Rajiv Gandhi University of Knowledge Technologies- AP.

(Catering to the Educational Needs of Gifted Rural Youth of Andhra Pradesh)

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5 Partial derivatives of Homogeneous functions, Euler's theorem, and Harmonic function

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Functions of several Variables:

A function which depends on more than one independent variables are called functions of several variables.

Example: Volume of the triangle $V = \pi r^2 h$

Limit of functions of two variables: Let f(x, y) be a function of two independent variables x and y. f(x,y) is said to have the limit L as (x, y) tends to (x_0, y_0), if for every $\epsilon > 0$ (however small it may be), there exist $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ whenever $|x - x_0| < \delta, |y - y_0| < \delta$ and it is denoted by $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ (or) $\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = L$

Note:

1. The limit of the function f(x, y) if exist, is always finite and unique. If not unique then limit does not exist.
2. In two variables function f(x, y) we can approach the point (x_0, y_0) from any direction/path such as by taking a line y=mx, y=mx^2, curve 2y-3x=mx^3, etc. If limit does not unique through any one of the path/directions, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.
3. If $x = r\cos\theta$ & $y = r\sin\theta$ then $x^2 + y^2 = r^2$ *and* $\theta = \tan^{-1}\frac{y}{x}$ then the definition of the limit is $|f(r,\theta) - L| < \epsilon$ whenever $r < \delta$, *indipendent of* θ

   $\lim_{r\to 0} f(x,y) = L$

Properties of Limits of Functions of Two Variables:

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$ then the following rules hold:

1. $\lim_{(x,y)\to(x_0,y_0)} (f \pm g) = L \pm M$
2. $\lim_{(x,y)\to(x_0,y_0)} (f * g) = L * M$
3. $\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right) = \frac{L}{M}$ *where* $M \neq 0$
4. $\lim_{(x,y)\to(x_0,y_0)} (kf) = kL$
5. $\lim_{(x,y)\to(x_0,y_0)} (f)^{\frac{r}{s}} = L^{\frac{r}{s}}$ *where* $s \neq 0$

1. Find the limit $\lim_{(x,y)\to(1,2)} \left(\frac{2x^2y}{x^2+y^2+1}\right)$

Sol: $\lim_{\substack{x\to 1 \\ y\to 2}} \left(\frac{2x^2y}{x^2+y^2+1}\right) = \lim_{y\to 2} \left(\frac{2(1)^2y}{(1)^2+y^2+1}\right) = \lim_{y\to 2} \left(\frac{2y}{y^2+2}\right) = \frac{4}{6} = \frac{2}{3}$

2. Find the limit $\lim_{(x,y)\to(0,0)} \left(\frac{x^3y^2+x^2y^3-3}{2-xy}\right)$

Sol: $\lim_{\substack{x\to 0 \\ y\to 0}} \left(\frac{x^3y^2+x^2y^3-3}{2-xy}\right) = \lim_{y\to 0} \left(\frac{0+0-3}{2-0}\right) = \lim_{y\to 0} \left(\frac{-3}{2}\right) = -\frac{3}{2}$

3. Check the existence of the limit of $f(x,y)=\frac{x^8y^3}{(x^4+y^3)^3}$ at (0,0).

Sol: Given that $f(x,y)=\frac{x^8y^3}{(x^4+y^3)^3}$

$\boxed{x^4+y^3=0 \Rightarrow y^3=mx^4}$

The limit of f(x,y) along the curve $y=mx^4$ *and* $m\in R$ is

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^8y^3}{(x^4+y^3)^3}\right)=\lim_{x\to 0}\left(\frac{x^8(mx^4)}{(x^4+mx^4)^3}\right)=\lim_{x\to 0}\left(\frac{m}{(1+m)^3}\right)=\frac{m}{(1+m)^3}$$

It is dependent on m and hence limit is not unique. The limit of the f(x,y) does not exist.

4. Check whether the limit of the function $f(x,y)=\frac{x(x^2-y^2)}{x+y}$ exist or not at (0,0).

Sol: Given that $f(x,y)=\frac{x(x^2-y^2)}{x+y}$

$x+y=0 \Rightarrow y=mx$

Take the limit along the curve $y=mx$ *and* $m\in R$ is

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x(x^2-y^2)}{x+y}\right)=\lim_{x\to 0}\left(\frac{x(x^2-(mx)^2)}{x+mx}\right)=\lim_{x\to 0}\left(\frac{x^2(1-m^2)}{1+m}\right)=0=L$$

The limit value exists and unique for the path y=mx. But by the definition, limit should be existed for all paths. Here we don't consider the other paths (we have not checked) Thus we cannot say by this test that limit exists or not.

To check whether the limit exist or not, we can proceed always like the following way.

Let $\epsilon>0, \exists\, \partial>0 \ni |x-x_0|<\delta, |y-y_0|<\delta \Rightarrow |x-0|<\delta, |y-0|<\delta \Rightarrow |x|<\delta, |y|<\delta$

$$|f(x,y)-L|=\left|\frac{x(x^2-y^2)}{x+y}-0\right|=|x|\left|\frac{(x^2-y^2)}{x+y}\right|=|x|\left|\frac{(x-y)x+y}{x+y}\right|$$

$$|f(x,y)-0|=|x||x-y|\le |x|\{|x|+|y|\}=\delta(2\delta)=\epsilon$$

$$|f(x,y)-0|<\epsilon \text{ when ever } |x-0|<\delta, |y-0|<\delta$$

$$\boxed{\begin{gathered}|xy|=|x||y|\\ |x-y|\le|x|+|y|\end{gathered}}$$

Hence limit of the function exists and is 0. i.e $\lim_{\substack{x\to 0\\ y\to 0}}\frac{x(x^2-y^2)}{x+y}=0$

5. Prove that $\lim_{(x,y)\to(0,0)}\left(y\sin\left(\frac{1}{x}\right)+x\sin\left(\frac{1}{y}\right)\right)=0$

Sol: Let $\epsilon>0, \exists\, \partial>0 \ni |x-x_0|<\delta, |y-y_0|<\delta \Rightarrow |x-0|<\delta, |y-0|<\delta \Rightarrow |x|<\delta, |y|<\delta$

$$|f(x,y)-L|=|f(x,y)-0|=\left|y\sin\left(\frac{1}{x}\right)+x\sin\left(\frac{1}{y}\right)-0\right|\le\left|y\sin\left(\frac{1}{x}\right)\right|+\left|x\sin\left(\frac{1}{y}\right)\right|$$

$$|f(x,y)-0|\le|y|\left|\sin\left(\frac{1}{x}\right)\right|+|x|\left|\sin\left(\frac{1}{y}\right)\right|$$

$$|f(x,y)-0|\le|y|+|x|=\delta+\delta=2\delta=\epsilon$$

Hence we have $|f(x,y)-0|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

$$\boxed{\begin{gathered}|xy|=|x||y|\\ |x+y|\le|x|+|y|\\ \left|\sin\left(\frac{1}{x}\right)\right|\le 1\end{gathered}}$$

Thus $\lim_{(x,y)\to(0,0)}\left(y\sin\left(\frac{1}{x}\right)+x\sin\left(\frac{1}{y}\right)\right)=0$

6. Evaluate the limit $\lim\limits_{(x,y)\to(0,0)}\left(\frac{2x^2y}{x^4+y^2}\right)$

$x^4 + y^2 = 0$

$y^2 = -x^4 \Rightarrow y = mx^2$

Sol: Take the limit along the curve $y = mx^2$

$$\lim_{(x,y)\to(0,0)}\left(\frac{2x^2y}{x^4+y^2}\right) = \lim_{x\to 0}\left(\frac{2x^2.mx^2}{x^4+(mx^2)^2}\right) = \lim_{x\to 0}\left(\frac{2m}{1+m^2}\right) = \frac{2m}{1+m^2}$$

It is dependent on m and hence limit is not unique

The limit of the function does not exist.

7. Check existence of the limit $\lim\limits_{(x,y)\to(0,0)}\left(\frac{xy^3}{x^2+y^6}\right)$

$x^2 + y^6 = 0$

$\Rightarrow y^6 = x^2 \Rightarrow y^3 = mx;\ m \in R$

Sol: Take the limit along the curve $y^3 = mx$

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy^3}{x^2+y^6}\right) = \lim_{x\to 0}\left(\frac{x(mx)}{x^2+(mx)^2}\right) = \lim_{x\to 0}\left(\frac{m}{1+m^2}\right) = \frac{m}{1+m^2}$$

Hence limit is not unique. limit does not exist.

8. Check existence of the limit $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x^2}{x^2-y}\right)$

Sol: Take the limit along the curve $y = mx^2$

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2}{x^2-y}\right) = \lim_{x\to 0}\left(\frac{x^2}{x^2-mx^2}\right) = \frac{1}{1-m}$$

$$\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2-y} = \begin{cases}\infty; & if\ m = 1\\ \dfrac{1}{1-m}; & if\ m \neq 1\end{cases}$$

It is dependent on m and hence limit is not unique. The limit of the function does not exist.

9. Evaluate the following limit $\lim\limits_{(x,y)\to(0,0)}(x+y)\left(\frac{y+(x+y)^2}{y-(x+y)^2}\right)$

$y - (x+y)^2 = 0$

$\Rightarrow my = (x+y)^2$

Sol: Take the limit along the curve $(x+y)^2 = my$

$$\lim_{(x,y)\to(0,0)}(x+y)\left(\frac{y+(x+y)^2}{y-(x+y)^2}\right) = \lim_{y\to 0}\left(\sqrt{my}\right)\left(\frac{y+my}{y-my}\right) = \lim_{y\to 0}\left(\sqrt{my}\right)\left(\frac{1+m}{1-m}\right) = \begin{cases}\infty; if\ m = 1\\ 0; if\ m \neq 1\end{cases}$$

Hence limit is not unique. limit does not exist.

(OR)

Let $\epsilon > 0, \exists\, \partial > 0 \ni |x - x_0| < \delta, |y - y_0| < \delta \Longrightarrow |x - 0| < \delta, |y - 0| < \delta \Longrightarrow |x| < \delta, |y| < \delta$

$$|f(x,y) - L| = |f(x,y) - 0| = \left|(x+y)\left(\frac{y+(x+y)^2}{y-(x+y)^2}\right) - 0\right| = |x+y|\left|\frac{y+(x+y)^2}{y-(x+y)^2}\right|$$

$$|f(x,y) - 0| = (|x| + |y|)\left|\frac{y+(x+y)^2}{y-(x+y)^2}\right| > |x| + |y| = \delta + \delta = 2\delta = \epsilon \qquad \text{since } \left|\frac{y+(x+y)^2}{y-(x+y)^2}\right| > 1$$

Hence we have $|f(x,y) - 0| \nless \epsilon$ when ever $|x - 0| < \delta, |y - 0| < \delta$

Hence limit does not exist at (0,0).

10. Evaluate the following limit $\lim\limits_{(x,y)\to(0,0)}\left(\frac{xy^2}{x^3+2y-3x}\right)$

$$\boxed{\begin{array}{l} x^3+2y-3x=0 \\ \Rightarrow 2y-3x=mx^3 \end{array}}$$

Sol: Take the limit along the curve $2y-3x=mx^3$

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy^2}{x^3+2y-3x}\right)=\lim_{x\to 0}\left(\frac{\frac{x(3x+mx^3)^2}{4}}{x^3+mx^3}\right)=\lim_{x\to 0}\left(\frac{x^3(3+mx^2)^2}{4x^3(1+m)}\right)=\frac{9}{4(1+m)}$$

Which attains different values as m varies. Thus limit of the function does not exist.

11. Evaluate the following limit $\lim\limits_{(x,y)\to(0,0)}\left(\frac{x^3}{x^3+y^3-x}\right)$

$$\boxed{\begin{array}{l} x^3+y^3-x=0 \\ \Rightarrow y^3-x=mx^3 \end{array}}$$

Sol: Take the limit along the curve $y^3-x=mx^3$

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^3}{x^3+y^3-x}\right)=\lim_{x\to 0}\left(\frac{x^3}{x^3+mx^3}\right)=\frac{1}{1+m}$$

Which attains different values as m varies. Thus limit of the function does not exist

12. Prove that $\lim\limits_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right)=0$

Sol: Let $\epsilon>0, \exists\, \partial>0 \ni |x-x_0|<\delta, |y-y_0|<\delta \Longrightarrow |x-0|<\delta, |y-0|<\delta \;\Longrightarrow |x|<\delta, |y|<\delta$

$$|f(x,y)-L|=|f(x,y)-0|=\left|(xy)\,\frac{x^2-y^2}{x^2+y^2}-0\right|=|x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right|$$

$$|f(x,y)-0|\le|x||y|=\delta.\delta=\epsilon \qquad\qquad \text{since } \left|\frac{x^2-y^2}{x^2+y^2}\right|<1$$

Hence we have $|f(x,y)-\;0|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$ Thus $\lim\limits_{(x,y)\to(0,0)}\left(xy\,\frac{x^2-y^2}{x^2+y^2}\right)=0$

(OR)

Let $x=r\cos\theta$ & $y=r\sin\theta$ then $x^2+y^2=r^2$ and $r<\delta$

$$|f(x,y)-L|=|f(x,y)-0|=\left|(xy)\,\frac{x^2-y^2}{x^2+y^2}\right|=\left|(r\cos\theta\; r\sin\theta)\left(\frac{(r\cos\theta\,)^2-(r\sin\theta)^2}{r^2}\right)\right|$$

$$|f(x,y)-0|=\left|r^2\left(\frac{2}{2}\cos\theta\,\sin\theta\right)\,r^2\left(\frac{\cos^2\theta-\sin^2\theta}{r^2}\right)\right|=\frac{r^2}{2}\,sin2\theta.\cos2\theta<\frac{r^2}{2}=\frac{\delta^2}{2}=\epsilon$$

$\forall\,\epsilon>0, \exists\,\partial>0 \ni\; |f(x,y)-0|<\epsilon$ *when* $r<\delta$

i.e; $\lim\limits_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right)=\lim\limits_{r\to o}\frac{(r\cos\theta)(r\sin\theta)(r^2\sin^2\theta-r^2\cos^2\theta)}{r^2}=\lim\limits_{r\to o}\frac{r^2}{2}\,sin2\theta.\cos2\theta=0$

13. Prove that $\lim\limits_{(x,y)\to(0,0)} \left(x \sin\left(\frac{1}{y}\right) + \frac{xy}{\sqrt{x^2+y^2}}\right) = 0$

Sol: Let $\epsilon > 0, \exists\, \partial > 0 \ni |x - x_0| < \delta, |y - y_0| < \delta \Longrightarrow |x - 0| < \delta, |y - 0| < \delta \;\Longrightarrow |x| < \delta, |y| < \delta$

$$|f(x,y) - L| = |f(x,y) - 0| = \left| x\sin\left(\frac{1}{y}\right) + \frac{xy}{\sqrt{x^2+y^2}} - 0 \right|$$

$$|f(x,y) - 0| = |x| \left|\sin\left(\frac{1}{y}\right)\right| + |x||y| \left|\frac{1}{\sqrt{x^2+y^2}}\right|$$

$|f(x,y) - 0| \leq |x|(1) + |x||y|(1)$ since $\left|\sin\left(\frac{1}{y}\right)\right| < 1$ *and* $\left|\frac{1}{\sqrt{x^2+y^2}}\right| < 1$

$|f(x,y) - 0| \leq \delta + \delta = 2\delta = \epsilon$

Hence we have $|f(x,y) - 0| < \epsilon$ when ever $|x - 0| < \delta, |y - 0| < \delta$

Thus $\lim\limits_{(x,y)\to(0,0)} \left(x\sin\left(\frac{1}{y}\right) + \frac{xy}{\sqrt{x^2+y^2}}\right) = 0$

Exercise Problems: Check whether the limit of the following functions exist or not.

(a) $\lim\limits_{(x,y)\to(0,0)} \left(\frac{x-2y}{x+y}\right)$

(b) $\lim\limits_{(x,y)\to(0,0)} \left(\frac{xy^2}{x^2+y^4}\right)$

(c) $\lim\limits_{(x,y)\to(0,0)} \left(\frac{xy^2}{x^2+y^2}\right)$

(d) $\lim\limits_{(x,y)\to(0,0)} \left(\frac{x^2+y^2}{x^2y^2(x-y)^2}\right)$

Continuity of a function of two variables:

A function f(x, y) is said to be continuous at a point (x_0, y_0) if

(a) f(x, y) is defined at (x_0, y_0)

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$ exist

(c) $\lim_{(x,y)\to(0,0)} f(x,y) = f(x_0, y_0)$

(OR)

A function f(x, y) is said to be continuous at a point (x_0, y_0), if for every $\epsilon > 0$ (however small it may be), there exist $\delta > 0$ such that $|f(x,y) - f(x_0,y_0)| < \epsilon$ whenever $|x - x_0| < \delta, |y - y_0| < \delta$ and it is denoted by $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Note:

1. If a function is not continuous at a point, then we say that function is discontinuous at that point

2. Polynomials in two variables are continuous functions.

3. Rational functions, i.e., ratios of polynomials, are continuous functions provided they are well defined.

4. constant multiples, sum, difference, product, quotient, and rational powers of continuous functions are continuous whenever they are well defined

Note:

1. e^{x-y} is continuous at all points in the plane.
2. $\cos\left(\frac{xy}{1+x^2}\right)$ and $\ln(1 + x^2 + y^2)$ are continuous on R²
3. At which points is $\tan^{-1}\frac{y}{x}$ continuous?

   Well, the function y/x is continuous everywhere except when x = 0

   The function $\tan^{-1}$ is continuous everywhere on R So, $\tan^{-1}\frac{y}{x}$ is continuous everywhere except when x = 0
4. The function $(x^2 + y^2 + z^2 - 1)^{-1}$ is continuous everywhere except on the sphere $x^2 + y^2 + z^2 = 1$ where it is not defined

14. Show that function f is discontinuous at (0,0). $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Take the limit along the curve $y = mx$

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy}{x^2+y^2}\right)=\lim_{x\to0}\left(\frac{x(mx)}{x^2+m^2x^2}\right)=\frac{m}{1+m^2}$$

Which attains different values as m varies. Thus limit of the function does not exist.

Hence function is not continuous at (0,0)

15. Check whether function is continuous at (0,0) or not. $f(x,y)=\begin{cases}\sin\left(\frac{xy}{x^2+y^2}\right) & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Take the limit along the curve $y = mx$

$$\lim_{(x,y)\to(0,0)}\sin\left(\frac{xy}{x^2+y^2}\right)=\lim_{x\to0}\sin\left(\frac{x(mx)}{x^2+m^2x^2}\right)=\sin\left(\frac{m}{1+m^2}\right)$$

Which attains different values as m varies. Hence limit of the f(x,y) is not unique.

Thus limit does not exist.

Hence given function f(x, y) is not continuous at (0,0)

16. Prove that the function is continuous at origin. $f(x,y)=\begin{cases}y\sin\left(\frac{1}{x}\right) & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Let $\epsilon>0,\exists\,\partial>0\ni|x-x_0|<\delta,|y-y_0|<\delta\Rightarrow|x-0|<\delta,|y-0|<\delta\ \Rightarrow|x|<\delta,|y|<\delta$

$|f(x,y)-f(0,0)|=\left|y\sin\left(\frac{1}{x}\right)-0\right|=|y|\left|\sin\left(\frac{1}{x}\right)\right|\le|y|=\delta=\epsilon$ Since $\left|\sin\left(\frac{1}{x}\right)\right|<1$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta,|y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)}f(x,y)=0=f(0,0)$

Hence f is continuous at (0,0)

17. Discuss the continuity of the function at (0,0) $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & ;(x,y)\neq(0,0)\\ 1 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=1

Take the limit along the curve $y = mx$

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy}{\sqrt{x^2+y^2}}\right)=\lim_{x\to0}\left(\frac{x(mx)}{\sqrt{x^2+m^2x^2}}\right)=\lim_{x\to0}\left(\frac{mx^2}{x\sqrt{1+m^2}}\right)=0$$

Hence $\lim_{(x,y)\to(0,0)}f(x,y)=0$ but f(0,0)=1

$\lim_{(x,y)\to(0,0)}f(x,y)=0\neq f(x,y)$

Hence function is not continuous at (0,0)

18. Show that function f is continuous at (0,0). $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that $f(0,0)=0$

Let $\epsilon>0, \exists\, \partial>0 \ni |x-x_0|<\delta, |y-y_0|<\delta \Rightarrow |x-0|<\delta, |y-0|<\delta \;\Rightarrow |x|<\delta, |y|<\delta$

$|f(x,y)-f(0,0)|=\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|=|x|\left|\frac{y}{\sqrt{x^2+y^2}}\right|=|x||y|\left|\frac{1}{\sqrt{x^2+y^2}}\right|\le|x||y|$ Since $\left|\frac{y^2}{x^2+y^2}\right|<1$

$|f(x,y)-f(0,0)|<|x||y|=\delta^2=\epsilon$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$

Hence f is continuous at (0,0)

19. Show that f is continuous at (0,0). $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that $f(0,0)=0$

Let $\epsilon>0, \exists\, \partial>0$ such that $|x-0|<\delta, |y-0|<\delta \;\Rightarrow |x|<\delta, |y|<\delta$

$|f(x,y)-f(0,0)|=\left|\frac{xy(x^2-y^2)}{x^2+y^2}-0\right|=|x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right|\le|x||y|<\delta.\delta=\delta^2=\epsilon$ Since $\left|\frac{x^2-y^2}{x^2+y^2}\right|<1$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$, Hence f is continuous at (0,0)

20. Discuss the continuity of the function at origin. $f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Take the limit along the curve $y=mx$

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^3-y^3}{x^2+y^2}\right)=\lim_{x\to0}\left(\frac{x^3-(mx)^3}{x^2+(mx)^2}\right)=0$$

The limit value exists and unique for the path y=mx. But we cannot say by this test that limit exists or not.

To check whether the limit exist or not, we can proceed always like the following way.

Sol: Let $\epsilon>0, \exists\, \partial>0$ such that $|x-0|<\delta, |y-0|<\delta \;\Rightarrow |x|<\delta, |y|<\delta$

$$|f(x,y)-f(0,0)|=\left|\frac{x^3-y^3}{x^2+y^2}-0\right|=\left|\frac{x^3}{x^2+y^2}\right|+\left|\frac{y^3}{x^2+y^2}\right|=|x|\left|\frac{x^2}{x^2+y^2}\right|+|y|\left|\frac{y^2}{x^2+y^2}\right|$$

$|f(x,y)-f(0,0)|=\le|x|+|y|=2\delta=\epsilon$ Since $\left|\frac{x^2}{x^2+y^2}\right|<1$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$ Hence f is continuous at (0,0)

21. Show that function f is discontinuous at (0,0). $f(x,y)=\begin{cases}\frac{x^2}{x^2+y^4-x} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Take the limit along the curve $y^4-x=mx^2$

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2}{x^2+y^4-x}\right)=\lim_{x\to0}\left(\frac{x^2}{x^2+mx^2}\right)=\frac{1}{1+m}$$

Limit do not exist at (0,0). Hence function is not continuous at (0,0)

22. Show that function f is discontinuous at (0, 0). $f(x,y)=\begin{cases}\frac{x^4y^4}{(x^2+y^4)^3} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Take the limit along the curve $y^4=mx^2$

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^4y^4}{(x^2+y^4)^3}\right)=\lim_{x\to0}\left(\frac{x^4(mx^2)}{(x^2+(mx^2))^3}\right)=\frac{m}{(1+m)^3}$$

$(x^2+y^4)^3=0$

$\Rightarrow x^2+y^4=0$

$\Rightarrow y^4=mx^2$

Limit do not exist at (0,0). Hence function is not continuous at (0,0)

23. Show that function f is discontinuous at (0,0). $f(x,y)=\begin{cases}\frac{xy^2}{x^3+y^3} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Take the limit along the curve **y**= mx

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy^2}{x^3+y^3}\right)=\lim_{x\to0}\left(\frac{x\,m^2x^2}{x^3+m^3x^3}\right)=\frac{m^2}{1+m^3}$$

Limit do not exist at (0,0). Hence function is not continuous at (0,0)

24. Discuss the continuity of the function at origin. $f(x,y)=\begin{cases}\sin x\sin\left(\frac{1}{y}\right) & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Sol: Given that f(0,0)=0

Let $\epsilon>0,\exists\,\partial>0$ such that $|x-0|<\delta,|y-0|<\delta\implies|x|<\delta,|y|<\delta$

$$|f(x,y)-f(0,0)|=\left|\sin x\sin\left(\frac{1}{y}\right)-0\right|=|\sin x|\left|\sin\left(\frac{1}{y}\right)\right|\leq|\sin x|<|x|<\delta=\epsilon$$

Since $\left|\sin\left(\frac{1}{y}\right)\right|\leq1$ and $|\sin x|<|x|$

$|f(x,y)-0|\leq\epsilon$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta,|y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)}f(x,y)=0=f(0,0)$

Hence f is continuous at (0,0)

25. Discuss the continuity of the following function at the point (0,0)

$$f(x,y)=\begin{cases}\dfrac{x^2+y^2}{\tan(xy)}, & (x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

Sol: Given that f(0,0)=0

Take the limit along the curve **y**= mx

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2+y^2}{\tan(xy)}\right)=\lim_{x\to 0}\left(\frac{x^2+m^2x^2}{\tan(x.mx)}\right)=\lim_{x\to 0}\left(\frac{x^2(1+m^2)}{\tan(mx^2)}\right)$$

$$\Longrightarrow \lim_{x\to 0}\left(\frac{1+m^2}{\frac{\tan(mx^2)}{x^2}}\right)=(1+m^2)\frac{1}{m}\frac{1}{\lim\limits_{x\to 0}\left(\frac{\tan(mx^2)}{mx^2}\right)}=\frac{1+m^2}{m}$$

Limit do not exist at (0,0). Hence function is not continuous at (0,0)

Partial Derivatives:

We know that, if y is a continuous function of the independent variable x, and Δy the increment in y corresponding to an increment Δx in x, then $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ if exists, is called the differential coefficient or the derivative of y with respect to x.

Let f(x,y) be a function of two independent variables x and y and let Δf be the increment in f corresponding to an increment in x, y remaining constant. If f is a continuous function of x, the limit $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ if exists is called the partial derivative of f with respect to x, and is represented by $\frac{\partial f}{\partial x}$ *or* f_x.

$$\therefore \frac{\partial f}{\partial x} = u_x = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \right)$$

In the same way, if u is a continuous function of y, the increment Δf in f, corresponding to an increment Δy in y, x remaining constant, then the limit $\lim_{\Delta y \to 0} \frac{\Delta f}{\Delta y}$ if exists is called the partial derivative of f with respect to y, and is represented by $\frac{\partial f}{\partial y}$ *or* f_y.

$$\therefore \frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \left(\frac{f(x, \ y + \Delta y) - f(x, y)}{\Delta y} \right)$$

Note:

1. The partial derivative of f(x, y) with respect to x at the point (x_0, y_0) is
$$\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} = f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, \ \ y_0) - f(x_0, y_0)}{h}$$
provided the limit exists.
2. The partial derivative of f(x,y) with respect to y at the point (x_0, y_0) is
$$\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} = f_y(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, \ y_0 + k) - f(x_0, y_0)}{k}$$
provided the limit exists.
3. The partial derivative $\frac{\partial f}{\partial x}$ at (x_0, y_0) gives the rate of change of f(x, y) w.r.to x, when y is held fixed at y_0 . This is the rate of change of f(x, y) in the direction of the positive x-axis at (x_0, y_0)
4. The partial derivative $\frac{\partial f}{\partial y}$ at (x_0, y_0) gives the rate of change of f(x, y) w.r.to y, when x is held fixed at x_0 . This is the rate of change of f(x, y) in the direction of the positive y-axis at (x_0, y_0)

Second and higher order partial derivatives:

Let u=f(x,y) be a function of two independent variables x and y. The first order partial derivative of u denoted by $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$. We can also find the second and higher order derivatives

$$\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x\,\partial y}, \quad \frac{\partial^2 u}{\partial y\,\partial x} \ldots\ldots\ldots..$$

They are evaluated as follows.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \;\&\; \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \& \;\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

These derivatives are usually denoted by $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$. Generally, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

The third and higher orders of the partial derivatives can be obtained similarly.

26. Find the value of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at the point (4,-5), if $f(x,y) = x^2 + 3xy + y - 1$

Sol: Given that $f(x,y) = x^2 + 3xy + y - 1$

Differentiate w.r.t x we get $\frac{\partial f}{\partial x} = 2x + 3y \Longrightarrow \left(\frac{\partial f}{\partial x}\right)_{(4,-5)} = -7$

Differentiate w.r.t y we get $\frac{\partial f}{\partial y} = 3x + 1 \Longrightarrow \left(\frac{\partial f}{\partial y}\right)_{(4,-5)} = 13$

(OR)

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, \;\; y_0) - f(x_0, y_0)}{h}$$

$$f_x(4,-5) = \lim_{h\to 0} \frac{f(4 + h, -5) - f(4,-5)}{h}$$

$$f_x(4,-5) = \lim_{h\to 0} \frac{\{(4 + h)^2 + 3(4 + h)(-5) + (-5) - 1\} - (-50)}{h}$$

$$f_x(4,-5) = \lim_{h\to 0} \frac{h^2 - 7h}{h} = \lim_{h\to 0} \frac{h(h - 7)}{h} = \lim_{h\to 0}(h - 7) = -7$$

$$f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, \; y_0 + k) - f(x_0, y_0)}{k} \Longrightarrow f_y(4,-5) = \lim_{k\to 0} \frac{f(4,-5 + k) - f(4,-5)}{k}$$

$$f_y(4,-5) = \lim_{k\to 0} \frac{16 + 3(4)(k - 5) + (k - 5) - 1 - (-50)}{k} = \lim_{h\to 0} \frac{13k}{k} = 13$$

27. Find the value of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ if $f(x,y) = y\sin(xy)$

Sol: Given that $f(x,y) = y\sin(xy)$

$$\frac{\partial f}{\partial x} = y\cos(xy)\frac{\partial}{\partial x}(xy) \Longrightarrow \frac{\partial f}{\partial x} = y\cos(xy)\,(y) = y^2\cos(xy)$$

$$\frac{\partial f}{\partial y} = y\cos(xy)\frac{\partial}{\partial x}(xy) + \sin(xy) \Longrightarrow \frac{\partial f}{\partial y} = y\cos(xy)\,(x) + \sin(xy) = xy\cos(xy) + \sin(xy)$$

28. If $f(x,y)=\begin{cases}\frac{x^2-y^2}{x-y}; \text{if } f(x,y)\neq(1,-1)\\ 0 \quad ; \text{if } f(x,y)=(1,-1)\end{cases}$ Find $f_x(1,-1)$ and $f_y(1,-1)$.

Sol: Given that f(1,-1)=0

$$f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,\ y_0)-f(x_0,y_0)}{h}$$

$$f_x(1,-1)=\lim_{h\to 0}\frac{f(1+h,-1)-f(1,-1)}{h}$$

$$f_x(1,-1)=\lim_{h\to 0}\left(\frac{\frac{(1+h)^2-(-1)^2}{(1+h)-(-1)}-0}{h}\right)=\lim_{h\to 0}\left(\frac{1+h^2+2h-1}{h(2+h)}\right)=\lim_{h\to 0}\left(\frac{h(h+2}{h(2+h)}\right)=1$$

$$f_y(x_0,y_0)=\lim_{k\to 0}\frac{f(x_0,\ y_0+k)-f(x_0,y_0)}{k}=\lim_{k\to 0}\frac{f(1,-1+k)-f(1,-1)}{k}$$

$$f_y(1,-1)=\lim_{k\to 0}\left(\frac{\frac{1-(-1+k)^2}{1-(-1+k)}-0}{k}\right)=\lim_{k\to 0}\left(\frac{1-(1+k^2-2k}{k(2-k)}\right)=\lim_{k\to 0}\left(\frac{-k^2+2k}{k(2-k)}\right)=1$$

29. Let $f(x,y)=\begin{cases}\frac{\sin(x^3+y^4)}{x^2+y^2}; \text{if } f(x,y)\neq 0\\ 0 \quad ; \text{if } f(x,y)=0\end{cases}$ compute f_x & f_y at (0,0)

Sol:

$$f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,\ y_0)-f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{\sin(h^3)}{h^2}-0}{h}=\lim_{h\to 0}\frac{\sin(h^3)}{h^3}=1$$

$$f_y(x_0,y_0)=\lim_{k\to 0}\frac{f(x_0,\ y_0+k)-f(x_0,y_0)}{k}$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{\frac{\sin(k^4)}{k^2}-0}{k}=\lim_{k\to 0}\frac{\sin(k^4)}{k^3}=\lim_{k\to 0}\left(\frac{\sin(k^4)}{k^4}\right)k=0$$

Hence $f_x(0,0)\neq f_x(0,0)$

30. find the value of $(a)\frac{\partial f}{\partial x}(x,0)\quad(b)\frac{\partial f}{\partial x}(0,y)\quad(c)\frac{\partial f}{\partial y}(x,0)\quad(d)\frac{\partial f}{\partial y}(0,y)$ if

$f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}; \text{if } f(x,y)\neq 0\\ 0 \quad ; \text{if } f(x,y)=0\end{cases}$

Sol:

$$\frac{\partial f}{\partial x}(x,0)=\lim_{h\to 0}\frac{f(x+h,0)-f(x,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$\frac{\partial f}{\partial x}(0,y)=\lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h}=\lim_{h\to 0}\frac{\frac{hy(h^2-y^2)}{h^2+y^2}-0}{h}=\lim_{h\to 0}\frac{\frac{hy(h^2-y^2)}{h^2+y^2}-0}{h}=\lim_{h\to 0}\frac{y(h^2-y^2)}{h^2+y^2}=-y$$

$$\frac{\partial f}{\partial y}(x,0)=\lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k}=\lim_{k\to 0}\frac{\frac{xk(x^2-k^2)}{x^2+k^2}-0}{k}=\lim_{k\to 0}\frac{\frac{xk(x^2-k^2)}{x^2+k^2}-0}{k}=\lim_{k\to 0}\frac{x(x^2-k^2)}{x^2+k^2}=x$$

$$\frac{\partial f}{\partial y}(0,y)=\lim_{k\to 0}\frac{f(0,y+k)-f(0,y)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

31. find the slope of the line tangent to the surface $f(x,y)=2x+3y-4$ at the point (2,-1) and lying in the (a) plane x=2 (b) plane y=-1.

Sol: Given that $f(x,y)=2x+3y-4$

(a) Along a plane x=2: $f=3y \Longrightarrow \frac{\partial f}{\partial y}=3$

Slope of f(x,y) at (2,-1) is 3

(b) Along a plane y=-1: $f=2x-7 \Longrightarrow \frac{\partial f}{\partial x}=2$

Slope of f(x,y) at (2,-1) is 2

32. find the slope of the line tangent to the surface $f(x,y)=x^2+y^3$ at the point (-1,1) and lying in the (a) plane x=-1 (b) plane y=1.

Sol: Given that $f(x,y)=x^2+y^3$

(a) Along a plane x=1: $f(y)=1+y^3 \Longrightarrow \frac{\partial f}{\partial y}=3y^2$

Slope of f(x,y) at (-1,1) is 3

(b) Along a plane y=1: $f(x)=x^2+1 \Longrightarrow \frac{\partial f}{\partial x}=2x$

Slope of f(x,y) at (-1,1) is -2

33. Show that the function $f(x,y)=\begin{cases}\frac{xy^2}{x^2+y^4}; & if\ f(x,y)\neq 0\\ 0 & ; if\ f(x,y)=0\end{cases}$ is not continuous at (0,0) but its partial derivatives f_x, f_y exist at (0,0)

Sol: Given that f(0,0)=0

$$\boxed{\begin{array}{l}x^2+y^4=0\\ y^2=mx\end{array}}$$

Take the limit along the curve $y^2=mx$

$$\lim_{(x,y)\to(0,0)}\left(\frac{x(mx)}{x^2+m^2x^2}\right)=\frac{m}{1+m^2}$$

Hence limit is not unique thus limit does not exist at (0,0).

Hence function is not continuous at (0,0)

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

Thus the partial derivative f_x, f_y exist at (0,0) but f(x,y) is not continuous at (0,0)

34. **Show that the function** $f(\mathrm{x},\mathrm{y}) = \begin{cases} \frac{xy}{\mathrm{x}^2+2\mathrm{y}^2}; if\ f(x,y) \neq 0 \\ 0 \quad ; if\ f(x,y) = 0 \end{cases}$ **is not continuous at (0,0) but its partial derivatives** f_x, f_y **exist at (0,0)**

Differentiable function:

A function f(x, y) is said to be differentiable at a point (x_0, y_0), if $f_x(x_0, y_0)$ *and* $f_y(x_0, y_0)$ exist

and $\lim\limits_{(h,k)\to(x_0,y_0)} \frac{f(x_0+h,\, y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$ exist

Note:

1. If the first order partial derivatives f_x & f_y of a function f(x,y) are continuous at a point (x_0, y_0), then f(x,y) is differentiable at (x_0, y_0).

2. If a function f(x,y) is differentiable at (x_0, y_0), then f(x,y) is continuous at(x_0, y_0).

3. If f(x,y) is not continuous at(x_0, y_0) then f(x,y) is not differentiable at (x_0, y_0).

Note: Let f (x y) be a function of two independent variables x and y and dx, dy are the increments in x and y respectively then the differential of f(x,y) is called the **total differential** and is denoted by $\mathbf{df} = \frac{\partial f}{\partial x}\mathbf{dx} + \frac{\partial f}{\partial y}\mathbf{dy}$

35. Show that the function $f(x,y) = \begin{cases} \frac{xy+y^2}{x^2+y^2}; & if\ f(x,y) \neq 0 \\ 0 & ; if\ f(x,y) = 0 \end{cases}$ is not differentiable at the origin.

Sol: Given that f(0,0)=0

Take the limit along the curve $y = mx$

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy+y^2}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x(mx)+(mx)^2}{x^2+m^2x^2}\right) = \frac{m+m^2}{1+m^2}$$

The different of m we get different limits Thus limit of the function does not exist.

Hence function is not continuous at (0,0) thus f(x,y) is not differentiable at (0,0)

36. The dimensions of a rectangular box are measured to be 75cm, 60cm, and 40 cm, and each measurement is correct to within 0.2cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Sol: Given That x=75, y=60, z=40cm and $dx = dy = dz = 0.2$

The volume of the box is V = xyz

$$\frac{\partial V}{\partial x} = yz, \frac{\partial V}{\partial y} = xz, \frac{\partial V}{\partial z} = xy$$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

The largest error in cubic cm is $dV = (60 * 40 * 0.2) + (40 * 75 * 0.2) + (75 * 60 * 0.2) = 1980$

Notice that the relative error is 1980/(75* 60* 40) which is about 1%

37. Show that $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$ is continuous but not differentiable at (0,0)

Sol: Given that $f(0,0)=0$

Let $\epsilon>0, \exists\, \partial>0$ such that $|x-0|<\delta, |y-0|<\delta \Rightarrow |x|<\delta, |y|<\delta$

$|f(x,y)-f(0,0)| = \left|\frac{xy}{\sqrt{x^2+y^2}}-0\right| = |x||y|\left|\frac{1}{\sqrt{x^2+y^2}}\right| \le |x||y| < \delta.\delta=\epsilon$ Since $\left|\frac{1}{x^2+y^2}\right|<1$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$, Hence f is continuous at (0,0)

$$\Rightarrow f_x(x_0,y_0)=\lim_{h\to0}\frac{f(x_0+h,\ y_0)-f(x_0,y_0)}{h}=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0$$

$$\Rightarrow f_y(x_0,y_0)=\lim_{k\to0}\frac{f(x_0,\ y_0+k)-f(x_0,y_0)}{k}=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to0}\frac{0-0}{k}=0$$

$$\Rightarrow \lim_{(h,k)\to(x_0,y_0)}\frac{f(x_0+h,\ y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)}\frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} \Rightarrow \lim_{(h,k)\to(0,0)}\frac{\frac{hk}{\sqrt{h^2+k^2}}-0-h.0-k.0}{\sqrt{h^2+k^2}}$$

$$=\lim_{(h,k)\to(0,0)}\frac{hk}{h^2+k^2}$$

Take the limit along the curve $h=mk$

$$\lim_{(h,k)\to(0,0)}\frac{hk}{h^2+k^2}=\lim_{k\to0}\left(\frac{(mk)k}{m^2k^2+k^2}\right)=\lim_{k\to0}\left(\frac{mk^2}{k^2(m^2+1)}\right)=\frac{m}{m^2+1}$$

The different of m we get different limits Thus limit of the function is not unique.

Hence f(x,y) is not differentiable at (0,0)

38. Show that $f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$ is continuous but not differentiable at origin although partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ exists at (0,0)

Sol: Given that $f(0,0)=0$

Let $\epsilon>0, \exists\, \partial>0$ such that $|x-0|<\delta, |y-0|<\delta \Rightarrow |x|<\delta, |y|<\delta$

$|f(x,y)-f(0,0)| = \left|\frac{x^3-y^3}{x^2+y^2}-0\right| = \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| |x| \left|\frac{x^2}{x^2+y^2}\right| + |y|\left|\frac{y^2}{x^2+y^2}\right| \le |x|+|y| = 2\delta=\epsilon$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$ Hence f is continuous at (0,0)

$$\Rightarrow f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,\ \ y_0)-f(x_0,y_0)}{h} = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3-0}{h^2+0}-0}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$$\Rightarrow f_y(x_0,y_0) = \lim_{k\to 0}\frac{f(x_0,\ y_0+k)-f(x_0,y_0)}{k} = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{\frac{0-k^3}{0+k^2}-0}{k} = \lim_{k\to 0}\frac{-k}{k} = -1$$

$$\Rightarrow \lim_{(h,k)\to(x_0,y_0)}\frac{f(x_0+h,\ \ y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)}\frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} \Rightarrow \lim_{(h,k)\to(0,0)}\frac{\frac{h^3-k^3}{h^2+k^2}-0-h.1-k.(-1)}{\sqrt{h^2+k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)}\frac{\frac{h^3-k^3}{h^2+k^2}-h+k}{\sqrt{h^2+k^2}} \Rightarrow \lim_{(h,k)\to(0,0)}\frac{(h^3-k^3)+(k-h)(h^2+k^2)}{(h^2+k^2)\sqrt{h^2+k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)}\frac{(h^3-k^3)+(k-h)(h^2+k^2)}{\sqrt{(h^2+k^2)^3}} \ldots\ldots\ldots..(1)$$

Take the limit along the curve $h = mk$

$$\lim_{(h,k)\to(0,0)}\frac{(m^3k^3-k^3)+(k-mk)(m^2k^2+k^2)}{\sqrt{(m^2k^2+k^2)^3}} = \lim_{k\to 0}\left(\frac{k^3\{(m^3-1)+(1-m)(m^2+1)\}}{k^3\sqrt{(m^2+1)^3}}\right)$$

$$= \frac{\{(m^3-1)+(1-m)(m^2+1)\}}{\sqrt{(m^2+1)^3}}$$

The different of m we get different limits. Thus limit does not exist.

Hence f(x,y) is not differentiable at (0,0)

(OR)

From (1)

$$\lim_{(h,k)\to(0,0)}\frac{(h^3-k^3)+(k-h)(h^2+k^2)}{\sqrt{(h^2+k^2)^3}}$$

Let $h = r\cos\theta, k = r\sin\theta \Rightarrow h^2+k^2 = r^2$

$$\Rightarrow \lim_{r\to o}\frac{(r^3\cos^3\theta - r^3\sin^3\theta)+(r\sin\theta - r\cos\theta)(r^2)}{r^3}$$

$$\Rightarrow \lim_{r\to o}\frac{r^3\{(\cos^3\theta-\sin^3\theta)+(\sin\theta-\cos\theta)\}}{r^3}$$

$$\Rightarrow \lim_{r\to o}(\cos^3\theta-\sin^3\theta)+(\sin\theta-\cos\theta)$$

$$\Rightarrow (\cos^3\theta-\sin^3\theta)+(\sin\theta-\cos\theta)$$

Limit value depend on θ so that imit does not exist. f(x,y) is not differentiable

39. Show that $f(x,y)=\begin{cases} x^2 \sin\left(\frac{1}{x}\right)+y^2 \sin\left(\frac{1}{y}\right); & (x,y)\neq(0,0) \\ 0; & (x,y)=(0,0)\end{cases}$ **is differentiable at origin.**

Sol:

$$\Rightarrow f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,\ y_0)-f(x_0,y_0)}{h} \Rightarrow \frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{h^2 \sin\left(\frac{1}{h}\right)-0}{h}=0$$

$$\Rightarrow f_y(x_0,y_0)=\lim_{k\to 0}\frac{f(x_0,\ y_0+k)-f(x_0,y_0)}{k} \Rightarrow \frac{\partial f}{\partial y}(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{k\to 0}\frac{k^2 \sin\left(\frac{1}{y}\right)-0}{k}=0$$

Hence f_x & f_y exist at (0,0)

$$\Rightarrow \lim_{(h,k)\to(x_0,y_0)}\frac{f(x_0+h,\ y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

$$\lim_{(h,k)\to(0,0)}\frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}=\lim_{(h,k)\to(0,0)}\frac{h^2 \sin\left(\frac{1}{h}\right)+k^2 \sin\left(\frac{1}{k}\right)}{\sqrt{h^2+k^2}}$$

Let $h=r\cos\theta$ and $k=r\sin\theta \Rightarrow h^2+k^2=r^2$

$$\Rightarrow \lim_{r\to o}\frac{r^2\cos^2\theta \sin\left(\frac{1}{r\cos\theta}\right)+r^2\sin^2\theta \sin\left(\frac{1}{r\sin\theta}\right)}{\sqrt{r^2}}$$

$$\Rightarrow \lim_{r\to o}\frac{r^2\left(\cos^2\theta \sin\left(\frac{1}{r\cos\theta}\right)+\sin^2\theta \sin\left(\frac{1}{r\sin\theta}\right)\right)}{r}=0$$

Hence f(x,y) is differentiable at (0,0)

40. Show that f is continuous and differentiable at (0,0) $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$.

Sol: Given that $f(0,0)=0$

Let $\epsilon>0$ to be given such that $|x-0|<\delta, |y-0|<\delta$

$$|f(x,y)-f(0,0)|=\left|\frac{xy(x^2-y^2)}{x^2+y^2}-0\right|=|x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right|\le |x||y|<\delta+\delta=2\delta=\epsilon \quad \text{Since } \left|\frac{x^2-y^2}{x^2+y^2}\right|<1$$

Hence we have $|f(x,y)-f(0,0)|<\epsilon$ when ever $|x-0|<\delta, |y-0|<\delta$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$, Hence f is continuous at (0,0)

$$\Rightarrow f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,\ y_0)-f(x_0,y_0)}{h} \Rightarrow \frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$\Rightarrow f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0,\ y_0 + k) - f(x_0, y_0)}{k} \Rightarrow \frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$\Rightarrow \lim_{(h,k)\to(x_0,y_0)} \frac{f(x_0 + h,\ \ y_0 + k) - f(x_0, y_0) - hf_x(x_0, y_0) - kf_y(x_0, y_0)}{\sqrt{h^2 + k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2 + k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{\frac{hk(h^2 - k^2)}{h^2 + k^2} - 0 - h.0 - k.0}{\sqrt{h^2 + k^2}} \Rightarrow \lim_{(h,k)\to(0,0)} \frac{hk(h^2 - k^2)}{(h^2 + k^2)\sqrt{h^2 + k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{hk(h^2 - k^2)}{(h^2 + k^2)\sqrt{h^2 + k^2}}$$

Let $h = r\cos\theta$ and $k = r\sin\theta \ \Rightarrow h^2 + k^2 = r^2$

$$\Rightarrow \lim_{r\to o} \frac{(r\cos\theta)(r\sin\theta)(r^2\sin^2\theta - r^2\cos^2\theta)}{r^2.\sqrt{r^2}} = \lim_{r\to o} \frac{r^4\{(\cos\theta)(\sin\theta)(\sin^2\theta - \cos^2\theta)\}}{r^3} = 0$$

Thus f(x,y) is differentiable at (0,0)

41. Show that function $f(x,y) = \begin{cases} \frac{x^2-y^2}{x-y} & ; (x,y) \neq (1,-1) \\ 0 & ; (x,y) = (1,-1) \end{cases}$ is continuous and differentiable at (1,-1)

Sol:

Continuity at (1,-1):

Given that f(1,-1)=0

$$\lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} \left(\frac{x^2 - y^2}{x - y}\right) = \lim_{(x,y)\to(1,-1)} (x + y) = 1 + (-1) = 0$$

$$\lim_{(x,y)\to(1,-1)} f(x,y) = 0 = f(1,-1)$$

Hence f(x,y) is continuous at(1,1)

Partial derivatives at (1,-1):

$$\Rightarrow f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h,\ \ y_0) - f(x_0, y_0)}{h} = \lim_{h\to 0} \frac{f(1 + h, -1) - f(1, -1)}{h}$$

$$\Rightarrow f_x(x_0, y_0) = \lim_{h\to 0} \frac{1}{h}\left(\frac{(1 + h)^2 - 1}{(1 + h) + 1}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{h^2 + 2h}{h + 2}\right) = 1$$

$$\Rightarrow f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0,\ y_0 + k) - f(x_0, y_0)}{k} = \lim_{k\to 0} \frac{f(1, k - 1) - f(1, -1)}{k} = \lim_{h\to 0} \frac{2 - h}{2 - h} = 1$$

Therefore, the partial derivatives exists at (1,–1)

Differentiability at (1,-1):

$$\Rightarrow \lim_{(h,k)\to(x_0,y_0)} \frac{f(x_0+h,\ \ y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{f(1+h,-1+k)-f(1,-1)-hf_x(1,-1)-kf_y(1,-1)}{\sqrt{h^2+k^2}}$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{\{(1+h)+(-1+k)\}-0-h.1-k.1}{\sqrt{h^2+k^2}} \Rightarrow \lim_{(h,k)\to(0,0)} \frac{0}{\sqrt{h^2+k^2}} = 0$$

Hence, , is differentiable at (1,-1)

Note:

1. A function $f(x,y) = \frac{x^a y^b}{x^c + y^d}$ *where* $\boldsymbol{a, b, c, d > 0}$ then

   (a) continuous at (0,0) if $\frac{a}{c} + \frac{b}{d} > 1$

   (b) Differentiable at (0,0) if $\frac{a}{c} + \frac{b}{d} > 1 + \frac{1}{c}$

   (c) if a>0 and b>0 then the value of $f_x(0,0) = 0$ *and* $f_y(0,0) = 0$

2. If f(x, y) is symmetric function then $f_x = f_y$

3. A function $f(x) = x^m \sin\left(\frac{1}{x^n}\right)$ *or* $x^m \cos\left(\frac{1}{x^n}\right)$ then

   (a) continuous if $for\ all\ m, n > 0$

   (b) Differentiable if only $m > n$

4. A function $f(x,y) = \frac{f_1(x,y)}{f_2(x,y)}$ and

   (a) If $\deg(Numirator) >$ deg(denominator) then f is continuous at point (x, y)

   (b) If $\deg(Numirato) \leq$ deg(enominator) then f is not continuous at point (x, y)

Objective Type Questions:

1. $\boldsymbol{f(x,y) = \begin{cases} \frac{x^4 y^2}{x^2+y^4}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}}$

Sol: a=4, b=2, c=2, d=4

$\frac{a}{c} + \frac{b}{d} = \frac{4}{2} + \frac{2}{4} > 1$ hence f(x, y) is continuous at (0,0).

$\frac{a}{c} + \frac{b}{d} > 1 + \frac{1}{c} \Longrightarrow = \frac{4}{2} + \frac{2}{4} > 1 + \frac{1}{2}$ hence f(x, y) is differentiable at (0,0)

If f(x,y) is differentiable at then directional derivative exist. But convers not sure

Since a>0 and b>0 hence $f_x(0,0) = 0$ *and* $f_y(0,0) = 0$

2. $\boldsymbol{f(x,y) = \begin{cases} \frac{xy}{x^2+y^4}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}}$

Sol: a=1, b=1, c=2, d=4

$\frac{a}{c} + \frac{b}{d} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \ngtr 1$ hence f(x, y) is not continuous at (0,0).

$\frac{a}{c} + \frac{b}{d} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ *and* $1 + \frac{1}{c} = 1 + \frac{1}{2} = \frac{3}{2} \Longrightarrow \frac{3}{4} \ngtr \frac{3}{2}$ hence f(x, y) is not differentiable at (0,0)

Since a>0 and b>0 hence $f_x(0,0) = 0$ *and* $f_y(0,0) = 0$

3. $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}$

Sol: a=2, b=1, c=2, d=2

$\frac{a}{c} + \frac{b}{d} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} > 1$ hence f(x, y) is continuous at (0,0).

$\frac{a}{c} + \frac{b}{d} = \frac{3}{2}$ and $1 + \frac{1}{c} = 1 + \frac{1}{2} = \frac{3}{2} \Longrightarrow \frac{3}{2} \ngtr \frac{3}{2}$ hence f(x, y) is not differentiable at (0,0)

Since a>0 and b>0 hence $f_x(0,0) = 0$ and $f_y(0,0) = 0$

4. $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}$

Sol: a=1, b=2, c=2, d=2

$\frac{a}{c} + \frac{b}{d} = \frac{1}{2} + \frac{2}{2} = \frac{3}{2} > 1$ hence f(x, y) is continuous at (0,0).

$\frac{a}{c} + \frac{b}{d} = \frac{3}{2}$ and $1 + \frac{1}{c} = 1 + \frac{1}{2} = \frac{3}{2} \Longrightarrow \frac{3}{2} \ngtr \frac{3}{2}$ hence f(x, y) is not differentiable at (0,0)

Since a>0 and b>0 hence $f_x(0,0) = 0$ and $f_y(0,0) = 0$

5. $f(x,y) = \begin{cases} \frac{x^3y^3}{x^6+y^6}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}$

Sol: a=3, b=3, c=6, d=6 and f(x,y) is symmetric function

$\frac{a}{c} + \frac{b}{d} = \frac{3}{6} + \frac{3}{6} = 1 \ngtr 1$ hence f(x, y) is not continuous at (0,0). And f(x,y) not differentiable at (0,0)

i.e $\frac{a}{c} + \frac{b}{d} = 1$ and $1 + \frac{1}{c} = \frac{7}{6} \Longrightarrow 1 \ngtr \frac{7}{6}$ hence f(x, y) is not differentiable at (0,0)

Since a>0 and b>0 hence $f_x(0,0) = 0$ and $f_y(0,0) = 0$

6. $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^4}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}$

Sol: a=2, b=2, c=2, d=4 and f(x,y) is symmetric function

$\frac{a}{c} + \frac{b}{d} = \frac{2}{2} + \frac{2}{4} = \frac{3}{2} > 1$ hence f(x, y) is continuous at (0,0).

$\frac{a}{c} + \frac{b}{d} = \frac{3}{2}$ and $1 + \frac{1}{c} = \frac{3}{2} \Longrightarrow \frac{3}{2} \ngtr \frac{3}{2}$ hence f(x, y) is not differentiable at (0,0)

Since a>0 and b>0 hence $f_x(0,0) = 0$ and $f_y(0,0) = 0$

7. $f(x,y) = \begin{cases} \frac{x^5y^5}{x^2+y^2}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}$

Sol: a=5, b=5, c=2, d=2 and f(x,y) is symmetric function

$\frac{a}{c}+\frac{b}{d}=\frac{5}{2}+\frac{5}{2}=5>1$ hence f(x, y) is continuous at (0,0).

$\frac{a}{c}+\frac{b}{d}=5\ and\ 1+\frac{1}{c}=\frac{3}{2} \Rightarrow 5>\frac{3}{2}$ hence f(x, y) is differentiable at (0,0) and all directional derivative exists.

Since a>0 and b>0 hence $f_x(0,0)=0\ and\ f_y(0,0)=0$

8. $f(x,y) = \begin{cases} \frac{x^3y}{x^2+y^2}; & if\ f(x,y) \neq 0 \\ 0\ ; & if\ f(x,y) = 0 \end{cases}$

Sol: a=3, b=1, c=2, d=2

$\frac{a}{c}+\frac{b}{d}=\frac{3}{2}+\frac{1}{2}=2>1$ hence f(x, y) is continuous at (0,0).

$\frac{a}{c}+\frac{b}{d}=2\ and\ 1+\frac{1}{c}=\frac{3}{2} \Rightarrow 2>\frac{3}{2}$ hence f(x, y) is differentiable at (0,0) and all directional derivative exists.

Since a>0 and b>0 hence $f_x(0,0)=0\ and\ f_y(0,0)=0$

9. If $f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^\alpha} & ;\ if\ f(x,y) \neq (0,0) \\ 0\ ; & if\ f(x,y) = (0,0) \end{cases}$ **then which of the following is TRUE.**

(a) For $\alpha = 1$, f is continuous but not differentiable.

(b) For $\alpha = \frac{1}{2}$, f is continuous and differentiable.

(c) For $\alpha = \frac{1}{4}$, f is continuous and differentiable.

(d) For $\alpha = \frac{3}{4}$, f is neither continuous nor differentiable.

Sol: $a=1, b=1, c=2\alpha, d=2\alpha$

Continuous $\frac{a}{c}+\frac{b}{d}>1=\frac{1}{2\alpha}+\frac{1}{2\alpha}>1 \Rightarrow \frac{1}{\alpha}>1 \Rightarrow \alpha<1$

Differentiable $\frac{a}{c}+\frac{b}{d}>\ 1+\frac{1}{c} \Rightarrow \frac{1}{2\alpha}+\frac{1}{2\alpha}>1+\frac{1}{2\alpha} \Rightarrow \frac{1}{2\alpha}>1 \Rightarrow \alpha<\frac{1}{2}$

Hence (C) is correct answer.

10. If $f(x,y)=\begin{cases}\frac{x^2|x|^{\beta}y}{x^4+y^2} & ;\ if\ f(x,y)\neq(0,0)\\ 0\ ; & if\ f(x,y)=(0,0)\end{cases}$ then f(x,y) is.

(a) f is continuous for $\beta=0$

(b) f is continuous for $\beta>0$

(c) f is continuous for $\beta<0$

(d) not differentiable for any β

Sol: $a=2+\beta, b=1, c=4, d=2$

Continuous $\frac{a}{c}+\frac{b}{d}>1=\frac{2+\beta}{4}+\frac{1}{2}>1\Rightarrow 2+\beta+2>4\Rightarrow\beta>0$

Differentiable $\frac{a}{c}+\frac{b}{d}>\ 1+\frac{1}{c}\Rightarrow\frac{2+\beta}{4}+\frac{1}{2}>1+\frac{1}{4}\Rightarrow 2+\beta+2>5\Rightarrow\beta>1$

Hence (B) is correct answer.

11. If $f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2}; & if\ f(x,y)\neq 0\\ 0\ ; & if\ f(x,y)=0\end{cases}$ then

(a) f_x, f_y exist at (0,0) and f is continuous at (0,0)

(b) f_x , f_y exist at (0,0) and f is discontinuous at (0,0)

(c) f_x , f_y does not exist at (0,0) and f is continuous at (0,0)

(d) f_x , f_y does not exist at (0,0) and f is discontinuous at (0,0)

Sol: a=1, b=1, c=2, d=2 and f(x,y) is symmetric function

$\frac{a}{c}+\frac{b}{d}=\frac{1}{2}+\frac{1}{1}=1\ngtr 1$, f(x, y) is not continuous at (0,0). So that f(x, y) is not differentiable at (0,0)

Since a>0 and b>0 hence $f_x(0,0)=f_y(0,0)=0$

Hence (B) is correct answer.

12. If $f:R^2\longrightarrow R$ defined by $f(x,y)=\begin{cases}xy\frac{(x^2-y^2)}{x^2+y^2}; if\ f(x,y)\neq 0\\ 0\qquad ; if\ f(x,y)=0\end{cases}$ then at (0,0)

(a) f is not continuous

(b) f is continuous and both f_x, f_y exist

(c) f is differentiable

(d) f_x, f_y exist but f is not differentiable

Sol:

$$f(x,y)=xy\frac{(x^2-y^2)}{x^2+y^2}=\frac{x^3y}{x^2+y^2}-\frac{xy^3}{x^2+y^2}=f_1(x,y)-f_2(x,y)$$

$f_1=\frac{x^3y}{x^2+y^2}=\frac{x^ay^b}{x^c+y^d}\Rightarrow a=3, b=1, c=2, d=2$

$f_1\Rightarrow\frac{a}{c}+\frac{b}{d}=\frac{3}{2}+\frac{1}{2}=2>1$ Hence $f_1(x,y)$ is continuous.

$f_1=\frac{a}{c}+\frac{b}{d}>\ 1+\frac{1}{c}\Rightarrow\frac{3}{2}+\frac{1}{2}>\ 1+\frac{1}{2}\Rightarrow 2>\frac{2}{2}$ Hence $f_1(x,y)$ is differentiable.

$f_{1x} = 0 \ \& \ f_{1y} = 0$

$$f_2 = \frac{xy^3}{x^2+y^2} = \frac{x^a y^b}{x^c + y^d} \Rightarrow a = 1, b = 3, c = 2, d = 2$$

$f_2 \Rightarrow \frac{a}{c} + \frac{b}{d} = \frac{1}{2} + \frac{3}{2} = 2 > 1$ Hence $f_2(x,y)$ is continuous.

$f_2 = \frac{a}{c} + \frac{b}{d} > 1 + \frac{1}{c} \Rightarrow \frac{1}{2} + \frac{3}{2} > 1 + \frac{1}{2} \Rightarrow 2 > \frac{3}{2}$ Hence $f_1(x,y)$ is differentiable.

$f_{2x} = 0 \ \& \ f_{2y} = 0$

Hence (B & C) are correct answers.

13. If $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}; & \text{if } f(x,y) \neq 0 \\ 0 ; & \text{if } f(x,y) = 0 \end{cases}$ then

(a) f is discontinuous at (0,0)

(b) f is continuous at (0,0)

(c) $f_x(0,0) = 0, f_y(0,0) = 0$

(d) $f_x(0,0) = 1, f_y(0,0) = 0$

Sol: $f(x,y) = \frac{x^3}{x^2+y^2} = \frac{x^a y^b}{x^c+y^d} \Rightarrow a = 3, b = 0 \ngtr 0, c = 2, d = 2$

Since Degree of numerator > Degree of Denominator. Hence f(x,y) is continuous

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, \ y_0) - f(x_0,y_0)}{h} \Rightarrow \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$f_x(0,0) = \frac{h-0}{h} = 1$$

$$f_y(x_0,y_0) = \lim_{k\to 0} \frac{f(x_0, \ y_0+k) - f(x_0,y_0)}{k} \Rightarrow \frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$f_y(0,0) = \frac{0-0}{=k} = 0$$

Hence (B & D) are correct answers.

(14) If the function $f: R \longrightarrow R$ defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right); & \text{if } x \neq 0 \\ 0 ; & \text{if } x = 0 \end{cases}$ then which of the following statements are true.

(a) $\lim_{x\to 0} f(x)$ exist

(b) f(x) is continuous at x=0

(c) f(x) is differentiable at x=0

(d) $\lim_{x\to 0} f^1(x)$ *exist*

(e) f^1 (x) is continuous at x=0

(f) f^1 (x) is differentiable at x=0

Sol: Since A function $f(x,y) = x^m \sin\left(\frac{1}{x^n}\right)$ *or* $x^m \cos\left(\frac{1}{x^n}\right)$ then

(a) continuous if $for\ all\ m, n > 0$

(b) Differentiable if only $m > n$

Given that m=2, n=1 hence m, n >0 and m>n so that f is continuous and differentiable

Since f is continuous so that limit exists.

i.e $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} \frac{x\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0} x \,.\, \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 0 * 1 = 0$ limit exist

$$f(\mathrm{x}) = x^2 \sin\left(\frac{1}{x}\right)$$

$$f^1(\mathrm{x}) = 2\mathrm{x}\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2\mathrm{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$$\lim_{x\to 0} f^1(x) = \lim_{x\to 0}\left\{2\mathrm{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right\} = \lim_{x\to 0}\left\{2\mathrm{x}\sin\left(\frac{1}{x}\right)\right\} - \lim_{x\to 0}\left\{\cos\left(\frac{1}{x}\right)\right\}$$

Since $\lim_{x\to 0}\left\{\cos\left(\frac{1}{x}\right)\right\}$ does not exist

$\lim_{x\to 0} f^1(x)$ does not exist.

$$f^1(\mathrm{x}) = 2\mathrm{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

Now $x^0\cos\left(\frac{1}{x}\right) = x^m \sin\left(\frac{1}{x^n}\right)$ $or\ x^m\cos\left(\frac{1}{x^n}\right)$ then $m = 0 \ngtr 0, n = 1 > 0\ \&\ m \ngtr n$.

Hence $x^0\cos\left(\frac{1}{x}\right)$ is not continuous and not differentiable so that $f^1(\mathrm{x})$ not continuous and not differentiable.

Homogeneous function: A function f (x, y) is said to be a homogeneous function in x and y of degree n if it can be written in any one of the following forms

(a) $f(kx, ky) = k^n f(x, y), k > 0$

(b) $f(x, y) = x^n\, g\left(\frac{y}{x}\right)$ or $y^n\, g\left(\frac{x}{y}\right)$

Example: (1) $f(x, y) = x^2 + y^2 + 2xy$ is a homogeneous function of degree 2 in x and y.

(2) If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then u is a homogeneous function of degree 0.

Euler's Theorem:

1. If $f(x, y)$ is a homogeneous function of degree n in x and y having continuous partial derivatives then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

Note: if $f(x_1, x_2, x_3, \dots \dots, x_n)$ is a homogeneous function of degree n in $x_1, x_2, x_3, \dots \dots, x_n$ then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots \dots \dots \dots + x_n\frac{\partial f}{\partial x_n} = nf$$

2. If $f(x, y)$ is a homogeneous function of degree n in x and y then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x.\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Note: If u is not homogeneous but f(u) is homogeneous then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x.\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = f(u)\{f^I(u) - 1\}$$

42. If $u = \frac{y}{x} + \frac{z}{x}$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Sol: $u = \frac{y}{x} + \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2} + \frac{-z}{x^2} \Rightarrow x\frac{\partial u}{\partial x} = \frac{-y}{x} + \frac{-z}{x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x} \Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{x}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x} \Rightarrow z\frac{\partial u}{\partial z} = \frac{z}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{-y}{x} + \frac{-z}{x} + \frac{y}{x} + \frac{z}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

(OR)

$$u(kx, ky) = \frac{ky}{kx} + \frac{kz}{kx} = k^0\left(\frac{y}{x} + \frac{z}{x}\right) = k^0\, u(x, y)$$

Hence u is a homogeneous function in x, y, z of degree n=0.

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0(u)$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

43. If $u=\tan^{-1}\left(\frac{x}{y}\right)$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Sol: $u=\tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)\Longrightarrow x\frac{\partial u}{\partial x}=\frac{xy}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right)\Longrightarrow y\frac{\partial u}{\partial y}=\frac{-xy}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+=\frac{xy}{x^2+y^2}-\frac{xy}{x^2+y^2}=0$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

(OR) $u(kx,ky)=\tan^{-1}\left(\frac{kx}{ky}\right)=k^0\left(\tan^{-1}\left(\frac{x}{y}\right)\right)=k^0u(x,y)$

Hence u is a homogeneous function of degree n=0.

By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0(u)$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

44. If $u=\log\left(\frac{x^3+y^3}{x+y}\right)$ then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

Sol: Given that $u=\log\left(\frac{x^3+y^3}{x+y}\right)\Longrightarrow e^u=\frac{x^3+y^3}{x+y}$

$$e^{u(kx,ky)}=\frac{(kx)^3+(ky)^3}{kx+ky}=k^2\left(\frac{x^3+y^3}{x+y}\right)=k^2e^u$$

Hence e^u is a homogeneous function in x, y of degree n=2.

By Euler's theorem, $x\frac{\partial}{\partial x}e^u+y\frac{\partial}{\partial y}e^u=2e^u$

$$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=2e^u\Longrightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2$$

45. If $u=\sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3tanu$

Sol: Given that $sinu=\frac{x^2y^2}{x+y}$

$$sinu(kx,ky)=\frac{(kx)^2(ky)^2}{kx+ky}=k^3\left(\frac{x^2y^2}{x+y}\right)=k^3sinu$$

Hence sinu is a homogeneous function in x, y of degree n=3.

By Euler's theorem, $x\frac{\partial}{\partial x}(\mathrm{sinu})+y\frac{\partial}{\partial y}(sinu)=3sinu$

$$x\,cosu\frac{\partial u}{\partial x}+ycosu\frac{\partial u}{\partial y}=3sinu$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\frac{sinu}{cosu}=3\,tanu$$

46. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\, tanu$

Sol: Given that $\mathrm{u} = \sin^{-1}\left(\frac{\mathrm{x+y}}{\sqrt{\mathrm{x}}+\sqrt{\mathrm{y}}}\right) \Rightarrow sinu = \frac{\mathrm{x+y}}{\sqrt{\mathrm{x}}+\sqrt{\mathrm{y}}}$

$$sinu(kx, ky) = \frac{(kx) + (ky)}{\sqrt{\mathrm{kx}} + \sqrt{\mathrm{ky}}} = k^{\frac{1}{2}}\left(\frac{\mathrm{x + y}}{\sqrt{\mathrm{x}} + \sqrt{\mathrm{y}}}\right) = k^{\frac{1}{2}} sinu$$

Hence sinu is a homogeneous function in x, y of degree n=1/2.

By Euler's theorem, $x\frac{\partial}{\partial \mathrm{x}}\mathrm{sinu} + y\frac{\partial}{\partial \mathrm{y}} sinu = \frac{1}{2} sinu$

$$\mathrm{x\ cosu}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{ycosu}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = \frac{1}{2}\mathrm{sinu}$$

$$\mathrm{x}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = 2\frac{\mathrm{sinu}}{\mathrm{cosu}} = \frac{1}{2}\ \mathrm{tanu}$$

47. If $\mathrm{u} = \sec^{-1}\left(\frac{\mathrm{x+2y+3z}}{\sqrt{\mathrm{x}^8}+\sqrt{\mathrm{y}^8}+\sqrt{\mathrm{z}^8}}\right)$ then find $x\frac{\partial u}{\partial \mathrm{x}} + y\frac{\partial u}{\partial \mathrm{y}} + z\frac{\partial u}{\partial \mathrm{z}}$

Sol: Given that $\mathrm{u} = \sec^{-1}\left(\frac{\mathrm{x+2y+3z}}{\sqrt{\mathrm{x}^8}+\sqrt{\mathrm{y}^8}+\sqrt{\mathrm{z}^8}}\right) \Rightarrow secu = \frac{\mathrm{x+2y+3z}}{\sqrt{\mathrm{x}^8}+\sqrt{\mathrm{y}^8}+\sqrt{\mathrm{z}^8}}$

$$\mathrm{secu(kx, ky)} = \frac{\mathrm{kx + 2ky + 3kz}}{\sqrt{(\mathrm{kx})^8} + \sqrt{(\mathrm{ky})^8} + \sqrt{(\mathrm{kz})^8}} = \mathrm{k}^{-3}\left(\frac{\mathrm{x + 2y + 3z}}{\sqrt{\mathrm{x}^8} + \sqrt{\mathrm{y}^8} + \sqrt{\mathrm{z}^8}}\right) = \mathrm{k}^{-3}\mathrm{secu}$$

Hence secu is a homogeneous function in x, y of degree n=-3.

By Euler's theorem, $\mathrm{x}\frac{\partial}{\partial \mathrm{x}}\mathrm{secu} + \mathrm{y}\frac{\partial}{\partial \mathrm{y}}\mathrm{secu} + \mathrm{z}\frac{\partial}{\partial \mathrm{z}}\mathrm{secu} = -3\ \mathrm{secu}$

$$\mathrm{x\ secu.tanu}\ \frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y\ secu\ tanu}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{z\ secu\ tanu}\frac{\partial \mathrm{u}}{\partial \mathrm{z}} = -3\ \mathrm{secu}$$

$$\mathrm{x}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{z}\frac{\partial \mathrm{u}}{\partial \mathrm{z}} = -3\frac{\mathrm{secu}}{\mathrm{secu\ tanu}} = -3\ \mathrm{cotu}$$

48. If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$ then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Sol: Given that $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

$$u(kx, ky) = (kx)^2\tan^{-1}\frac{ky}{kx} - (ky)^2\tan^{-1}\frac{kx}{ky}$$

$$\mathrm{u(kx, ky)} = \mathrm{k}^2\left(x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}\right) = \mathrm{k}^2\mathrm{u}$$

Hence u is a homogeneous function in x, y of degree n=2.

By Euler's theorem, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u$

49. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$

Sol: Given that $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right) \Longrightarrow \tan u = \frac{x^3+y^3}{x+y}$

$$\tan u(kx, ky) = \frac{(kx)^3 + (ky)^3}{kx + ky} = k^2\left(\frac{x^3 + y^3}{x + y}\right) = k^2 tanu$$

Hence tanu is a homogeneous function in x, y of degree n=2

By Euler's theorem, $x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}tanu = 2\,tanu$

$$x\,sec^2u\frac{\partial u}{\partial x} + y\,sec^2u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{sec^2u} = 2\,sinu\,cosu = sin2u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \dots\dots\dots(1)$$

Differentiating (1) partially w.r.t. x we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x.\partial y} = 2cos2u\,\frac{\partial u}{\partial x}$$

Multiplying by x we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x.\partial y} = 2cos2u\,x\frac{\partial u}{\partial x}\dots\dots\dots..(2)$$

Differentiating (1) partially w.r.t. y we get

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2cos2u\,\frac{\partial u}{\partial y}$$

Multiplying by y we get

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2cos2u\,y\frac{\partial u}{\partial y}\dots\dots\dots..(3)$$

Adding (2) and (3) we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x.\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2cos2u\,x\frac{\partial u}{\partial x} + 2cos2u\,y\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2cos2u\,x\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + (\sin 2u) = 2cos2u\,(\sin 2u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u(2cos2u - 1)2sin2u\,cos2u - sin2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = sin4u - sin2u\ \ (or)2sinu\,cos3u$$

50. If $u = \tan^{-1}\frac{y^2}{x}$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u\, sin^2 u$

Sol: Given that $u = \tan^{-1}\left(\frac{y^2}{x}\right) \Longrightarrow \tan u = \frac{y^2}{x}$

$$\tan u(kx, ky) = \frac{(ky)^2}{kx} = k^1\left(\frac{y^2}{x}\right) = k^1 tanu$$

Hence tanu is a homogeneous function in x, y of degree n=1

By Euler's theorem, $x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}tanu = 2\, tanu$

$$x\, sec^2 u\frac{\partial u}{\partial x} + y\, sec^2 u\frac{\partial u}{\partial y} = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{sec^2 u} = sinu\, cosu = \frac{2}{2}sinu\, cosu = \frac{1}{2}sin2u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u \ldots\ldots\ldots (1)$$

Differentiating (1) partially w.r.t. x we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x.\partial y} = \frac{1}{2}cos2u\,\frac{\partial u}{\partial x}$$

Multiplying by x we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x.\partial y} = \frac{1}{2}cos2u\, x\frac{\partial u}{\partial x} \ldots\ldots\ldots..(2)$$

Differentiating (1) partially w.r.t. y we get

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{1}{2}cos2u\,\frac{\partial u}{\partial y}$$

Multiplying by y we get

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = \frac{1}{2}cos2u\, y\frac{\partial u}{\partial y} \ldots\ldots\ldots..(3)$$

Adding (2) and (3) we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x.\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = cos2u\, x\frac{\partial u}{\partial x} + cos2u\, y\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = cos2u\,\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + \left(\frac{1}{2}\sin 2u\right) = cos2u\,\left(\frac{1}{2}\sin 2u\right) = \left(\frac{1}{2}\sin 2u\right)(cos2u - 1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}sin2u\,(-2\sin^2 u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u\,\sin^2 u$$

Exercise Problems:

1. If $u = \sec^{-1}\left(\frac{x^2+y^2}{x+y}\right)$,then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = cotu$
2. If $u = \sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{2}}$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{144}\text{tanu}(sec^2u + 12)$
3. If $u = \sqrt{y^2 - x^2}\sin^{-1}\left(\frac{y}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

Laplace Equation: If u is a function of two variables x and y, then the partial differential equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mathbf{0}$ is called Laplace's equation in two variables or Two-dimensional Laplace's equation.

Harmonic function: A function u(x,y) is called harmonic, if it satisfies Laplace's equation, The operator ∇^2 is called the Laplacian and $\nabla^2 u$ is called the Laplacian of u.

1. Verify $u(x,y) = x^3-3xy^2-5y$ is harmonic in the entire complex plane.

Sol: Given that $u = x^3-3xy^2-5y$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Longrightarrow \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy - 5 \Longrightarrow \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

Hence u(x,y) is harmonic function.

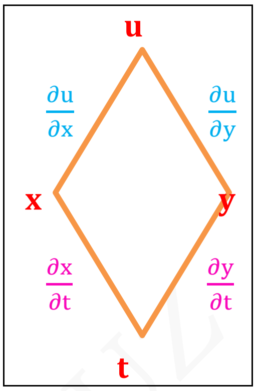
Examples:

2. Linear functions: Any function of the form f(x, y) = ax + by is harmonic.
3. Polynomials: Functions like $f(x, y) = x^2 - y^2$ and f(x, y) = 2xy are harmonic.
4. Exponential and Trigonometric combinations: A function such as $f(x, y) = e^x\cos y$ is a classic example.
5. Logarithmic functions: The function $f(x, y) = \ln(x^2 + y^2)$ is harmonic on any domain not containing the origin.

Chain Rule or Total Derivative of the Composite function and Implicit function:

Let u=f(x,y) is differentiable functions of x & y and x, y are differentiable functions of an independent variable t {i.e; x=x(t), y=y(t)} then the total derivative of u with respect to t is given by $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$

which expresses the rate of change of u w.r.t. t along the curve x=x(t), y=y(t)

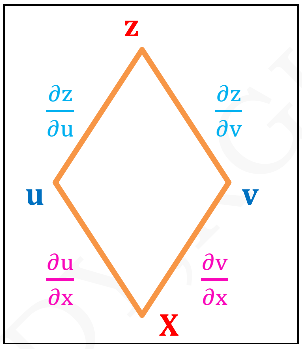


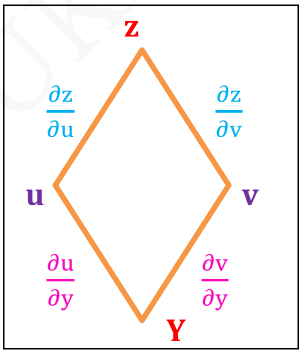
Note: If u=f(x,y,z) is differentiable and x, y, z are differentiable functions of t then u is differential function of t and $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$

Composite functions: Let z = f (u,v) and u and v are themselves functions of the independent variables x, y so that u = φ(x, y) and v = ψ(x, y) is called composite function. Then the partial derivative of z w.r.t. x and y, given by

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

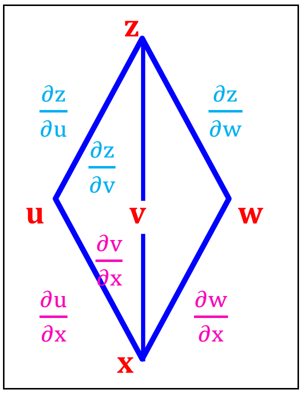


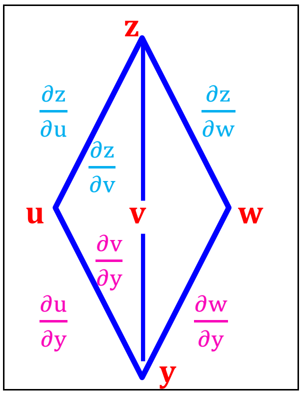


Note: z= f (u,v,w) and u,v,w are themselves functions of the independent variables x, y then the partial derivative of z w.r.t. x and y, given by

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$





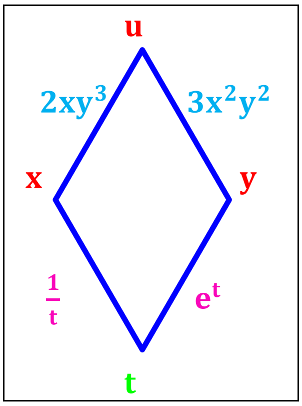
Implicit functions: Let f(x,y)=c where c is a constant, define y as an implicit function of x. The derivative of f(x,y)=c is given by $\frac{dy}{\partial x}=-\frac{f_x}{f_y}$, Similarly $\frac{dx}{\partial y}=-\frac{f_y}{f_x}$

$$\frac{d^2y}{dx^2}=-\frac{f_{xx}(f_y)^2-2f_{yx}.f_x.f_y+f_{yy}(f_x)^2}{(f_y)^3}$$

Note: If the equation f(x,y,z)=0 such that x,y are independent and z is dependent, then

$$\frac{\partial z}{\partial x}=-\frac{f_z}{f_x}\ \&\ \frac{\partial z}{\partial y}=-\frac{f_z}{f_y}$$

51. If $\mathbf{u = x^2y^3, x = logt, y = e^t}$ **then find** $\frac{\mathbf{du}}{\mathbf{dt}}$



Sol: we have $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$u = x^2y^3 \Rightarrow \frac{\partial u}{\partial x} = 2xy^3 \ and \ \frac{\partial u}{\partial y} = 3x^2y^2$$

$$x = logt \Rightarrow \frac{\partial x}{\partial t} = \frac{1}{t} \quad \text{and} \quad y = e^t \Rightarrow \frac{\partial t}{\partial t} = e^t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = (2xy^3)\left(\frac{1}{t}\right) + (3x^2y^2)(e^t)$$

$$\frac{du}{dt} = 2logt\,(e^t)^3\left(\frac{1}{t}\right) + 3(logt)^2(e^t)^2(e^t)$$

$$\frac{du}{dt} = \frac{2}{t}logt\ e^{3t} + e^{3t}(logt)^2$$

52. If $\mathbf{f = x\,cosy + e^x\,siny}$ **and** $\mathbf{x = t^2 + 1}, \boldsymbol{y} = \mathbf{t^3} + \boldsymbol{t}$ **then find** $\frac{\mathbf{df}}{\mathbf{dt}}$ **at** $\mathbf{t = 0}$

Sol: We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$f = x\,cosy + e^x\,siny$ & $x = t^2 + 1, y = t^3 + t$

when t=o then x=1, y=0

$f = x\,cosy + e^x\,siny$

$$\Rightarrow \frac{\partial f}{\partial x} = cosy + e^x\,siny \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(1,0)} = 1 + 0 = 1$$

$$\Rightarrow \frac{\partial f}{\partial y} = -x\,siny + e^x\,cosy \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(1,0)} = 0 + e = e$$

$$x = t^2 + 1 \Rightarrow \frac{\partial x}{\partial t} = 2t \Rightarrow \left(\frac{\partial x}{\partial t}\right)_{t=0} = 0$$

$$y = t^3 + t \Rightarrow \frac{\partial y}{\partial t} = 3t^2 + 1 \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = 1$$

$$\left(\frac{\partial f}{\partial t}\right)_{t=0} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 1(0) + e(1) = e$$

53. If $\mathbf{f = x^2 + y^2}$ **and** $\mathbf{x = cost + sint}$ & $\mathbf{y = cost - sint}$ **then find** $\frac{\mathbf{df}}{\mathbf{dt}}$ **at** $\mathbf{t = 0}$

Sol: We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$f = x^2 + y^2$ and $x = cost + sint, y = cost - sint$. when t=o then x=1, y=1

$f = x^2 + y^2$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(1,1)} = 2$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2y \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(1,1)} = 2$$

$$x = \cos t + \sin t \Rightarrow \frac{\partial x}{\partial t} = -\sin t + \cos t \Rightarrow \left(\frac{\partial x}{\partial t}\right)_{t=0} = 1$$

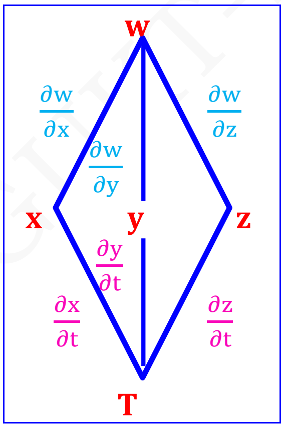
$$y = \cos t - \sin t \Rightarrow \frac{\partial y}{\partial t} = -\sin t - \cos t \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = -1$$

$$\left(\frac{\partial f}{\partial t}\right)_{t=0} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2(1) + 2(-1) = 0$$

54. If $\mathbf{w = z - \sin xy, x = t, y = \log t, z = e^{t-1}}$ **then find** $\frac{dw}{dt}$ **at** $\mathbf{t = 1}$

Sol: Given that $w = z - \sin xy, x = t, y = \log t, z = e^{t-1}$

We have $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} w = z - \sin(xy)$



$$\Rightarrow \frac{\partial w}{\partial x} = -y\cos(xy)$$

$$\Rightarrow \frac{\partial w}{\partial y} = -x\cos(xy)$$

$$\Rightarrow \frac{\partial w}{\partial z} = 1$$

$$x = t \Rightarrow \frac{\partial x}{\partial t} = 1$$

$$y = \log t \Rightarrow \frac{\partial y}{\partial t} = \frac{1}{t}$$

$$z = e^{t-1} \Rightarrow \frac{\partial z}{\partial t} = e^{t-1}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \big(-y\cos(xy)\big)(1) + \big(-x\cos(xy)\big)(1) + (1)(e^{t-1})$$

Since $x = t, y = \log t, z = e^{t-1}$

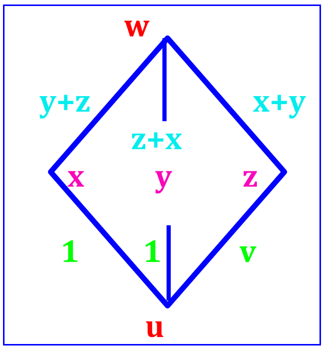
If t=1 then x=1, y=0, z=1

$$\frac{dw}{dt} = (0)(1) + (-\cos 0)(1) + (1)(1) = 0$$

55. If $\mathbf{w = xy + yz + zx}$ **and** $\mathbf{x = u + v}$ & $\mathbf{y = u - v}$ **and** $\mathbf{z = uv}$ **then find** $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$

Sol: Given that $w = xy + yz + zx$ and $x = u + v$ & $y = u - v$

By chain rule We have $\frac{\partial w}{du} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$



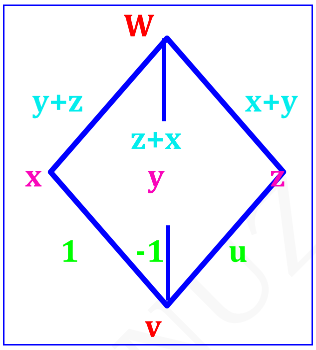
$$\Rightarrow \frac{\partial w}{\partial x} = y + z \ \& \ \frac{\partial w}{\partial y} = z + x \ \& \ \frac{\partial w}{\partial z} = y + x$$

$$\Rightarrow \frac{\partial x}{\partial u} = 1 \ \& \ \frac{\partial x}{\partial v} = 1$$

$$\Rightarrow \frac{\partial y}{\partial u} = 1 \ \& \ \frac{\partial y}{\partial v} = -1$$

$$\Rightarrow \frac{\partial z}{\partial u} = v \ \& \ \frac{\partial z}{\partial v} = u$$

By chain rule We have $\frac{\partial w}{du} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$



$$\frac{\partial w}{du} = (y+z)(1) + (z+x)(1) + (x+y)(v) = x + y + 2z + (x+y)v$$

$$\frac{\partial w}{du} = x + y + 2z + (x+y)v = 2u + 2uv + 2uv = 2u + 4uv$$

$$\frac{\partial w}{du} = 2u + 4uv$$

By chain rule We have $\frac{\partial w}{dv} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} + \frac{\partial w}{\partial z}\frac{dz}{dv}$

$$\frac{\partial w}{dv} = (y+z)(1) + (z+x)(-1) + (x+y)(u) = y - x + (x+y)u$$

$$\frac{\partial w}{dv} = y - x + (x+y)u = -2v + 2u^2$$

56. If $z = 4e^x$ logy and $x = \log(r\cos\theta), y = r\sin\theta$ then find $\frac{\partial z}{dr}, \frac{\partial z}{d\theta}$

Sol: Given that $z = 4e^x$ logy and $x = \log(r\cos\theta), y = r\sin\theta$

$$\frac{\partial z}{dx} = 4e^x \log y \text{ and } \frac{\partial z}{dy} = \frac{4e^x}{y}$$

$$\frac{\partial x}{\partial r} = \frac{\cos\theta}{r\cos\theta} = \frac{1}{r} \ \& \ \frac{\partial y}{dr} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -\frac{r\sin\theta}{r\cos\theta} = -\tan\theta \ \& \ \frac{\partial y}{d\theta} = r\cos\theta$$

By chain rule We have $\frac{\partial z}{dr} = \frac{\partial z}{\partial x}\frac{dx}{dr} + \frac{\partial z}{\partial y}\frac{dy}{dr}$

$$\frac{\partial z}{\partial r} = (4e^x \log y)\left(\frac{1}{r}\right) + \left(\frac{4e^x}{y}\right)(\sin\theta) = 4e^x\left(\frac{\log y}{r} + \frac{\sin\theta}{y}\right)$$

By chain rule We have $\frac{\partial z}{d\theta} = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta}$

$$\frac{\partial z}{\partial \theta} = (4e^x \log y)(-\tan\theta) + \left(\frac{4e^x}{y}\right)(r\cos\theta) = 4e^x\left(-\log y \tan\theta + \frac{r\cos\theta}{y}\right)$$

57. Find $\frac{dy}{dx}$ at (1,1) if $x^3 - 2y^2 + xy = 0$

Sol: Given that $x^3 - 2y^2 + xy = 0$ and Let $f(x,y) = x^3 - 2y^2 + xy$

$f_x = 3x^2 + y$ and $f_y = -4y + x$

By the definition of implicit function we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = \frac{3x^2 + y}{-4y + x} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -2$$

(OR)

Given that $x^3 - 2y^2 + xy = 0$

Differentiate w.r.to x we get $3x^2 - 4y\frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) = 0$

$$3x^2 + y + \frac{dy}{dx}(x - 4y) = 0$$

$$\frac{dy}{dx} = \frac{3x^2 + y}{x - 4y}$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = -2$$

58. Find $\frac{dy}{dx}$ if x and y are connected by the relation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Sol: Given that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Let $f(x,y) = x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}}$

$f_x = \frac{2}{3}x^{\frac{-1}{3}}$ and $f_y = \frac{2}{3}y^{\frac{-1}{3}}$

By the definition of implicit function we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = \frac{\frac{2}{3}x^{\frac{-1}{3}}}{\frac{2}{3}y^{\frac{-1}{3}}} = \left(\frac{x}{y}\right)^{\frac{-1}{3}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

59. if $x^y = y^x$ then find $\frac{dy}{dx}$

Sol: Given that $x^y = y^x$

Let $f(x,y) = x^y - y^x$

$f_x = yx^{y-1} - y^x logy$

$f_y = x^y logx - xy^{x-1}$

By the definition of implicit function we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = -\frac{yx^{y-1} - y^x logy}{x^y logx - xy^{x-1}} = -\frac{yx^{y-1} - x^y logy}{y^x logx - xy^{x-1}} = -\frac{x^y}{y^x}\frac{yx^{-1} - logy}{logx - xy^{-1}} = -\frac{\frac{1}{x}(y - xlogy)}{\frac{1}{y}(ylogx - x)}$$

$$\frac{dy}{dx} = \frac{y\,(y - xlogy)}{x\,(ylogx - x)}$$

60. Find $\frac{dy}{dx}$ if $f(x,y) = log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

Sol: Given that $f(x,y) = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

$$f_x = \frac{2x}{x^2 + y^2} + \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{2x - y}{x^2 + y^2}$$

$$f_y = \frac{2y}{x^2 + y^2} + \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{2y + x}{x^2 + y^2}$$

By the definition of implicit function we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = -\frac{\frac{2x - y}{x^2 + y^2}}{\frac{2y + x}{x^2 + y^2}} = \frac{y - 2x}{2y + x}$$

61. Find $\frac{\partial z}{dx}$ & $\frac{\partial z}{dx}$ at (1,1,1) if $z^3 - xy + yz + y^3 - 2 = 0$

Sol: Given that $f(x, y) = z^3 - xy + yz + y^3 - 2$

$f_x = -y \Longrightarrow$

$f_y = -x + z + 3y^2$

$f_z = 3z^2 + y$

By the definition of implicit function we have $\frac{\partial z}{dx} = -\frac{f_x}{f_z}$ & $\frac{\partial z}{dy} = -\frac{f_y}{f_z}$

$$\frac{\partial z}{dx} = -\frac{-y}{3z^2 + y} \Longrightarrow \left(\frac{\partial z}{dx}\right)_{(1,1,1)} = \frac{1}{4}$$

$$\frac{\partial z}{dy} = -\frac{-x + z + 3y^2}{3z^2 + y} \Longrightarrow \left(\frac{\partial z}{dx}\right)_{(1,1,1)} = \frac{-3}{4}$$

62. Find $\frac{du}{dx}$ if $u = cos(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$

Sol: Given that $u = \cos(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$

We have $du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} \quad \ldots \ldots . . (1)$$

$u = \cos(x^2 + y^2)$

$$\Longrightarrow \frac{\partial u}{\partial x} = -2x sin(x^2 + y^2)$$

$$\Longrightarrow \frac{\partial u}{\partial y} = -2y sin(x^2 + y^2)$$

$a^2x^2 + b^2y^2 = c^2$

$let\ f = a^2x^2 + b^2y^2 - c^2$

$$\Longrightarrow \frac{\partial f}{\partial x} = 2a^2x \ \& \ \frac{\partial f}{\partial y} = 2b^2y$$

By the definition of implicit function we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = -\frac{2a^2 x}{2b^2 y} = -\frac{a^2 x}{b^2 y}$$

From (1)

$$\frac{du}{dx} = \{-2x\, sin(x^2 + y^2)\}(1) + \{-2y\, sin(x^2 + y^2)\}\left\{-\frac{a^2 x}{b^2 y}\right\}$$

$$\frac{du}{dx} = sin(x^2 + y^2)\left\{-2x + 2y\frac{a^2 x}{b^2 y}\right\}$$

$$\frac{du}{dx} = sin(x^2 + y^2)\left\{\frac{2(a^2 - b^2)x}{b^2}\right\}$$

63. if $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$ then find the value of $\frac{dz}{dx}$ when x=y=a

Sol: Given that $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$

We have $du = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \ldots\ldots(1)$$

$$z = \sqrt{x^2 + y^2}$$

$$\Longrightarrow \frac{\partial z}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\Longrightarrow \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$x^3 + y^3 + 3axy = 5a^2$$

Let $f = x^3 + y^3 + 3axy = 5a^2$

$$\Longrightarrow \frac{\partial f}{\partial x} = 3x^2 - 3ay \ \& \ \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

By the definition of implicit function we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = -\frac{x^2 - ay}{y^2 - ax}$$

From (1)

$$\frac{du}{dx} = \left\{\frac{x}{\sqrt{x^2 + y^2}}\right\}(1) + \left\{\frac{x}{\sqrt{x^2 + y^2}}\right\}\left\{-\frac{x^2 - ay}{y^2 - ax}\right\}$$

$$\left(\frac{du}{dx}\right)_{x=y=a} = \frac{1}{\sqrt{2a^2}} + \frac{1}{\sqrt{2a^2}}\left\{-\frac{2a^2}{2a^2}\right\} = 0$$

Exercise Problems:

1. if $u = \frac{p-q}{q-r}, p = x + y + z, q = x - y + z, r = x + y - z$ Evaluate $\frac{\partial \mathrm{u}}{\partial \mathrm{x}}, \frac{\partial \mathrm{u}}{\partial \mathrm{y}}$ at the given point $(\sqrt{3}, 2, 1)$
2. If H=f(x,y,z) where x=u+v+w, y=uv+vw+wu, z=uvw show that $u\frac{\partial \mathrm{H}}{\partial \mathrm{u}} + v\frac{\partial \mathrm{H}}{\partial \mathrm{v}} + w\frac{\partial \mathrm{H}}{\partial \mathrm{w}} = x\frac{\partial \mathrm{H}}{\partial \mathrm{x}} + 2y\frac{\partial \mathrm{H}}{\partial \mathrm{y}} + 3z\frac{\partial \mathrm{H}}{\partial \mathrm{z}}$
3. Find $\frac{dy}{dx}$ if $x^2 + xy + 3 = xy^2 + \sqrt{x}$
4. If $\mathrm{w} = \mathrm{f}(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ then find $\frac{1}{\mathrm{x}}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{1}{\mathrm{y}}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \frac{1}{\mathrm{z}}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}$

Jacobian: If u and v are continuous functions of two independent variables x and y having first order partial derivatives, then the determent $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u and v with respect to x and y and denoted by $J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Note: If u, v and w, are functions of x, y and z, then $J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Jacobi implicit function: If u, v, w, are implicit functions of x, y and z, $i.e;\ f_i(u,v,w,x,y,z) = 0$ then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)\ /\ \partial(x,y,z)}{\partial(f_1,f_2,f_3)\ /\ \partial(u,v,w)}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)\ /\ \partial(u,v,w)}{\partial(f_1,f_2,f_3)\ /\ \partial(x,y,z)}$$

Properties of the Jacobian:

1. If u and v are functions of independent variables r and s and r and s are functions of the variables x and y, then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} * \frac{\partial(r,s)}{\partial(x,y)}$
2. If u and v are functions in x & y and x,y are functions in u & v then $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$

   $$Let\ J = \frac{\partial(u,v)}{\partial(x,y)} \,\&\, J^I = \frac{\partial(x,y)}{\partial(u,v)} \Longrightarrow JJ^I = 1$$
3. If u and v are functions of x and y and if u, v are functionally dependent then Jacobian vanish.

   $i.e;\ \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Note: if u and v are functionally dependent then u and v are connected by a relation F(u,v)=0

64. If $u = x^2 - 2xy$ & $v = x + y + z$ and $w = x - 2y + 3z$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol:

$u = x^2 - 2xy$ $\quad$ $v = x + y + z$ $\quad$ $w = x - 2y + 3z$

$\frac{\partial u}{\partial x} = u_x = 2x$ $\quad$ $\frac{\partial v}{\partial x} = v_x = 1$ $\quad$ $\frac{\partial w}{\partial x} = w_x = 1$

$\frac{\partial u}{\partial y} = u_y = 2$ $\quad$ $\frac{\partial v}{\partial y} = v_y = 1$ $\quad$ $\frac{\partial w}{\partial y} = w_y = -2$

$\frac{\partial u}{\partial z} = u_z = 0$ $\quad$ $\frac{\partial v}{\partial z} = v_z = 1$ $\quad$ $\frac{\partial w}{\partial z} = w_z = 3$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 2x & 2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 2x(3+2) - 2(3-1) + 0$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 10x + 4$$

65. If $x = r\cos\theta, y = r\sin\theta$ then prove that $\frac{\partial(r,\theta)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(r,\theta)} = 1$

Sol:

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ and $\quad$ $x^2 + y^2 = r^2$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial x}{\partial r} = \cos\theta$ $\quad$ $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\quad$ $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Since $x = r\cos\theta$ & $y = r\sin\theta \implies x^2 + y^2 = r^2$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r^2 = x^2 + y^2$ $\quad$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$ $\quad$ $\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$

$2r\frac{\partial r}{\partial y} = 2y \implies \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$ $\quad$ $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x^2}\right) = \frac{x}{x^2+y^2} = -\frac{\cos\theta}{r}$

$$J^I = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}(\cos^2\theta + \sin^2\theta) = \frac{1}{r}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r.\frac{1}{r} = 1$$

Hence $J.J^I = 1$

(OR)

Let $f_1 = x - r\cos\theta$ $\quad f_2 = y - r\sin\theta$

$\frac{\partial f_1}{\partial x} = 1$ $\quad \frac{\partial f_2}{\partial x} = 0$

$\frac{\partial f_1}{\partial y} = 0$ $\quad \frac{\partial f_2}{\partial y} = 1$

$\frac{\partial f_1}{\partial r} = -\cos\theta$ $\quad \frac{\partial f_2}{\partial r} = -\sin\theta$

$\frac{\partial f_1}{\partial \theta} = r\sin\theta$ $\quad \frac{\partial f_2}{\partial \theta} = -r\cos\theta$

$$\partial(f_1, f_2)/\partial(x, y) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\partial(f_1, f_2)/\partial(r, \theta) = \begin{vmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = (-1)^2 \frac{\partial(f_1, f_2)\,/\,\partial(r,\theta)}{\partial(f_1, f_2)\,/\,\partial(x,y)} = \frac{r}{1} = r$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1, f_2)\,/\,\partial(x,y)}{\partial(f_1, f_2)\,/\,\partial(r,\theta)} = \frac{1}{r}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r.\frac{1}{r} = 1$$

66. If $u = x + y$ & $v = xy$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

Sol:

$u = x + y$ $\quad v = xy$

$\frac{\partial u}{\partial x} = u_x = 1$ $\quad \frac{\partial v}{\partial x} = v_x = y$

$\frac{\partial u}{\partial y} = u_y = 1$ $\quad \frac{\partial v}{\partial y} = v_y = x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix} = x - y$$

Since $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{x - y}$$

67. If $u = 2xy, v = x^2 - y^2$ & $x = r\cos\theta, y = r\sin\theta$ thenfind $\frac{\partial(u,v)}{\partial(r,\theta)}$

$u = 2xy$ $\quad v = x^2 - y^2$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad$ and $x^2 + y^2 = r^2$

$\frac{\partial u}{\partial x} = 2y$ $\quad \frac{\partial v}{\partial x} = 2x$ $\quad \frac{\partial x}{\partial r} = \cos\theta$ $\quad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial u}{\partial y} = 2x$ $\quad \frac{\partial v}{\partial y} = -2y$ $\quad \frac{\partial x}{\partial \theta} = -r\sin\theta$ $\quad \frac{\partial y}{\partial \theta} = r\cos\theta$

Since we have $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} * \frac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2 + y^2) = -4r^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} * \frac{\partial(x,y)}{\partial(r,\theta)} = (-4r^2)r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4r^3$$

68. If $u = xy + yz + zx$ & $v = x^2 + y^2 + z^2$ & $w = x + y + z$ then show that u, v, w, are functionally dependent and find relation between them.

Sol:

$u = xy + yz + zx$ $\quad$ $v = x^2 + y^2 + z^2$ $\quad$ $w = x + y + z$

$u_x = y + z$ $\quad$ $v_x = 2x$ $\quad$ $w_x = 1$

$u_y = x + z$ $\quad$ $v_y = 2y$ $\quad$ $w_y = 1$

$u_z = x + y$ $\quad$ $v_z = 2z$ $\quad$ $w_z = 1$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Hence u,v,w are functionally dependent.

$w^2 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = u + 2v$

$w^2 = u + 2v$

Exercise problems:

1. If $u = xyz, v = x^2 + y^2 + z^2, w = x + y + z$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

2. If $u = x + y + z, uv = y + z, uvw = z$ then prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

3. The transformation equations in spherical polar coordinates is

   $x = r\sin\theta\cos\phi$ & $y = r\sin\theta\ \sin\phi$ & $z = r cos\theta$. Then show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 sin\theta$

4. In cylindrical polar coordinates, $x = r\cos\theta, y = r\sin\theta, z = z$. Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

5. If $x = u(1 - v)$ & $y = uv$ then prove that $\frac{\partial(x,y)}{\partial(u,v)} * \frac{\partial(u,v)}{\partial(x,y)} = 1$

6. If $x = e^u\cos u$ & $y = e^u\sin u$ then prove that $\frac{\partial(x,y)}{\partial(u,v)} * \frac{\partial(u,v)}{\partial(x,y)} = 1$

Verify which of the following are functionally dependent or not and find relation between them.

7. If $u = \frac{x+y}{1-xy}$ & $v = \tan^{-1}x + \tan^{-1}y$

   Ans: $v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(u) \Rightarrow v = \tan^{-1}(u)$

8. If $u = \frac{x+y}{x-y}$ & $v = \frac{xy}{(x-y)^2}$

   Ans: $u^2 = \left(\frac{x+y}{x-y}\right)^2 = \frac{(x-y)^2}{(x-y)^2} + \frac{4xy}{(x-y)^2} = 1 + 4v \Rightarrow u^2 = 1 + 4v$

9. If $u = \frac{x}{y}, v = \frac{x+y}{x-y}$

   Ans: $v = \frac{x+y}{x-y} = \frac{\frac{x}{y}+1}{\frac{x}{y}-1} = \frac{u+1}{u-1} \Rightarrow v = \frac{u+1}{u-1}$

10. If $u = \sin^{-1}x + \sin^{-1}y$ & $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

    Ans: $u = \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sin^{-1}v$

11. If $u = y + z, v = x + 2z^2, w = x - 4yz - 2y^2$

    Ans: $w = x - 4yz - 2y^2 = v - 2z^2 - 4yz - 2y^2 = v - 2(y+z)^2 = v - 2u^2$

12. If $u = x + y + z, v = x^2 + y^2 + z^2, w = x^3 + y^3 + z^3 - 3xyz$

    Ans: $w = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

    $$w = u(v - xy - yz - zx) = u\left(v - \frac{u^2 - v}{2}\right) \Rightarrow 2w = u(3v - u^2)$$

13. If $u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}$

    Ans: $uv + vw + uw = -1$

14. If $u=x+2y+z$, $v=x-2y+3z$ and $w=2xy-xz+4yz-2z^2$

    Ans: $u^2-v^2=4w$.