack on Descartes' derivations, Fermat used stronger and more direct language than he had in 1637. For example, he once again attacked Descartes' proof of the sine law, but, says Harvard professor emeritus A. I. Sabra, "not any longer because it is not conclusive, as he believed twenty years earlier, but simply because it now appears to him to be founded on an assumption that is 'neither an axiom, nor is . . . legitimately deduced from any primary truth.'"[49]

In the course of his reasoning, however, Fermat came up with an important idea, which has come to be called Fermat's principle of least time. In essence, it states that nature follows the shortest path, or the least time possible, for a process in nature. Based on it, and on his own, somewhat different assumptions from Descartes', he in 1661 mathematically derived his own sine law. In essence, he stated that in refraction, it was the optical distance–the products of the distances the light traveled and the corresponding refractive indices–that is a minimum.

Ironically, although some of his objections to Descartes' work were valid, his sine law turned out, most writers feel, to agree exactly with Descartes': sin $i = n$ sin r.

Sabra, however, argues that the sine law did *not* match exactly. He writes that although both laws assert the constancy of the ratio of the sines, Fermat thought that Descartes, in the course of his reasoning, had used the construction $n = v_i/v_r$ (= sin i/sin r), whereas it was actually $n = v_r/v_i$.[50]

It didn't matter, though. The first reasonable determination of the speed of light was not to take place until 1657, so experimental verification of Fermat's theory was not yet possible anyway. Clerselier argued that it was ridiculous to believe that nature could change its mind as light travels from one medium to the next.

Fermat's approach had led him to deduce that light speed is finite, and that light travels faster in air than in water. Both of these conclusions were brilliant insights and just the opposite of Descartes' results. Science eventually came down on Fermat's side. His principle–later expanded to include maxima as well as minima–is now thought of as a basic law of optics, but none of this could be known at the time.

Sabra concludes, "In the face of this unexpected result [identical laws of refraction], he [Fermat] was willing to abandon the battlefield, as he said, leaving to Descartes the glory of having first made the discovery of an important truth, and himself being content to have provided the first real demonstration of it. With this conditional declaration of peace in his last letter to Clerselier of 21 May 1662, the discussion came to an end."[51]

Comparison

In the two decades between the two episodes, Descartes had pretty much steered clear of mathematics. Fermat, on the other hand, had been busy indeed, which makes Descartes' denigration of Fermat's mathematical abilities both sad and strange. By the 1630s, Fermat had already shown his colors, though not widely and not publicly. That Descartes didn't like his colors was Descartes' problem, not

Fermat's. For in the intervening years, Fermat continued his mathematical efforts and came up with several major advances. First, his principle of least time provided a good foundation for several areas in physics. In fact, the whole of geometrical optics could be based on a properly modified form of his principle.

Fermat also made major advances in number theory and probability and provided groundwork for the calculus, which was to follow soon (see chapter 3 in this book). His work, in fact, strongly influenced a number of later mathematicians, including Jean Bernoulli (see chapter 4). Thanks to his reluctance to publish, however, his own thesis on analytic geometry was not published until 1679, which was 14 years after his death.[52]

So he had some small recognition, but basically it all came after he died in 1665. By 1662, perhaps discouraged that there seemed to be little interest in his new work, he had essentially retired from the field altogether. His activity in number theory during the last 15 years of his life found no resonance among his colleagues, and when he died, there was little of the honor he deserved. Part of the reason was that his feelings about publication had apparently not changed. He wrote to a friend, "I much prefer to know the truth with certainty, rather than to take more time in debates and superfluous and useless contentions."[53]

On the other hand, during these last years the new editions of Descartes' *Geometry* were being published, and, with the clarification and the simplification, his treatise gained new adherents and influence. His science, too, was in ascendance. Ironically, however, it was both men's work in analytical geometry that eventually led to the development of the calculus later in the century, and this in turn led to the decline of Cartesian science.

There is general agreement that Fermat's basic approach to analytic geometry is significantly closer to our own than Descartes' was. Yet to the end, Fermat championed Viète's cumbersome notation, while Descartes' symbolic notation is quite modern.

Both men made major contributions to the advance of mathematics in the 17th century. It is sad that the process generated so much heartache and bitterness.