

$$\frac{\partial w}{\partial x} = 2y - z, \frac{\partial w}{\partial y} = 2x + 4z, \frac{\partial w}{\partial z} = -x + 4y - 4z.$$

$$\begin{aligned} \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix} \\ &= 1(-2(-x + 4y - 4z) - 3(2x + 4z)) - 2(-x + 4y - 4z - 3(2y - z)) \\ &\quad + 1(2x + 4z + 2(2y - z)) \\ &= 2x - 8y + 8z - 6x - 12z + 2x - 8y + 8z + 12y - 6z + 2x + 4z + 4y - 2z = 0. \end{aligned}$$

Hence, u, v, w are not independent.

Now $u + v = 2x + 4z, u - v = 4y - 2z$.

$$\begin{aligned} (u + v)(u - v) &= 2(x + 2z).2(2y - z) \\ u^2 - v^2 &= 4(2xy - xz + 4yz - 2z^2) \\ u^2 - v^2 &= 4w. \end{aligned}$$

2.6 Taylor's expansion for functions of two variables

We know that for a function $f(x)$ of one single variable x , the Taylor's expansion is

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Now let $f(x, y)$ be a function of two independent variables x, y defined in a region R of the xy -plane and let (a, b) be a point in R . Suppose $f(x, y)$ has all its partial

derivatives in a neighbourhood of (a, b) then

$$\begin{aligned}
 f(a+h, b+k) &= f(a, b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(a, b) \\
 &\quad + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f(a, b) + \frac{1}{3!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^3 f(a, b) + \cdots + \\
 &= f(a, b) + \left(h f_x(a, b) + k f_y(a, b)\right) \\
 &\quad + \frac{1}{2!} \left(h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)\right) \\
 &\quad + \frac{1}{3!} \left(h^3 f_{xxx}(a, b) + 3h^2 k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) \right. \\
 &\quad \left. + k^3 f_{yyy}(a, b)\right) + \cdots .
 \end{aligned}$$

Put $x = a + h, y = b + k$ then $h = x - a, k = y - b$.

\therefore The Taylor's series can be written as

$$\begin{aligned}
 f(x, y) &= f(a, b) + \left((x-a)f_x(a, b) + (y-b)f_y(a, b)\right) \\
 &\quad + \frac{1}{2!} \left((x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right) \\
 &\quad + \frac{1}{3!} \left((x-a)^3 f_{xxx}(a, b) + 3(x-a)^2(y-b)f_{xxy}(a, b) \right. \\
 &\quad \left. + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)\right) + \cdots .
 \end{aligned}$$

This is known as the Taylor's expansion of $f(x, y)$ in the neighbourhood of (a, b) or about the point (a, b) .

Put $a = 0, b = 0$. we get

$$\begin{aligned}
 f(x, y) &= f(0, 0) + \left(x f_x(0, 0) + y f_y(0, 0)\right) \\
 &\quad + \frac{1}{2!} \left(x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right) \\
 &\quad + \frac{1}{3!} \left(x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\right) + \cdots .
 \end{aligned}$$

This is called Maclaurin's series for $f(x, y)$ in powers of x and y .

Worked Examples

Example 2.63. Expand $e^x \sin y$ in powers of x and y as far as the terms of third degree. [Jun 2013]

Solution. $f(x, y) = e^x \sin y$ $f(0, 0) = 0$

$$f_x(x, y) = e^x \sin y \quad f_x(0, 0) = 0$$

$$f_y(x, y) = e^x \cos y \quad f_y(0, 0) = 1$$

$$f_{xx}(x, y) = e^x \sin y \quad f_{xx}(0, 0) = 0$$

$$f_{xy}(x, y) = e^x \cos y \quad f_{xy}(0, 0) = 1$$

$$f_{yy}(x, y) = -e^x \sin y \quad f_{yy}(0, 0) = 0$$

$$f_{xxx}(x, y) = e^x \sin y \quad f_{xxx}(0, 0) = 0$$

$$f_{xxy}(x, y) = e^x \cos y \quad f_{xxy}(0, 0) = 1$$

$$f_{xyy}(x, y) = -e^x \sin y \quad f_{xyy}(0, 0) = 0$$

$$f_{yyy}(x, y) = -e^x \cos y \quad f_{yyy}(0, 0) = -1.$$

$$\begin{aligned} \text{Now } f(x, y) &= f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}(x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)) \\ &\quad + \frac{1}{3!}(x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)) + \cdots \\ &= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) \\ &\quad + \frac{1}{6}(x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot 0 + y^3(-1)) + \cdots \\ &= y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \cdots \end{aligned}$$

Example 2.64. Expand $e^x \log_e(1 + y)$ in powers of x and y upto terms of third degree. [Jan 2014, Dec 2011, Jan 2003]

Solution.

$$f_{yyy} = \frac{2e^x}{(1+y)^3}$$

$$\begin{array}{ll}
 f(x, y) = e^x \log_e(1+y) & f(0, 0) = 0 \\
 f_x(x, y) = e^x \log_e(1+y) & f_x(0, 0) = 0 \\
 f_y(x, y) = \frac{e^x}{1+y} & f_y(0, 0) = 1 \\
 f_{xx} = e^x \log(1+y) & f_{xx}(0, 0) = 0 \\
 f_{xy} = \frac{e^x}{1+y} & f_{xy}(0, 0) = 1 \\
 f_{yy} = \frac{-e^x}{(1+y)^2} & f_{yy}(0, 0) = -1 \\
 f_{xxx} = e^x \log(1+y) & f_{xxx}(0, 0) = 0 \\
 f_{xxy} = \frac{e^x}{1+y} & f_{xxy}(0, 0) = 1 \\
 \text{-----} & \text{-----} \\
 f_{xyy} = \frac{-e^x}{(1+y)^2} & f_{xyy}(0, 0) = -1 \\
 \text{-----} & \text{-----} \\
 f_{yyy} = \frac{2e^x}{(1+y)^3} & f_{yyy}(0, 0) = 2
 \end{array}$$

By Maclaurin's series we have,

$$\begin{aligned}
 f(x, y) &= f(0, 0) + xf_x(0, 0) + yf_y(0, 0) \\
 &\quad + \frac{1}{2!}(x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)) \\
 &\quad + \frac{1}{3!}(x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) \\
 &\quad + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)) + \dots \\
 e^x \log_e(1+y) &= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2(-1)) + \\
 &\quad \frac{1}{6}[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2(-1) + y^3(2)] + \dots \\
 &= y + xy - \frac{y^2}{2} + \frac{x^2 y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} - \dots
 \end{aligned}$$

Example 2.65. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$ upto third degree terms. [Jun 2012]

Solution. $f(x, y) = x^2y + 3y - 2$. $f(1, -2) = 1 \times (-2) + 3(-2) - 2$

$$= -2 - 6 - 2 = -10.$$

$$f_x(x, y) = 2xy.$$

$$f_x(1, -2) = -4.$$

$$f_y(x, y) = x^2 + 3.$$

$$f_y(1, -2) = 4.$$

$$f_{xx}(x, y) = 2y.$$

$$f_{xx}(1, -2) = -4.$$

$$f_{xy}(x, y) = 2x.$$

$$f_{xy}(1, -2) = 2.$$

$$f_{yy}(x, y) = 0.$$

$$f_{yy}(1, -2) = 0.$$

$$f_{xxx}(x, y) = 0.$$

$$f_{xxx}(1, -2) = 0.$$

$$f_{xxy}(x, y) = 2.$$

$$f_{xxy}(1, -2) = 2.$$

$$f_{xyy}(x, y) = 0.$$

$$f_{xyy}(1, -2) = 0.$$

$$f_{yyy}(x, y) = 0.$$

$$f_{yyy}(1, -2) = 0.$$

By Taylor's theorem we have,

$$f(x, y) = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$$

$$+ \frac{1}{2!} \left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right]$$

$$+ \frac{1}{3!} \left[(x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) \right.$$

$$\left. + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b) + \dots \right]$$

$$x^2y + 3y - 2 = f(1, -2) + (x - 1)f_x(1, -2) + (y + 2)f_y(1, -2)$$

$$+ \frac{1}{2!} \left[(x - 1)^2 f_{xx}(1, -2) + 2(x - 1)(y + 2)f_{xy}(1, -2) + (y + 2)^2 f_{yy}(1, -2) \right]$$

$$+ \frac{1}{3!} \left[(x - 1)^3 f_{xxx}(1, -2) + 3(x - 1)^2(y + 2)f_{xxy}(1, -2) \right.$$

$$\left. + 3(x - 1)(y + 2)^2 f_{xyy}(1, -2) + (y + 2)^3 f_{yyy}(1, -2) \right] + \dots$$

$$\begin{aligned}
&= -10 - 4(x-1) + 4(y+2) + \frac{1}{2} \left[-4(x-1)^2 + 2(x-1)(y+2) \right] \\
&\quad + \frac{1}{6} \left[6(x-1)^2(y+2) \right] + \dots \\
&= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + (x-1)(y+2) + (x-1)^2(y+2) + \dots
\end{aligned}$$

Example 2.66. Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3y^2$ in powers of $(x+2)$ and $y-1$ upto third degree terms. [Jan 2012, Jun 2010, Jan 2010]

Solution.

$f(x, y) = x^2y^2 + 2x^2y + 3y^2.$	$f(-2, 1) = 4 + 8 + 3 = 15.$
$f_x(x, y) = 2xy^2 + 4xy.$	$f_x(-2, 1) = -4 - 8 = -12.$
$f_y(x, y) = 2x^2y + 2x^2 + 6y.$	$f_y(-2, 1) = 8 + 8 + 6 = 22.$
$f_{xx}(x, y) = 2y^2 + 4y.$	$f_{xx}(-2, 1) = 2 + 4 = 6.$
$f_{xy}(x, y) = 4xy + 4x.$	$f_{xy}(-2, 1) = -8 - 8 = -16.$
$f_{yy}(x, y) = 2x^2 + 6.$	$f_{yy}(-2, 1) = 8 + 6 = 14.$
$f_{xxx}(x, y) = 0.$	$f_{xxx}(-2, 1) = 0.$
$f_{xxy}(x, y) = 4y + 4.$	$f_{xxy}(-2, 1) = 4 + 4 = 8.$
$f_{xyy}(x, y) = 4x.$	$f_{xyy}(-2, 1) = -8.$
$f_{yyy}(x, y) = 0.$	$f_{yyy}(-2, 1) = 0.$

By Taylor's theorem we have,

$$\begin{aligned}
f(x, y) &= f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) \\
&\quad + \frac{1}{2!} \left[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right] \\
&\quad + \frac{1}{3!} \left[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2(y-b)f_{xxy}(a, b) \right. \\
&\quad \left. + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b) \right] + \dots
\end{aligned}$$

$$\begin{aligned}
x^2y^2 + 2x^2y + 3y^2 &= 15 - 12(x+2) + 22(y-1) \\
&\quad + \frac{1}{2} \left[6(x+2)^2 - 2 \times 16(x+2)(y-1) + 14(y-1)^2 \right] \\
&\quad + \frac{1}{6} \left[24(x+2)^2(y-1) - 24(x+2)(y-1)^2 \right] + \cdots \\
&= 15 - 12(x+2) + 22(y-1) + 3(x+2)^2 - 16(x+2)(y-1) + 7(y-1)^2 \\
&\quad + 4(x+2)^2(y-1) - 4(x+2)(y-1)^2 + \cdots
\end{aligned}$$

Example 2.67. Use Taylor's formula to expand the function defined by $f(x, y) = x^3 + y^3 + xy^2$ in powers of $(x-1)$ and $(y-2)$. [May 2011]

Solution.

$$\begin{aligned}
f(x, y) &= x^3 + y^3 + xy^2. & f(1, 2) &= 1 + 8 + 4 = 13. \\
f_x(x, y) &= 3x^2 + y^2. & f_x(1, 2) &= 3 + 4 = 7. \\
f_y(x, y) &= 3y^2 + 2xy. & f_y(1, 2) &= 12 + 4 = 16. \\
f_{xx}(x, y) &= 6x. & f_{xx}(1, 2) &= 6. \\
f_{xy}(x, y) &= 2y. & f_{xy}(1, 2) &= 4. \\
f_{yy}(x, y) &= 6y + 2x. & f_{yy}(1, 2) &= 12 + 2 = 14. \\
f_{xxx}(x, y) &= 6. & f_{xxx}(1, 2) &= 6. \\
f_{xxy}(x, y) &= 0. & f_{xxy}(1, 2) &= 0. \\
f_{xyy}(x, y) &= 2. & f_{xyy}(1, 2) &= 2. \\
f_{yyy}(x, y) &= 6. & f_{yyy}(1, 2) &= 6.
\end{aligned}$$

By Taylor's theorem we have,

$$\begin{aligned}
f(x, y) &= f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) \\
&\quad + \frac{1}{2!} \left[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3!} \left[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2(y-b) f_{xxy}(a, b) \right. \\
& \left. + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b) \right] + \dots \\
x^3 + y^3 + xy^2 &= 13 + 7(x-1) + 16(y-2) + \frac{1}{2} \left[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2 \right] \\
& + \frac{1}{6} \left[6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3 \right] + \dots \\
&= 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2 \\
& + (x-1)^3 + (x-1)(y-2)^2 + (y-2)^3 + \dots
\end{aligned}$$

Example 2.68. Expand $e^{-x} \log y$ as a Taylor's series in powers of x and $y-1$ upto third degree terms. [Jun 2011]

Solution.

$$f(x, y) = e^{-x} \log y. \quad f(0, 1) = 0.$$

$$f_x(x, y) = -e^{-x} \log y. \quad f_x(0, 1) = 0.$$

$$f_y(x, y) = \frac{e^{-x}}{y}. \quad f_y(0, 1) = 1.$$

$$f_{xx}(x, y) = e^{-x} \log y. \quad f_{xx}(0, 1) = 0.$$

$$f_{xy}(x, y) = -\frac{e^{-x}}{y}. \quad f_{xy}(0, 1) = -1.$$

$$f_{yy}(x, y) = -\frac{e^{-x}}{y^2}. \quad f_{yy}(0, 1) = -1.$$

$$f_{xxx}(x, y) = -e^{-x} \log y. \quad f_{xxx}(0, 1) = 0.$$

$$f_{xxy}(x, y) = \frac{e^{-x}}{y}. \quad f_{xxy}(0, 1) = 1.$$

$$f_{xyy}(x, y) = \frac{e^{-x}}{y^2}. \quad f_{xyy}(0, 1) = 1.$$

$$f_{yyy}(x, y) = \frac{2e^{-x}}{y^3}. \quad f_{yyy}(0, 1) = 2.$$

By Taylor's theorem we have,

$$\begin{aligned}
 f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\
 &\quad + \frac{1}{2!} \left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right] \\
 &\quad + \frac{1}{3!} \left[(x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) \right. \\
 &\quad \left. + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b) \right] + \cdots \\
 e^{-x} \log y &= y - 1 + \frac{1}{2!} [-2x(y - 1) - (y - 1)^2] + \frac{1}{3!} [3x^2(y - 1) + 3x(y - 1)^2 + 2(y - 1)^3] + \cdots
 \end{aligned}$$

Example 2.69. Expand $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ about $(1, 1)$ upto the second degree terms. Hence compute $f(1.1, 0.9)$ approximately. [Jan 2005]

Solution. Given, $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$(a, b) = (1, 1) \quad a = 1, b = 1$$

$$f(1, 1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$f_x = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{-y}{x^2} \right) = -\frac{x^2 y}{(x^2 + y^2)x^2} = \frac{-y}{x^2 + y^2} \quad f_x(1, 1) = \frac{-1}{2}$$

$$f_y = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x} \right) = \frac{x^2}{(x^2 + y^2)x} = \frac{x}{x^2 + y^2} \quad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx} = -y(-1)(x^2 + y^2)^{-2} 2x = \frac{2xy}{(x^2 + y^2)^2} \quad f_{xx}(1, 1) = \frac{2}{4} = \frac{1}{2}$$

$$f_{xy} = \frac{x^2 + y^2 - x2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad f_{xy}(1, 1) = 0$$

$$f_{yy} = x(-1)(x^2 + y^2)^{-2} 2y = \frac{-2xy}{(x^2 + y^2)^2} \quad f_{yy}(1, 1) = \frac{-1}{2}$$

By Taylor's theorem we have

$$\begin{aligned}
 f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\
 &\quad + \frac{1}{2!} \left((x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right) + \cdots
 \end{aligned}$$

$$\begin{aligned}
\tan^{-1}\left(\frac{y}{x}\right) &= f(1, 1) + (x-1)f_x(1, 1) + (y-1)f_y(1, 1) \\
&\quad + \frac{1}{2!}\left((x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1)f_{xy}(1, 1) + (y-1)^2 f_{yy}(1, 1)\right) \\
&= \frac{\pi}{4} + (x-1)\left(\frac{-1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2}\left((x-1)^2 \frac{1}{2} + 2(x-1)(y-1)0\right. \\
&\quad \left.+ (y-1)^2 \frac{(-1)}{2}\right) + \dots \\
&= \frac{\pi}{4} - \frac{1}{2}(x-1-y+1) + \frac{1}{2}\left(\frac{1}{2}(x-1)^2 - \frac{1}{2}(y-1)^2\right) + \dots \\
&= \frac{\pi}{4} - \frac{1}{2}(x-y) + \frac{1}{4}(x^2 - y^2 - 2x + 2y) + \dots \\
\tan^{-1}\left(\frac{y}{x}\right) &= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \dots
\end{aligned}$$

$$\begin{aligned}
f(1.1, 0.9) &= \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2 \text{ approximately} \\
&= \frac{\pi}{4} - 0.1 = 0.685 \text{ approximately.}
\end{aligned}$$

Example 2.70. Find the Taylor's series expansion of $e^x \sin y$ at the point $\left(-1, \frac{\pi}{4}\right)$ upto third degree terms. [Jan 2009]

Solution.

$$\begin{aligned}
f(x, y) &= e^x \sin y & f\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}} \\
f_x(x, y) &= e^x \sin y & f_x\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}} \\
f_y(x, y) &= e^x \cos y & f_y\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}} \\
f_{xx}(x, y) &= e^x \sin y & f_{xx}\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}} \\
f_{xy}(x, y) &= e^x \cos y & f_{xy}\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}} \\
f_{yx}(x, y) &= e^x \cos y & f_{yx}\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}} \\
f_{yy}(x, y) &= -e^x \sin y & f_{yy}\left(-1, \frac{\pi}{4}\right) &= \frac{-1}{e\sqrt{2}} \\
f_{xxx}(x, y) &= e^x \sin y & f_{xxx}\left(-1, \frac{\pi}{4}\right) &= \frac{1}{e\sqrt{2}}
\end{aligned}$$

$$f_{xxy} = e^x \cos y \quad f_{xxy}\left(-1, \frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}}$$

$$f_{xyy} = -e^x \sin y \quad f_{xyy}\left(-1, \frac{\pi}{4}\right) = \frac{-1}{e\sqrt{2}}$$

$$f_{yyx} = -e^x \sin y \quad f_{yyx}\left(-1, \frac{\pi}{4}\right) = \frac{-1}{e\sqrt{2}}$$

$$f_{yyy} = -e^x \sin y \quad f_{yyy}\left(-1, \frac{\pi}{4}\right) = \frac{-1}{e\sqrt{2}}$$

By Taylor's theorem we have

$$\begin{aligned} f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\ &\quad + \frac{1}{2!} \left((x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right) \\ &\quad + \frac{1}{3!} \left((x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) \right. \\ &\quad \left. + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b) \right) + \cdots . \\ e^x \sin y &= f\left(-1, \frac{\pi}{4}\right) + (x + 1)f_x\left(-1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)f_y\left(-1, \frac{\pi}{4}\right) \\ &\quad + \frac{1}{2!} \left((x + 1)^2 f_{xx}\left(-1, \frac{\pi}{4}\right) + 2(x + 1)\left(y - \frac{\pi}{4}\right)f_{xy}\left(-1, \frac{\pi}{4}\right) \right. \\ &\quad \left. + \left(y - \frac{\pi}{4}\right)^2 f_{yy}\left(-1, \frac{\pi}{4}\right) \right) \\ &\quad + \frac{1}{6} \left((x + 1)^3 f_{xxx}\left(-1, \frac{\pi}{4}\right) + 3(x + 1)^2\left(y - \frac{\pi}{4}\right)f_{xxy}\left(-1, \frac{\pi}{4}\right) \right. \\ &\quad \left. + 3(x + 1)\left(y - \frac{\pi}{4}\right)^2 f_{xyy}\left(-1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^3 f_{yyy}\left(-1, \frac{\pi}{4}\right) \right) + \cdots . \\ &= \frac{1}{e\sqrt{2}} + \frac{1}{e\sqrt{2}}(x + 1) + \frac{1}{e\sqrt{2}}\left(y - \frac{\pi}{4}\right) + \frac{1}{2\sqrt{2}e}(x + 1)^2 \\ &\quad + \frac{1}{\sqrt{2}e}(x + 1)\left(y - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}e}\left(y - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}e}(x + 1)^3 \\ &\quad + \frac{\sqrt{2}}{e}(x + 1)^2\left(y - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{e}(x + 1)\left(y - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}e}\left(y - \frac{\pi}{4}\right)^3 + \cdots . \end{aligned}$$

Example 2.71. Expand $e^x \cos y$ near the point $(1, \frac{\pi}{4})$ by Taylor's series as far as quadratic terms. [Jan 1996]

Solution.

$$\begin{aligned} f(x, y) &= e^x \cos y & f(1, \frac{\pi}{4}) &= \frac{e}{\sqrt{2}} \\ f_x(x, y) &= e^x \cos y & f_x(1, \frac{\pi}{4}) &= e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}} \\ f_y(x, y) &= -e^x \sin y & f_y(1, \frac{\pi}{4}) &= -\frac{e}{\sqrt{2}} \\ f_{xx}(x, y) &= e^x \cos y & f_{xx}(1, \frac{\pi}{4}) &= \frac{e}{\sqrt{2}} \\ f_{xy}(x, y) &= -e^x \sin y & f_{xy}(1, \frac{\pi}{4}) &= \frac{-e}{\sqrt{2}} \\ f_{yy}(x, y) &= -e^x \cos y & f_{yy}(1, \frac{\pi}{4}) &= \frac{-e}{\sqrt{2}} \end{aligned}$$

By Taylor's theorem we have

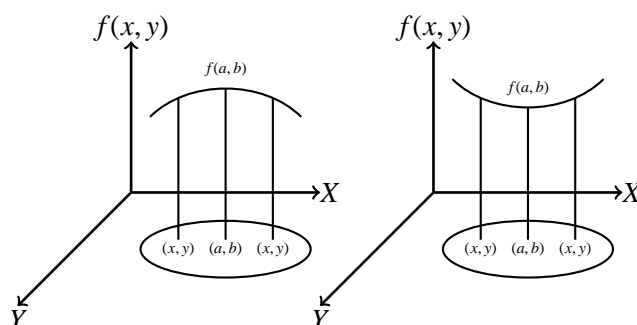
$$\begin{aligned} f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\ &\quad + \frac{1}{2!} \left((x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right) \\ e^x \cos y &= f(1, \frac{\pi}{4}) + (x - 1)f_x(1, \frac{\pi}{4}) + (y - \frac{\pi}{4})f_y(1, \frac{\pi}{4}) \\ &\quad + \frac{1}{2!} \left((x - 1)^2 f_{xx}(1, \frac{\pi}{4}) + 2(x - 1)(y - \frac{\pi}{4})f_{xy}(1, \frac{\pi}{4}) \right. \\ &\quad \left. + (y - \frac{\pi}{4})^2 f_{yy}(1, \frac{\pi}{4}) \right) + \dots \\ &= \frac{e}{\sqrt{2}} + (x - 1)\frac{e}{\sqrt{2}} + (y - \frac{\pi}{4})\frac{(-e)}{\sqrt{2}} \\ &\quad + \frac{1}{2} \left((x - 1)^2 \frac{e}{\sqrt{2}} + 2(x - 1)(y - \frac{\pi}{4})\frac{(-e)}{\sqrt{2}} + (y - \frac{\pi}{4})^2 \frac{(-e)}{\sqrt{2}} \right) + \dots \\ &= \frac{e}{\sqrt{2}} \left(1 + (x - 1) - (y - \frac{\pi}{4}) + \frac{1}{2}(x - 1)^2 - (y - \frac{\pi}{4})(x - 1) - \frac{1}{2}(y - \frac{\pi}{4})^2 \right) + \dots \end{aligned}$$

2.7 Maxima and Minima for functions of two variables

Definition. Let $f(x, y)$ be a continuous function defined in a closed and bounded domain D of the xy plane and let (a, b) be an interior point of D .

(i) $f(a, b)$ is said to be a local maximum value of $f(x, y)$ at the point (a, b) if there exists a neighborhood N of (a, b) such that $f(x, y) < f(a, b)$ for all points (x, y) in N .

(ii) $f(a, b)$ is said to be a local minimum if $f(x, y) > f(a, b)$ for all points (x, y) in N other than (a, b) .



Local maximum or local minimum values are called extreme values.

Stationary point of $f(x, y)$

A point (a, b) satisfying $f_x = 0$ and $f_y = 0$ is called a stationary point of $f(x, y)$.

Necessary conditions for Maximum or minimum

If $f(a, b)$ is an extreme value of $f(x, y)$ at (a, b) , then (a, b) is a stationary point of $f(x, y)$ if f_x and f_y exist at (a, b) and $f_x(a, b) = 0, f_y(a, b) = 0$.

Sufficient conditions for extreme values of $f(x, y)$

Let (a, b) be a stationary point of the differentiable function $f(x, y)$.

i.e., $f_x(a, b) = 0, f_y(a, b) = 0$.

Let us define $f_{xx}(a, b) = r, f_{xy}(a, b) = s, f_{yy}(a, b) = t$.

(i) If $rt - s^2 > 0$ and $r < 0$, then $f(a, b)$ is a maximum value.

(ii) If $rt - s^2 > 0$ and $r > 0$ then $f(a, b)$ is a minimum value.

(iii) If $rt - s^2 < 0$, then $f(a, b)$ is not an extreme value but (a, b) is a saddle point of

$f(x, y)$.

(iv) If $rt - s^2 = 0$, then no conclusion is possible and further investigation is required.

Working rule to find maxima and minima of $f(x, y)$

step (1). Find $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ and solve for $f_x = 0$ and $f_y = 0$ as simultaneous equations in x and y .

Let $(a, b), (a_1, b_1), \dots$ be the solutions which are stationary points of $f(x, y)$.

step (2). Find $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$.

step (3). Evaluate r, s, t at each stationary point.

At the point (a, b) if

(i) $rt - s^2 > 0$ and $r < 0$ then $f(a, b)$ is a maximum value of $f(x, y)$.

(ii) $rt - s^2 > 0$ and $r > 0$ then $f(a, b)$ is a minimum value of $f(x, y)$.

(iii) $rt - s^2 < 0$ then (a, b) is called a saddle point.

(iv) $rt - s^2 = 0$, no conclusion can be made, further investigation is required.

Critical Point

A point (a, b) is a critical point of $f(x, y)$ if $f_x = 0$ and $f_y = 0$ at (a, b) or f_x and f_y do not exist at (a, b) .

Maxima or Minima occur at a critical point.

Worked Examples

Example 2.72. Examine $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme values.

[Jun 2013, Jan 2012, May 2011, Jun 2010]

Solution. Given $f(x, y) = x^3 + y^3 - 12x - 3y + 20$.

$$\begin{aligned} f_x &= 3x^2 - 12 & f_y &= 3y^2 - 3 \\ r &= f_{xx} = 6x & s &= f_{xy} = 0 & t &= f_{yy} = 6y \end{aligned}$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$f_x = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$

$$f_y = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$$

Stationary points are $(2, 1)$, $(2, -1)$, $(-2, 1)$ and $(-2, -1)$.

$$rt - s^2 = 6x6y - 0 = 36xy.$$

$$\text{At } (2, 1), rt - s^2 = 36 \times 2 \times 1 = 72 > 0.$$

$$\text{At } (2, 1), r = 6(2) = 12 > 0.$$

$\therefore (2, 1)$ is a minimum point.

$$\text{Minimum value is } f(2, 1) = 8 + 1 - 24 - 3 + 20 = 29 - 27 = 2.$$

$$\text{At } (2, -1), rt - s^2 = 36 \times 2 \times -1 = -72 < 0.$$

$\therefore (2, -1)$ is a saddle point.

$$\text{At } (-2, 1), rt - s^2 = 36 \times (-2) \times 1 = -72 < 0.$$

$\therefore (-2, 1)$ is a saddle point.

$$\text{At } (-2, -1), rt - s^2 = 36 \times -2 \times -1 = 72 > 0.$$

$$r = 6(-2) = -12 < 0.$$

$\therefore (-2, -1)$ is a maximum point.

$$\text{Maximum value } f(-2, -1) = -8 - 1 + 24 + 3 + 20 = 47 - 9 = 38.$$

Example 2.73. Examine $f(x, y) = x^3 + y^3 - 3axy$ for maximum and minimum values. [Jan 1999]

Solution.

$$\text{Given } f(x, y) = x^3 + y^3 - 3axy.$$

$$f_x = 3x^2 - 3ay$$

$$f_y = 3y^2 - 3ax$$

$$r = f_{xx} = 6x$$

$$s = f_{xy} = -3a \quad t = f_{yy} = 6y.$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$f_x = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow ay = x^2 \Rightarrow y = \frac{x^2}{a}.$$

$$f_y = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow \frac{x^4}{a^2} - ax = 0.$$

$$x\left(\frac{x^3}{a^2} - a\right) = 0 \Rightarrow x(x^3 - a^3) = 0 \Rightarrow x = 0 \text{ or } x = a.$$

$$x = 0 \Rightarrow y = 0$$

$$x = a \Rightarrow y = \frac{a^2}{a} = a.$$

Stationary points are $(0, 0)$ and (a, a) .

$$\text{At } (0, 0), r - s^2 = 6x6y - 9a^2 = 36xy - 9a^2 = -9a^2 < 0.$$

$$r = 6x = 0.$$

\therefore No maximum or minimum at $(0, 0)$.

$\therefore (0, 0)$ is a saddle point.

$$\text{At } (a, a), r - s^2 = 36a^2 - 9a^2 = 27a^2 > 0 \text{ if } a \neq 0.$$

$$r = 6x = 6a.$$

If $a < 0, r < 0$.

$\therefore (a, a)$ is a maximum point if $a < 0$.

If $a > 0, r > 0$.

$\therefore (a, a)$ is a minimum point if $a > 0$.

$$\text{Maximum value} = a^3 + a^3 - 3a^3 = -a^3 \text{ if } a < 0$$

$$\text{Minimum value} = -a^3 \text{ if } a > 0.$$

Example 2.74. Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.

[Jan 2014]

Solution. Given: $f(x, y) = x^3y^2(1 - x - y) = x^3y^2 - x^4y^2 - x^3y^3$.

$$f_x = y^2[x^3(-1) + (1 - x - y)3x^2] = x^2y^2[-x + 3 - 3x - 3y]$$

$$= x^2y^2[-4x - 3y + 3].$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2.$$

$$r = f_{xx} = y^2[-12x^2 - 6xy + 6x].$$

$$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2.$$

$$t = f_{yy} = 2x^3 - 2x^4 - 6x^3y.$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$f_x = 0 \Rightarrow x^2y^2[-4x - 3y + 3] = 0.$$

$$\Rightarrow x = 0, y = 0, 4x + 3y = 3.$$

$$f_y = 0 \Rightarrow x^3 y(2 - 2x - 3y) = 0.$$

$$x = 0, y = 0, 2x + 3y = 2.$$

$$\text{Solving } 4x + 3y = 3 \quad (1)$$

$$2x + 3y = 2 \quad (2)$$

we get $2x = 1 \Rightarrow x = \frac{1}{2}$.

When $x = \frac{1}{2}$, (1) $\Rightarrow 2 + 3y = 3 \Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3}$.

\therefore The stationary points are $(0, 0), (\frac{1}{2}, \frac{1}{3})$.

At $(0, 0)$, $rt - s^2 = 0 \cdot 0 - 0 = 0$.

We can not say maximum or minimum. Further investigation is required.

At $(\frac{1}{2}, \frac{1}{3})$,

$$r = \frac{1}{9} \left(-12 \times \frac{1}{4} - 6 \times \frac{1}{2} \times \frac{1}{3} + 6 \times \frac{1}{2} \right)$$

$$= \frac{1}{9} (-3 - 1 + 3) = \frac{1}{9} (-1) = -\frac{1}{9}.$$

$$t = 2 \frac{1}{8} - 2 \frac{1}{16} - 6 \frac{1}{8} \frac{1}{3} = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}.$$

$$s^2 = \left(6 \frac{1}{4} \frac{1}{3} - 8 \frac{1}{8} \frac{1}{3} - 9 \frac{1}{4} \frac{1}{9} \right)^2 = \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4} \right)^2$$

$$= \left(\frac{1}{4} - \frac{1}{3} \right)^2 = \left(-\frac{1}{12} \right)^2 = \frac{1}{144}.$$

$$rt - s^2 = \left(-\frac{1}{9} \right) \left(-\frac{1}{8} \right) - \frac{1}{144}$$

$$= \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} > 0 \text{ and } r < 0.$$

$\therefore (\frac{1}{2}, \frac{1}{3})$ is a maximum point.

Maximum value is $f(\frac{1}{2}, \frac{1}{3}) = \frac{1}{8} \frac{1}{9} \left(1 - \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{72} \left(\frac{6-3-2}{6} \right) = \frac{1}{72 \times 6} = \frac{1}{432}$.

Example 2.75. Find the extreme values of the function $f(x, y) = x^4 + y^2 + x^2 y$.

Solution. Given: $f(x, y) = x^4 + y^2 + x^2 y$.

$$f_x = 4x^3 + 2xy. \quad r = f_{xx} = 12x^2 + 2y.$$

$$f_y = 2y + x^2. \quad s = f_{xy} = 2x. \quad t = f_{yy} = 2.$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$f_x = 0 \Rightarrow 4x^3 + 2xy = 0 \Rightarrow 2x(y + 2x^2) = 0 \Rightarrow x = 0, y + 2x^2 = 0 \Rightarrow y = -2x^2.$$

$$f_y = 0 \Rightarrow 2y + x^2 = 0 \Rightarrow -4x^2 + x^2 = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0.$$

When $x = 0, y = 0$.

\therefore The only stationary point is $(0, 0)$.

$$rt - s^2 = (12x^2 + 2y)2 - 4x^2.$$

At $(0, 0)$, $rt - s^2 = 0$.

We can not say maximum or minimum.

We shall investigate the nature of the function in a neighbourhood of $(0, 0)$.

We have $f(0, 0) = 0$. In a neighbourhood of $(0, 0)$ on the x -axis, take the point $(h, 0)$,

$$f(h, 0) = h^4 > 0.$$

On the y -axis take the point $(0, k)$, $f(0, k) = k^2 > 0$.

On $y = mx$, for any m , take the point (h, mh) .

$$f(h, mh) = h^4 + m^2h^2 + mh^3 = h^2[h^2 + m^2 + mh].$$

For the quadratic in m , $m^2 + mh + h^2$

$$\text{discriminant} = B^2 - 4AC = h^2 - 4.1.h^2 = -3h^2 < 0 \text{ if } m \neq 0.$$

$\therefore f(h, mh) > 0$ for all $m \neq 0$.

\therefore In a neighbourhood of $(0, 0)$ for all points (x, y) , $f(x, y) > 0$.

$\therefore f(0, 0)$ is minimum and the minimum value = 0.

Example 2.76. Find the maximum and minimum values of $x^2 - xy + y^2 - 2x + y$.

[Jun 2012, Jun 2010]

Solution. $f(x, y) = x^2 - xy + y^2 - 2x + y$.

$$f_x = 2x - y - 2.$$

$$f_y = -x + 2y + 1.$$

$$r = f_{xx} = 2.$$

$$s = f_{xy} = -1.$$

$$t = f_{yy} = 2.$$

For stationary points, solve $f_x = 0, f_y = 0$

$$2x - y - 2 = 0. \quad (1)$$

$$-x + 2y + 1 = 0. \quad (2)$$

$$(1) \Rightarrow y = 2x - 2.$$

$$(2) \Rightarrow -x + 2(2x - 2) + 1 = 0$$

$$-x + 4x - 4 + 1 = 0$$

$$3x - 3 = 0$$

$$3x = 3$$

$$x = 1.$$

$$\therefore y = 2 - 2 = 0.$$

The stationary point is $(1, 0)$.

$$rt - s^2 = 4 + 1 = 5 > 0 \quad \text{and} \quad r = 2 > 0.$$

$\therefore (1, 0)$ is a minimum point.

Minimum value of $f = 1 - 2 = -1$.

Example 2.77. Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

[Jan 2012]

Solution. $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$$f_x = 3x^2 - 3.$$

$$f_y = 3y^2 - 12.$$

$$r = f_{xx} = 6x.$$

$$s = f_{xy} = 0.$$

$$t = f_{yy} = 6y.$$

For stationary points, solve $f_x = 0, f_y = 0$.

$$3x^2 - 3 = 0. \quad 3y^2 - 12 = 0.$$

$$3x^2 = 3. \quad 3y^2 = 12.$$

$$x^2 = 1. \quad y^2 = 4.$$

$$x = \pm 1. \quad y = \pm 2.$$

The stationary points are $(1, 2), (-1, 2), (1, -2), (-1, -2)$.

At $(1, 2)$, $rt - s^2 = 36xy = 36 \times 1 \times 2 = 72 > 0$.

$$r = 6 > 0.$$

$\therefore (1, 2)$ is a minimum point.

Minimum value of $f = 1 + 8 - 3 - 24 + 20 = 2$.

At $(-1, 2)$, $rt - s^2 = 36xy = 36 \times (-1) \times 2 = -72 < 0$.

$\therefore (-1, 2)$ is a saddle point.

At $(1, -2)$, $rt - s^2 = 36xy = 36 \times 1 \times (-2) = -72 < 0$.

$\therefore (1, -2)$ is a saddle point.

At $(-1, -2)$, $rt - s^2 = 36xy = 72 > 0$.

$$r = 6 \times (-1) = -6 < 0.$$

$\therefore (-1, -1)$ is a maximum point.

Maximum value of $f = -1 - 8 + 3 + 24 + 20 = 38$.

Maxima = 38.

Minima = 2.

Example 2.78. Test for maxima and minima of the function $f(x, y) = x^3y^2(6 - x - y)$.

[Jan 2013]

Solution. $f(x, y) = x^3y^2(6 - x - y)$

$$= 6x^3y^2 - x^4y^2 - x^3y^3.$$

$$f_x = 18x^2y^2 - 4x^3y^2 - 3x^2y^3.$$

$$f_y = 12x^3y - 2x^4y - 3x^3y^2.$$

$$r = f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3.$$

$$s = f_{xy} = 36x^2y - 8x^3y - 9x^2y^2.$$

$$t = f_{yy} = 12x^3 - 2x^4 - 6x^3y.$$

For stationary points, $f_x = 0, f_y = 0$.

$$f_x = 0 \Rightarrow 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(18 - 4x - 3y) = 0$$

$$\text{i.e., } 4x + 3y = 18. \quad (1)$$

$$f_y = 0 \Rightarrow 12x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(12 - 2x - 3y) = 0$$

$$2x + 3y = 12. \quad (2)$$

$$(1) - (2) \Rightarrow 2x = 6$$

$$x = 3.$$

$$(1) \Rightarrow 12 + 3y = 18$$

$$3y = 6$$

$$y = 2.$$

The stationary point is $(3, 2)$

At $(3, 2)$,

$$\begin{aligned}
 r &= 36 \times 3 \times 4 - 12 \times 9 \times 4 - 6 \times 3 \times 8 \\
 &= 432 - 432 - 144 \\
 &= -144 < 0.
 \end{aligned}$$

$$\begin{aligned}
 t &= 12 \times 9 - 2 \times 81 - 6 \times 27 \times 2 \\
 &= 108 - 162 - 324 \\
 &= -378.
 \end{aligned}$$

$$\begin{aligned}
 s &= 34 \times 9 \times 2 - 8 \times 27 \times 2 - 9 \times 9 \times 4 \\
 &= 612 - 432 - 324 \\
 &= -144.
 \end{aligned}$$

$$\begin{aligned}
 rt - s^2 &= (-144)(-378) - (-144)^2 \\
 &= 54432 - 20736 \\
 &= 33696 > 0
 \end{aligned}$$

Since $rt - s^2 > 0$ and $r < 0$, $(3, 2)$ is a maximum point.

\therefore Maximum value of $f = 27 \times 4(6 - 3 - 2) = 108$.

Example 2.79. Examine for minimum and maximum values $\sin x + \sin y + \sin(x + y)$.

Solution. We have $f(x, y) = \sin x + \sin y + \sin(x + y)$.

$$f_x = \cos x + \cos(x + y) \quad f_y = \cos y + \cos(x + y).$$

$$r = f_{xx} = -\sin x - \sin(x + y)$$

$$s = f_{xy} = -\sin(x + y)$$

$$t = f_{yy} = -\sin y - \sin(x + y).$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$\text{i.e., } \cos x + \cos(x + y) = 0. \quad (1)$$

$$\cos y + \cos(x + y) = 0. \quad (2)$$

$$(1) - (2) \implies \cos x - \cos y = 0.$$

$$\text{i.e., } \cos x = \cos y$$

$$\implies x = y$$

$$\text{Now (1)} \implies \cos x + \cos 2x = 0$$

$$\text{i.e., } \cos 2x = -\cos x.$$

$$\cos 2x = \cos(\pi - x)$$

$$\implies 2x = \pi - x.$$

$$\text{i.e., } 3x = \pi$$

$$x = \frac{\pi}{3}.$$

When $x = \frac{\pi}{3}, y = \frac{\pi}{3}$.

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a stationary point.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = -\sin \frac{\pi}{3} - \sin \frac{2\pi}{3} = \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0.$$

$$s = -\sin \frac{2\pi}{3} = -\frac{\sqrt{3}}{2}, t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}.$$

$$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

Since $rt - s^2 > 0$ and $r < 0$, $f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\therefore \text{Maximum value} = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

Example 2.80. Find the maximum and minimum values of $\sin x \sin y \sin(x + y)$, $0 < x, y < \pi$. [Jan 1997]

Solution. Given $f(x, y) = \sin x \sin y \sin(x + y)$.

$$f_x = \sin y [\sin x \cos(x + y) + \sin(x + y) \cos x] = \sin y \sin(2x + y).$$

$$r = f_{xx} = 2 \sin y \cos(2x + y).$$

$$f_y = \sin x [\sin y \cos(x + y) + \sin(x + y) \cos y] = \sin x \sin(x + 2y).$$

$$s = f_{xy} = \sin x \cos(x + 2y) + \sin(x + 2y) \cos x = \sin(2x + 2y).$$

$$t = f_{yy} = 2 \sin x \cos(x + 2y).$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$f_x = 0 \Rightarrow \sin y \sin(2x + y) = 0.$$

$$f_y = 0 \Rightarrow \sin x \sin(x + 2y) = 0.$$

Since, $x, y \neq 0 \& \neq \pi \Rightarrow \sin x \neq 0, \sin y \neq 0$.

$$\therefore \sin(2x + y) = 0 \text{ and } \sin(x + 2y) = 0.$$

Since, $0 < x < \pi, 0 < 2x < 2\pi$

$$0 < y < \pi \Rightarrow 0 < 2x + y < 3\pi.$$

Similarly $0 < x + 2y < 3\pi$. Since, $\sin(2x + y) = 0 \Rightarrow 2x + y = \pi \text{ or } 2\pi$.

Similarly $x + 2y = \pi \text{ or } 2\pi$.

$$\text{If } 2x + y = \pi \quad (1)$$

$$\text{and } x + 2y = \pi \quad (2)$$

then $x - y = 0 \Rightarrow x = y$.

$$\therefore (1) \Rightarrow 3x = \pi \Rightarrow x = \frac{\pi}{3}.$$

$$\therefore y = \frac{\pi}{3}.$$

\therefore one stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

If $2x + y = \pi$ and $x + 2y = 2\pi$ then, $x - y = -\pi \Rightarrow x = y - \pi$.

$$\therefore 2(y - \pi) + y = \pi \Rightarrow 3y - 2\pi = \pi \Rightarrow 3y = 3\pi \Rightarrow y = \pi,$$

which is not admissible since $y \neq \pi$.

Similarly, $2x + y = 2\pi$ and $x + 2y = \pi$ is also not possible.

Now take $2x + y = 2\pi$ and $x + 2y = 2\pi$

$$\Rightarrow x - y = 0 \Rightarrow x = y$$

$$\Rightarrow 3x = 2\pi \Rightarrow x = \frac{2\pi}{3}$$

$$\therefore y = \frac{2\pi}{3}.$$

\therefore Another stationary point is $(\frac{2\pi}{3}, \frac{2\pi}{3})$.

At $(\frac{\pi}{3}, \frac{\pi}{3})$.

$$r = 2 \sin \frac{\pi}{3} \cos \pi = 2 \frac{\sqrt{3}(-1)}{2} < 0.$$

$$s = \sin \frac{4\pi}{3} = \sin \left(\pi + \frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

$$t = 2 \sin \frac{\pi}{3} \cos \pi = \frac{2 \sqrt{3}(-1)}{2} = -\sqrt{3}.$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2} \right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

$\therefore (\frac{\pi}{3}, \frac{\pi}{3})$ is a maximum point.

$$\begin{aligned} \text{Maximum value} &= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} \cdot \sin \frac{\pi}{3} \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \sin \left(\pi - \frac{\pi}{3} \right) \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}. \end{aligned}$$

At $(\frac{2\pi}{3}, \frac{2\pi}{3})$

$$r = 2 \sin \frac{2\pi}{3} \cos \frac{6\pi}{3} = 2 \frac{\sqrt{3}}{2} \cdot 1 = \sqrt{3} > 0.$$

$$t = 2 \sin \frac{2\pi}{3} \cos \frac{6\pi}{3} = \sqrt{3}.$$

$$s = \sin \frac{8\pi}{3} = \sin \left(3\pi - \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}.$$

$$rt - s^2 = \sqrt{3} \sqrt{3} - \left(\frac{\sqrt{3}}{2} \right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

$\therefore (\frac{2\pi}{3}, \frac{2\pi}{3})$ is a minimum point.

$$\text{Minimum value} = f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \sin \frac{2\pi}{3} \sin \frac{2\pi}{3} \sin \frac{4\pi}{3} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{3}}{2} \right) = \frac{-3\sqrt{3}}{8}.$$

Example 2.81. In a plane triangle, find the maximum value of $\cos A \cos B \cos C$.

[Jan 2000]

Solution. The angles of the $\Delta^{le} ABC$ satisfy $0 < A, B, C < \pi$ and $A + B + C = \pi$,

$\implies C = \pi - (A + B)$. Replacing C , we get

$$f(A, B) = \cos A \cos B \cos(\pi - (A + B)) = -\cos A \cos B \cos(A + B), \quad 0 < A, B < \pi.$$

$$f_A = -\cos B [\cos A (-\sin(A + B)) + \cos(A + B)(-\sin A)] = \cos B \sin(2A + B).$$

$$f_B = -\cos A[-\sin(A + 2B)] = \cos A \sin(A + 2B).$$

$$r = f_{AA} = 2 \cos B \cos(2A + B).$$

$$s = f_{AB} = \cos A \cos(A + 2B) + \sin(A + 2B)(-\sin A) = \cos(2A + 2B).$$

$$t = f_{BB} = 2 \cos A \cos(A + 2B).$$

For stationary points, solve $f_A = 0, f_B = 0$.

$$\text{i.e., } \cos B \sin(2A + B) = 0 \text{ and } \cos A \sin(A + 2B) = 0.$$

$$\cos B = 0 \text{ or } \sin(2A + B) = 0 \text{ and } \cos A = 0 \text{ or } \sin(A + 2B) = 0$$

$$\Rightarrow B = \frac{\pi}{2} \text{ or } 2A + B = \pi \text{ or } 2\pi$$

and

$$A = \frac{\pi}{2} \text{ or } A + 2B = \pi \text{ or } 2\pi.$$

Different possibilities

case (i) Let $B = \frac{\pi}{2}$ and $A = \frac{\pi}{2}$.

$$\Rightarrow A + B = \pi$$

$$\Rightarrow C = 0 \text{ not possible.}$$

case (ii) If $B = \frac{\pi}{2}$ and $A + 2B = \pi$.

$$\Rightarrow A + \pi = \pi \Rightarrow A = 0 \text{ not possible.}$$

case (iii) If $B = \frac{\pi}{2}$ and $A + 2B = 2\pi$.

$$\Rightarrow A + \pi = 2\pi \Rightarrow A = \pi \text{ not possible.}$$

case (iv) $A = \frac{\pi}{2}, 2A + B = \pi$.

$$\Rightarrow \pi + B = \pi \Rightarrow B = 0 \text{ not possible.}$$

case (v) $A = \frac{\pi}{2}, 2A + B = 2\pi \Rightarrow B = \pi$ not possible.

case (vi) If $2A + B = \pi, A + 2B = \pi$.

$$\text{Subtracting } A - B = 0 \Rightarrow A = B.$$

$$\therefore 3A = \pi \Rightarrow A = \frac{\pi}{3}.$$

$$\Rightarrow B = \frac{\pi}{3}, C = \frac{\pi}{3}.$$

case (vii) If $2A + B = \pi$ and $A + 2B = 2\pi \Rightarrow A - B = -\pi$ not possible.

Finally $2A + B = 2\pi$ and $A + 2B = 2\pi \Rightarrow A - B = 0 \Rightarrow A = B$.

$$3A = 2\pi \Rightarrow A = \frac{2\pi}{3}, B = \frac{2\pi}{3}.$$

$$A + B = \frac{2\pi}{3} + \frac{2\pi}{3} = 4\frac{\pi}{3} > \pi \text{ not possible.}$$

\therefore The only stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$$r = 2 \cos \frac{\pi}{3} \cos \pi = 2 \frac{1}{2}(-1) < 0.$$

$$t = 2 \cos \frac{\pi}{3} \cos \pi = -1.$$

$$s = \cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right) = -\frac{1}{2}.$$

$$rt - s^2 = (-1)(-1) - \left(-\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0.$$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a maximum point.

\therefore Maximum value of f is $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{3} \cos \frac{\pi}{3} = \frac{1}{8}$.

Example 2.82. A flat circular plate is heated so that the temperature at any point (x, y) is $U(x, y) = x^2 + 2y^2 - x$. Find the coldest point on the plate. [Jan 2005]

Solution. Given $U = x^2 + 2y^2 - x$.

$$U_x = 2x - 1 \qquad U_y = 4y.$$

$$r = U_{xx} = 2. \qquad s = U_{xy} = 0. \qquad t = U_{yy} = 4.$$

$$rt - s^2 = 8 > 0 \text{ and } r = 2 > 0.$$

\therefore All points are minimum points.

At minimum, $U_x = 0$ and $U_y = 0$.

$$U_x = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}.$$

$$U_y = 0 \Rightarrow 4y = 0 \Rightarrow y = 0.$$

\therefore The minimum point is $\left(\frac{1}{2}, 0\right)$.

\therefore The coldest point on the plate is $\left(\frac{1}{2}, 0\right)$.

2.8 Constrained Maxima and Minima - Lagrange's Method

Lagrange's Method

Let $f(x, y, z)$ be the function for which the extreme values are to be found subject to the condition

$$\phi(x, y, z) = 0. \quad (1)$$

Construct the auxiliary function $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$, where λ is an undetermined parameter independent of x, y, z which is called the Lagrange's multiplier. Any relative extremum of $f(x, y, z)$ subject to (1) must occur at a stationary point of $F(x, y, z)$.

The stationary points of F are given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = 0$.

$$\Rightarrow f_x + \lambda\phi_x = 0, f_y + \lambda\phi_y = 0, f_z + \lambda\phi_z = 0, \phi(x, y, z) = 0.$$

$$\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} = \frac{f_z}{\phi_z} = -\lambda \text{ and } \phi(x, y, z) = 0.$$

Solving these equations we can find the values of x, y, z which are stationary points of F and the values of f at these points give the maximum and minimum values of $f(x, y, z)$.

Worked Examples

Example 2.83. Find the maximum value of $x^m y^n z^p$ subject to $x + y + z = a$.

[Jan 2009]

Solution. Let $f = x^m y^n z^p$.

$$\phi = x + y + z - a = 0. \quad (1)$$

We have to maximise f subject to (1).

Let $F = f + \lambda\phi$ where λ is the Lagrange's multiplier.

$$F = x^m y^n z^p + \lambda(x + y + z - a).$$

$$F_x = mx^{m-1}y^n z^p + \lambda, F_y = nx^m y^{n-1} z^p + \lambda, F_z = px^m y^n z^{p-1} + \lambda.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$$F_x = 0 \Rightarrow mx^{m-1}y^n z^p = -\lambda.$$

$$F_y = 0 \Rightarrow nx^m y^{n-1} z^p = -\lambda.$$

$$F_x = 0 \Rightarrow px^m y^n z^{p-1} = -\lambda.$$

From the above three equations we get

$$mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}.$$

Dividing by $x^m y^n z^p$ we get, $\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$.

$$x = \frac{ma}{m+n+p}, y = \frac{na}{m+n+p}, z = \frac{pa}{m+n+p}.$$

The stationary point is $\left(\frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}\right)$.

$$\therefore \text{Max. value of } f = \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

Example 2.84. Find the minimum value of $x^2 y z^3$ subject to $2x + y + 3z = a$.

[Jan 2007]

Solution. Given $f = x^2 y z^3$.

$$\phi = 2x + y + 3z - a = 0. \quad (1)$$

Let $F = f + \lambda \phi$ where λ is the Lagrange's multiplier.

$$F = x^2 y z^3 + \lambda(2x + y + 3z - a).$$

$$F_x = 2xyz^3 + 2\lambda, F_y = x^2 z^3 + \lambda, F_z = 3x^2 y z^2 + 3\lambda.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$$F_x = 0 \Rightarrow 2xyz^3 + 2\lambda = 0 \Rightarrow xyz^3 = -\lambda.$$

$$F_y = 0 \Rightarrow x^2 z^3 = -\lambda.$$

$$F_z = 0 \Rightarrow 3x^2 y z^2 + 3\lambda = 0 \Rightarrow x^2 y z^2 = -\lambda.$$

Therefore

$$xyz^3 = x^2 z^3 = x^2 y z^2$$

$$xyz^3 = x^2z^3 \Rightarrow y = x.$$

$$x^2z^3 = x^2yz^2 \Rightarrow y = z.$$

$$x = y = z.$$

$$(1) \Rightarrow 2x + x + 3x = a \Rightarrow 6x = a \Rightarrow x = \frac{a}{6} = y = z.$$

\therefore The stationary point is $\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$.

$$\text{Minimum value} = \left(\frac{a}{6}\right)^2 \frac{a}{6} \left(\frac{a}{6}\right)^3 = \frac{a^6}{6^6} = \left(\frac{a}{6}\right)^6.$$

Example 2.85. If $u = x^2 + y^2 + z^2$ where $ax + by + cz - p = 0$, find the stationary value of u . [Jan 2006]

Solution. Given $f = x^2 + y^2 + z^2$.

$$\phi = ax + by + cz - p = 0. \quad (1)$$

Let $F = f + \lambda\phi$ where λ is the Lagrange's multiplier.

$$F = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p).$$

$$F_x = 2x + a\lambda, F_y = 2y + b\lambda, F_z = 2z + c\lambda.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$$2x + \lambda a = 0 \Rightarrow x = \frac{-a\lambda}{2} \Rightarrow ax = \frac{-a^2\lambda}{2}.$$

Similarly

$$by = \frac{-b^2\lambda}{2}, cz = \frac{-c^2\lambda}{2}.$$

$$(1) \Rightarrow \frac{-a^2\lambda}{2} - \frac{b^2\lambda}{2} - \frac{c^2\lambda}{2} = p$$

$$\frac{a^2 + b^2 + c^2}{2} \lambda = -p \Rightarrow \lambda = \frac{-2p}{a^2 + b^2 + c^2}.$$

$$\therefore x = \frac{ap}{a^2 + b^2 + c^2}, y = \frac{bp}{a^2 + b^2 + c^2}, z = \frac{cp}{a^2 + b^2 + c^2}.$$

Stationary value of u is

$$\begin{aligned} u &= \frac{a^2 p^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2 p^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2 p^2}{(a^2 + b^2 + c^2)^2} \\ &= \frac{p^2(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}. \end{aligned}$$

Example 2.86. The temperature T at any point (x, y, z) in space is $T = 400xyz^2$.

Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

[Jan 2005]

Solution. We have to maximize

$$T = 400xyz^2 \quad (1)$$

$$\text{subject to} \quad \phi = x^2 + y^2 + z^2 - 1 = 0. \quad (2)$$

Consider $F = T + \lambda\phi$, where λ is the Lagrange multiplier.

$$F = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1).$$

$$F_x = 400yz^2 + 2\lambda x, F_y = 400xz^2 + 2\lambda y, F_z = 400xyz + 2\lambda z.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$$F_x = 0 \Rightarrow 400yz^2 + 2\lambda x = 0 \Rightarrow -\lambda = \frac{200yz^2}{x}.$$

$$F_y = 0 \Rightarrow 400xz^2 + 2\lambda y = 0 \Rightarrow -\lambda = \frac{200xz^2}{y}.$$

$$F_z = 0 \Rightarrow 800xyz + 2\lambda z = 0 \Rightarrow -\lambda = 400xy.$$

$$\therefore \frac{200yz^2}{x} = \frac{200xz^2}{y} = 400xy.$$

$$\text{Taking } \frac{200yz^2}{x} = \frac{200xz^2}{y} \text{ we get } x^2 = y^2 \Rightarrow y = \pm x.$$

$$\text{Taking } \frac{200yz^2}{x} = 400xy \text{ we get } \frac{z^2}{x} = 2x \Rightarrow z^2 = 2x^2 \Rightarrow z = \pm \sqrt{2}x.$$

$$\text{Taking } \frac{200xz^2}{y} = 400xy \text{ we get } \frac{z^2}{y} = 2y \Rightarrow z^2 = 2y^2 \Rightarrow z = \pm \sqrt{2}y.$$

Substituting in $x^2 + y^2 + z^2 = 1$ we get

$$x^2 + x^2 + 2x^2 = 1 \Rightarrow 4x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

$$\therefore y = \pm \frac{1}{2}, z = \pm \sqrt{2} \frac{1}{2} = \pm \frac{1}{\sqrt{2}}.$$

The stationary points are given by $x = \pm \frac{1}{2}, y = \pm \frac{1}{2}, z = \pm \frac{1}{\sqrt{2}}$.

To have maximum value, we must have xy positive.

\therefore The points are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}), (\frac{1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}}), (\frac{-1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}), (\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{\sqrt{2}})$.

$$\therefore \text{Maximum } T = 400 \frac{1}{2} \frac{1}{2} \frac{1}{2} = 50^\circ C.$$

Example 2.87. Find the shortest and longest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$ using Lagrange's method of constrained maxima and minima. [Jun 2011, Jan 2002]

Solution. Let $P(x, y, z)$ be any point on the sphere.

Let A be the point $(1, 2, -1)$.

$$AP = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}.$$

$$\text{Let } f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2. \quad (1)$$

AP is minimum or maximum if f is minimum or maximum.

\therefore The problem is now reduced to minimise or maximise f subject to

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 24 = 0.$$

Consider the auxiliary function

$F = f + \lambda \phi = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$ where λ is the Lagrange's multiplier.

$$F_x = 2(x-1) + 2\lambda x, F_y = 2(y-2) + 2\lambda y, F_z = 2(z+1) + 2\lambda z.$$

To find the stationary points solve

$$F_x = 0, F_y = 0, F_z = 0, \phi = 0.$$

$$F_x = 0 \Rightarrow 2(x-1) + 2\lambda x = 0 \Rightarrow x-1 = -\lambda x \Rightarrow -\lambda = \frac{x-1}{x} = 1 - \frac{1}{x}.$$

$$F_y = 0 \Rightarrow 2(y-2) + 2\lambda y = 0 \Rightarrow y-2 = -\lambda y \Rightarrow -\lambda = \frac{y-2}{y} = 1 - \frac{2}{y}.$$

$$F_z = 0 \Rightarrow 2(z+1) + 2\lambda z = 0 \Rightarrow z+1 = -\lambda z \Rightarrow -\lambda = \frac{z+1}{z} = 1 + \frac{1}{z}.$$

$$\therefore 1 - \frac{1}{x} = 1 - \frac{2}{y} = 1 + \frac{1}{z}.$$

Taking $1 - \frac{1}{x} = 1 - \frac{2}{y}$ we get $\frac{1}{x} = \frac{2}{y} \Rightarrow y = 2x$.

Taking $1 - \frac{1}{x} = 1 + \frac{1}{z}$ we get $z = -x$.

Taking $1 - \frac{2}{y} = 1 + \frac{1}{z}$ we get $z = \frac{-y}{2} = -x$.

We have $x^2 + y^2 + z^2 = 24 \Rightarrow x^2 + 4x^2 + x^2 = 24 \Rightarrow 6x^2 = 24 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

When $x = 2, y = 4, z = -2$, the first point is $(2, 4, -2)$. Let this point be P_1 .

When $x = -2, y = -4, z = 2$, the second point is $(-2, -4, 2)$. Let this point be P_2 .

$$P_1A = \sqrt{1+4+1} = \sqrt{6}, P_2A = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$$

\therefore Shortest distance = $\sqrt{6}$

Longest distance = $3\sqrt{6}$.

Example 2.88. A rectangular box open at the top is to have a volume of $32cc$. Find the dimensions of the box which requires least amount of material for its construction. [Jun 2012, Dec 2011, Jun 2010, Jan 2005]

Solution. Let the dimensions of the box be Length = x , Breadth = y , height = z .

Given: Volume = $32cc$.

$$\Rightarrow xyz = 32, x, y, z > 0. \quad (1)$$

We have to minimize the amount of material used for the construction of the box.

Let S be the surface area of the box whose top is open

$$\therefore S = xy + 2xz + 2yz \quad (2)$$

By Lagrange's method

Let $F = s + \lambda\phi = xy + 2yz + 2xz + \lambda(xyz - 32)$ where λ is the Lagrange's multiplier.

$$F_x = y + 2z + \lambda yz, F_y = x + 2z + \lambda xz, F_z = 2y + 2x + \lambda xy.$$

To find the stationary points, solve

$$F_x = 0, F_y = 0, F_z = 0, \phi = 0$$

$$F_x = 0 \Rightarrow y + 2z + \lambda yz = 0$$

$$\Rightarrow y + 2z = -\lambda yz$$

$$\begin{aligned}
 &\Rightarrow \frac{y}{yz} + \frac{2z}{yz} = -\lambda \\
 &\Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda
 \end{aligned} \tag{3}$$

$$F_y = 0 \Rightarrow x + 2z + \lambda xz = 0$$

$$\Rightarrow x + 2z = -\lambda xz$$

$$\text{i.e., } xy + 2yz = -\lambda xyz.$$

$$\begin{aligned}
 &\Rightarrow \frac{xy}{xyz} + \frac{2yz}{xyz} = -\lambda \\
 &\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda
 \end{aligned} \tag{4}$$

$$F_z = 0 \Rightarrow 2y + 2x + \lambda xy = 0$$

$$\Rightarrow 2x + 2y = -\lambda xy$$

$$2xz + 2yz = -\lambda xyz.$$

$$\begin{aligned}
 &\Rightarrow \frac{2xz}{xyz} + \frac{2yz}{xyz} = -\lambda \\
 &\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda
 \end{aligned} \tag{5}$$

$$(3) - (4) \Rightarrow$$

$$\begin{aligned}
 &\frac{2}{y} - \frac{2}{x} = 0 \\
 &\frac{1}{y} = \frac{1}{x} \\
 &\Rightarrow x = y
 \end{aligned} \tag{6}$$

$$(3) - (5) \Rightarrow$$

$$\begin{aligned}
 &\frac{1}{z} - \frac{2}{x} = 0 \\
 &\frac{1}{z} = \frac{2}{x} \\
 &\Rightarrow x = 2z
 \end{aligned} \tag{7}$$

From (6) and (7) we obtain $x = y = 2z$.

$$(1) \Rightarrow 2z \cdot 2z \cdot z = 32 \Rightarrow 4z^3 = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2.$$

$\therefore x = 4, y = 4.$

The stationary point is $(4, 4, 2).$

The dimensions are $4\text{cm}, 4\text{cm}, 2\text{cm}.$

Example 2.89. Find the dimensions of the rectangular box open at the top of maximum capacity whose surface area is 432 sq.m. [Jun 2013]

Solution. Let the dimensions of the box be $x, y, z.$

Given, surface area = 432.

$$xy + 2xz + 2yz = 432 \quad (1)$$

Let V be the volume of the box.

We have to maximize $V.$

$$V = xyz \quad (2)$$

By lagrange's method

$F = V + \lambda\phi$, where λ is the lagrange multiplier.

$$F = xyz + \lambda(xy + 2xz + 2yz - 432).$$

$$F_x = yz + \lambda(y + 2z).$$

$$F_y = xz + \lambda(x + 2z).$$

$$F_z = xy + \lambda(2x + 2y).$$

$$F_\lambda = xy + 2xz + 2yz - 432.$$

For stationary points, $F_x = 0, F_y = 0, F_z = 0, F_\lambda = 0.$

$$F_x = 0 \Rightarrow yz + \lambda(y + 2z) = 0$$

$$\Rightarrow xyz + \lambda(xy + 2xz) = 0. \quad (1)$$

$$F_y = 0 \Rightarrow xz + \lambda(x + 2z) = 0$$

$$\Rightarrow xyz + \lambda(xy + 2yz) = 0. \quad (2)$$

$$F_z = 0 \Rightarrow xy + \lambda(2x + 2y) = 0$$

$$xyz + \lambda(2xz + 2yz) = 0. \quad (3)$$

$$(1) + (2) + (3) \Rightarrow$$

$$3xyz + \lambda(2xy + 4xz + 4yz) = 0$$

$$3xy + 2\lambda(xy + 2xz + 2yz) = 0$$

$$3xyz + 2\lambda \times 432 = 0$$

$$3xyz = -864\lambda$$

$$\lambda = -\frac{3xyz}{864} = -\frac{xyz}{288}$$

Substituting the value of λ in $F_x = 0$ we get

$$yz - \frac{xyz}{288}(y + 2z) = 0$$

$$1 - \frac{x}{288}(y + 2z) = 0$$

$$1 = \frac{x}{288}(y + 2z)$$

$$xy + 2xz = 288 \quad (4)$$

$$F_y = 0 \Rightarrow xz - \frac{xyz}{288}(x + 2z) = 0$$

$$1 - \frac{y}{288}(x + 2z) = 0$$

$$1 - \frac{y}{288}(x + 2z) = 0, 1 = \frac{y}{288}(x + 2z)$$

$$xy + 2yz = 288 \quad (5)$$

$$F_z = 0 \Rightarrow xy - \frac{xyz}{288}(2x + 2y) = 0$$

$$1 - \frac{z}{288}(2x + 2y) = 0, 1 = \frac{z}{288}(2x + 2y)$$

$$2xz + 2yz = 288 \quad (6)$$