Rajiv Gandhi University of Knowledge Technologies-AP

Catering to the Educational Needs of Gifted Rural Youth of Andhra Pradesh

VECTOR SPACE

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AP-521202

Vector space:

Let P(x, y, z) be any point in plane.

P is a certain distance in a certain direction from the origin.

The distance is characterized by the length and the direction of the line segment from the origin O to P is called a position vector and denoted by $\overline{\boldsymbol{OP}} = \boldsymbol{xi} + \boldsymbol{yj} + \boldsymbol{zk} = \overline{\boldsymbol{a}}$

Thus, each vector is representing an ordered triad of three real numbers $\bar{\mathbf{a}} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$

The set of all vectors of ordered tried is denoted by R^3.

$\therefore\ \mathbf{R^3} = \{\ \bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{\mathbf{d}}, \dots\dots\dots / \ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}\ \}$

$\therefore\ \mathbf{R^3} = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) / \ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}\ \}$

Set of ordered n-tuples:

If $a_1, a_2, \dots\dots\dots a_n$ are the elements of field F then $v=\left(a_1, a_2, a_3, \dots\dots\dots, a_n\right)$ is called an n-tuples.

Set of these types of n tuples are denoted by $\mathbf{F^n} = \{v_1, v_2, v_3, \dots\dots\dots, v_n \ / \ a_i \in F\}$

$F^n = \left(a_1, a_2, a_3, \dots\dots\dots, a_n\right) / \ a_i \in F$

Standard sum of n tuples:

Let $v_1=\left(a_1, a_2, a_3, \dots\dots\dots, a_n\right)$ and $v_2=\left(b_1, b_2, b_3, \dots\dots\dots, b_n\right)$ $v_1, v_2 \in F^n$ then

$v_1+v_2=\{(a_1+b_1), (a_2+b_2), \dots\dots\dots (a_n+b_n\}$.

Standard Scalar Multiplication:

Let $v=\left(a_1, a_2, a_3, \dots\dots\dots, a_n\right) \in F^n$ and k be any scalar then $kv=\left(ka_1, ka_2, ka_3, \dots\dots\dots, ka_n\right)$ is called the scalar multiplication of n tuples.

Vector spaces: A non-empty set V is vector space over a field F with two operations, vector addition and scalar multiplication such that the following axioms hold,

Where the elements of V are called vectors and the elements of F are called scalars, and V is also called linear space over the field F, denoted by V(F) or L(F)

Axioms for vector additions:

1) $\forall$ u, v $\in$V $\Rightarrow$u+v $\in$ V

2) $\forall$ u, v, w $\in$V $\Rightarrow$ u+(v+w)=(u+v)+w

3) $\forall$ u$\in$V $\exists$ 0$\in$V $\Rightarrow$ u+0 = 0+u = u

4) $\forall$ u$\in$ V $\exists$ -u$\in$ V $\Rightarrow$ u+(-u) = 0

5) $\forall$ u, v $\in$V $\Rightarrow$ u+v =v+u

Axioms for scalar Multiplications:

6) $\forall$ a$\in$F & u$\in$V $\Rightarrow$ au$\in$V

7) $\forall$ a, b$\in$ F & u$\in$ V $\Rightarrow$ a(bu) = (ab)u.

8) $\forall$ u$\in$ V $\exists$1$\in$V $\Rightarrow$1.u = u.1=u

9) $\forall$ a$\in$F & u, v $\in$V $\Rightarrow$ a(u+v) = au+av

10) $\forall$ a, b$\in$F & u$\in$V $\Rightarrow$ (a + b)u = au + b

Note:

1. The sets Q, R, C are fields.

2. When F = R we call V as a real vector space and when F = C we call it as complex vector space

Note: if F_1 and F_2 are two fields, such that $F_1 \subseteq F_2$ then ${F_2}^n(F_1)$ is a vector space, and The vector space $F^n(F)$ is also denoted by $V_n(F)$.

1) $C^n(C)$ or $V_3(C)$ is a vector space.

2) $C^n(R)$ is a vector space

3) $C^n(Q)$ is a vector space

4) $R^n(C)$ is not a vector space

5) $R^n(R)$ or $V_3(R)$ is a vector space.

6) $R^n(Q)$ is a vector space

7) $Q^n(C)$ is not a vector space

8) $Q^n(R)$ is not a vector space. **Ex**: $a = \sqrt{3} \in R$ and $v = (2,3) \in Q$ but $a.v = [2\sqrt{3}, 3\sqrt{3}] \notin Q$

9) $Q^n(Q)$ or $V_3(Q)$ is a vector space.

10) The set of all polynomials $P_n(F)$ is vector space over field F.

11) The set of all Matrices $M_{mxn}(F)$ is vector space over field F.

1. Show that V={ (x, x, y)/x, y ∈ R } is a vector space using the operation of $R^3(R)$

1) Let $u=(x_1, x_1, y_1)$, $v=(x_2, x_2, y_2) \in V \Rightarrow u+v = \{ x_1+x_2, x_1+x_2, y_1+y_2 \} \in V$ & $av=(ax, ax, ay) \in V$

2) Clearly $u,v,w \in V \Rightarrow u+(v+w)=(u+v)+w$

3) Let $u=(x,x,y) \in V$ there exist $0=(0,0,0) \in V \Rightarrow u+0=0+u=u$

4) $u=(x,x,y) \in V$ there exist $-u=(-x,-x,-y) \Rightarrow u+(-u)=0$

5) Clearly $u,v \in V \Rightarrow u+v=v+u$

6) Let $u=(x,x,y) \in V$ and $a \in F=R$ then $au=a(x,x,y)=(ax,ax,ay) \in V$

7) Clearly for every $a, b \in F=R$ & $u \in V \Rightarrow a(bu)=(ab)u$.

8) Clearly for every $u \in V$ there exist $1 \in V$ such that $1.u = u.1=u$

9) Clearly for every $a \in F=R$ and $u, v \in V \Rightarrow a(u+v) = au+av$

10) Clearly for every $a,b \in F=R$ and $u \in V \Rightarrow (a + b)u = au + bu$

Hence, V is vector space.

2. If $(a_1, a_2)+(b_1, b_2)=(a_1+b_1, a_2+b_2)$ and $a(a_1,a_2)=(aa_1,0)$ then verify $Q^2(Q)$ is vector space or not? Give an example.

Sol: Given that $(a_1,a_2)+(b_1,b_2)=(a_1+b_1, a_2+b_2)$ and $a(a_1,a_2)=(aa_1,0)$

If $v \in Q$ and $1 \in Q$ then $1.v \neq v$

Ex: $v=(x, y)\in Q$ and $1\in Q \Rightarrow 1.v=1(x,y)=(1.x,0)=(x,0) \neq v$

Hence, $Q^2(Q)$ is not a vector space.

3. If $(a_1,a_2)+(b_1,b_2)=(a_1+b_1, a_2+b_2)$ and $a(a_1,a_2)=(aa_1,a_2)$ then verify $R^2(R)$ is vector space or not? Give an example

Sol: Given that $(a_1,a_2)+(b_1,b_2)=(a_1+b_1, a_2+b_2)$ and $a(a_1,a_2)=(aa_1,a_2)$

If $v\in R$ and $a,b\in R$ then $(a+b)v\neq av+bv$

Ex: Let $v=(1,2)$ and $a=2$, $b=3$

$\Rightarrow(a+b)v=(2+3)(1,2)=5(1,2)=(5.1, 2)=(5,2)$

$\Rightarrow av+bv=2(1,2)+3(1,2)=(1,2)+(3,2)=(4,4)$

Hence $(a+b)v\neq av+bv$, Hence $Q^2(Q)$ is not a vector space.

4. If $(a_1,a_2)+(b_1,b_2)=(a_1+b_1, 0)$ and $a(a_1,a_2)=(aa_1,aa_2)$ then verify $V_2(R)$ is vector space or not? Give an example

Sol: Given that $(a_1,a_2)+(b_1,b_2)=(a_1+b_1, 0)$ and $a(a_1, a_2)=(aa_1, aa_2)$

If $v\in R$ and $0\in R$ then $v+0\neq 0$

Ex: $v=(2,3)\in R$ and $0\in R$

$\Rightarrow v+0=(2,3)+(0,0)=(2+0,0)=(2,0) \neq 0$

Hence, $(a+b)\, v\neq av+bv$, Hence $V_2(R)$ is not a vector space.

Vector Subspace:

Let V(F) be a vector space. A non-empty subset W of V is said to be a subspace of V if W itself is a vector space over F with the same vector addition and scalar multiplication as for V.

Note:

1) For every vector space V, the entire space V and zero space {0} are always subspaces of V, and are called improper (Trivial) subspaces.

2) All subspaces other than these two are called proper (Non-Trivial) subspaces.

Results:

1) A necessary and sufficient condition for sub space:

Let W be a non-empty subset of a vector space V(F). W is a subspace of V ⇒For all a, b∈F and u, v∈W **(i)** u+v∈W **(ii)** au ∈W. **(OR)** au+bv ∈W **(OR)** au+b∈W.

2) The intersection of an arbitrary family of subspaces of a vector space is again a subspace of that vector space.

3) The union of arbitrary family of subspaces of a vector space V(F) may not be a subspace of V(F)

4) The union of two subspaces is a subspace iff one is contained in the other.

5) If $W = (x_1, x_2, \dots \dots x_n)/\ a_1x_1 + a_2x_{2+\dots\dots\dots\dots+}a_nx_n = 0 \subseteq V_n(R)$ (i.e homogenious linear equation) then W is always a subspace of $V_n(F)$.

1. Which of the following are subspaces?

a) $\mathbf{W = \{(x, y, z)\ :\ xy = 0\} \subseteq V_3(\boldsymbol{R})}$

b) $\mathbf{W = \{(x, y, z)\ :\ x = y + 1\} \subseteq V_3(\boldsymbol{R})}$

c) $\mathbf{W = \{(x, y, z)\ :\ x^2 = y\ \} \subseteq V_3(\boldsymbol{R})}$

d) $\mathbf{W = \{(x, y, z)\ :\ |\boldsymbol{x}| = y\} \subseteq V_3(\boldsymbol{R})}$

e) $\mathbf{W = \{(x, y, z)\ :\ x \geq 0\} \subseteq V_3(\boldsymbol{R})}$

f) $\boldsymbol{W = \{(x, y) : x, y \geq 0\} \subseteq V_2(R)}$

g) $\mathbf{W = (x, x^2, y) \subseteq V_3(\boldsymbol{R})}$

h) $\mathbf{W = \{(x, y, z)\ :\ x, y, z \in Q\} \subseteq \boldsymbol{Q}(\boldsymbol{R})}$

Sol:

a) $u = (2,0,3) \in W$ and $v = (0,3,1) \in W$ then $u + v = (2,3,4) \notin W$

Hence, W is not a vector subspace

b) $u = (3,2,1) \in W$ and $v = (4,3,1) \in W$ then $u + v = (7,5,2) \notin W$

Hence, W is not a vector subspace.

c) $u = (2.4,1) \in W$ and $v = (3,9,1) \in W$ then $u + v = (5,13,2) \notin W$

Hence, W is not a vector subspace.

d) $u = (2,2,1) \in W$ and $\mathrm{v} = (-3,3,1) \in \mathrm{W}$ then $u + v = (-1,5,2) \notin W$

Hence, W is not a vector subspace.

e) $u = (2,1,3) \in W$ and $v = (0,3,1) \in W$ then $u + v = (2,4,4) \in W$ but

$u = (2,1,3) \in W$ & $a = -1 \in R \implies au = (-2, -1, -3) \notin W$

Hence, W is not a vector subspace

f) Let u=(1,2)∈V and v=(−1, − 2)∈V , then u+v=(0,0)∉ V

Hence, W is not a vector subspace of R^2

g) $u = (2,4,3) \in W$ and $v = (3,9,1) \in W$ then $u + v = (5,11,4) \notin W$

Hence, W is not a vector subspace

h) $u = (1,2,3) \in W\ and\ a = \sqrt{2} \in R$ then $au = \sqrt{2}(1,2,3) \notin W$

Hence, W is not a vector subspace

2. Which of the following are subspaces?

a) $\mathbf{W = \{(x, y, z)\ :\ x^2 + y^2 + z^2 = 0\}; x, y, z \in R \subseteq V_3(\boldsymbol{R})}$

b) $\boldsymbol{W = \{(x, y, z)\ /\ x = 2y\} \subseteq V_3(R)}$

c) $\mathbf{W = \{(x, y, z)\ :\ x = y = z\} \subseteq V_3(\boldsymbol{R})}$

d) $\mathbf{W = \{(x, y, z)\ :\ y = 2z\} \subseteq V_3(\boldsymbol{R})}$

Sol:

(a) $W = \{(0,0,0) \ : \ x^2 + y^2 + z^2 = 0\}; x, y, z \in R \Rightarrow W = \{\bar{0}\}$ Hence, W is trivial subspace or improper subspace of V

(b) Let $u = (2y_1, y_1, z_1) \in W$ & $v = (2y_2, y_2, z_2) \in W$ then $u + v = \{2(y_1 + y_2), (y_1 + y_2), (z_1 + z_2)\} \in W$

And $a \in R$, $au = (2ay_1, ay_1, az_1) \in W$

Ex: $u = (4,2,1) \in W$ and $v = (6,3,2) \in W$ then $u + v = (10,5,3) \in W$ and $a \in R$, $au \in W$ Hence W is vector subspace.

Note: x-2y+0z=0 is a homogeneous linear condition in x,y, z. Hence, W is a subspace

(c) $u = (x, x, x) \in W$ and $v = (y, y, y) \in W$ then $u + v = (x + y, x + y, x + y) \in W$ and $a \in R$, $au \in W$ Hence, W is a subspace.

Ex: $u = (2,2,2) \in W$ and $v = (3,3,3) \in W$ then $u + v = (5,5,5) \in W$ and $a \in R$, $au \in W$ Hence, W is a subspace.

(d) Let $u = (x_1, 2z_1, z_1) \in W$ & $v = (x_2, 2z_2, z_2) \in W$ then $u + v = \{(x_1 + x_2), 2(z_1 + z_2), (z_1 + z_2)\} \in W$

And $a \in R$, $au = (ax_1, 2az_1, az_1) \in W$

Hence, W is a subspace.

Ex: $u = (1,4,2) \in W$ and $v = (2,6,3) \in W$ then $u + v = (3,10,5) \in W$ and $a \in R$, $au \in W$

Hence, W is a subspace.

Note: 0x+y-2z=0 is a homogenous linear condition in x,y,z. hence W is a subspace

3. For V3(R) which of the following is a sub space.

1. $W = \{(x, y, z) \ : \ 2x = 3z\}$

2. $W = \{(x, y, z) \ : \ 3x + y - z = 0\}$

3. $W = \{(x, y, z) \ : \ x - 3y + 4z = 0\}$

4. $W = \{(x, y, z) \ / \ x = 2y\}$

5. $W = \{(x, y, 0) \ / \ x, y \in R\}$

6. $W = \{(x, 2x, y) \ / \ x, y \in R\}$

7. $W = \{(x, x + y, 3x) \ / \ x, y \in R\}$

8. $W = \{(x, x + 1, y) \ / \ x, y \in R\}$

9. $W = \{(x, y, x + y - 4) \ / \ x, y \in R\}$

10. $W = \{(x, y, z) \ / \ xy = xz\}$

11. $W = \{(x, y, z) \ / \ x + y + z = 0\}$

12. $W = \{(x, y, z) \ : \ z = 2021\}$

13. $W = \{(x, y, z) \ : \ x + y + z = 2021\}$

14. $W = \{(x, y, z) \ : \ x + y = z^2\}$

15. $W = \{(x, y, z) \ : \ x + y = 2021z\}$

16. $W = \{(x, y, z) \ : \ x < y\}$

17. $W = \{(x, y, z) \ : \ x^2 = y^2\}$

18. $W = \{(x, y, z) \ : \ x = 0 \text{ or } y = 0\}$

19. $W = \{(x, y, z) \ : \ 2019\,x + 2020\,y = 2021z\}$

20. $W = \left\{(x, y, z) \ : \ \frac{x}{y} = 2\right\}$

21. $W = \left\{(x, y, z) \ : \ \frac{y}{z} = 2\right\}$

22. $W = \{(x, y, z) \ : \ x = 4z\}$

23. $W = \{(x, y, z) \ : \ |x| = y\}$

24. $W = \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 0\}$

25. $W = \{(x, y, z) \ : \ x + y = 0 \text{ or } y = 2\}$

26. $W = \{(x, y, z) \ : \ |x| + |y| = |z|\}$

4. **For $M_{m\times n}(R)$ which of the following is a sub space.**

a) $\mathbf{W = \{A: A \text{ is a singular matrix}\} \subseteq M_{m\times n}}$

b) $\mathbf{W = \{A: A \text{ is a non} - \text{singular matrix}\} \subseteq M_{m\times n}}$

c) $\mathbf{W = \{A: A \text{ is a idempotent matrix}\} \subseteq M_{m\times n}}$

d) $\mathbf{W = \{A: A \text{ is a involutary matrix}\} \subseteq M_{m\times n}}$

e) $\mathbf{W = \{A: A \text{ is a nilpotent matrix}\} \subseteq M_{m\times n}}$

f) $\mathbf{W = \{A: A \text{ is a symmetric matrix}\} \subseteq M_{m\times n}}$

g) $\mathbf{W = \{A: A \text{ is a Skew symmetric matrix}\} \subseteq M_{m\times n}}$

h) $\mathbf{W = \{A: A \text{ is orthogonal triangle matrix}\} \subseteq M_{m\times n}}$

i) $\mathbf{W = \{A: A \text{ is upper triangle matrix}\} \subseteq M_{m\times n}}$

j) $\mathbf{W = \{A: A \text{ is lower triangle matrix}\} \subseteq M_{m\times n}}$

k) $\mathbf{W = \{A: A \text{ is scaler matrix}\} \subseteq M_{m\times n}}$

l) $\mathbf{W = \{A: A \text{ is diagonal matrix}\} \subseteq M_{m\times n}}$

m) $\mathbf{W = \{A: \text{Trace}A = 0\} \subseteq M_{m\times n}}$

n) If $\mathbf{W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b = c - d\right\} \subseteq M_{2\times 2}}$

(a) $\mathbf{W = \{A: A \text{ is a singular matrix}\} \subseteq M_{m\times n}}$

Sol: $A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix}$ then $A + B = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} \notin W$

W is not a subspace.

(b) $\mathbf{W = \{A: A \text{ is a non} - \text{singular matrix}\} \subseteq M_{m\times n}}$

Sol: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix}$ then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \notin W$

W is not a subspace.

(c) $\mathbf{W = \{A: A \text{ is a idempotent matrix}\} \subseteq M_{m\times n}}$

Sol: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = A, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow B^2 = B$ then $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow (A + B)^2 \neq A + B \notin W$

W is not a subspace

(d) $\mathbf{W = \{A: A \text{ is a involutary matrix}\} \subseteq M_{m\times n}}$

Sol: $A^2 = I$ & $B^2 = I$, $A, B \in W$ then $A + B = 2I \Rightarrow (A + B)^2 = 4I \neq I \notin W$

W is not a subspace

(e) $\mathbf{W = \{A: A \text{ is a nilpotent matrix}\} \subseteq M_{m\times n}}$

Sol: $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = 0, B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \Rightarrow B^2 = 0$ then $A + B = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \Rightarrow (A + B)^n \neq 0 \notin W$

W is not a subspace

(f) **$W = \{A: A \text{ is a symmetric matrix}\} \subseteq M_{m\times n}$**

Sol: $a, b \in R$ *and* $A, B \in W$ then$A^T = A, \quad B^T = B$ then

$(aA + bB)^T = (aA)^T + (bB)^T = aA^T + bB^T = aA + bB \in W$

W is a subspace.

(g) **$W = \{A: A \text{ is a skew symmetric matrix}\} \subseteq M_{m\times n}$**

Sol: $a, b \in R$ *and* $A, B \in W$ then$A^T = -A, \quad B^T = -B$ then

$(aA + bB)^T = (aA)^T + (bB)^T = aA^T + bB^T = -(aA + bB) \in W$

W is a subspace.

(h) **$W = \{A: A \text{ is orthogonal triangle matrix}\} \subseteq M_{m\times n}$**

Sol: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow AA^T = IB = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow BB^T = I$ then

$A + B = \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} \Rightarrow (A + B)(A + B)^T \neq I \notin W$

W is not a subspace

(i) **$W = \{A: A \text{ is upper triangle matrix}\} \subseteq M_{m\times n}$**

Sol: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \& B = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$ then $A + B = \begin{bmatrix} 2 & 5 \\ 0 & 8 \end{bmatrix} \in W$ and a∈R, au∈W

W is a subspace

(j) **$W = \{A: A \text{ is a lower triangle matrix}\} \subseteq M_{m\times n}$**

Sol: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \& B = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$ then $A + B = \begin{bmatrix} 2 & 0 \\ 5 & 8 \end{bmatrix} \in W$ and a∈R, au∈W

Hence W is a subspace

(k) **$W = \{A: A \text{ is a scalar matrix}\} \subseteq M_{m\times n}$**

Sol: $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \& B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ then $A + B = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \in W$ and a∈R, au∈W

Hence W is a subspace

(l) **$W = \{A: A \text{ is a diagonal matrix}\} \subseteq M_{m\times n}$**

Sol: $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \& B = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ then $A + B = \begin{bmatrix} 3 & 0 \\ 5 & 8 \end{bmatrix} \in W$ and a∈R, au∈W

Hence W is a subspace

(m) **$W = \{A: \text{Trace}(A) = 0\} \subseteq M_{m\times n}$**

Sol: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then Trace(A)=a+d since Trac(A)=0 , a+d=0 which is homogenious linear condition. Hence W is a subspace

(n) **If $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b = c - d\right\} \subseteq M_{2\times 2}$**

Sol: since a+b=c-d is homogeneous linear condition

Hence W is a vector subspace

5. For P(R) which of the following is subspace of R

a. $W = \{f(x) : \text{deg. of } f(x) = 2\} \subseteq P(R)$

b. $W = \{f(x) : \text{deg. of } f(x) > 2\} \subseteq P(R)$

c. $W = \{f(x) : \text{deg. of } f(x) \geq 2\} \subseteq P(R)$

d. $W = \{f(x) : \text{deg. of } f(x) < 2\} \subseteq P(R)$

e. $W = \{f(x) : \text{deg. of } f(x) \leq 5\}$

f. $W = \{f(x) : \text{ sum of all coefficients } = 0\}$

g. $W = \{f(x) : \text{sum of co} - \text{eff of even powes of x}\} = \text{sum of co} - \text{eff of odd powes of x}\}$

Note: If V is the set of all real valued continuous functions over R, then set of solutions W of differential equations is a vector subspace.

Linear combination of vectors:

Let V(F) be a vector space. A vector $v \in V$ is said to be Linear Combination (L.C.) of the vectors $v_1, v_2, v_3, \ldots\ldots\ldots v_n \in V$ if v can be expressed as $v = a_1v_1 + a_2v_2 + a_3v_3 \ldots\ldots + a_nv_n$

where $a_1, a_2, a_3 \ldots\ldots a_n \in F$ are called coordinates.

Ex: If $v_1=(1,2)$ and $v_2=(3,0)$ are two vectors in $V_2(R)$ then express $v=(6,6)$ as a linear combination of v_1 and v_2

Sol: $v = a_1v_1 + a_2 v_2$

$(6,6) = a_1(1,2) + a_2(3,0) = (a_1+3a_2, 2a_1)$

$a_1+3a_2=6$ and $2a_1=6 \Rightarrow a_1=3$ and $a_2=1$ hence $v=3v_1+1v_2$

Linear Span:

If $S=\{u_1, u_2, u_3 \ldots\ldots u_n\}$ is any set of vectors in a vector space V(F), the set of all possible linear combination of these vectors is called their span and is denoted by Span(S) or L(S) or [S]

i.e $L(S) = \{v_i /\ v_i = a_1u_1 + a_2u_2 + \cdots + a_nu_n : a_i \in F \text{ and } u_i \in S\} \subseteq V$

Generating set (Spanning set):

A non-empty set S of vector space V is called a generating set of V if every element $v \in V$ is a linear combination of finite number of elements of S i.e. L(S)=V and it is denoted by $<S>$

1. Express $v=(1,-2,5)$ as a linear combination of the vectors $v_1=(1,1,1)$, $v_2=(1,2,3)$, $v_3=(2,-1,1)$

Sol: $v = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$

$(1,-2,5) = (a_1+a_2+2a_3, a_1+2a_2-a_3, a_1+3a_2+a_3)$

$a_1+a_2+2a_3=1$, $a_1+2a_2-a_3=-2$, $a_1+3a_2+a_3=5$

Solve these equations we get $a_1=-6$, $a_2=3$ and $a_3=2$

Hence $V=-6v_1+3v_2+2v_3$

2. Show that the vector $v=(2,-5,3)$ cannot be expressed as a linear combination of the vector $v_1=(1,-3,2)$, $v_2=(2,-4,-1)$, $v_3=(1,-5,7)$

Sol: $v = a_1 v_1 + a_2v_2 + a_3v_3$

$(2,-5,3) = a_1(1,-3,2) + a_2(2,-4,-1) + a_3(1,-5,7)$

$a_1+2a_2+a_3=2$, $-3a_1-4a_2-5a_3=-5$, $2a_1-a_2+7a_3=3$

The above equations are inconsistent so no solution

Hence, v cannot be expressed as a linear combination of v_1, v_2, v_3.

3. Express the vector (4,5,5) as a linear combination of the vectors (1,2,3), (-1,1,4), (3,3,2).

Sol: $V = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(4, 5, 5) = a_1 (1,2,3) + a_2 (-1,1,4) + a_3 (3,3,2)$

$4 = a_1 - a_2 + 3a_3,$

$5 = 2a_1 + a_2 + 3a_3,$

$5 = 3a_1 + 4a_2 + 2a_3$

This system of equations has many solutions $a_1 = -2k+3$, $a_2 = k-1$, $a_3 = k$

$(4,5,5) = (-2k+3)(1,2,3) + (k-1)(-1,1,4) + k(3,3,2)$

For example,

If k=3 gives $(4,5,5) = -3(1,2,3) + 2(-1,1,4) + 3(3,3,2)$

If k=-1 gives $(4,5,5) = 5(1,2,3) + (-2)(-1,1,4) + (-1)(3,3,2)$, ………………..

4. Show that the vector $x^2+x+2 \subseteq P_2(R)$ cannot be expressed as a linear combination of the vector $\{1+x, 2-x^2, x^2+2x\} \subseteq P_2(R)$

Sol: Given that $v = x^2+x+2 = (1,1,2)$ and $v_1 = 1+x = (0,1,1)$, $v_2 = 2-x^2 = (-1,0,2)$, $v_3 = x^2+2x = (1,2,0)$

$v = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(1,1,2) = a_1(0,1,1) + a_2(-1,0,2) + a_3(1,2,0) = \{-a_2+a_3, a_1+2a_3, a_1+2a_2\}$

$1 = a_1 + a_3$, $1 = a_1 + 2a_3$, $2 = a_1 + 2a_2$

The above system of equations are inconsistent

Hence, above system has no solution

x^2+x+2 cannot be expressed as a linear combination of $1+x$, $2-x^2$, x^2+2x

5. If $S = \{1, x-1, x^2-1, x^3+1\} \subseteq P_3(R)$ then find the coordinates of $x^3+x^2+7x+5 \subseteq P_3(R)$

Ans: $a_1 = 12$, $a_2 = 1$, $a_3 = 7$, $a_4 = 1$

6. Express the vector $v = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Sol: Given that $v = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = (3,-1,1,-2)$ and $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = (1,1,0,-1)$, $v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = (1,1,-1,0)$ and $v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = (1,-1,0,0)$

$v = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(3,-1,1,-2) = a_1(1,1,0,-1) + a_2(1,1,-1,0) + a_3(1,-1,0,0) = \{a_1+a_2+a_3, a_1+a_2, -a_2, -a_1\}$

$a_1+a_2+a_3 = 1$, $a_1+a_2 = -1$, $-a_2 = 1$, $-a_1 = -2$

$a_1 = 2$, $a_2 = -1$, but $a_1+a_2 \neq -1$ hence v cannot be expressed as a linear combination of v_1, v_2, v_3.

7. Find the linear span of the vectors (1,0,0), (0,1,0) and (0,0,1) in $V_3(R)$

Let S= $\{v_1,v_2,v_3\} \subseteq V_3(R)$ where $v_1=(1,0,0)$, $v_2=(0,1,0)$, $v_3=(0,0,1)$

Let $(x, y, z) \in V$

$(x, y, z)=a(1,0,0)+b(0,1,0)+c(0,0,1)$.

$(x, y, z)=(a, b, c)$

$x=a, y=b, z=c$

Hence, $(x, y, z)=x(1,0,0)+y(0,1,0)+z(0,0,1)$.

$L(S)=\{v_1, v_2, v_3, \ldots\ldots\ldots v_n\}=V$

Hence, S is generating set

8. If S= {(1, 2, 1), (3, 4, 1)} ⊆$V_3(R)$ then verify which of the following vector is in L(S).

(i)(0,0,0), (ii)(1,3,2), (iii)(3,2,1), (iv)(0,1,1)

Sol: Let $(x,y,z) \in V_3(R)$

$(x,y,z)=a(1,2,1)+b(3,4,1)=(a+3b\ ,\ 2a+4b,\ a+b)$

(i) If $(x,y,z)=(0,0,0)$ then $(0,0,0)=a(1,2,1)+b(3,4,1)=(a+3b\ ,\ 2a+4b,\ a+b)$

$a+3b=0,\ 2a+4b=0,\ a+b=0$

$a=0, b=0, c=0$

$(0,0,0)=0(1,2,1)+0(3,4,1)$

Hence, (0,0,0) can be expressed as a linear combination of (1,2,1) and (3,4,1)

(ii) If $(x,y,z)=(1,3,2)$ then $(1,3,2)=a(1,2,1)+b(3,4,1)=(a+3b\ ,\ 2a+4b,\ a+b)$

$a+3b=1,\ 2a+4b=3,\ a+b=2$

Solve the above equations we get infinite solutions

Hence, (1,3,2) can be expressed as a linear combination of (1,2,1) and (3,4,1)

(iii) If $(x,y,z)=(3,2,1)$ then $(3,2,1)=a(1,2,1)+b(3,4,1)=(a+3b\ ,\ 2a+4b,\ a+b)$

$a+3b=3,\ 2a+4b=2,\ a+b=1$

The above system is inconsistent and hence it has no solution.

Hence, (3,2,1) can not be expressed as a linear combination of (1,2,1) and (3,4,1)

(iv) If $(x,y,z)=(0,1,1)$ then $(0,1,1)=a(1,2,1)+b(3,4,1)=(a+3b\ ,\ 2a+4b,\ a+b)$

$a+3b=0,\ 2a+4b=1,\ a+b=1$

Solve the above equations we get infinite solutions

Hence, (0,1,1) can be expressed as a linear combination of (1,2,1) and (3,4,1)

(OR)

Let $(x,y,z) \in V_3(R)$

$(x,y,z)=a(1,2,1)+b(3,4,1)=(a+3b\ ,\ 2a+4b,\ a+b)$

$x=a+3b,\ y=2a+4b,\ z=a+b$

$L(S)=\{(x,y,z)/x+z=y\}$

If $(x,y,z)=(0,0,0)\Rightarrow 0=0+0$

If $(x,y,z)=(1,3,2)\Rightarrow 3=1+2$

If $(x,y,z)=(3,2,1)\Rightarrow 1\neq 3+1$

If $(x,y,z)=(0,1,1)\Rightarrow 1=0+1$

$L(S)=\{ (0,0,0), (1,3,2), (0,1,1)\}$

9. If $S=\{ 1+x, 2-x^2 ,x^2+2x \} \subseteq P_2(R)$ then which of the following vector is in Linear L(S)

(a) x^2+x+2 (b) x^2+x-1 (c) x^2-x+1 (d) x^2-x-2

Sol: Given that $S=\{ 1+x, 2-x^2 ,x^2+2x \}=\{ (0,1,1), (-1,0,2),(1,2,0) \}$

Let $(u,v,w)\in P_2(R)$

$(u,v,w)=a(0,1,1)+b (-1,0,2)+c (1,2,0)$

$(u,v,w)=\{ -b+c, a+2c, a+2b \}$

$u=-b+c, v=a+2c, w=a+2b.$

$L(S)= \{(u,v,w)/2u=v-w\}$

$L(S)=\{(2,1,-1)\}$

Satisfying vector is x^2+x-1

Exercise Problems:

1. Suppose that V1 = (2, 1, 0, 3), V2 = (3, −1, 5, 2) and V3 = (−1, 0, 2, 1) Which of the following is not there in the span{V1, V2, V3}? (a) (2,3,-7,3) (b) (1,1,1,1) (c) (0,0,0,0) (d) (4,6,-14,6)

2. Show that the vector (3,-4,-6) cannot be expressed as a linear combination of the vectors (1,2,3), (-1,-1,-2) and(1,4,5).

3. Express the vector v = (4,– 5,9,– 7) as a linear combination of vectors v_1=(1,1,-2,1), v_2=(3,0,4,-1), v_3=(-1,2,5,2)

4. Is it possible to express the vector v = (2,-5,4) as a linear combination of v_1= (1,-3,2) and v_2=(2,-1, 1)

5. Consider the vector v= (1, -2, k) inR^3(R) For what value of k (if any) the vector v can be expressed as a linear combination of vectors v_1=(3,0,-2) and v_2=(2,-1,-5)?

6. Find the condition on a , b , c such that the matrix $\begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ is a linear combination of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

7. If S= {(1,0,1), (0,1,1)} $\subseteq V_3$(R) then which of the following vector is in Linear Span.

(a)(2,1,1) (b) (1,1,2) (c) (1,2,1) (d) (2,2,2)

Ans: L(S)= (x,y,z)={(a, b, a+b)}=(1,1,2)

8. If S={(0,1,-1),(3,2,1), (1,0,1)} $\subseteq V_3$(R) then which of the following vector is in L(S).

(a)(1,4,3) (b)(1,3,4) (c)(4,3,1) (d)(3,4,1)

Ans: L(S)={ (x,y,z)/y+z=x}={(4,3,1)}

9. If S={(3,2,0),(1,1,-1),(2,1,1)} $\subseteq V_3$(R) then which of the following vector is not in L(S).

(a)(1,1,1) (b)(7,5,-1) (c)(0,0,0) (d)(0,5,-15)

Ans: L(S)={ (x,y,z)/3y+z=2x}={(7,5,-1),(0,0,0),(0,5,-15)} and not satisfying vector is (1,1,1)

10. Is the vector (3,-1,0,-1) in the subspace of R^4 spanned by the vectors v_1=(2,-1,3,2), v_2=(-1,1,1,-3) and v_3=(1,1,9,-5) ?

Note:

1) Every non-empty set S is always contained in its linear span.

   Let $v \in S$, we can write $v = 1.v \in L(S) \Rightarrow S \subseteq L(S)$

2) The linear span of empty set ϕ is defined to be $\{0\}$ i.e. span $(\emptyset) = \{0\} \subseteq V$.

3) In a vector space V, L(S) is the smallest sub–space containing S.

4) L(S) =S iff S is a **subspace** of V and L{L(S)} =S

5) Let S be non-empty subset of the vector space V(F).The linear span L(S) is the intersection of all subspace of V which contains S.

6) Let W_1 and W_2 be **subspace** of vector space V then $L(W_1 \cup W_2) = W_1 + W_2$

7) If S and T are two non-empty **subsets** of a vector space V(F), then **(a)** $S \subseteq T \Rightarrow L(S) \subseteq L(T)$ **(b)**$L(S \cup T) = L(S) + L(T)$ **(c)** $L(L(S)) = L(S)$

NOTE:

a. If $S=\{ (x,0)/ x \in R \}$ then Linear span L(S) is X-axis in $V_2(R)$

b. If $S=\{ (0,y)/y \in R \}$ then Linear span L(S) is Y-axis in $V_2(R)$

c. If $S=\{ (x,y)/x,y \in R \}$ then Linear span L(S) is XY-axis in $V_2(R)$

d. If $S=\{ (x,0,0)/ x \in R \}$ then Linear span L(S) is X-axis in $V_3(R)$

e. If $S=\{ (0,y,0)/ y \in R \}$ then Linear span L(S) is Y-axis in $V_3(R)$

f. If $S=\{ (0,0,z)/ z \in R \}$ then Linear span L(S) is Z-axis in $V_3(R)$

g. If $S=\{ (x,y,0)/x,y \in R \}$ then Linear span L(S) is XY- Plane in $V_3(R)$

h. If $S=\{ (x,0,z)/x,z \in R \}$ then Linear span L(S) is XZ- Plane in $V_3(R)$

i. If $S=\{ (0,y,z)/y,z \in R \}$ then Linear span L(S) is YZ-Plane in $V_3(R)$

j. If $S=\{ (x,y,z)/x,y,z \in R \}$ then Linear span L(S) is $V_3(R)$

Linearly independent (L.I.) vectors:

Let S={ $v_1, v_2 \ldots\ldots\ldots, v_n$ } $\subseteq$ V(F) is a non-empty subset of V(F), if there exist scalars $a_1=a_2=a_3=\ldots\ldots\ldots=a_n=0$ such that $a_1v_1+ a_2v_2+\ldots\ldots.+ a_nv_n= 0$ then the set S is called linearly independent set or $v_1, v_2 \ldots\ldots\ldots, v_n$ are called linearly independent vectors.

Linearly dependent (L.D.) vectors:

Let S={ $v_1, v_2 \ldots\ldots\ldots, v_n$ } $\subseteq$ V(F) is a non-empty subset of V(F), if there exist scalars $a_1, a_2 \ldots\ldots\ldots, a_n \in F$ at least one non-zero such that $a_1v_1+ a_2v_2+\ldots\ldots.+ a_nv_n= 0$ then the set S is called linearly dependent set or $v_1, v_2 \ldots\ldots\ldots, v_n$ are called linearly dependent vectors.

Working Steps to determine the given vectors are L.D or L.I:

1) The given vectors are $v_1, v_2, \ldots\ldots v_n$
2) Construct a matrix A by writing the given vectors in the row (column)
3) Reduce the matrix A into row echelon form by using elementary row operation.
4) Number of non – zero rows gives the rank of A i.e. $\rho(A)$.
5) If $\rho(A)$ = number of vectors, then given vectors are L.I. and
6) if$\rho(A)$ < number of vectors, then given vectors are L.D.

NOTE:

A. If the determent of the matrix is zero then the given set of vectors are **L.D**
B. If the determent of the matrix is non-zero then the given set of vectors are **L.I**

Results:

1. The empty set is defined to be linearly independent (L.I). Hence every linearly dependent (L.D) sets must be non-empty
2. A set containing only the zero vector is linearly dependent.
3. A set containing only a single non–zero vector is linearly independent.
4. A set, which contains at least one null vector is linearly dependent.
5. If two vectors are L.D. then one of them is scalar multiple of other.
6. If S is a non-empty subset of V(F) then S is L.D then one of the vector of S can be written as linear combination of the remaining vectors of S, otherwise it is L.I
7. Every super set of a linearly dependent set is linearly dependent.
8. Every subset of a linearly independent set is linearly independent
9. The vectors $(a_1, a_2), (b_1, b_2)$ of $V_2(F)$ are L.D if $a_1b_2 - a_2b_1 = 0$

1. Determine whether the following vectors in $V_3(R)$ is linearly dependent or independent (1,2,3), (4,1,5), (-4,6,2)

Sol: Let $v_1=(1,2,3)$, $v_2=(4,1,5)$, $v_3=(-4,6,2)$

$av_1 + bv_2 + cv_3 = 0$ where $a,b,c \in R$

$a(1,2,3)+b(4,1,5)+c(-4,6,2)=(0,0,0)$

$a+4b-4c=0$ ……..(1)

$2a+b+6c=0$……..(2)

$3a+5b+2c=0$…….(3) solve (1) (2) and (3) we get a=-4, b=2, c=1

Hence, the vectors are linearly dependent

(OR) We construct a matrix A whose rows are given vectors i.e. $A = \begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{vmatrix} \neq 0$ Hence, the vectors are linearly dependent

(OR)

$$A = \begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \Longrightarrow A = \begin{bmatrix} 1 & 4 & -4 \\ 0 & -7 & 14 \\ 0 & -7 & 14 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \Longrightarrow A = \begin{bmatrix} 1 & 4 & -4 \\ 0 & -7 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in row echelon form.

$\rho(A) = 2 <$ number of given vectors.

Hence, the given vectors are L.D

2. Determine whether the following vector are linearly dependent or independent (1,-2,5,-3), (2,3,-1,4) , (3,8,-3,-5).

Sol: We construct a matrix A whose rows are given vectors i.e. $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & -1 & 4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \Longrightarrow A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -11 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \Longrightarrow A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -11 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Which is in row echelon form. $\rho(A) = 3 =$ number of given vectors. Hence, the given vectors are L.I

3. Determine whether the polynomials x^3-5x^2-2x+3, x^3-1, x^3+2x+4 are linearly dependent or independent.

Sol: $A = \begin{bmatrix} 1 & -5 & -2 & 3 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 2 & 4 \end{bmatrix}$

$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \Rightarrow A = \begin{bmatrix} 1 & -5 & -2 & 3 \\ 0 & 5 & 2 & -4 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ and $R_3 \to R_3 - R_2 \Rightarrow A = \begin{bmatrix} 1 & -5 & -2 & 3 \\ 0 & 5 & 2 & -4 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

This is in row echelon form. Here ρ(A)=3=number vectors. Hence, the given vectors are L.I.

4. Which of the following set is linearly dependent (a) S_1={a+b, b+c, c+a} (b) S_2={a+b, b+c, c-a} (c) S_3={a-b, b-c, c-a} (d) S_4={a+b-c, a-b+c, 10a }

Sol: (a) $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \neq 0$, Hence S_1 is L.I

(b) $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0$, Hence S_2 is LD

(c) $|A| = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 0$, Hence S_3 is LD

(d) $|A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 10 & 0 & 0 \end{vmatrix} = 0$, Hence S_4 is LD

5. For M_{2x2}(R)which of the following set is LI set

$(\boldsymbol{a}) \boldsymbol{S_1} = \left\{ \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \right\}$

$(\boldsymbol{b}) \boldsymbol{S_2} = \left\{ \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix} \right\}$

$(\boldsymbol{c}) \boldsymbol{S_3} = \left\{ \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\}$

$(\boldsymbol{d})\ \boldsymbol{S_4} = \left\{ \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\}$

Sol:

(a) Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, Rank A< number of vectors (4), Hence S_1 is LD

(OR) $v_1=v_2+2v_3+3v_4$ i.e; If S is a non-empty subset of V(F) then S is L.D then one of the vector of S can be written as linear combination of the remaining vectors of S

(b) $v_3=v_1+v_2$, Hence S_2 is LD.

(c)We cannot write v_1 as multiple of v_2, Hence S_3 is LI

(d)It has null matrix, Hence S_4 is LD.

Wronskian value: If f(x), g(x),h(x) are three continuous functions then Wronskian value of f(x), g(x) and h(x) is denoted by W(f(x),g(x),h(x)) and is defined as $W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$

Note:

a) If Wronskian value is zero then given function is L.D

b) If Wronskian value is non-zero then given function is L.I

6. Which of the following set is L.I.

(a)$\boldsymbol{S_1 = \{sin^2x, cos^2x, \cos 2x\}}$**,**

(b)$\boldsymbol{S_2 = \{\sin x, \cos x, \cos(x+1)\}}$**,**

(c)$\boldsymbol{S_3 = \{1, cosx, sinx\}}$

(d) $\boldsymbol{S_4 = \{cos2x, cos^2x, 2022\}}$

Sol:

(a) $W = \begin{vmatrix} sin^2x & cos^2x & cos2x \\ sin2x & -sin2x & 2cos2x \\ 2cos2x & 2cos2x & -4sin2x \end{vmatrix} = 0$, Hence, S_1 is L.D set. **(OR)** $cos2x = cos^2x - sin^2x$, S_1 is L.D

(b)$\cos(x+1) = cosx.1 + sinx.1$, Hence S_2 is L.D

(c)$W = \begin{vmatrix} 1 & cosx & sinx \\ 0 & -sinx & cosx \\ 0 & -cosx & -sinx \end{vmatrix} = 1 \neq 0$, Hence S_3 is L.I

(d) $cos2x = 2cos^2x - 1$, Hence S_4 is L.D

7. Which of the following statement is True or False (a){1,i } is a LD set on C(R) (b) {1,i } is a LI set on C(R) (c) {1,i } is a LD set on C(C) (d) {1,i } is a LI set on C(C)

Sol: (a) a.1+b.i=0 so a=0,b=0 and a,b∈R so it is L.I so it is False (b) it is true.

(b) a.1+b.i=0 so a=1,b=i and a,b∈C so it is L.D so it is True (d) it is false

Exercise Problems:

1. For $V_3(R)$ which of the following set is linearly dependent or independent

1) $S_1=\{(1,2,3), (3,2,1)\}$

2) $S_2=\{(1,2,3), (3,2,1), (1,1,1)\}$

3) $S_3=\{(1,2,3), (3,2,1), (0,0,0)\}$

4) $S_4=\{(1,2,3), (3,2,1), (1,-1,1), (1,1,1)\}$

5) $S_5=\{(1,2,-3),(1,-3,2),(2,-1,5)\}$

2. Determine whether the following set of vectors are L.I or L.D

6) $S=\{(1,1,1), (1,2,3), (0,1,2)\}$in $R^3(R)$

7) $S=\{(1,-2,1), (2,1,-1), (7,-4,1)\}$ in $R^3(R)$

8) $S=\{(1,-2,3), (2,3,4), (0,1,2)\}$ in $R^3(R)$

9) $S=(1,3,2),(1,-7,-8),((2,1,-1)$ in R^3 (R)

10) $S=(-1,2,1),(3,0,-1),(-5,4,3)$ in R^3 (R)

11) $S=(2,-3,7),(0,0,0),(3,-1,4)$ in R^3 (R)

12) $S=(1,2,-1),(2,1,-1)(7,-4,1)$ in R^3 (R)

13) $S=(2,1,5)(3,-5,8)(4,-2,1)(0,0,1)$ in R^3 (R)

14) $S=\{(1,2,3,4), (0,1,0,0), (0,0,1,1), (0,1,0,0)\}$ in $R^4(R)$

15) $S=\{(2,3,-1,-1), (1,-1,-2,-4), (3,1,3,-2), (6,3,0,-7)\}$in $R^4(R)$

16) $S=\{(1,1,0,0), (0,1,-1,0), (0,0,0,3)\}$in $R^4(R)$

17) $S=(2,1,1,1)(1,3,1,-2)(1,2,-1,3)$ in V^4 (Q)

18) $S=(0,1,0,1)(1,2,3,-1)(1,0,1,0)(0,3,2,0)$ in V^4 (Q)

19) $S=(1,2,-1,1)(0,1,-1,2)(2,1,0,3)((1,1,0,0)$ in V^4 (Q)

20) $S=(1,1,2,4)(2,-1,-5,2)(1,-1,-4,0)(2,1,1,6)$ in V^4 (Q)

21) $S_5=\{(1,2,-3,1), (3,7,1,-2), (1,3,7,-4)\}\subseteq V_4(R)$

22) $S=\{(0,1,0,1,1), (1,0,1,0,1), (0,1,0,1,1), (1,1,1,1,1)$ in $\}R^5(Q)$

3. Which of the following set is linearly dependent or independent?

23) $S=\{1+x, 1-x, 2+3x\}\subseteq P_1(R)$

24) $S=\{1+x, 1+x^2, x^2-x\}\subseteq P_2(R)$

25) $S=\{1+x, x+x^2, x^2+1\}\subseteq P_2(R)$

26) $S=\{1-x, x-x^2, x^2-1\}\subseteq P_2(R)$

27) $S=\{1, x+x^2, x-x^2, 3x\}\subseteq P_2(R)$

28) $S=\{x^2+x+1, x, 1\}\subseteq P_2(R)$

29) $S=\{x^3-x, x^3+1, x^3-1\}\subseteq P_3(R)$

30) $S=\{x^3+2x+1, x^3-x+1, x+1\} \subseteq P_3(R)$

31) $S=\{x^4+x^3, x^4-1, x^3+x^2, x^2+x, x+1\}\subseteq P_4(R)$

4. Verify the following set is LI or LD

32) $S = \{1, sin^2x, \cos 2x\}$

33) $S = \{\sin x, \cos x, \sin(x+1)\}$

34) $S = \{1, sinx, cos2x\}$

35) $S = \{1, sin^2x, cos^2x\}$

36) $S = \{x, sinx, cosx\}$

37) $S = \{logx, logx^2, logx^3\}$

38) $S = \{e^x, e^{-x}, sinhx\}$

39) $S = \{e^x, coshx, sinhx\}$

Basis of Vector Space:

A subset S of a vector space V(F) is said to be basis of V(F) if

a) S contains linearly independent vectors

b) S generates V(F), i.e L(S)=V, i.e Every vector in V can be expressed as a linear combination of the elements of S.

Dimensions of Vector Space:

The numbers of elements in any basis of a vector space V(F) is called dimension of V and is denoted by dimV, If dimV=n, then V is called n dimensional vector space.

Note: A vector space of finite dimension is called finite dimensional vector space

Theorem: if W_1, W_2 are two subspace of a finite dimensional vector space V(F) then $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1\cap W_2)$

Note 1: if W_1, W_2 are two subspace of a finite dimensional vector space V(F), if $\dim(W_1\cap W_2)=0$ then $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)$ then V is called direct sum of W_1 & W_2 and written as $V=W_1\oplus W_2$

Note2: $\dim(V/W)=\dim V-\dim W$

NOTE:

1. The basis for a vector space is not unique.
2. If V is a finitely generated vector space, then any two basis of V have same number of vectors.
3. If V(F) is a finite dimensional vector space, then it has a basis.
4. A set containing more vectors than its dimension is always linearly dependent.
5. Let V be a "n" dimensional vector space and S is a set of "n" linearly independent vectors in V, then S is a basis for V.

   (i.e: A L.I. set of n vectors in an n-dimensional vector space is always a basis)
6. Let V be a n dimensional vector space is S is a set of n vectors spans V, then S is a basis for V.
7. Every basis is spanning set but every, spanning set need not be base set.

Note:

- An arbitrary vector in V(F) is $(a)\in V$, where $a\in F$, Its standard basis is S={1}, Its dimension is =1
- If $(a,b)\in V_2(F)$, where $a,b\in F$, its standard basis is S={(1,0),(0,1)}, Its dimension dimV=2
- If $(a,b,c)\in V_3(F)$ where $a,b,c \in F$, Its standard basis is {(1,0,0),(0,1,0), (0,0,1)}, Its dimension dimV=3
- If $(a,b,c,d)\in V_4(F)$ where $a,b,c,d\in F$, Its standard basis is S={(1,0,0,0),(0,1,0,0), (0,0,1,0),(0,0,0,1)} Its dimension dimV=4

- If $a+bx+cx^2 \in P_2(R)$, where $a,b,c \in R$, Its standard basis is $\{1,x,x^2\}$, Its dimension is 3
- If $a_0+a_1x+a_2x^2+\cdots+a_n x^n \in P_n(R)$, where $a_i \in R$, Its standard basis is $\{1,x,x^2,\ldots x^n\}$, Its dimension is n+1
- if $(a_1, a_2, a_3, a_4 \ldots a_n) \in V_n(F)$, where $a_i \in F$, Its standard basis is $\{(1,0,0,\ldots 0),(0,1,0\ldots 0),\ldots (0,0,\ldots,1)\}$, Its dimension is =n
- if $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in M2x3(F)$, where $a,b,c,d,e,f \in F$, Its standard basis is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Its dimension is =6
- Dimension of vector space of $M_{m\times n}(F)$ is=$m\times n$.
- Dimension of vector space of all $m\times n$ matrices whose elements are complex over a real field is 2mn
- An arbitrary vector in C(R) is (a+ib) where $a,b \in R$, Its standard basis is $\{1,i\}$, Its dimension is 2
- If $\{(a+ib), (c+id)\} \in C^2(R)$, where $a,b,c,d \in R$, Its standard basis is $\{(1,0),(0,1),(0,i),(i,0)\}$, Its dimension is =4
- Dimension of $C^n(R)$ is =2n, Dimension of R(Q) is $=\infty$
- Dimension of vector space of symmetric matrix of nxn is $\frac{n(n+1)}{2}$
- Dimension of vector space of skew-symmetric matrix of nxn is $\frac{n(n-1)}{2}$
- Dimension of vector space of upper triangular, lower triangular, matrix of nxn is $\frac{n(n+1)}{2}$
- Dimension of vector space of diagonal matrix of nxn is n
- Dimension of vector space of scalar matrix of nxn is one

5. Show that the set {(0,5), (2,0)} is basis for R^2

Sol: Let S= {v_1, v_2 } where v_1=(0,5) and v_2=(2,0)

$a_1v_1+ a_2v_2= 0$

$a_1(0,5)+a_2(2,0)= 0$

$2a_2=0 \Rightarrow a_2=0$

$5a_1=0 \Rightarrow a_1=0$

Hence S is L.I set.

Let (x, y) $\in V_2(R)$

(x, y)=a(0,5)+b(2,0)=(2b, 5a)

x=2b⇨b=x/2

y=5a ⇨a=y/5

(x, y)=(y/5) (0,1)+(x/2) (1,0)

L(S)=V so that S is generating set.

Hence S is Basis set.

6. Prove that the set S={1+x, 1-x, -x^2+x+2} is a basis for P_2(R).

Sol: Given that S={1+x, 1-x, -x^2+x+2} ={(0,1,1), 0,-1,1), (-1,1,2)}

Let $A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix} \neq 0$, ⇒Hence S is L.I set.

Let $(\mathrm{p,q,r})\epsilon \mathrm{P}_2(\mathrm{R})$

$(\mathrm{p,q,r}) = \mathrm{a}(0,1,1) + \mathrm{b}(0,-1,1) + \mathrm{C}(-1,1,2) = (-c, a-b+c, a+b+2c)$

$p = -c \,\&\, q = a-b+c \text{ and } r = a+b+2c$

$q = a-b+c \text{ and } r = a+b+2c \implies 2a+3c = q+r \implies a = \frac{1}{2}(q+r+3p)$

$b = a+c-q \Rightarrow b = \frac{1}{2}(q+r+3p) - p - q \Rightarrow b = \frac{1}{2}(p-q+r)$

$(\mathrm{p,q,r}) = \frac{1}{2}(3p+q+r)(0,1,1) + \frac{1}{2}(p-q+r)(0,-1,1) - \mathrm{p}(-1,1,2)$

L(S)=V so that S is generating set.

Hence S is Basis set.

7. Prove that the set {(1,3,1), (2,1,0), (4,2,1)} is a basis for R^3

8. Which of the following set is a basis for $V_2(R)$

(a)S_1= {(1,-1), (-1,1)} (b) (c)S_3= {(0,5), (2,0)} (d)S_4= {(1,2), (3,4)}

Sol:

(a)Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow |A| = 0$, Hence, S_1 is L.D

$\therefore$ S_1 is not a basis set

(b)Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |A| \neq 0$, Hence S_2 is L.I

Let $(x, y) \in V_2(R)$

(x, y)=a(0,1)+b(1,0)=(b, a)

x=b, y=a

(x, y)=y(0,1)+x(1,0)

L(S)=V

Hence, S_2 is generating set.

$\therefore$ S_2 is a basis set

(C)A= $\begin{vmatrix} 0 & 5 \\ 2 & 0 \end{vmatrix} \neq 0$, $\Rightarrow$Hence S_3 is L.I and (x, y)=a(0,5)+b(2,0)=(2b, 5a) $\Rightarrow$ x=2b, y=5a

(x,y)=(y/5)(0,5)+(x/2)(2,0) $\Rightarrow$ L(s)=V, Hence S_3 is generating set.

$\therefore$ S_3 is a basis set.

(c) $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \neq 0$, $\Rightarrow$Hence it is L.I and ……..generating set, hence S_4 is a basis set

9. Which of the following is the basis for $P_2(R)$, (a) S_1={1+x, 1-x, $-x^2$+x+2} (b) S_2= {1,1+x, 1-x^2}

(c) S_3={ 1+x, 1-x, 5+x }

Sol:

(a) A= $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix} \neq 0$, $\Rightarrow$Hence it is L.I

(b) $A = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{vmatrix} \neq 0$, $\Rightarrow$Hence it is L.I

(c) A= $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 0$, $\Rightarrow$Hence it is L.D

Hence, {1+x 1-x, 5+x } is not a basis set.

10. If W is a subspace of $V_4(R)$, W=(1,-2,5,-3),(2,3,-1,-4),(3,8,-3,-5) then find basis of W and its dimensions.

Sol: $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & -1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$

$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \implies A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -11 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$

$R_3 \to R_3 - 2R_2 \implies A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -11 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

Basis of W is S= { (1,-2,5,-3), (0,7,-11,2), (0,0,4,0) } and dim(W)=3

11. Let W be the vector sub space generated by the polynomials x^3+2x^2-2x+1, x^3+3x^2-x+4, $2x^3+x^2-7x-7$ then find basis and dimensions.

Sol: Let A= $\begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$

$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \implies W = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix}$

$R_3 \to R_3 + 3R_2 \implies W = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis of W is S= {x^3+2x^2-2x+1, x^2+x+3} and dim(W)=2

12. If W={ (a,b,c) /a+b+c=0 } $\subseteq V_3(F)$ then find dim(W)

Sol: Give that a+b+c=0 $\Rightarrow$ a=-b-c

(a,b,c)={ -b-c, b, c }=b(-1,1,0)+c(-1,0,1)

Hence, (a,b,c) can be written as linear combination of (-1,1,0), (-1,0,1), i.e L(S)=V

Basis set for W is S={ (-1,1,0) ,(-1,0,1)} and dim(W)=2

Note: dim(W)=number of elements –number of independent conditions

13. If W_1={ (a,b,c,d)/b-2c+d=0 } $\subseteq V_4(F)$ and W_2={ (a,b,c,d)/a=d, b=2c } $\subseteq V_4(F)$ then find dim(W_1), dim(W_2), dim(W1+W2).

Sol:

$\Rightarrow$Given that W_1= (a,b,c,d) and b-2c+d=0 $\Rightarrow$b=2c-d

(a,b,c,d)=(a, 2c-d, c, d)=a (1,0,0,0)+c (0,2,1,0)+d (0,-1,0,1)

Hence (a,b,c,d) can be written as linear combination of (1,0,0,0),(0,2,1,0), (0,-1,0,1) , i.e L(S)=V

Basis for W_1 is S={ (1,0,0,0),(0,2,1,0), (0,-1,0,1)} and dim(W_1)=3

$\Rightarrow$ Given that W_2= (a,b,c,d) and a=d, b=2c,

(a,b,c,d)=(d, 2c, c, d)=c(0,2,1,0)+d(1,0,0,1)

Hence (a,b,c,d) can be written as linear combination of (0,2,1,0),(1,0,0,1) , i.e L(S)=V

Basis for W_2 is S={ (0,2,1,0),(1,0,0,1)} and dim(W_2)=2

⇒ Now $W_1 \cap W_2$=(a,b,c,d) and b-2c+d=0,a=d, b=2c

($W_1 \cap W_2$)=(a,b,c,d)/b=2c, a=d=0}

(a,b,c,d)=(0, 2c, c, 0)=c (0,2,1,0)

Hence (a,b,c,d) can be written as linear combination of (0,2,1,0), i.e L(S)=V

Basis for $W_1 \cap W_2$ is S=(0,2,1,0) and dim($W_1 \cap W_2$)=1

From dimension theorem dim(W_1+W_2)=dim(W_1)+dim(W_2)- dim($W_1 \cap W_2$)

dim(W_1+W_2)=3+2-1=4

(OR)

dim(W_1)=4-1=3

dim(W_2)=4-2=2

dim($W_1 \cap W_2$)=4-3=1

dim(W_1+W_2)=dim(W_1)+dim(W_2)-dim($W_1 \cap W_2$)=3+2-1=4

14. Let P_3 be the space of real polynomial of degree at most 3. Let W_1={f(x)/f(1)=0} and W_2={f(x)/f^1(1)=0} then find basis and dimensions of W_1, W_2, W_1+W_2

Sol: Given that $W_1 = \{f(x)/f(1) = 0\}$ and $W_2 = \{f(x)/f^I(1) = 0\}$

Let $f(x) = ax^3 + bx^2 + cx + d \,\epsilon P_3$ & $f^I(x) = 3ax^2 + 2bx + c$

$f(1) = a + b + c + d$ & $f^I(1) = 3a + 2b + c$

Now $W_1 = \{f(x)/f(1) = 0\}$ $\Rightarrow W_1 = \{ax^3 + bx^2 + cx + d/f(1) = 0\}$

$W_1 = \{ax^3 + bx^2 + cx + d/a + b + c + d = 0\}$ $\Rightarrow W_1 = \{ax^3 + bx^2 + cx + d/d = -(a + b + c\}$

$W_1 = \{ax^3 + bx^2 + cx - (a + b + c)\}$ $\Rightarrow W_1 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$

The basis for $W_1 = \{(x^3 - 1), (x^2 - 1), (x - 1)\}$ and dimW_1=3

Now $W_2 = \{f(x)/f^I(1) = 0\}$ $\Rightarrow W_2 = \{ax^3 + bx^2 + cx + d/f(1) = 0\}$

$W_2 = \{ax^3 + bx^2 + cx + d/3a + 2b + c = 0\}$ $\Rightarrow W_2 = \{ax^3 + bx^2 + cx + d/c = -3a - 2b\}$

$W_2 = \{ax^3 + bx^2 + (3a - 2b)x + d\}$ $\Rightarrow W_2 = a(x^3 - 3x) + b(x^2 - 2x) + d$

The basis for $W_2 = \{(x^3 - 3x), (x^2 - 2x), 1\}$ and dimW_2=3

Now $W_1 \cap W_2 = \{f(x)/f(1) = 0 \ \& f^I(1) = 0\}$

$W_1 \cap W_2 = \{ax^3 + bx^2 + cx + d/a + b + c + d = 0 \,\&\, 3a + 2b + c = 0\}$

$a + b + c + d = 0 \,\&\, 3a + 2b + c = 0$

$a + b + c + d = 0 \,\&\, c = -3a - 2b$

$a + b - 3a - 2b + d = 0$

$d = 2a + b \,\&\, c = -3a - 2b$

$W_1 \cap W_2 = \{ax^3 + bx^2 - (3a + 2b)x + (2a + b)\}$

$W_1 \cap W_2 = \{a(x^3 - 3x + 2) + b(x^2 - 2x + 1)\}$

The basis for $W_1 \cap W_2 = \{(x^3 - 3x + 2), (x^2 - 2x + 1)\}$ and $\dim(W_1 \cap W_2) = 2$

Since $\dim(W_1 + W_2) = \dim W_1 + \dim W_1 - \dim(W_1 \cap W_2)$

$\dim(W_1 + W_2) = 3 + 3 - 2 = 4$

15. If W_1={ (a,b,c,d)/a+b+c=0, 2b+d=0,2a+2c-d=0} and W_2={(a,b,c,d)/a+c+d=0,a+c-2d=0, b-d=0} then find dim(W_1), dim(W_2), dim(W_1+W_2)

Sol:

⇒ Given that W_1=(a,b,c,d) and a+b+c=0,2b+d=0,2a+2c-d=0

Note: Here dimW_1 is nothing but number of linearly independent solutions of system of equations a+b+c=0, 2b+d=0, 2a+2c-d=0, which is n-r (i.e number of unknowns-rank of matrix)

$$W_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1 \Rightarrow W_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -2 & 0 & -1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \Rightarrow W_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank (W_1)=2 and n=4 ⇒ dimW_1=n-r=4-2=2

⇒ Given that W_2=(a,b,c,d) and a+c+d=0, a+c-2d=0, b-d=0}

$$W_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\mathrm{R}_3 \to \mathrm{R}_3 - \mathrm{R}_1 \Rightarrow \mathrm{W}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\mathrm{R}_3 \to \mathrm{R}_3 \leftrightarrow \mathrm{R}_2 \Rightarrow W_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Rank(W_2)=3 and n=4 ⇒dimW_2=n-r=4-3=1

⇒ $W_1 \cap W_2$=(a,b,c,d) and a+b+c=0, 2b+d=0, 2a+2c-d=0, a+c+d=0, a+c-2d=0, b-d=0.

$$W_1 \cap W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1,\quad R_4 \to R_4 - R_1,\quad R_5 \to R_5 - R_1 \Rightarrow W_1 \cap W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2,\quad R_4 \to 2R_4 + R_2,\quad R_5 \to 2R_5 + R_2,\quad R_6 \to 2R_6 - R_2 \Rightarrow W_1 \cap W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$R_5 \to R_5 + R_4,\quad R_6 \to R_6 + R_4 \Rightarrow W_1 \cap W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(W_1 \cap W_2)=3$ and $n=4 \Rightarrow$ Hence $\dim(W_1 \cap W_2)=4-3=1$

$\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)=2+1-1=2$

16. If $W_1=\{(a,b,0,c) / a+2b=0\} \subseteq V_4(F)$ and $W_2=\{(a,0,b,c) / 2b+3c=0\} \subseteq V_4(F)$ then find $\dim(W_1)$, $\dim(W_2)$, $\dim(W_1+W_2)$.

Sol:

$\Rightarrow$ Given that $W_1=(a,b,0,c)$ and $a+2b=0$

$W_1=(a,b,0,c)$ and $a=-2b$

$(a,b,0,c)=(-2b,b,0,c)=b(-2,1,0,0)+c(0,0,0,1)$

Basis of W_1 is $S=\{(-2,1,0,0), (0,0,0,1)\}$ and $\dim(W_1)=2$

$\Rightarrow$ Given that $W_2=(a,0,b,c)$ and $2b+3c=0$

$W_2=(a,0,b,c)$ and $b=-3c/2$

$(a,0,b,c)=(a,0,-3c/2,c)=a\,(1,0,0,0)+c\,(0,0,-3/2,1)$

Basis of W_2 is $S=\{(1,0,0,0), (0,0,-3,2)\}$ and $\dim(W_2)=2$

$\Rightarrow$ Now $W_1 \cap W_2=\{(a,b,0,c) / a+2b=0\} \cap \{(a,0,b,c) / 2b+3c=0\}$

$W_1 \cap W_2=(-2b,b,0,c) \cap (a,0,-3c/2,c)=0$

$\dim(W_1 \cap W_2)=0$

$\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)=2+2-0=4$

17. If W={ (1,2,-1), (-3,-6,3), (2,1,3), (8,7,7) } Find its basis and dimensions.

Sol: Given that $W = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -6 & 3 \\ 2 & 1 & 3 \\ 8 & 7 & 7 \end{bmatrix}$

$$R_2 \to R_2 + 3R_1$$
$$R_3 \to R_3 - 2R_1 \implies W = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -3 & 5 \\ 0 & -9 & 15 \end{bmatrix} \implies W = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & -9 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$
$$R_4 \to R_4 - 8R_1$$

$$R_3 \to R_3 - 3R_2 \implies W = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

S={ (1,2,-1) (0,-3,5) } is basis of W and dimW=2

18. Find the basis and dimensions of a subspace spanned by the vectors W={(1,2,0),(-1,0,1), (0,2,1)}

Sol: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$$R_2 \to R_2 + R_1 \implies A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \implies A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of W is S= {(1,2,0),(0,2,1)} and dimW=2

19. Consider the following row vectors v_1=(1,1,0,1,0,0), v_2=(1,1,0,0,1,0), v_3=(1,1,0,0,0,1), v_4=(1,0,1,1,0,0), v_5=(1,0,1,0,1,0), v_6=(1,0,1,0,0,1) then find the dimension of the vector space spanned by these row vectors.

Sol: $$W = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow W = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ S={ (1,1,0,1,0,0), (0,-1,1,0,0,0), (0,0,0,-1,1,0) ,(0,0,0,0,-1,1) } are the basis of W, dim W=4

20. Which of the following subset is a basis of R^4

B_1={ (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1) }

B_2={ (1,0,0,0), (1,2,0,0), (1,2,3,0), (1,2,3,4) }

B_3={ (1,2,0,0), (0,0,1,1), (2,1,0,0), (-5,5,0,0) }

(a) B_1 and B_2 but not B_3 (b) B_1, B_2 ,and B_3 (c) B_1 and B_3 but not B_2 (d) only B_1

21. Find the dimension of the following vectors

a) **If W={ (a,b,c) /a-b=2c } $\subseteq$ V_3(F) then find dim(W)**

b) **If W={ (a,b,c,d) /a+b=0, c=d } $\subseteq$ V_4(F) then find dim(W)**

c) **If W={ (p,q,r,s,t) /p+q=0, q=r, t=0 } $\subseteq$ V_5(F) then find dim(W)**

d) **If** $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: Trc(W) = 0 \subseteq M_{2x2}(F)$**then find dim(W)**

a) dim(W)=3-1=2

b) dim(W)=4-2=2

c) dim(W)=5-3=2

d) dim(W)=4-1=3

22. If W_1={ (a,b,c) /a=b } $\subseteq$ V_3(F) and W_2={ (a,b,c) /b+c=0 } $\subseteq$ V_3(F) then find dim(W_1), dim(W_2), dim(W_1+W_2).

Sol: dim(W_1)=3-1=2

dim(W_2)=3-1=2

dim($W_1 \cap W_2$)=3-2=1

dim(W_1+W_2)=dim(W_1)+dim(W_2)-dim($W_1 \cap W_2$)

dim(W_1+W_2)=2+2-1=3

Linear Transformation (LT) or (Homomorphism): Let V(F) & W(F) be two vectors space and $T:V(F)\rightarrow W(F)$ be a function if

(a) $T(u+v)=T(u)+T(v)$ for all $u,v \in V$

(b) $T(au)=a\,T(u)$ for all $u\in V$ & $a\in F$

(OR) $T(au+bv)=aT(u)+bT(v)$ for all $u,v\in V$ and $a,b\in F$ then T is called a Linear Transformation or Homomorphism from V(F) to W(F)

Monomorphism: A linear transformation $T: V(F)\rightarrow W(F)$ is said to be Monomorphism if T is one –one.

Epimorphism: A linear transformation $T: V(F)\rightarrow W(F)$ is said to be Epimorphism if T is onto.

Isomorphism: A homomorphism $T: V(F)\rightarrow W(F)$ is said to be isomorphism if T is both one –one & onto.

Note:

(a) If $\dim V > \dim W$ then T never one-one

(b) If $\dim V < \dim W$ then T never onto

(c) If $\dim V \neq \dim W$ then T never be bisection.

Properties of homomorphism (L.T): If $T: V(F)\rightarrow W(F)$ is a linear transformation then

(a) $T(0_v)=0_w$

(b) $T(-u) = -T(u)$

(c) $T(u_1-u_2)= T(u_1)-T(u_2)$ where $u_1,u_2\in V$

(d) $T(u_1+u_2+u_3+\text{----------}+u_n)= T(u_1)+T(u_2)+ T(u_3)+\ldots\ldots+ T(u_n)$, $u_i\in V$,

(e) $T(a_1 u_1+ a_2u_2 + a_3u_3+\text{----------}+ a_n u_n)= a_1 T(u_1)+ a_2 T(u_2)+ a_3 T(u_3)+\ldots\ldots+ a_n T(u_n)$, $u_i \in V$, $a_i\in V$

Null space or Kernal of homomorphism:

If $T: V(F)\rightarrow W(F)$ is a linear transformation with 0_v and 0_w are zero vectors of V(F) & W(F) respectively, then the set of pre-images (invers images) of 0_w is called the null space of T and is denoted by N(T).

$N(T)=\{ v\in V \,/T(v)= 0_w \} \subseteq V(F)$

Nullity of a Linear Transformation:

The dimension of Null space N(T) or Ker(T) of a linear transformation is called Nullity of Linear Transformation and is denoted by $\eta(T)$. i.e $\eta(T)= \dim N(T)$

Note: $\eta(T)\leq \dim V$

Range of Linear Transformation:

If T: V(F)→W(F) is a linear transformation then the set of images of all vectors of V(F) is called Range of T, it is denoted by R(T).

$R(T)=\{T(v) / v\in V\} \subseteq W(F)$

Rank of Linear Transformation:

The dimension of range space R(T) of a linear transformation is called Rank of Linear Transformation and is denoted by $\rho(T)$. i.e $\rho(T) = \dim R(T)$

Note:

a) $\rho(T)\leq \dim W$

b) if $\rho(T) = \dim W$ then T is onto

Rank Nullity Theorem or (Sylvester Law): If T: V(F)→W(F) is a linear transformation, if V is finite dimensional vector space then $\rho(T) + \eta(T)=\dim V$, i.e $\dim R(T) + \dim N(T)=\dim V$ (dim of domain)

Note: $\rho(T)=\min\{\dim V, \dim W\}$

Singular Transformation: A linear transformation T:V(F) →W(F) is said to be singular if the null space of T consists of at least one non-zero vector

i.e; if there exist a vector $v\in V$ and $T(v)=0_w$ for $v\neq 0_v \Rightarrow \eta(T) \neq 0$, then T is singular

Non-Singular Transformation: A linear transformation T:V(F) →W(F) is said to be Non-singular if the null space of T consists one zero vector alone.

i.e; if there exist a vector $v\in V$ and $T(v)=0_w$ for $v=0_v \Rightarrow N(T)=\{\bar{O}\} \Rightarrow \eta(T)=0$, then T is Non-singular

Note:

1) If T:V(F) →W(F) is an Isomorphism iff T is non-singular L.T

2) If T:V(F) →W(F) is an not Isomorphism iff T is singular L.T

3) If $\dim V \neq \dim W$ then the L.T, T:V(F) →W(F) is always singular L.T

4) If $\dim V = \dim W$ then the L.T, T:V(F) →W(F) is either non-singular L.T or L.T only.

1. If the mapping T: $R^3 \to R^2$ is defined by T(x, y, z)=(x-y, x-z) then show that T is a linear transformation.

Sol: Let $u=(x_1, y_1, z_1)$ and $v=(x_2, y_2, z_2)$ are two vectors of $V_3(R)$ and $a, b\in R$

$T(au+bv)=T\{a (x_1, y_1, z_1) +b (x_1, y_2, z_2) \}= T\{ (ax_1, ay_1, az_1) +(bx_2, by_2, bz_2) \}$

$T(au+bv)=T(ax_1+ bx_2 , ay_1+ by_2 , az_1+ bz_2)$

$T(au+bv)= (ax_1+bx_2 -ay_1 -by_2 , ax_1+ bx_2 -az_1- bz_2)$ since T(x, y, z)=(x-y, x-z)

$T(au+bv)= a(x_1-y_1, x_1-z_1) + b(x_2-y_2, x_2-z_2)= a\,T(x_1, y_1, z_1) + b\,T(x_2, y_2, z_2)$

$T(au+bv)= a\,T(u) + b\,T(v)$ Hence, T is a linear transformation from $V_3(R)$ to $V_2(R)$

2. The mapping $T: V_3(R) \to V_1(R)$ is defined by $T(x, y, z)=x^2+y^2+z^2$, can T be a linear transformation ?

Sol: Let $u=(x_1, y_1, z_1)$ and $v=(x_2, y_2, z_2)$ be two vectors of $V_3(R)$ and $a,b\in R$

L.H.S$=T(au+bv)=T(ax_1+ bx_2, ay_1+ by_2, az_1+ bz_2)$

$T(au+bv)=(ax_1+ bx_2)^2+(ay_1+ by_2)^2+(az_1+ bz_2)^2$

RHS$= aT(u)+bT(v)=a(x_1)^2+ a(y_1)^2+ a(z_1)^2+ a(x_2)^2+ a(y_2)^2+ a(z_2)^2$

$aT(u)+bT(v)=a(x_1)^2+ b(x_2)^2+ a(y_1)^2 + b(y_2)^2+a(z_1)^2 +b(z_2)^2$

$\therefore\ T(au+bv)\neq aT(u)+bT(v)$

T is not a linear Transformation from $V_3(R)$ to $V_1(R)$

3. Is the mapping $T: R^3\to R^2$ is defined by $T(x, y, z)= (|x|,0)$ a linear transformation.

Sol: Let $u=(x_1, y_1, z_1)$, $v=(x_2, y_2, z_2)$ be two vectors of $V_3(R)$ and $a,b\in R$

$T(au+bv)= T\{ (ax_1, ay_1, az_1) +(bx_2, by_2, bz_2) \}=T(ax_1+ bx_2, ay_1+ by_2, az_1+ bz_2)$

$T(au+bv)=\{|ax_1+ bx_2|, 0 \}$

$aT(u)+bT(v)= aT(x_1,y_1,z_1) +bT(x_2,y_2,z_2)= a(|x_1|,0) + b(|x_2|,0)=\{ a|x_1|+ b|x_2|, 0 \}$

$T(au+bv)\neq aT(u)+bT(v)$

T is not a linear Transformation from $V_3(R)$ to $V_2(R)$

4. Let P_n be the vector space of real polynomial functions of degree less than are equal to n. Show that the following transformation $T: P_2 \to P_1$ is linear. $T(ax^2+bx+c)= (a+b)x+c$

Sol: Let $u = \mathbf{ax^2 + bx + c}$ & $\mathbf{v = dx^2 + ex + f}$

5. If T is a linear map on $V_2(R)$ and $T(1,0)=(2,3)$, $T(1,1)=(5,7)$ then find $T(x,y)$

Sol: Let $S=\{ (1,0),(1,1) \}$ clearly $\det(S)\neq 0$, hence S is LI

$(x,y)=a(1,0)+b(1,1)$……………….(1)

$(x,y)=(a+b, b) \Rightarrow x=a+b$ and $y=b$

$\therefore\ a=x-y$ and $y=b$

Applying "T" on (1) $T(x,y)=a\,T(1,0)+b\,T(1,1)$

$T(x,y)=a\,(2,3)+b\,(5,7)=\{2a+5b, 3a+7b\}=\{2(x-y)+5y, 3(x-y)+7y\}=\{ 2x+3y, 3x+4y \}$

6. Find the linear transformation from $T: R^2\to R^2$ and $T(2,3)=(4,5)$, $T(1,0)=(0,0)$.

Let $S=\{ (2,3),(1,0) \}$ clearly $\det(s)\neq 0$ so S is LI

$(x,y)=a(2,3)+b(1,0)$……………….(1)

$(x,y)=(2a+b, 3a)$

$x=2a+b$, $y=3a \Rightarrow a=y/3$ and $b=(3x-2y)/3$

Applying T on (1), $T(x,y)=a\,T(2,3)+bT(1,0)=a\,(4,5)+b\,(0,0)=(4a, 5a)=(4y/3, 5y/3)$

Verification Method:

(a)$(2y/3, 0)$ (b)$\{(3x+2y)/3, (2x+3y)/3\}$,(c)$(4y/3, 5y/3)$ (d)$(0,0)$

Sol: $T(2,3)=(4,5)$, $T(1,0)=(0,0)$ satisfying only (C)

Exercise Problems:

1. Which of the following maps are linear transformations?

1) $T: V_1\to V_3$ defined by $T(x)=(x,2x,3x)$

2) $T: R^2\to R^2$ defined by $T(x,y)=(2x+3y,3x-4y)$

3) $T: R^3\to R^3$ defined by $T(x,y,z)=(x+1,y\ z)$

4) $T: R^2\to R^2$ defined by $T(x,y)=(x^3,y^3)$

5) $T: R^3\to R^2$ defined by $T(x,y,z)=(\sin x, \log yz)$

6) $T: R^3\to R^2$ defined by $T(x,y,z)=(|x|,y+z)$

7) $T: R^3\to R^3$ defined by $T(x,y,z)=(x+y+z, x+y, x)$

8) $T: R^3\to R^3$ defined by $T(x,y,z)=(x+y, y+z, z+x)$

9) $T: R^3\to R^3$ defined by $T(x,y,z)=(x-y, y-z, 0)$

10) $T: R^2\to R^2$ defined by $T(x,y)=(x^2,y^3)$

11) $T: R^2\to R^2$ defined by $T(x,y)=(1/x, \sqrt{y})$

12) $T: R^2\to R^2$ defined by $T(x,y)=(x+1, y-1)$

13) $T: R^2\to R^2$ defined by $T(x,y)=(0,0)$

14) $T: P(R)\to P(R)$ defined by $T\{f(x)\}=x\,f(x)$

15) $T: P\to P$ defined by $T\{f(x)\}=x+f(x)$

16) $T: P\to P$ defined by $T\{f(x)\}=x^2+f(x)$

17) $T: P\to P$ defined by $T(x)=x^2+x$

18) Let P_n be the vector space of real polynomial functions of degree<=n. Show that the following transformation $T: P_2 \to P_1$ is linear. $T(ax^2+bx+c)=(a+b)x+c$

2. Find a linear transformation for the following

19) $T: R^2\to R^2$ such that $T(1,0)=(1,1)$ and $T(0,1)=(-1,2)$

20) $T: R^2\to R^2$ such that $T(1,2)=(3,0)$ and $T(2,1)=(1,2)$

21) $T: R^3\to R^3$ such that $T(0,1,2)=(3,1,2)$ and $T(1,1,1)=(2,2,2)$

22) $T: R^2\to R^3$ such that $T(1,2)=(3,-1,5)$ and $T(0,1)=(2,1,-1)$

23) $T: R^2\to R^3$ such that $T(2,-5)=(-1,2,3)$ and $T(3,4)=(0,1,5)$

3. Find the kernel and range of the linear operator $T(x,y,z)=(x,y,0)$

Sol:

We know that $Ker(T)=N(T)=\{v\in V/T(v)=0\}$

Let $v=(x, y, z)\in V_3(R)$

Since $T(v)=0 \Rightarrow T(x, y, z)=0$

$(x, y, 0)=(0, 0, 0) \Rightarrow x=0$ and $y=0$

$Ker(T)=N(T)=\{(x, y, z) / x=0 \;\&\; y=0\}$

$Ker(T)=N(T)=\{(0, 0, z)/z\in R\}$

$\dim N(T)=1=\eta(T)$

We know that $Rang(T)= R(T)=\{T(v) / v\in V\}$

Let $v=(x, y, z)\in V_3(R)$

$T(v)=T(x,y,z)=(x,y,0)$

Range (T) is the set of all vectors that lie in the x-y plane.

$T(v)=x(1,0,0)+y(0,1,0)$

Basis set is $S=\{(1,0,0)\ (0,1,0)\}$ and $\dim R(T)=2=\rho(T)$

4. If $T: R^3 \to R^2$ is defined by $T(x,y,z)=(x+y,z)$, then find the kernel and range of the linear transformation. Show that dim ker(T)+dim range (T)=dim domain (T)

Sol:

We know that $Ker(T)=N(T)=\{v\in V/T(v)=0\}$

Let $v=(x, y, z)\in R^3$

Since $T(v)=0 \Rightarrow T(x, y, z)=0$

$(x+y, z)=(0,0) \Rightarrow x+y=0$ and $z=0 \Rightarrow x=-y$ and $z=0$

$Ker(T)=N(T)=\{(x, y, z) / x=-y \;\&\; z=0\}$

$Ker(T)=N(T)=\{(-y, y, 0)/y\in R\}$

$\dim N(T)=1=\eta(T)$

We know that $Rang(T)= R(T)=\{T(v) / v\in V\}$

Let $v=(x, y, z)\in R^3$

$T(v)=T(x,y,z)=(x+y,z)$

$T(v)=x(1,0)+y(1,0)+z(0,1)$

Basis set is $S=\{(1,0)\ (0,1)\}$ and $\dim R(T)=2=\rho(T)$

Since the $\dim V=3$

Hence $\dim R(T) + \dim N(T)=\dim V$

5. If T: $R^2 \to R^3$ is defined by T(x,y)=(3x,x-y,y), then find the kernel and range of the linear transformation. Show that dim ker(T)+dim range (T)=dim domain (T)

Sol:

We know that Ker(T)=N(T)={v∈V/T(v)= O}

Let v=(x,y)∈ R^2

Since T(v)= O ⇒ T(x, y)=0

(3x, x-y, y)=(0,0,0) ⇒3x=0, x-y=0 and y=0 ⇒ x=y=0

Ker(T)=N(T)={(x,y)/ x=0 & y=0}

Ker(T)={(0,0)}

dimN(T)=0=η(T)

We know that Rang(T)= R(T)={T(v) /v∈V }

Let v=(x, y)∈R^2

T(v)=T(x,y)=(3x, x-y, y)

T(v)=x(3,1,0)+y(0,-1,1)

Basis set is S={(3,1,0) (0,-1,1)} and dimR(T)=2= ρ(T)

Since the dim domain=2

Hence dimR(T) + dimN(T)=dimV

6. If T: $V_2(R) \to V_3(R)$ is defined by T(x,y)=(x, x+y, x-y) then T is –

(a)A Linear Transformation only

(b)1-1 Linear Transformation

(c)An onto Linear Transformation

(d)Isomorphism

Sol:

(a) Let u=(x_1, y_1) and v=(x_2, y_2) are two vectors of V_2(R) and a, b∈R then

T(au+bv)=T{a(x_1, y_1) +b(x_1, y_2)}=T{(ax_1, ay_1)+(bx_2, by_2)}=T(ax_1+ bx_2 , ay_1+ by_2)

T(au+bv)=(ax_1+bx_2 , ax_1+ bx_2 +ay_1+ by_2, ax_1+ bx_2 -ay_1- by_2)   since T(x, y, z)=(x, x+y, x-y)

aT(u)+bT(v)= aT(x_1, y_1) + b(x_2, y_2)=T(ax_1 , ay_1)+T(bx_2 ,by_2)= {ax_1, ax_1 + ay_1, ax_1-ay_1}+{ bx_2, bx_2+by_2, bx_2-by_2}=)=(ax_1+bx_2 , ax_1+ bx_2 +ay_1+ by_2, ax_1+ bx_2 -ay_1- by_2)

T(au+bv)= aT(u)+bT(v) Hence, T is a linear transformation from V_3(R) to V_2(R)

(b) **For 1-1**

N(T)={ v∈V /T(v)= O_w }

Let v=(x,y) ∈V_2(R)⇒T(x,y)=0_w

(x, x+y, x-y)=(0,0,0)

x=0, x+y=0, x-y=0 so x=0, y=0

N(T)={(x,y) / x=0,y=0 }

N(T)={(0,0)}⇒N(T)={Ō}⇒ η(T)=0 hence, T is 1-1

(c)For onto: ρ(T) +η(T)=dim V ⇒ρ(T) +0=2⇒ρ(T)=2≠ codomine=3=dimV_3(R)

Hence T is not onto

(OR)

R(T)={T(v) / v∈V }= T(x,y)=(x, x+y, x-y)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{matrix} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{matrix} \Longrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$R_3 \longrightarrow R_3 + R_2 \Longrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Rank A=2=ρ(T)

(d) Since T is not 1-1, hence T is not a bijection, ∴ T is not a Isomorphism.

7. If T: V_2(R) → V_3(R) is defined by T(x,y)=(x+y, 0,0) then T is –

(a)A Linear Transformation only

(b) A 1-1 Linear Transformation

(c)An onto Linear Transformation

(d) An Isomorphism

Sol: Clearly, it is homogenious linear expression hence T is L.T

dimv=2 and dimw=3 so dimv<dimw hence T is not onto

For 1-1

N(T)={ u∈V /T(u)= 0_w }

let (x,y) ∈ V_2(R) ⇒ T(x,y)=0_w

(x+y,0,0)=(0,0,0)

x+y=0 so y=-x

N(T)={(x,y) / y=-x } ⇒dimN(T)=2-1=1 ⇒ η(T)≠0 hence T is not 1-1

Hence, T is Linear Transformation only

8. If T: V_3(R) → V_2(R) is defined by T(x,y,z)=(x+2y, y-3z) then T is –

(a)T is Linear Transformation only (b) Monomorphism (c)Epimorphism (d) T is Isomorphism

Sol: Clearly, it is homogenious linear expression hence T is L.T

dimv=3 and dimw=2 so dimv>dimw hence T is not 1-1

For onto

N(T)={ u∈V /T(u)= 0_w }

Let $(x,y,z) \in V_3(R) \Rightarrow T(x,y,z)=0_w$

$(x+2y,y-3z)=(0,0)$

$x+2y=0$ and $y-3z=0 \Rightarrow x=-2y$, $y=3z$ so $x=-6z$

$N(T)=\{(x,y,z) \,/\, x=-6z , y=3z \}$

$\dim N(T)=3-2=1 \Rightarrow \eta(T)=1$

$\Rightarrow \rho(T) +\eta(T)=\dim V$

$\Rightarrow \rho(T) +1=3 \Rightarrow \rho(T)= 2 =\dim V_2(R)$ Hence T is onto

T is L.T and onto so it is Epimorphism

9. If $T: V_3(R) \rightarrow V_2(R)$ is defined by $T(x,y,z)=(x, 2x)$ then T is –

(a)T is Linear Transformation only (b) Monomorphism (c) Epimorphism (d) T is Isomorphism

Sol: Clearly, it is homogenious linear expression hence T is L.T

dimv=3 and dimw=2 so dimv>dimw hence T is not 1-1

For onto

$\Rightarrow T(x,y,z)=0$

$\Rightarrow (x, 2x)=(0,0) \Rightarrow x=0$

$\Rightarrow N(T)=\{(x,y,z) \,/\, x=0\}$

$\Rightarrow \dim N(T)=3-1=2 \Rightarrow \eta(T)=2$

$\Rightarrow \rho(T) +\eta(T)=\dim V \Rightarrow \rho(T) +2=3$

$\Rightarrow \rho(T)= 1 \neq \dim V_2(R)$ Hence T is not onto

T is L.T only.

10. If $T: V_2(R) \rightarrow V_2(R)$ is defined by $T(x,y)=(x+y, x-y)$ then T is –

(a)T is Linear Transformation only (b)Monomorphism (c)Epimorphism (d) Isomorphism

Sol: Clearly, it is homogenious linear expression hence T is L.T

dimv=2 and dimw=2

$\Rightarrow T(x,y)=0_w$

$\Rightarrow (x+y , x-y)=(0,0) \Rightarrow x=0, y=0$

$\Rightarrow N(T)=\{(x,y,z) \,/\, x=0,y=0\}$

$\Rightarrow \dim N(T)=2-2=0 \Rightarrow \eta(T)=0$

Hence, T is one-one

$\Rightarrow \rho(T) +\eta(T)=\dim V \Rightarrow \rho(T) +0=2$

$\Rightarrow \rho(T)=2=\dim V_2(R)$ Hence T is onto

T is Isomorphism

Note:

I. A linear Transformation T: $V_2(R) \to V_2(R)$ is defined by T(x,y)=(ax+by, cx+dy)

Then T is Isomorphism ⇒ detT≠0 i.e $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

II. A linear Transformation T: $V_3(R) \to V_3(R)$ is defined by T(x,y,z)=($a_1x+b_1y+c_1z$, $a_2x+b_2y+c_2z$,

$a_3x+b_3y+c_3z$) Then T is Isomorphism ⇒ detT≠0 i.e $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

11. Which of the following linear maps is an Isomorphism on V_2(R)

(a)T(x,y)=(x-y , y-x)

(b)T(x,y)=(x+y , 2x-y)

(c) T(x,y)=(x , 2x)

(d)T(x,y)=(2y , 2120y)

Sol:

(a) $\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$

(b) $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \neq 0$

(c) $\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$

(d) $\begin{vmatrix} 0 & 2 \\ 0 & 2020 \end{vmatrix} = 0$

T(x,y)=(x+y , 2x-y) is Isomorphism

12. Which of the following linear maps is an Isomorphism on V_3(R) (a)T(x,y,z)=(y+z, z+x, x+y), (b)T(x,y,z)=(y-z, z-x, x-y),(c) T(x,y,z)=(x-y, y-z, z-x), (d)T(x,y,z)=(x+2y+3z, 3x+2y+z, x+y+z)

Sol: (a) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \neq 0$, (b) $\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0$, (c) $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} \neq 0$, (d) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

(a)&(c) are Isomorphism

13. If T is a linear transformation on V_3(R) defined by T(x,y,z)=(y,z,0) then T^3(2019,2020,2021)

Sol:

T^3(2019,2020,2021)= T^2 {T(2019,2020,2021)}= T^2(2020,2021,0)=

T{T(2019,2020,0)}=T(2020,0,0)=(0,0,0)

14. If T is a linear transformation on $V_3(R)$ defined by T(x,y,z)=(y,z,0) then T is

(a)Idempotent L.T

(b) Nilpotent L.T with index 2

(c)Nilpotent L.T with index 3

(d) Involuatry L.T

Sol: T(x,y,z)=(y,z,0)

T^2 (x,y,z)= T{T(x,y,z)}=T(y,z,0)=(z,0,0)

T^3 (x,y,z)= T{ T^2 (x,y,z) }=T (z,0,0)=(0,0,0)

T^3=0, Hence, T is Nilpotent L.T with index 3

15. If T is a linear transformation on $V_3(R)$ such that T(x,y,z)=(0,y,z) then T is

(a)Idempotent, (b) Nilpotent, (c) Identity, (d) Involuatry

Sol: T(x,y,z)=(0,y,z)

T^2 (x,y,z)= T{T(x,y,z)}=T(0,y,z)=(0,y,z)=T(x,y,z)

T^2=T hence T is idempotent L.T

16. If T: V(F) → W(F) is L.T such that ρ(T) = η(T) then

(a)dimV is even (b)dimV is odd (c)dimW is even (d)dimW is odd

Sol: Let ρ(T)=n and η(T)=n

⇒ dimV=ρ(T) + η(T)=n+n=2n

⇒dim V is even

17. If S & T are two linear maps on $V_2(R)$ such that S(x,y)=(y,-x) & T(x,y)=(-y,-x) then

(a)S^2=I& T^2=I (b) S^2=I & T^2=-I (c) S^2=-I & T^2=-I (d) S^2=-I & T^2=I

Sol: S^2(x,y)=S{S(x,y)}=S{y,-x}=(-x,-y)=-(x,y)=-I ⇒ S^2=-I

T^2(x,y)=T{T(x,y)}=T{-y,-x}=(x,y)=I ⇒ T^2=I

18. If S & T are two linear maps on $V_2(R)$ such that S(x,y)=(y,-x) & T(x,y)=(-y,-x) then find

(a)TS(x,y) (b) ST(x,y) (c)(2S-3T)(x,y)

Sol: (a) TS(x,y)=T{S(x,y)}=T{y,-x}=(x,-y)

(b) ST(x,y)=S{T(x,y)}=S{-y,-x}=(-x,y)

(c)(2S-3T)(x,y)=2S(x,y)-3T(x,y)=2(y,-x)-3(-y,-x)=(5y,x)

Invertible Linear Transformation:

Let T: V→W be one-one onto mapping then the mapping T^{-1}:W→V defined by $T^{-1}(v)=u \Leftrightarrow T(u)=v$, u∈V, v∈W is called the invers mapping of T

Note: if T is invertible ⇔T is non-singular ⇔T is an Isomorphism

19. If T is a linear map on $V_2(R)$ such that T(x,y)=(x+y,-y) then find T^{-1}(x,y).

Sol: Let $T^{-1}(x,y)=(a,b)$

T(a,b)=(x,y)

(a+b,-b)=(x,y)

b=-y & a+b=x so a=x+y and b=-y

$T^{-1}(x,y)=(x+y,-y)$

20. If T is a linear map on $V_3(R)$ such that T(x,y)=(x+y+z ,x+y ,x) then find T^{-1}(x,y,z)

Sol: Let $T^{-1}(x,y,z)=(a,b,c)$

T(a,b,c)=(x,y,z)

(a+b+c ,a+b, a)=(x,y,z)

a+b+c=x, a+b=y, a=z

b=y-z and c=x-y

$T^{-1}(x,y,z)=(z, y-z, x-y)$

21. If T is a linear map on $V_3(R)$ such that $T(e_1)=e_1,T(e_2)=e_2,T(e_3)=e_3$ then find T^{-1}(3,-4,7)

Sol: Let $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$

$T(e_1)=e_1=(1,0,0)$ and $T(e_2)=e_2=(0,1,0)$ and $T(e_3)=e_3=(0,0,1)$

$(x,y,z)=x\, e_1 + y\, e_2 + z\, e_3$

$T(x,y,z)=x\, T(e_1)+ y\, T(e_2)+ z\, T(e_3)$

T(x,y,z)=x(1,0,0)+ y0,1,0)+ z(0,0,1)

T(x,y,z)=(x,y,z)

(x,yz)= T^{-1}(x,y,z)

T^{-1}(3,-4, 7)=(3,-4,7)

22. If T is a linear map on $V_3(R)$ such that $T(e_1)=e_1+ e_2$,$T(e_2)=e_2+ e_3$,$T(e_3)= e_1+ e_2 +e_3$ then find T^{-1}

Sol: Let $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$

$T(e_1)=e_1+ e_2 =(1,1,0)$

$T(e_2)=e_2+ e_3 =(0,1,1)$

$T(e_3)= e_1+ e_2 +e_3 =(1,1,1)$

$(x,y,z)=x\, e_1 + y\, e_2 + z\, e_3$

$T(x,y,z)=x\, T(e_1)+ y\, T(e_2)+ z\, T(e_3)$

T(x,y,z)=x(1,1,0)+y(0,1,1)+z(1,1,1)=(x+z , x+y+z , y+z)

Let $T^{-1}(x,y,z)=(a,b\ c) \Rightarrow T(a,b,c)=(x,y,z)$

$(a+c,\ a+b+c,\ b+c\ )=(x,y,z)$

$a+c=x$, $a+b+c=y$, $b+c=z$

$a=y-z$, $c=x-y+z$, $b=y-x$

$T^{-1}(x,y,z)=(y-z,\ y-x\ ,\ x-y+z)$

Exercise Problems: **If T is a linear map on $V_3(R)$ is invertible then find T^{-1}**

1) $T(x,y,z)=(2x\ ,4x-y\ ,2x+3y-z)$
2) $T(x,y,z)=(x+y+z\ ,y+z\ ,z)$
3) $T(a,b,c)=(\ a-3b-2c\ ,\ b-4c,\ c\ )$
4) $T(a,b,c)=(\ a-2b-c\ ,\ b-c,\ a\ )$
5) $T(a,b,c)=(\ 3a\ ,\ a-b,\ 2a+b+c\ )$
6) $T(e_1)=e_1+e_2$,$T(e_2)= e_1-e_2+e_3$, $T(e_3)=3e_1+4e_3$
7) $T(e_1)=e_1-e_2$,$T(e_2)= e_2+e_3$, $T(e_3)=e_1+e_2-e_3$
8) $T(e_1)=e_1+e_2-e_3$,$T(e_2)= 3e_1-5e_3$, $T(e_3)=3e_1-2e_3$

Matrix of a linear Transformation:

Let T: V(F)→W(F) be a linear Transformation, dimV=m and dimW=n.

Let B_1={ v_1 , v_2 , v_3 ,.........v_m } be the ordered basis of V and B_2={ w_1 , w_2 , w_3 ,.......w_n } be the ordered basis of W.

For every v∈V ⇒T(v)∈W and T(v) can be expressed as a linear combinations of elements of B_2.

Let there exist a_{ij} ∈F such that

$T(v_1)= a_{11}\, w_1+ a_{11}\, w_2+ a_{13}\, w_3$ +------------- $+a_{1n}\, w_n$

$T(v_2)= a_{21}\, w_1+ a_{22}\, w_2+ a_{23}\, w_3$ +------------- $+a_{2n}\, w_n$

$T(v_m)= a_{m1}\, w_1+ a_{m2}\, w_2+ a_{m3}\, w_3$+------------$+a_{mn}\, w_n$

Writing the co-ordinates of $T(v_1)$, $T(v_2)$, $T(v_3)$....... $T(v_n)$ successively as column of a matrix we get

$$\Rightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} & \ldots\ldots a_{m1} \\ a_{12} & a_{22} & a_{32} & \ldots\ldots a_{m2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{1n} & a_{2n} & a_{3n} & \ldots\ldots a_{nm} \end{bmatrix}_{nXm} = (a_{ij})_{n\times m}$$

The matrix representation $(aij)_{mXn}$ is called the matrix of the linear transformation T w.r.to the bases B_1 and B_2. Symbolically [T: B_1:B_2] or $[T]=[aij]_{nXm}$

Model-: If B_2 is standard Basis

1. If T: V_2→V_3 defined by T(x,y)=(x+2y , 2x-y , 7y) find [T: B_1:B_2] where B_1& B_2 are the standard bases of V_2 and V_3.

Sol: Let B_1 is the standard bases of V_2 and B_2 is the standard bases of V_3.

B_1={ (1,0)(0,1) } and B_2={ (1,0,0), (0,1,0), (0,0,1) }

T(x,y)=(x+2y , 2x-y , 7y)

⇒T(1,0)=(1,2,0)=**a**(1,0,0)+**b**(0,1,0)+**c**(0,0,1)=(a,b,c)

a=1, b=2, c=0

T(1,0)=(1,2,0)=**1**(1,0,0)+**2**(0,1,0)+**0**(0,0,1)

The coordinate vectors of (1,2,0) w.r.to B_2 is (1,2,0)

⇒T(0,1)=(2,-1,7) =**a**(1,0,0)**+b**(0,1,0)+**c**(0,0,1)=(a,b,c)

a=2, b=-1, c=7

T(0,1)=(2,-1,7) =**2**(1,0,0)**-1**(0,1,0)+**7**(0,0,1)=(a,b,c)

The coordinate vectors of (1,-1,7) w.r.to B_2 is (1,-1,7)

The matrix relative to B_1 and B_2 is $[T: B1: B2] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$

2. If T: $V_2 \to V_3$ defined by T(x,y)=(2x+3y , 5x-7y , 9x-17y) find [T: B_1:B_2] where B_1& B_2 are the standard bases of V_2 and V_3.

Sol: B_1={ (1,0)(0,1) } & B_2={ (1,0,0), (0,1,0), (0,0,1) }

T(x,y)=(2x+3y , 5x-7y , 9x-17y)

⇒T(1,0)=(2,5,9)

⇒T(0,1)=(3,-7,-17)

The matrix relative to B_1 and B_2 is $[T: B1: B2] = \begin{bmatrix} 2 & 3 \\ 5 & -7 \\ 9 & -17 \end{bmatrix}$

3. If T: $V_3 \to V_2$ defined by T(x,y,z)=(x+2y-3z ,4x+9z) find [T: B_1:B_2] where B_1& B_2 are the standard bases of V_3 and V_2.

Sol:B_1={ (1,0,0)(0,1,0) ,(0,0,1)} & B_2={ (1,0,), (0,1)}

T(x,y)=(x+2y-3z ,4x+9z)

⇒T(1,0,0)=(1,4)

⇒T(0,1,0)=(2,0)

⇒T(0,0,1)=(-3,9)

The matrix relative to B_1 and B_2 is $[T: B1: B2] = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 9 \end{bmatrix}$

4. If T: $V_3 \to V_3$ defined by T(x,y,z)=(x, y+z , x+y-z) then find matrix of T w.r.to standard basis.

Sol: B_1={ (1,0,0)(0,1,0) ,(0,0,1)} & B_2={ (1,0,0), (0,1,0),(0,0,1)}

T(x,y,z)=(x, y+z , x+y-z)

⇒T(1,0,0)=(1,0,1) ⇒T(0,1,0)=(0,1,1) ⇒T(0,0,1)=(0,1,-1)

$$[T: B1: B2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

5. If T: $V_2 \to V_3$ defined by T(x,y)=(x-y, 2y, 8x) find the matrix of T relative to basis B_1={(1,1), (1,-1)} & B_2={e_1,e_2,e_3}.

Sol: B_1 is the non-standard bases of V_2 and B_2 is the standard bases of V_3.

B_1={ (1,1)(1,-1) } & B_2={ (1,0,0), (0,1,0), (0,0,1) }

T(x,y)=(x-y,2y , 8x)

⇒T(1,1)=(0,2,8)=**0**(1,0,0)+**2**(0,1,0)+**8**(0,0,1)

The coordinate vectors of (0,2,8) w.r.to B_2 is (60,2,8)

⇒T(1,-1)=(2,-2,8) =**2**(1,0,0)-**2**(0,1,0)**+8**(0,0,1)

The coordinate vectors of (2,-2,8) w.r.to B_2 is(2,-2,8)

The matrix relative to B_1 and B_2 is $[T: B1: B2] = \begin{bmatrix} 0 & 2 \\ 2 & -2 \\ 8 & 8 \end{bmatrix}$

6. If T: $V_3 \to V_2$ defined by T(x,y,z)=(y, 2z) then the matrix of T w.r.to basis B_1={(1,1,0),(1,0, 1), (0,1,1)} & B_2={e_1,e_2}.

Sol: B_1={ (1,1,0)(1,0,1),(0,1,1) } & B_2={ (1,0), (0,1)}

T(x,y,z)=(y, 2z)

⇒T(1,1,0)=(1,0)

⇒T(1,0,1)=(0,2)

⇒T(0,1,1)=(1,2)

$$[\mathrm{T:B1:B2}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Model-2: If B_2 is Non-standard Basis

7. If T: $V_3 \to V_2$ defined by T(x,y,z)=(x+z , 2y-3z) find the matrix of T w.r.to basis B_1={(1,1,1), (1,1,0), (1,0,0) } & B_2={ (1,0), (2,3) }.

Sol:

⇒T(1,1,1)=(2,-1)

Let The coordinate vectors of (2,-1) w.r.to B_2 is (a,b)

⇒ (2,-1)=a(1,0)+b(2,3)=(a+2b , 3b)

a+2b=2, 3b=1

b=1/3 and a=8/3

(2,-1)=(8/3)(1,0)+(1/3)(2,3)

The coordinate vectors of (2,-1) w.r.to B_2 is (8/3, 1/3)

⇒T(1,1,0)=(1 ,2)

Let The coordinate vectors of (1,2) w.r.to B_2 is (c,d)

⇒ (1,2)=c(1,0)+d(2,3)=(c+2d , 3d)

c+2d=1, 3d=2

d=2/3 and c=-1/3

(1, 2)=(2/3)(1,0)+(-1/3)(2,3)

The coordinate vectors of (1, 2,) w.r.to B_2 is (2/3, -1/3)

⇒T(1,0,0)=(1 ,0)

Let The coordinate vectors of (1,0) w.r.to B_2 is (e,f)

⇒ (1,0)=e(1,0)+f(2,3) =(e+2f , 3f)

e+2f=1, 3f=0

f=0 and e=1

(1, 0)=1(1,0)+0(2,3)

The coordinate vectors of (1,0,) w.r.to B_2 is (1,0)

The matrix relative to B_1 and B_2 is $[\mathrm{T:B1:B2}] = \begin{bmatrix} 8/3 & 2/3 & 1 \\ 1/3 & -1/3 & 0 \end{bmatrix}$

8. If T: $V_3 \to V_2$ defined by T(x,y,z)=(3x+2y-4z , x-5y+3z) find the matrix of T w.r.to basis B_1={(1,1,1),(1,1,0) , (1,0,0) } & B_2={ (1,3) , (2,5) }.

Sol: Let a,b∈V_2 and (a,b)=p(1,3)+q(2,5)=(p+2q, 3p+5q)

p=-5a+2b , q=3a-b

(a,b)=(5a+2b)(1,3)+(3a-b)(2,5)

⇒T(1,1,1)=(1,-1)=-7(1,3) +4 (2,5)

⇒T(1,1,0)=(5,-4))=-33(1,3) +19 (2,5)

⇒T(1,0,0)=(3,1))=-13(1,3) +8(2,5)

$$[\mathrm{T{:}\,B1{:}\,B2}] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

9. Consider the linear transformation T: $R^3 \to R^2$, defined by T(x,y,z)=(x+y, 2z) . Find the matrix of T with respect to the bases $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and $\{\mathbf{u_1^1}, \mathbf{u_2^1}\}$ of R^3 and R^2, where $\mathbf{u_1} = (\mathbf{1}, \mathbf{1}, \mathbf{0}), \mathbf{u_2} = (\mathbf{0}, \mathbf{1}, \mathbf{4}), \mathbf{u_3} = (\mathbf{1}, \mathbf{2}, \mathbf{3}), \mathbf{u_1^1} = (\mathbf{10}), \mathbf{u_2^1} = (\mathbf{0}, \mathbf{2})$

10. If D: $P_3(R) \to P_2(R)$ is defined by D{f(x)}=$f^1(x)$ then the matrix of "D" w.r.to standard basis.

Sol:B_1= {1,x,x^2,x^3) and B_2={1,x,x^2) are standard basis

D(1)=d/dx(1)=0=**0**.1+**0**.x+**0**.x^2

D(x)=d/dx(x)=1 =**0**.1+**1**.x+**0**.x^2

D(x^2)=d/dx(x^2)=2x =**0**.1+**2**.x+**0**.x^2

D(x^3)=d/dx(x^3)=3x^2 =**0**.1+**0**.x+**3**.x^2

$$[\mathrm{T{:}\,B1{:}\,B2}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

11. If I: $P_2(R) \to P_3(R)$ is defined by $I\{f(x)\} = \int_0^x f(x)dx$ then the matrix of "I" w.r.to standard basis.

Sol:B_1= {1,x,x^2) and B_2={1,x,x^2,x^3) are standard basis

$$\mathrm{I}\{1\} = \int_0^{\mathrm{x}} 1\,\mathrm{dx} = (\mathrm{x})_0{}^{\mathrm{x}} = \mathrm{x} = 0.1 + 1.\mathrm{x} + 0.\mathrm{x}^2 + 0.\mathrm{x}^3$$

$$\mathrm{I}\{\mathrm{x}\} = \int_0^{\mathrm{x}} \mathrm{x}\,\mathrm{dx} = \left(\frac{x^2}{2}\right)_0^{\mathrm{x}} = \frac{x^2}{2} = 0.1 + 0.\mathrm{x} + \frac{1}{2}.\mathrm{x}^2 + 0.\mathrm{x}^3$$

$$\mathrm{I}\{\mathrm{x}\} = \int_0^{\mathrm{x}} x^2\,\mathrm{dx} = \left(\frac{x^3}{3}\right)_0^{\mathrm{x}} = \frac{x^3}{3} = 0.1 + 0.\mathrm{x} + 0.\mathrm{x}^2 + \frac{1}{3}.\mathrm{x}^3$$

$$[\mathrm{T{:}\,B1{:}\,B2}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$