

$$\frac{\partial w}{\partial x} = 2y - z, \frac{\partial w}{\partial y} = 2x + 4z, \frac{\partial w}{\partial z} = -x + 4y - 4z.$$

$$\begin{aligned}\frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix} \\ &= 1(-2(-x + 4y - 4z) - 3(2x + 4z)) - 2(-x + 4y - 4z - 3(2y - z)) \\ &\quad + 1(2x + 4z + 2(2y - z)) \\ &= 2x - 8y + 8z - 6x - 12z + 2x - 8y + 8z + 12y - 6z + 2x + 4z + 4y - 2z = 0.\end{aligned}$$

Hence,  $u, v, w$  are not independent.

Now  $u + v = 2x + 4z, u - v = 4y - 2z$ .

$$\begin{aligned}(u + v)(u - v) &= 2(x + 2z).2(2y - z) \\ u^2 - v^2 &= 4(2xy - xz + 4yz - 2z^2) \\ u^2 - v^2 &= 4w.\end{aligned}$$

## 2.6 Taylor's expansion for functions of two variables

We know that for a function  $f(x)$  of one single variable  $x$ , the Taylor's expansion is

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

Now let  $f(x, y)$  be a function of two independent variables  $x, y$  defined in a region  $R$  of the  $xy$ -plane and let  $(a, b)$  be a point in  $R$ . Suppose  $f(x, y)$  has all its partial

derivatives in a neighbourhood of  $(a, b)$  then

$$\begin{aligned}
 f(a+h, b+k) &= f(a, b) + \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(a, b) \\
 &\quad + \frac{1}{2!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(a, b) + \frac{1}{3!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^3 f(a, b) + \dots + \\
 &= f(a, b) + \left( h f_x(a, b) + k f_y(a, b) \right) \\
 &\quad + \frac{1}{2!} \left( h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b) \right) \\
 &\quad + \frac{1}{3!} \left( h^3 f_{xxx}(a, b) + 3h^2 k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) \right. \\
 &\quad \left. + k^3 f_{yyy}(a, b) \right) + \dots .
 \end{aligned}$$

Put  $x = a + h, y = b + k$  then  $h = x - a, k = y - b$ .

$\therefore$  The Taylor's series can be written as

$$\begin{aligned}
 f(x, y) &= f(a, b) + \left( (x-a) f_x(a, b) + (y-b) f_y(a, b) \right) \\
 &\quad + \frac{1}{2!} \left( (x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right) \\
 &\quad + \frac{1}{3!} \left( (x-a)^3 f_{xxx}(a, b) + 3(x-a)^2 (y-b) f_{xxy}(a, b) \right. \\
 &\quad \left. + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b) \right) + \dots .
 \end{aligned}$$

This is known as the Taylor's expansion of  $f(x, y)$  in the neighbourhood of  $(a, b)$  or about the point  $(a, b)$ .

Put  $a = 0, b = 0$ . we get

$$\begin{aligned}
 f(x, y) &= f(0, 0) + \left( x f_x(0, 0) + y f_y(0, 0) \right) \\
 &\quad + \frac{1}{2!} \left( x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0) \right) \\
 &\quad + \frac{1}{3!} \left( x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0) \right) + \dots .
 \end{aligned}$$

This is called Maclaurin's series for  $f(x, y)$  in powers of  $x$  and  $y$ .

### Worked Examples

**Example 2.63.** Expand  $e^x \sin y$  in powers of  $x$  and  $y$  as far as the terms of third degree. [Jun 2013]

**Solution.**  $f(x, y) = e^x \sin y \quad f(0, 0) = 0$

$$f_x(x, y) = e^x \sin y \quad f_x(0, 0) = 0$$

$$f_y(x, y) = e^x \cos y \quad f_y(0, 0) = 1$$

$$f_{xx}(x, y) = e^x \sin y \quad f_{xx}(0, 0) = 0$$

$$f_{xy}(x, y) = e^x \cos y \quad f_{xy}(0, 0) = 1$$

$$f_{yy}(x, y) = -e^x \sin y \quad f_{yy}(0, 0) = 0$$

$$f_{xxx}(x, y) = e^x \sin y \quad f_{xxx}(0, 0) = 0$$

$$f_{xxy}(x, y) = e^x \cos y \quad f_{xxy}(0, 0) = 1$$

$$f_{xyy}(x, y) = -e^x \sin y \quad f_{xyy}(0, 0) = 0$$

$$f_{yyy}(x, y) = -e^x \cos y \quad f_{yyy}(0, 0) = -1.$$

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$$\begin{aligned} \text{Now } f(x, y) &= f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}(x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)) \\ &\quad + \frac{1}{3!}(x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)) + \dots \\ &= 0 + x.0 + y.1 + \frac{1}{2}(x^2.0 + 2xy.1 + y^2.0) \\ &\quad + \frac{1}{6}(x^3.0 + 3x^2 y.1 + 3xy^2.0 + y^3(-1)) + \dots \\ &= y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \dots. \end{aligned}$$

**Example 2.64.** Expand  $e^x \log_e(1+y)$  in powers of  $x$  and  $y$  upto terms of third degree. [Jan 2014, Dec 2011, Jan 2003]

**Solution.**

$$f_{yy} = \frac{2e^x}{(1+y)^3}$$

$f(x, y) = e^x \log_e(1 + y)$	
$f_x(x, y) = e^x \log_e(1 + y)$	$f(0, 0) = 0$
$f_y(x, y) = \frac{e^x}{1 + y}$	$f_x(0, 0) = 0$
$f_{xx} = e^x \log(1 + y)$	$f_y(0, 0) = 1$
$f_{xy} = \frac{e^x}{1 + y}$	$f_{xx}(0, 0) = 0$
$f_{yy} = \frac{-e^x}{(1 + y)^2}$	$f_{xy}(0, 0) = 1$
$f_{xxx} = e^x \log(1 + y)$	$f_{yy}(0, 0) = -1$
$f_{xxy} = \frac{e^x}{1 + y}$	$f_{xxx}(0, 0) = 0$
$f_{xyy} = \frac{-e^x}{(1 + y)^2}$	$f_{xxy}(0, 0) = 1$
	$f_{xyy}(0, 0) = -1$
	$f_{yyy}(0, 0) = 2$

By Maclaurin's series we have,

$$\begin{aligned} f(x, y) &= f(0, 0) + xf_x(0, 0) + yf_y(0, 0) \\ &\quad + \frac{1}{2!}(x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)) \\ &\quad + \frac{1}{3!}(x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) \\ &\quad + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)) + \dots . \end{aligned}$$

$$\begin{aligned} e^x \log_e(1 + y) &= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)) + \\ &\quad \frac{1}{6}[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot (2)] + \dots . \\ &= y + xy - \frac{y^2}{2} + \frac{x^2 y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} - \dots \end{aligned}$$

**Example 2.65.** Expand  $x^2y + 3y - 2$  in powers of  $x - 1$  and  $y + 2$  upto third degree terms. [Jun 2012]

**Solution.**  $f(x, y) = x^2y + 3y - 2$ .  $f(1, -2) = 1 \times (-2) + 3(-2) - 2$

$$= -2 - 6 - 2 = -10.$$

$f_x(x, y) = 2xy$ .	$f_x(1, -2) = -4$ .
$f_y(x, y) = x^2 + 3$ .	$f_y(1, -2) = 4$ .
$f_{xx}(x, y) = 2y$ .	$f_{xx}(1, -2) = -4$ .
$f_{xy}(x, y) = 2x$ .	$f_{xy}(1, -2) = 2$ .
$f_{yy}(x, y) = 0$ .	$f_{yy}(1, -2) = 0$ .
$f_{xxx}(x, y) = 0$ .	$f_{xxx}(1, -2) = 0$ .
$f_{xxy}(x, y) = 2$ .	$f_{xxy}(1, -2) = 2$ .
$f_{xyy}(x, y) = 0$ .	$f_{xyy}(1, -2) = 0$ .
$f_{yyy}(x, y) = 0$ .	$f_{yyy}(1, -2) = 0$ .

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By Taylor's theorem we have,

$$\begin{aligned} f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\ &\quad + \frac{1}{2!} \left[ (x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right] \\ &\quad + \frac{1}{3!} \left[ (x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) \right. \\ &\quad \left. + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b) + \dots \right] \\ x^2y + 3y - 2 &= f(1, -2) + (x - 1)f_x(1, -2) + (y + 2)f_y(1, -2) \\ &\quad + \frac{1}{2!} \left[ (x - 1)^2 f_{xx}(1, -2) + 2(x - 1)(y + 2)f_{xy}(1, -2) + (y + 2)^2 f_{yy}(1, -2) \right] \\ &\quad + \frac{1}{3!} \left[ (x - 1)^3 f_{xxx}(1, -2) + 3(x - 1)^2(y + 2)f_{xxy}(1, -2) \right. \\ &\quad \left. + 3(x - 1)(y + 2)^2 f_{xyy}(1, -2) + (y + 2)^3 f_{yyy}(1, -2) \right] + \dots \end{aligned}$$

$$\begin{aligned}
&= -10 - 4(x-1) + 4(y+2) + \frac{1}{2}[-4(x-1)^2 + 2(x-1)(y+2)] \\
&\quad + \frac{1}{6}[6(x-1)^2(y+2)] + \dots \\
&= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + (x-1)(y+2) + (x-1)^2(y+2) + \dots
\end{aligned}$$

**Example 2.66.** Find the Taylor's series expansion of  $x^2y^2 + 2x^2y + 3y^2$  in powers of  $(x+2)$  and  $y-1$  upto third degree terms. [Jan 2012, Jun 2010, Jan 2010]

**Solution.**

$$\begin{aligned}
f(x, y) &= x^2y^2 + 2x^2y + 3y^2 & f(-2, 1) &= 4 + 8 + 3 = 15. \\
f_x(x, y) &= 2xy^2 + 4xy & f_x(-2, 1) &= -4 - 8 = -12. \\
f_y(x, y) &= 2x^2y + 2x^2 + 6y & f_y(-2, 1) &= 8 + 8 + 6 = 22. \\
f_{xx}(x, y) &= 2y^2 + 4y & f_{xx}(-2, 1) &= 2 + 4 = 6. \\
f_{xy}(x, y) &= 4xy + 4x & f_{xy}(-2, 1) &= -8 - 8 = -16. \\
f_{yy}(x, y) &= 2x^2 + 6 & f_{yy}(-2, 1) &= 8 + 6 = 14. \\
f_{xxx}(x, y) &= 0 & f_{xxx}(-2, 1) &= 0. \\
f_{xxy}(x, y) &= 4y + 4 & f_{xxy}(-2, 1) &= 4 + 4 = 8. \\
f_{xyy}(x, y) &= 4x & f_{xyy}(-2, 1) &= -8. \\
f_{yyy}(x, y) &= 0 & f_{yyy}(-2, 1) &= 0.
\end{aligned}$$

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By Taylor's theorem we have,

$$\begin{aligned}
f(x, y) &= f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) \\
&\quad + \frac{1}{2!}[(x-a)^2f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2f_{yy}(a, b)] \\
&\quad + \frac{1}{3!}\left[(x-a)^3f_{xxx}(a, b) + 3(x-a)^2(y-b)f_{xxy}(a, b)\right. \\
&\quad \left.+ 3(x-a)(y-b)^2f_{xyy}(a, b) + (y-b)^3f_{yyy}(a, b)\right] + \dots
\end{aligned}$$

$$\begin{aligned}
x^2y^2 + 2x^2y + 3y^2 &= 15 - 12(x+2) + 22(y-1) \\
&\quad + \frac{1}{2} [6(x+2)^2 - 2 \times 16(x+2)(y-1) + 14(y-1)^2] \\
&\quad + \frac{1}{6} [24(x+2)^2(y-1) - 24(x+2)(y-1)^2] + \dots \\
&= 15 - 12(x+2) + 22(y-1) + 3(x+2)^2 - 16(x+2)(y-1) + 7(y-1)^2 \\
&\quad + 4(x+2)^2(y-1) - 4(x+2)(y-1)^2 + \dots
\end{aligned}$$

**Example 2.67.** Use Taylor's formula to expand the function defined by  $f(x, y) = x^3 + y^3 + xy^2$  in powers of  $(x-1)$  and  $(y-2)$ . [May 2011]

**Solution.**

$$\begin{array}{ll}
f(x, y) = x^3 + y^3 + xy^2 & f(1, 2) = 1 + 8 + 4 = 13. \\
f_x(x, y) = 3x^2 + y^2 & f_x(1, 2) = 3 + 4 = 7. \\
f_y(x, y) = 3y^2 + 2xy & f_y(1, 2) = 12 + 4 = 16. \\
f_{xx}(x, y) = 6x & f_{xx}(1, 2) = 6. \\
f_{xy}(x, y) = 2y & f_{xy}(1, 2) = 4. \\
f_{yy}(x, y) = 6y + 2x & f_{yy}(1, 2) = 12 + 2 = 14. \\
f_{xxx}(x, y) = 6 & f_{xxx}(1, 2) = 6. \\
f_{xxy}(x, y) = 0 & f_{xxy}(1, 2) = 0. \\
f_{xyy}(x, y) = 2 & f_{xyy}(1, 2) = 2. \\
f_{yyy}(x, y) = 6 & f_{yyy}(1, 2) = 6.
\end{array}$$

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By Taylor's theorem we have,

$$\begin{aligned}
f(x, y) &= f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) \\
&\quad + \frac{1}{2!} [(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3!} \left[ (x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) \right. \\
& \quad \left. + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b) \right] + \dots \\
x^3 + y^3 + xy^2 & = 13 + 7(x-1) + 16(y-2) + \frac{1}{2} \left[ 6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2 \right] \\
& \quad + \frac{1}{6} \left[ 6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3 \right] + \dots \\
& = 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2 \\
& \quad + (x-1)^3 + (x-1)(y-2)^2 + (y-2)^3 + \dots
\end{aligned}$$

**Example 2.68.** Expand  $e^{-x} \log y$  as a Taylor's series in powers of  $x$  and  $y-1$  upto third degree terms. [Jun 2011]

**Solution.**

$$f(x,y) = e^{-x} \log y. \quad f(0,1) = 0.$$

$$f_x(x,y) = -e^{-x} \log y. \quad f_x(0,1) = 0.$$

$$f_y(x,y) = \frac{e^{-x}}{y}. \quad f_y(0,1) = 1.$$

$$f_{xx}(x,y) = e^{-x} \log y. \quad f_{xx}(0,1) = 0.$$

$$f_{xy}(x,y) = -\frac{e^{-x}}{y}. \quad f_{xy}(0,1) = -1.$$

$$f_{yy}(x,y) = -\frac{e^{-x}}{y^2}. \quad f_{yy}(0,1) = -1.$$

$$f_{xxx}(x,y) = -e^{-x} \log y. \quad f_{xxx}(0,1) = 0.$$

$$f_{xxy}(x,y) = \frac{e^{-x}}{y}. \quad f_{xxy}(0,1) = 1.$$

$$f_{xyy}(x,y) = \frac{e^{-x}}{y^2}. \quad f_{xyy}(0,1) = 1.$$

$$f_{yyy}(x,y) = \frac{2e^{-x}}{y^3}. \quad f_{yyy}(0,1) = 2.$$

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By Taylor's theorem we have,

$$\begin{aligned}
 f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\
 &\quad + \frac{1}{2!} \left[ (x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right] \\
 &\quad + \frac{1}{3!} \left[ (x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) \right. \\
 &\quad \left. + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b) \right] + \dots \\
 e^{-x} \log y &= y - 1 + \frac{1}{2!}[-2x(y - 1) - (y - 1)^2] + \frac{1}{3!}[3x^2(y - 1) + 3x(y - 1)^2 + 2(y - 1)^3] + \dots
 \end{aligned}$$

**Example 2.69.** Expand  $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$  about  $(1, 1)$  upto the second degree terms.

Hence compute  $f(1.1, 0.9)$  approximately.

[Jan 2005]

**Solution.** Given,  $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$(a, b) = (1, 1) \quad a = 1, b = 1$$

$$f(1, 1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\begin{aligned}
 f_x &= \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{-y}{x^2} \right) = -\frac{x^2 y}{(x^2 + y^2)x^2} = \frac{-y}{x^2 + y^2} & f_x(1, 1) &= \frac{-1}{2} \\
 f_y &= \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right) = \frac{x^2}{(x^2 + y^2)x} = \frac{x}{x^2 + y^2} & f_y(1, 1) &= \frac{1}{2} \\
 f_{xx} &= -y(-1)(x^2 + y^2)^{-2} 2x = \frac{2xy}{(x^2 + y^2)^2} & f_{xx}(1, 1) &= \frac{2}{4} = \frac{1}{2} \\
 f_{xy} &= \frac{x^2 + y^2 - x2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} & f_{xy}(1, 1) &= 0 \\
 f_{yy} &= x(-1)(x^2 + y^2)^{-2} 2y = \frac{-2xy}{(x^2 + y^2)^2} & f_{yy}(1, 1) &= \frac{-1}{2}
 \end{aligned}$$


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By Taylor's theorem we have

$$\begin{aligned}
 f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\
 &\quad + \frac{1}{2!} \left( (x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right) + \dots
 \end{aligned}$$

$$\begin{aligned}
\tan^{-1}\left(\frac{y}{x}\right) &= f(1, 1) + (x-1)f_x(1, 1) + (y-1)f_y(1, 1) \\
&\quad + \frac{1}{2!}\left((x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1)f_{xy}(1, 1) + (y-1)^2 f_{yy}(1, 1)\right) \\
&= \frac{\pi}{4} + (x-1)\left(\frac{-1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2}\left((x-1)^2 \frac{1}{2} + 2(x-1)(y-1)0\right. \\
&\quad \left.+ (y-1)^2 \frac{(-1)}{2}\right) + \dots \\
&= \frac{\pi}{4} - \frac{1}{2}(x-1-y+1) + \frac{1}{2}\left(\frac{1}{2}(x-1)^2 - \frac{1}{2}(y-1)^2\right) + \dots \\
&= \frac{\pi}{4} - \frac{1}{2}(x-y) + \frac{1}{4}(x^2 - y^2 - 2x + 2y) + \dots \\
\tan^{-1}\left(\frac{y}{x}\right) &= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \dots .
\end{aligned}$$

$$\begin{aligned}
f(1.1, 0.9) &= \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2 \text{ approximately} \\
&= \frac{\pi}{4} - 0.1 = 0.685 \text{ approximately.}
\end{aligned}$$

**Example 2.70.** Find the Taylor's series expansion of  $e^x \sin y$  at the point  $(-1, \frac{\pi}{4})$  upto third degree terms. [Jan 2009]

**Solution.**

$$\begin{aligned}
f(x, y) &= e^x \sin y & f(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}} \\
f_x(x, y) &= e^x \sin y & f_x(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}} \\
f_y(x, y) &= e^x \cos y & f_y(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}} \\
f_{xx} &= e^x \sin y & f_{xx}(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}} \\
f_{xy} &= e^x \cos y & f_{xy}(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}} \\
f_{yx} &= e^x \cos y & f_{yx}(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}} \\
f_{yy} &= -e^x \sin y & f_{yy}(-1, \frac{\pi}{4}) &= \frac{-1}{e \sqrt{2}} \\
f_{xxx} &= e^x \sin y & f_{xxx}(-1, \frac{\pi}{4}) &= \frac{1}{e \sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
 f_{xxy} &= e^x \cos y & f_{xxy}(-1, \frac{\pi}{4}) &= \frac{1}{e\sqrt{2}} \\
 f_{xyy} &= -e^x \sin y & f_{xyy}(-1, \frac{\pi}{4}) &= \frac{-1}{e\sqrt{2}} \\
 f_{yyx} &= -e^x \sin y & f_{yyx}(-1, \frac{\pi}{4}) &= \frac{-1}{e\sqrt{2}} \\
 f_{yyy} &= -e^x \sin y & f_{yyy}(-1, \frac{\pi}{4}) &= \frac{-1}{e\sqrt{2}}
 \end{aligned}$$

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By Taylor's theorem we have

$$\begin{aligned}
 f(x, y) &= f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) \\
 &\quad + \frac{1}{2!} \left( (x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right) \\
 &\quad + \frac{1}{3!} \left( (x-a)^3 f_{xxx}(a, b) + 3(x-a)^2(y-b)f_{xxy}(a, b) \right. \\
 &\quad \left. + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b) \right) + \dots . \\
 e^x \sin y &= f\left(-1, \frac{\pi}{4}\right) + (x+1)f_x\left(-1, \frac{\pi}{4}\right) + (y-\frac{\pi}{4})f_y\left(-1, \frac{\pi}{4}\right) \\
 &\quad + \frac{1}{2!} \left( (x+1)^2 f_{xx}\left(-1, \frac{\pi}{4}\right) + 2(x+1)(y-\frac{\pi}{4})f_{xy}\left(-1, \frac{\pi}{4}\right) \right. \\
 &\quad \left. + (y-\frac{\pi}{4})^2 f_{yy}\left(-1, \frac{\pi}{4}\right) \right) \\
 &\quad + \frac{1}{6} \left( (x+1)^3 f_{xxx}\left(-1, \frac{\pi}{4}\right) + 3(x+1)^2(y-\frac{\pi}{4})f_{xxy}\left(-1, \frac{\pi}{4}\right) \right. \\
 &\quad \left. + 3(x+1)(y-\frac{\pi}{4})^2 f_{xyy}\left(-1, \frac{\pi}{4}\right) + (y-\frac{\pi}{4})^3 f_{yyy}\left(-1, \frac{\pi}{4}\right) \right) + \dots . \\
 &= \frac{1}{e\sqrt{2}} + \frac{1}{e\sqrt{2}}(x+1) + \frac{1}{e\sqrt{2}}(y-\frac{\pi}{4}) + \frac{1}{2\sqrt{2}e}(x+1)^2 \\
 &\quad + \frac{1}{\sqrt{2}e}(x+1)(y-\frac{\pi}{4}) - \frac{1}{2\sqrt{2}e}(y-\frac{\pi}{4})^2 + \frac{1}{6\sqrt{2}e}(x+1)^3 \\
 &\quad + \frac{\sqrt{2}}{e}(x+1)^2(y-\frac{\pi}{4}) - \frac{\sqrt{2}}{e}(x+1)(y-\frac{\pi}{4})^2 - \frac{1}{6\sqrt{2}e}(y-\frac{\pi}{4})^3 + \dots .
 \end{aligned}$$

**Example 2.71.** Expand  $e^x \cos y$  near the point  $(1, \frac{\pi}{4})$  by Taylor's series as far as quadratic terms.

[Jan 1996]

**Solution.**

$$\begin{aligned}
 f(x, y) &= e^x \cos y & f(1, \frac{\pi}{4}) &= \frac{e}{\sqrt{2}} \\
 f_x(x, y) &= e^x \cos y & f_x(1, \frac{\pi}{4}) &= e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}} \\
 f_y(x, y) &= -e^x \sin y & f_y(1, \frac{\pi}{4}) &= -\frac{e}{\sqrt{2}} \\
 f_{xx}(x, y) &= e^x \cos y & f_{xx}(1, \frac{\pi}{4}) &= \frac{e}{\sqrt{2}} \\
 f_{xy}(x, y) &= -e^x \sin y & f_{xy}(1, \frac{\pi}{4}) &= \frac{-e}{\sqrt{2}} \\
 f_{yy}(x, y) &= -e^x \cos y & f_{yy}(1, \frac{\pi}{4}) &= \frac{-e}{\sqrt{2}}
 \end{aligned}$$

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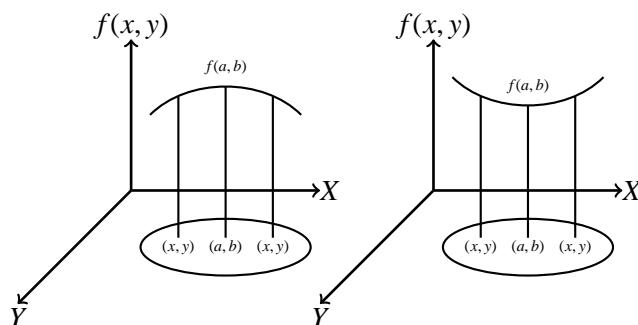
By Taylor's theorem we have

$$\begin{aligned}
 f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\
 &\quad + \frac{1}{2!} \left( (x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right) \\
 e^x \cos y &= f(1, \frac{\pi}{4}) + (x - 1)f_x(1, \frac{\pi}{4}) + (y - \frac{\pi}{4})f_y(1, \frac{\pi}{4}) \\
 &\quad + \frac{1}{2!} \left( (x - 1)^2 f_{xx}(1, \frac{\pi}{4}) + 2(x - 1)(y - \frac{\pi}{4})f_{xy}(1, \frac{\pi}{4}) \right. \\
 &\quad \left. + (y - \frac{\pi}{4})^2 f_{yy}(1, \frac{\pi}{4}) \right) + \dots \\
 &= \frac{e}{\sqrt{2}} + (x - 1)\frac{e}{\sqrt{2}} + (y - \frac{\pi}{4})\frac{(-e)}{\sqrt{2}} \\
 &\quad + \frac{1}{2} \left( (x - 1)^2 \frac{e}{\sqrt{2}} + 2(x - 1)(y - \frac{\pi}{4})\frac{(-e)}{\sqrt{2}} + (y - \frac{\pi}{4})^2 \frac{(-e)}{\sqrt{2}} \right) + \dots \\
 &= \frac{e}{\sqrt{2}} \left( 1 + (x - 1) - (y - \frac{\pi}{4}) + \frac{1}{2}(x - 1)^2 - (y - \frac{\pi}{4})(x - 1) - \frac{1}{2}(y - \frac{\pi}{4})^2 \right) + \dots .
 \end{aligned}$$

## 2.7 Maxima and Minima for functions of two variables

**Definition.** Let  $f(x, y)$  be a continuous function defined in a closed and bounded domain  $D$  of the  $xy$  plane and let  $(a, b)$  be an interior point of  $D$ .

- (i)  $f(a, b)$  is said to be a local maximum value of  $f(x, y)$  at the point  $(a, b)$  if there exists a neighborhood  $N$  of  $(a, b)$  such that  $f(x, y) < f(a, b)$  for all points  $(x, y)$  in  $N$ .
- (ii)  $f(a, b)$  is said to be a local minimum if  $f(x, y) > f(a, b)$  for all points  $(x, y)$  in  $N$  other than  $(a, b)$ .



Local maximum or local minimum values are called extreme values.

### Stationary point of $f(x, y)$

A point  $(a, b)$  satisfying  $f_x = 0$  and  $f_y = 0$  is called a stationary point of  $f(x, y)$ .

### Necessary conditions for Maximum or minimum

If  $f(a, b)$  is an extreme value of  $f(x, y)$  at  $(a, b)$ , then  $(a, b)$  is a stationary point of  $f(x, y)$  if  $f_x$  and  $f_y$  exist at  $(a, b)$  and  $f_x(a, b) = 0, f_y(a, b) = 0$ .

### Sufficient conditions for extreme values of $f(x, y)$

Let  $(a, b)$  be a stationary point of the differentiable function  $f(x, y)$ .

i.e.,  $f_x(a, b) = 0, f_y(a, b) = 0$ .

Let us define  $f_{xx}(a, b) = r, f_{xy}(a, b) = s, f_{yy}(a, b) = t$ .

- (i) If  $rt - s^2 > 0$  and  $r < 0$ , then  $f(a, b)$  is a maximum value.
- (ii) If  $rt - s^2 > 0$  and  $r > 0$  then  $f(a, b)$  is a minimum value.
- (iii) If  $rt - s^2 < 0$ , then  $f(a, b)$  is not an extreme value but  $(a, b)$  is a saddle point of

$f(x, y)$ .

(iv) If  $rt - s^2 = 0$ , then no conclusion is possible and further investigation is required.

### Working rule to find maxima and minima of $f(x, y)$

step (1). Find  $f_x = \frac{\partial f}{\partial x}$  and  $f_y = \frac{\partial f}{\partial y}$  and solve for  $f_x = 0$  and  $f_y = 0$  as simultaneous equations in  $x$  and  $y$ .

Let  $(a, b), (a_1, b_1), \dots$  be the solutions which are stationary points of  $f(x, y)$ .

step (2). Find  $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ .

step (3). Evaluate  $r, s, t$  at each stationary point.

At the point  $(a, b)$  if

(i)  $rt - s^2 > 0$  and  $r < 0$  then  $f(a, b)$  is a maximum value of  $f(x, y)$ .

(ii)  $rt - s^2 > 0$  and  $r > 0$  then  $f(a, b)$  is a minimum value of  $f(x, y)$ .

(iii)  $rt - s^2 < 0$  then  $(a, b)$  is called a saddle point.

(iv)  $rt - s^2 = 0$ , no conclusion can be made, further investigation is required.

### Critical Point

A point  $(a, b)$  is a critical point of  $f(x, y)$  if  $f_x = 0$  and  $f_y = 0$  at  $(a, b)$  or  $f_x$  and  $f_y$  do not exist at  $(a, b)$ .

Maxima or Minima occur at a critical point.

### Worked Examples

**Example 2.72.** Examine  $f(x, y) = x^3 + y^3 - 12x - 3y + 20$  for its extreme values.

[ Jun 2013, Jan 2012, May 2011, Jun 2010]

**Solution.** Given  $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ .

$$f_x = 3x^2 - 12 \quad f_y = 3y^2 - 3$$

$$r = f_{xx} = 6x \quad s = f_{xy} = 0 \quad t = f_{yy} = 6y$$

For stationary points, solve  $f_x = 0$  and  $f_y = 0$ .

$$f_x = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$

$$f_y = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$$

Stationary points are  $(2, 1), (2, -1), (-2, 1)$  and  $(-2, -1)$ .

$$rt - s^2 = 6x6y - 0 = 36xy.$$

At  $(2, 1)$ ,  $rt - s^2 = 36 \times 2 \times 1 = 72 > 0$ .

At  $(2, 1)$ ,  $r = 6(2) = 12 > 0$ .

$\therefore (2, 1)$  is a minimum point.

Minimum value is  $f(2, 1) = 8 + 1 - 24 - 3 + 20 = 29 - 27 = 2$ .

$$\text{At } (2, -1), rt - s^2 = 36 \times 2 \times -1 = -72 < 0.$$

$\therefore (2, -1)$  is a saddle point.

$$\text{At } (-2, 1), rt - s^2 = 36 \times (-2) \times 1 = -72 < 0.$$

$\therefore (-2, 1)$  is a saddle point.

$$\text{At } (-2, -1), rt - s^2 = 36 \times -2 \times -1 = 72 > 0.$$

$$r = 6(-2) = -12 < 0.$$

$\therefore (-2, -1)$  is a maximum point.

Maximum value  $f(-2, -1) = -8 - 1 + 24 + 3 + 20 = 47 - 9 = 38$ .

**Example 2.73.** Examine  $f(x, y) = x^3 + y^3 - 3axy$  for maximum and minimum values. [Jan 1999]

### Solution.

Given  $f(x, y) = x^3 + y^3 - 3axy$ .

$$f_x = 3x^2 - 3ay$$

$$f_y = 3y^2 - 3ax$$

$$r = f_{xx} = 6x$$

$$s = f_{xy} = -3a \quad t = f_{yy} = 6y.$$

For stationary points, solve  $f_x = 0$  and  $f_y = 0$ .

$$f_x = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow ay = x^2 \Rightarrow y = \frac{x^2}{a}.$$

$$f_y = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow \frac{x^4}{a^2} - ax = 0.$$

$$x\left(\frac{x^3}{a^2} - a\right) = 0 \Rightarrow x(x^3 - a^3) = 0 \Rightarrow x = 0 \text{ or } x = a.$$

$$x = 0 \Rightarrow y = 0$$

$$x = a \Rightarrow y = \frac{a^2}{a} = a.$$

Stationary points are  $(0, 0)$  and  $(a, a)$ .

At  $(0, 0)$ ,  $rt - s^2 = 6x6y - 9a^2 = 36xy - 9a^2 = -9a^2 < 0$ .

$$r = 6x = 0.$$

$\therefore$  No maximum or minimum at  $(0, 0)$ .

$\therefore (0, 0)$  is a saddle point.

At  $(a, a)$ ,  $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$  if  $a \neq 0$ .

$$r = 6x = 6a.$$

If  $a < 0$ ,  $r < 0$ .

$\therefore (a, a)$  is a maximum point if  $a < 0$ .

If  $a > 0$ ,  $r > 0$ .

$\therefore (a, a)$  is a minimum point if  $a > 0$ .

Maximum value =  $a^3 + a^3 - 3a^3 = -a^3$  if  $a < 0$

Minimum value =  $-a^3$  if  $a > 0$ .

**Example 2.74.** Discuss the maxima and minima of  $f(x, y) = x^3y^2(1 - x - y)$ .

[Jan 2014]

**Solution.** Given:  $f(x, y) = x^3y^2(1 - x - y) = x^3y^2 - x^4y^2 - x^3y^3$ .

$$\begin{aligned} f_x &= y^2[x^3(-1) + (1 - x - y)3x^2] = x^2y^2[-x + 3 - 3x - 3y] \\ &= x^2y^2[-4x - 3y + 3]. \end{aligned}$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2.$$

$$r = f_{xx} = y^2[-12x^2 - 6xy + 6x].$$

$$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2.$$

$$t = f_{yy} = 2x^3 - 2x^4 - 6x^3y.$$

For stationary points, solve  $f_x = 0$  and  $f_y = 0$ .

$$f_x = 0 \Rightarrow x^2y^2[-4x - 3y + 3] = 0.$$

$$\Rightarrow x = 0, y = 0, 4x + 3y = 3.$$

$$f_y = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0.$$

$$x = 0, y = 0, 2x + 3y = 2.$$

$$\text{Solving } 4x + 3y = 3 \quad (1)$$

$$2x + 3y = 2 \quad (2)$$

we get  $2x = 1 \Rightarrow x = \frac{1}{2}$ .

When  $x = \frac{1}{2}$ , (1)  $\Rightarrow 2 + 3y = 3 \Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3}$ .

$\therefore$  The stationary points are  $(0, 0), (\frac{1}{2}, \frac{1}{3})$ .

At  $(0, 0)$ ,  $rt - s^2 = 0.0 - 0 = 0$ .

We can not say maximum or minimum. Further investigation is required.

At  $(\frac{1}{2}, \frac{1}{3})$ ,

$$\begin{aligned} r &= \frac{1}{9} \left( -12 \times \frac{1}{4} - 6 \times \frac{1}{2} \times \frac{1}{3} + 6 \times \frac{1}{2} \right) \\ &= \frac{1}{9} (-3 - 1 + 3) = \frac{1}{9} (-1) - \frac{1}{9}. \\ t &= 2 \frac{1}{8} - 2 \frac{1}{16} - 6 \frac{1}{8} \frac{1}{3} = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}. \\ s^2 &= \left( 6 \frac{1}{4} \frac{1}{3} - 8 \frac{1}{8} \frac{1}{3} - 9 \frac{1}{4} \frac{1}{9} \right)^2 = \left( \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \right)^2 \\ &= \left( \frac{1}{4} - \frac{1}{3} \right)^2 = \left( -\frac{1}{12} \right)^2 = \frac{1}{144}. \\ rt - s^2 &= \left( -\frac{1}{9} \right) \left( -\frac{1}{8} \right) - \frac{1}{144} \\ &= \frac{1}{72} - \frac{1}{144} = \frac{2 - 1}{144} > 0 \text{ and } r < 0. \end{aligned}$$

$\therefore (\frac{1}{2}, \frac{1}{3})$  is a maximum point.

Maximum value is  $f(\frac{1}{2}, \frac{1}{3}) = \frac{1}{8} \frac{1}{9} \left( 1 - \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{72} \left( \frac{6 - 3 - 2}{6} \right) = \frac{1}{72 \times 6} = \frac{1}{432}$ .

**Example 2.75.** Find the extreme values of the function  $f(x, y) = x^4 + y^2 + x^2y$ .

**Solution.** Given:  $f(x, y) = x^4 + y^2 + x^2y$ .

$$f_x = 4x^3 + 2xy. \quad r = f_{xx} = 12x^2 + 2y.$$

$$f_y = 2y + x^2. \quad s = f_{xy} = 2x. \quad t = f_{yy} = 2.$$

For stationary points, solve  $f_x = 0$  and  $f_y = 0$ .

$$f_x = 0 \Rightarrow 4x^3 + 2xy = 0 \Rightarrow 2x(y + 2x^2) = 0 \Rightarrow x = 0, y + 2x^2 = 0 \Rightarrow y = -2x^2.$$

$$f_y = 0 \Rightarrow 2y + x^2 = 0 \Rightarrow -4x^2 + x^2 = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0.$$

When  $x = 0, y = 0$ .

$\therefore$  The only stationary point is  $(0, 0)$ .

$$rt - s^2 = (12x^2 + 2y)2 - 4x^2.$$

At  $(0, 0)$ ,  $rt - s^2 = 0$ .

We can not say maximum or minimum.

We shall investigate the nature of the function in a neighbourhood of  $(0, 0)$ .

We have  $f(0, 0) = 0$ . In a neighbourhood of  $(0, 0)$  on the  $x$ -axis, take the point  $(h, 0)$ ,  $f(h, 0) = h^4 > 0$ .

On the  $y$ -axis take the point  $(0, k)$ ,  $f(0, k) = k^2 > 0$ .

On  $y = mx$ , for any  $m$ , take the point  $(h, mh)$ .

$$f(h, mh) = h^4 + m^2h^2 + mh^3 = h^2[h^2 + m^2 + mh].$$

For the quadratic in  $m$ ,  $m^2 + mh + h^2$

$$\text{discriminant} = B^2 - 4AC = h^2 - 4.1.h^2 = -3h^2 < 0 \text{ if } m \neq 0.$$

$\therefore f(h, mh) > 0$  for all  $m \neq 0$ .

$\therefore$  In a neighbourhood of  $(0, 0)$  for all points  $(x, y)$ ,  $f(x, y) > 0$ .

$\therefore f(0, 0)$  is minimum and the minimum value = 0.

**Example 2.76.** Find the maximum and minimum values of  $x^2 - xy + y^2 - 2x + y$ .

[ Jun 2012, Jun 2010]

**Solution.**  $f(x, y) = x^2 - xy + y^2 - 2x + y$ .

$$f_x = 2x - y - 2.$$

$$f_y = -x + 2y + 1.$$

$$r = f_{xx} = 2.$$

$$s = f_{xy} = -1.$$

$$t = f_{yy} = 2.$$

For stationary points, solve  $f_x = 0, f_y = 0$

$$2x - y - 2 = 0. \quad (1)$$

$$-x + 2y + 1 = 0. \quad (2)$$

$$(1) \Rightarrow y = 2x - 2.$$

$$(2) \Rightarrow -x + 2(2x - 2) + 1 = 0$$

$$-x + 4x - 4 + 1 = 0$$

$$3x - 3 = 0$$

$$3x = 3$$

$$x = 1.$$

$$\therefore y = 2 - 2 = 0.$$

The stationary point is  $(1, 0)$ .

$$rt - s^2 = 4 + 1 = 5 > 0 \quad \text{and} \quad r = 2 > 0.$$

$\therefore (1, 0)$  is a minimum point.

Minimum value of  $f = 1 - 2 = -1$ .

**Example 2.77.** Find the extreme values of the function  $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ .

[Jan 2012]

**Solution.**  $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ .

$$f_x = 3x^2 - 3.$$

$$f_y = 3y^2 - 12.$$

$$r = f_{xx} = 6x.$$

$$s = f_{xy} = 0.$$

$$t = f_{yy} = 6y.$$

For stationary points, solve  $f_x = 0, f_y = 0$ .

$$3x^2 - 3 = 0. \quad 3y^2 - 12 = 0.$$

$$3x^2 = 3. \quad 3y^2 = 12.$$

$$x^2 = 1. \quad y^2 = 4.$$

$$x = \pm 1. \quad y = \pm 2.$$

The stationary points are  $(1, 2), (-1, 2), (1, -2), (-1, -2)$ .

At  $(1, 2)$ ,  $rt - s^2 = 36xy = 36 \times 1 \times 2 = 72 > 0$ .

$$r = 6 > 0.$$

$\therefore (1, 2)$  is a minimum point.

Minimum value of  $f = 1 + 8 - 3 - 24 + 20 = 2$ .

At  $(-1, 2)$ ,  $rt - s^2 = 36xy = 36 \times (-1) \times 2 = -72 < 0$ .

$\therefore (-1, 2)$  is a saddle point.

At  $(1, -2)$ ,  $rt - s^2 = 36xy = 36 \times 1 \times (-2) = -72 < 0$ .

$\therefore (1, -2)$  is a saddle point.

At  $(-1, -2)$ ,  $rt - s^2 = 36xy = 72 > 0$ .

$$r = 6 \times (-1) = -6 < 0.$$

$\therefore (-1, -2)$  is a maximum point.

Maximum value of  $f = -1 - 8 + 3 + 24 + 20 = 38$ .

Maxima = 38.

Minima = 2.

**Example 2.78.** Test for maxima and minima of the function  $f(x, y) = x^3y^2(6 - x - y)$ .

[Jan 2013]

**Solution.**  $f(x, y) = x^3y^2(6 - x - y)$

$$= 6x^3y^2 - x^4y^2 - x^3y^3.$$

$$f_x = 18x^2y^2 - 4x^3y^2 - 3x^2y^3.$$

$$f_y = 12x^3y - 2x^4y - 3x^3y^2.$$

$$r = f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3.$$

$$s = f_{xy} = 36x^2y - 8x^3y - 9x^2y^2.$$

$$t = f_{yy} = 12x^3 - 2x^4 - 6x^3y.$$

For stationary points,  $f_x = 0, f_y = 0$ .

$$f_x = 0 \Rightarrow 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(18 - 4x - 3y) = 0$$

$$\text{i.e., } 4x + 3y = 18. \quad (1)$$

$$f_y = 0 \Rightarrow 12x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(12 - 2x - 3y) = 0$$

$$2x + 3y = 12. \quad (2)$$

$$(1) - (2) \Rightarrow 2x = 6$$

$$x = 3.$$

$$(1) \Rightarrow 12 + 3y = 18$$

$$3y = 6$$

$$y = 2.$$

The stationary point is  $(3, 2)$

At  $(3, 2)$ ,

$$\begin{aligned}
 r &= 36 \times 3 \times 4 - 12 \times 9 \times 4 - 6 \times 3 \times 8 \\
 &= 432 - 432 - 144 \\
 &= -144 < 0.
 \end{aligned}$$

$$\begin{aligned}
 t &= 12 \times 9 - 2 \times 81 - 6 \times 27 \times 2 \\
 &= 108 - 162 - 324 \\
 &= -378.
 \end{aligned}$$

$$\begin{aligned}
 s &= 34 \times 9 \times 2 - 8 \times 27 \times 2 - 9 \times 9 \times 4 \\
 &= 612 - 432 - 324 \\
 &= -144.
 \end{aligned}$$

$$\begin{aligned}
 rt - s^2 &= (-144)(-378) - (-144)^2 \\
 &= 54432 - 20736 \\
 &= 33696 > 0
 \end{aligned}$$

Since  $rt - s^2 > 0$  and  $r < 0$ ,  $(3, 2)$  is a maximum point.

$\therefore$  Maximum value of  $f = 27 \times 4(6 - 3 - 2) = 108$ .

**Example 2.79.** Examine for minimum and maximum values  $\sin x + \sin y + \sin(x + y)$ .

**Solution.** We have  $f(x, y) = \sin x + \sin y + \sin(x + y)$ .

$$\begin{aligned}
 f_x &= \cos x + \cos(x + y) & f_y &= \cos y + \cos(x + y) \\
 r &= f_{xx} = -\sin x - \sin(x + y) \\
 s &= f_{xy} = -\sin(x + y) \\
 t &= f_{yy} = -\sin y - \sin(x + y).
 \end{aligned}$$

For stationary points, solve  $f_x = 0$  and  $f_y = 0$ .

$$\text{i.e., } \cos x + \cos(x + y) = 0. \quad (1)$$

$$\cos y + \cos(x + y) = 0. \quad (2)$$

$$(1) - (2) \implies \cos x - \cos y = 0.$$

$$\text{i.e., } \cos x = \cos y$$

$$\implies x = y$$

$$\text{Now (1)} \implies \cos x + \cos 2x = 0$$

$$\text{i.e., } \cos 2x = -\cos x.$$

$$\cos 2x = \cos(\pi - x)$$

$$\implies 2x = \pi - x.$$

$$\text{i.e., } 3x = \pi$$

$$x = \frac{\pi}{3}.$$

When  $x = \frac{\pi}{3}, y = \frac{\pi}{3}$ .

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$  is a stationary point.

At  $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = -\sin \frac{\pi}{3} - \sin \frac{2\pi}{3} = \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0.$$

$$s = -\sin \frac{2\pi}{3} = -\frac{\sqrt{3}}{2}, t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}.$$

$$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

Since  $rt - s^2 > 0$  and  $r < 0$ ,  $f(x, y)$  has a maximum value at  $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ .

$$\therefore \text{Maximum value} = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

**Example 2.80.** Find the maximum and minimum values of  $\sin x \sin y \sin(x + y)$ ,  $0 < x, y < \pi$ . [Jan 1997]

**Solution.** Given  $f(x, y) = \sin x \sin y \sin(x + y)$ .

$$f_x = \sin y [\sin x \cos(x + y) + \sin(x + y) \cos x] = \sin y \sin(2x + y).$$

$$r = f_{xx} = 2 \sin y \cos(2x + y).$$

$$f_y = \sin x [\sin y \cos(x+y) + \sin(x+y) \cos y] = \sin x \sin(x+2y).$$

$$s = f_{xy} = \sin x \cos(x+2y) + \sin(x+2y) \cos x = \sin(2x+2y).$$

$$t = f_{yy} = 2 \sin x \cos(x+2y).$$

For stationary points, solve  $f_x = 0$  and  $f_y = 0$ .

$$f_x = 0 \Rightarrow \sin y \sin(2x+y) = 0.$$

$$f_y = 0 \Rightarrow \sin x \sin(x+2y) = 0.$$

Since,  $x, y \neq 0 \& \neq \pi \Rightarrow \sin x \neq 0, \sin y \neq 0$ .

$$\therefore \sin(2x+y) = 0 \text{ and } \sin(x+2y) = 0.$$

Since,  $0 < x < \pi, 0 < 2x < 2\pi$

$$0 < y < \pi \Rightarrow 0 < 2x+y < 3\pi.$$

Similarly  $0 < x+2y < 3\pi$ . Since,  $\sin(2x+y) = 0 \Rightarrow 2x+y = \pi \text{ or } 2\pi$ .

Similarly  $x+2y = \pi \text{ or } 2\pi$ .

$$\text{If } 2x+y = \pi \quad (1)$$

$$\text{and } x+2y = \pi \quad (2)$$

then  $x-y=0 \Rightarrow x=y$ .

$$\therefore (1) \Rightarrow 3x = \pi \Rightarrow x = \frac{\pi}{3}.$$

$$\therefore y = \frac{\pi}{3}.$$

$\therefore$  one stationary point is  $(\frac{\pi}{3}, \frac{\pi}{3})$ .

If  $2x+y=\pi$  and  $x+2y=2\pi$  then,  $x-y=-\pi \Rightarrow x=y-\pi$ .

$$\therefore 2(y-\pi)+y=\pi \Rightarrow 3y-2\pi=\pi \Rightarrow 3y=3\pi \Rightarrow y=\pi,$$

which is not admissible since  $y \neq \pi$ .

Similarly,  $2x+y=2\pi$  and  $x+2y=\pi$  is also not possible.

Now take  $2x+y=2\pi$  and  $x+2y=2\pi$

$$\Rightarrow x-y=0 \Rightarrow x=y$$

$$\Rightarrow 3x=2\pi \Rightarrow x=\frac{2\pi}{3}$$

$$\therefore y = \frac{2\pi}{3}.$$

$\therefore$  Another stationary point is  $(\frac{2\pi}{3}, \frac{2\pi}{3})$ .

At  $(\frac{\pi}{3}, \frac{\pi}{3})$ .

$$r = 2 \sin \frac{\pi}{3} \cos \pi = 2 \frac{\sqrt{3}(-1)}{2} < 0.$$

$$s = \sin \frac{4\pi}{3} = \sin \left(\pi + \frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

$$t = 2 \sin \frac{\pi}{3} \cos \pi = \frac{2\sqrt{3}(-1)}{2} = -\sqrt{3}.$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

$\therefore (\frac{\pi}{3}, \frac{\pi}{3})$  is a maximum point.

$$\begin{aligned} \text{Maximum value} &= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} \cdot \sin \frac{\pi}{3} \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \sin\left(\pi - \frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}. \end{aligned}$$

At  $(\frac{2\pi}{3}, \frac{2\pi}{3})$

$$r = 2 \sin \frac{2\pi}{3} \cos \frac{6\pi}{3} = 2 \frac{\sqrt{3}}{2} \cdot 1 = \sqrt{3} > 0.$$

$$t = 2 \sin \frac{2\pi}{3} \cos \frac{6\pi}{3} = \sqrt{3}.$$

$$s = \sin \frac{8\pi}{3} = \sin\left(3\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

$$rt - s^2 = \sqrt{3} \sqrt{3} - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

$\therefore (\frac{2\pi}{3}, \frac{2\pi}{3})$  is a minimum point.

$$\text{Minimum value} = f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \sin \frac{2\pi}{3} \sin \frac{2\pi}{3} \sin \frac{4\pi}{3} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{8}.$$

**Example 2.81.** In a plane triangle, find the maximum value of  $\cos A \cos B \cos C$ .

[Jan 2000]

**Solution.** The angles of the  $\Delta^{le}ABC$  satisfy  $0 < A, B, C < \pi$  and  $A + B + C = \pi$ ,

$\implies C = \pi - (A + B)$ . Replacing  $C$ , we get

$$f(A, B) = \cos A \cos B \cos(\pi - (A + B)) = -\cos A \cos B \cos(A + B), \quad 0 < A, B < \pi.$$

$$f_A = -\cos B [\cos A(-\sin(A + B)) + \cos(A + B)(-\sin A)] = \cos B \sin(2A + B).$$

$$f_B = -\cos A[-\sin(A + 2B)] = \cos A \sin(A + 2B).$$

$$r = f_{AA} = 2 \cos B \cos(2A + B).$$

$$s = f_{AB} = \cos A \cos(A + 2B) + \sin(A + 2B)(-\sin A) = \cos(2A + 2B).$$

$$t = f_{BB} = 2 \cos A \cos(A + 2B).$$

For stationary points, solve  $f_A = 0, f_B = 0$ .

i.e.,  $\cos B \sin(2A + B) = 0$  and  $\cos A \sin(A + 2B) = 0$ .

$$\cos B = 0 \text{ or } \sin(2A + B) = 0 \text{ and } \cos A = 0 \text{ or } \sin(A + 2B) = 0$$

$$\implies B = \frac{\pi}{2} \text{ or } 2A + B = \pi \text{ or } 2\pi$$

and

$$A = \frac{\pi}{2} \text{ or } A + 2B = \pi \text{ or } 2\pi.$$

### Different possibilities

case (i) Let  $B = \frac{\pi}{2}$  and  $A = \frac{\pi}{2}$ .

$$\Rightarrow A + B = \pi$$

$\implies C = 0$  not possible.

case (ii) If  $B = \frac{\pi}{2}$  and  $A + 2B = \pi$ .

$$\Rightarrow A + \pi = \pi \Rightarrow A = 0 \text{ not possible.}$$

case (iii) If  $B = \frac{\pi}{2}$  and  $A + 2B = 2\pi$ .

$$\Rightarrow A + \pi = 2\pi \Rightarrow A = \pi \text{ not possible.}$$

case (iv)  $A = \frac{\pi}{2}, 2A + B = \pi$ .

$$\Rightarrow \pi + B = \pi \Rightarrow B = 0 \text{ not possible.}$$

case (v)  $A = \frac{\pi}{2}, 2A + B = 2\pi \Rightarrow B = \pi$  not possible.

case (vi) If  $2A + B = \pi, A + 2B = \pi$ .

Subtracting  $A - B = 0 \Rightarrow A = B$ .

$$\therefore 3A = \pi \Rightarrow A = \frac{\pi}{3}.$$

$$\implies B = \frac{\pi}{3}, C = \frac{\pi}{3}.$$

case (vii) If  $2A + B = \pi$  and  $A + 2B = 2\pi \Rightarrow A - B = -\pi$  not possible.

Finally  $2A + B = 2\pi$  and  $A + 2B = 2\pi \Rightarrow A - B = 0 \Rightarrow A = B$ .

$$3A = 2\pi \Rightarrow A = \frac{2\pi}{3}, B = \frac{2\pi}{3}.$$

$$A + B = \frac{2\pi}{3} + \frac{2\pi}{3} = 4\frac{\pi}{3} > \pi \text{ not possible.}$$

$\therefore$  The only stationary point is  $(\frac{\pi}{3}, \frac{\pi}{3})$ .

At  $(\frac{\pi}{3}, \frac{\pi}{3})$ ,

$$r = 2 \cos \frac{\pi}{3} \cos \pi = 2 \frac{1}{2}(-1) < 0.$$

$$t = 2 \cos \frac{\pi}{3} \cos \pi = -1.$$

$$s = \cos \left( \frac{4\pi}{3} \right) = \cos \left( \pi + \frac{\pi}{3} \right) = -\frac{1}{2}.$$

$$rt - s^2 = (-1)(-1) - \left( \frac{-1}{2} \right)^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0.$$

$\therefore (\frac{\pi}{3}, \frac{\pi}{3})$  is a maximum point.

$$\therefore \text{Maximum value of } f \text{ is } f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{3} \cos \frac{\pi}{3} = \frac{1}{8}.$$

**Example 2.82.** A flat circular plate is heated so that the temperature at any point  $(x, y)$  is  $U(x, y) = x^2 + 2y^2 - x$ . Find the coldest point on the plate. [Jan 2005]

**Solution.** Given  $U = x^2 + 2y^2 - x$ .

$$U_x = 2x - 1 \quad U_y = 4y.$$

$$r = U_{xx} = 2. \quad s = U_{xy} = 0. \quad t = U_{yy} = 4.$$

$$rt - s^2 = 8 > 0 \text{ and } r = 2 > 0.$$

$\therefore$  All points are minimum points.

At minimum,  $U_x = 0$  and  $U_y = 0$ .

$$U_x = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}.$$

$$U_y = 0 \Rightarrow 4y = 0 \Rightarrow y = 0.$$

$$\therefore \text{The minimum point is } \left( \frac{1}{2}, 0 \right).$$

$$\therefore \text{The coldest point on the plate is } \left( \frac{1}{2}, 0 \right).$$

## 2.8 Constrained Maxima and Minima - Lagrange's Method

### Lagrange's Method

Let  $f(x, y, z)$  be the function for which the extreme values are to be found subject to the condition

$$\phi(x, y, z) = 0. \quad (1)$$

Construct the auxiliary function  $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$ , where  $\lambda$  is an undetermined parameter independent of  $x, y, z$  which is called the Lagrange's multiplier. Any relative extremum of  $f(x, y, z)$  subject to (1) must occur at a stationary point of  $F(x, y, z)$ .

The stationary points of  $F$  are given by  $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = 0$ .

$$\Rightarrow f_x + \lambda\phi_x = 0, f_y + \lambda\phi_y = 0, f_z + \lambda\phi_z = 0, \phi(x, y, z) = 0.$$

$$\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} = \frac{f_z}{\phi_z} = -\lambda \text{ and } \phi(x, y, z) = 0.$$

Solving these equations we can find the values of  $x, y, z$  which are stationary points of  $F$  and the values of  $f$  at these points give the maximum and minimum values of  $f(x, y, z)$ .

### Worked Examples

**Example 2.83.** Find the maximum value of  $x^m y^n z^p$  subject to  $x + y + z = a$ .

[Jan 2009]

**Solution.** Let  $f = x^m y^n z^p$ .

$$\phi = x + y + z - a = 0. \quad (1)$$

We have to maximise  $f$  subject to (1).

Let  $F = f + \lambda\phi$  where  $\lambda$  is the Lagrange's multiplier.

$$F = x^m y^n z^p + \lambda(x + y + z - a).$$

$$F_x = mx^{m-1}y^n z^p + \lambda, F_y = nx^m y^{n-1} z^p + \lambda, F_z = px^m y^n z^{p-1} + \lambda.$$

To find the stationary points, solve  $F_x = 0, F_y = 0, F_z = 0, \phi = 0$ .

$$F_x = 0 \Rightarrow mx^{m-1}y^n z^p = -\lambda.$$

$$F_y = 0 \Rightarrow nx^m y^{n-1} z^p = -\lambda.$$

$$F_x = 0 \Rightarrow px^m y^n z^{p-1} = -\lambda.$$

From the above three equations we get

$$mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}.$$

Dividing by  $x^m y^n z^p$  we get,  $\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$ .

$$x = \frac{ma}{m+n+p}, y = \frac{na}{m+n+p}, z = \frac{pa}{m+n+p}.$$

The stationary point is  $\left( \frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p} \right)$ .

$\therefore$  Max. value of  $f = \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$ .

**Example 2.84.** Find the minimum value of  $x^2yz^3$  subject to  $2x + y + 3z = a$ .

[Jan 2007]

**Solution.** Given  $f = x^2yz^3$ .

$$\phi = 2x + y + 3z - a = 0. \quad (1)$$

Let  $F = f + \lambda\phi$  where  $\lambda$  is the Lagrange's multiplier.

$$F = x^2yz^3 + \lambda(2x + y + 3z - a).$$

$$F_x = 2xyz^3 + 2\lambda, F_y = x^2z^3 + \lambda, F_z = 3x^2yz^2 + 3\lambda.$$

To find the stationary points, solve  $F_x = 0, F_y = 0, F_z = 0, \phi = 0$ .

$$F_x = 0 \Rightarrow 2xyz^3 + 2\lambda = 0 \Rightarrow xyz^3 = -\lambda.$$

$$F_y = 0 \Rightarrow x^2z^3 = -\lambda.$$

$$F_z = 0 \Rightarrow 3x^2yz^2 + 3\lambda = 0 \Rightarrow x^2yz^2 = -\lambda.$$

Therefore

$$xyz^3 = x^2z^3 = x^2yz^2$$

$$xyz^3 = x^2z^3 \Rightarrow y = x.$$

$$x^2z^3 = x^2yz^2 \Rightarrow y = z.$$

$$x = y = z.$$

$$(1) \Rightarrow 2x + x + 3x = a \Rightarrow 6x = a \Rightarrow x = \frac{a}{6} = y = z.$$

$\therefore$  The stationary point is  $(\frac{a}{6}, \frac{a}{6}, \frac{a}{6})$ .

$$\text{Minimum value} = \left(\frac{a}{6}\right)^2 \frac{a}{6} \left(\frac{a}{6}\right)^3 = \frac{a^6}{6^6} = \left(\frac{a}{6}\right)^6.$$

**Example 2.85.** If  $u = x^2 + y^2 + z^2$  where  $ax + by + cz - p = 0$ , find the stationary value of  $u$ . [Jan 2006]

**Solution.** Given  $f = x^2 + y^2 + z^2$ .

$$\phi = ax + by + cz - p = 0. \quad (1)$$

Let  $F = f + \lambda\phi$  where  $\lambda$  is the Lagrange's multiplier.

$$F = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p).$$

$$F_x = 2x + a\lambda, F_y = 2y + b\lambda, F_z = 2z + c\lambda.$$

To find the stationary points, solve  $F_x = 0, F_y = 0, F_z = 0, \phi = 0$ .

$$2x + a\lambda = 0 \Rightarrow x = \frac{-a\lambda}{2} \Rightarrow ax = \frac{-a^2\lambda}{2}.$$

Similarly

$$by = \frac{-b^2\lambda}{2}, cz = \frac{-c^2\lambda}{2}.$$

$$(1) \Rightarrow \frac{-a^2\lambda}{2} - \frac{b^2\lambda}{2} - \frac{c^2\lambda}{2} = p$$

$$\frac{a^2 + b^2 + c^2}{2} \lambda = -p \Rightarrow \lambda = \frac{-2p}{a^2 + b^2 + c^2}.$$

$$\therefore x = \frac{ap}{a^2 + b^2 + c^2}, y = \frac{bp}{a^2 + b^2 + c^2}, z = \frac{cp}{a^2 + b^2 + c^2}.$$

Stationary value of  $u$  is

$$\begin{aligned} u &= \frac{a^2 p^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2 p^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2 p^2}{(a^2 + b^2 + c^2)^2} \\ &= \frac{p^2(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}. \end{aligned}$$

**Example 2.86.** The temperature  $T$  at any point  $(x, y, z)$  in space is  $T = 400xyz^2$ .

Find the highest temperature on the surface of the unit sphere  $x^2 + y^2 + z^2 = 1$ .

[Jan 2005]

**Solution.** We have to maximize

$$T = 400xyz^2 \quad (1)$$

$$\text{subject to} \quad \phi = x^2 + y^2 + z^2 - 1 = 0. \quad (2)$$

Consider  $F = T + \lambda\phi$ , where  $\lambda$  is the Lagrange multiplier.

$$F = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1).$$

$$F_x = 400yz^2 + 2\lambda x, F_y = 400xz^2 + 2\lambda y, F_z = 400xy + 2\lambda z.$$

To find the stationary points, solve  $F_x = 0, F_y = 0, F_z = 0, \phi = 0$ .

$$F_x = 0 \Rightarrow 400yz^2 + 2\lambda x = 0 \Rightarrow -\lambda = \frac{200yz^2}{x}.$$

$$F_y = 0 \Rightarrow 400xz^2 + 2\lambda y = 0 \Rightarrow -\lambda = \frac{200xz^2}{y}.$$

$$F_z = 0 \Rightarrow 800xyz + 2\lambda z = 0 \Rightarrow -\lambda = 400xy.$$

$$\therefore \frac{200yz^2}{x} = \frac{200xz^2}{y} = 400xy.$$

Taking  $\frac{200yz^2}{x} = \frac{200xz^2}{y}$  we get  $x^2 = y^2 \Rightarrow y = \pm x$ .

Taking  $\frac{200yz^2}{x} = 400xy$  we get  $\frac{z^2}{x} = 2x \Rightarrow z^2 = 2x^2 \Rightarrow z = \pm \sqrt{2}x$ .

Taking  $\frac{200xz^2}{y} = 400xy$  we get  $\frac{z^2}{y} = 2y \Rightarrow z^2 = 2y^2 \Rightarrow z = \pm \sqrt{2}y$ .

Substituting in  $x^2 + y^2 + z^2 = 1$  we get

$$x^2 + x^2 + 2x^2 = 1 \Rightarrow 4x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

$$\therefore y = \pm \frac{1}{2}, z = \pm \sqrt{2} \frac{1}{2} = \pm \frac{1}{\sqrt{2}}.$$

The stationary points are given by  $x = \pm \frac{1}{2}, y = \pm \frac{1}{2}, z = \pm \frac{1}{\sqrt{2}}$ .

To have maximum value, we must have  $xy$  positive.

$\therefore$  The points are  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$ .

$$\therefore \text{Maximum } T = 400 \frac{1}{2} \frac{1}{2} \frac{1}{2} = 50^0C.$$

**Example 2.87.** Find the shortest and longest distance from the point  $(1, 2, -1)$  to the sphere  $x^2 + y^2 + z^2 = 24$  using Lagrange's method of constrained maxima and minima. [Jun 2011, Jan 2002]

**Solution.** Let  $P(x, y, z)$  be any point on the sphere.

Let  $A$  be the point  $(1, 2, -1)$ .

$$AP = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}.$$

$$\text{Let } f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2. \quad (1)$$

$AP$  is minimum or maximum if  $f$  is minimum or maximum.

$\therefore$  The problem is now reduced to minimise or maximise  $f$  subject to

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 24 = 0.$$

Consider the auxiliary function

$$F = f + \lambda\phi = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24) \quad \text{where } \lambda \text{ is the Lagrange's multiplier.}$$

$$F_x = 2(x-1) + 2\lambda x, F_y = 2(y-2) + 2\lambda y, F_z = 2(z+1) + 2\lambda z.$$

To find the stationary points solve

$$F_x = 0, F_y = 0, F_z = 0, \phi = 0.$$

$$F_x = 0 \Rightarrow 2(x-1) + 2\lambda x = 0 \Rightarrow x-1 = -\lambda x \Rightarrow -\lambda = \frac{x-1}{x} = 1 - \frac{1}{x}.$$

$$F_y = 0 \Rightarrow 2(y-2) + 2\lambda y = 0 \Rightarrow y-2 = -\lambda y \Rightarrow -\lambda = \frac{y-2}{y} = 1 - \frac{2}{y}.$$

$$F_z = 0 \Rightarrow 2(z+1) + 2\lambda z = 0 \Rightarrow z+1 = -\lambda z \Rightarrow -\lambda = \frac{z+1}{z} = 1 + \frac{1}{z}.$$

$$\therefore 1 - \frac{1}{x} = 1 - \frac{2}{y} = 1 + \frac{1}{z}.$$

Taking  $1 - \frac{1}{x} = 1 - \frac{2}{y}$  we get  $\frac{1}{x} = \frac{2}{y} \Rightarrow y = 2x$ .

Taking  $1 - \frac{1}{x} = 1 + \frac{1}{z}$  we get  $z = -x$ .

Taking  $1 - \frac{2}{y} = 1 + \frac{1}{z}$  we get  $z = \frac{-y}{2} = -x$ .

We have  $x^2 + y^2 + z^2 = 24 \Rightarrow x^2 + 4x^2 + x^2 = 24 \Rightarrow 6x^2 = 24 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ .

When  $x = 2, y = 4, z = -2$ , the first point is  $(2, 4, -2)$ . Let this point be  $P_1$ .

When  $x = -2, y = -4, z = 2$ , the second point is  $(-2, -4, 2)$ . Let this point be  $P_2$ .

$$P_1A = \sqrt{1+4+1} = \sqrt{6}, P_2A = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$$

$\therefore$  Shortest distance =  $\sqrt{6}$

Longest distance =  $3\sqrt{6}$ .

**Example 2.88.** A rectangular box open at the top is to have a volume of  $32cc$ . Find the dimensions of the box which requires least amount of material for its construction. [Jun 2012, Dec 2011, Jun 2010, Jan 2005]

**Solution.** Let the dimensions of the box be Length =  $x$ , Breadth =  $y$ , height =  $z$ .

Given: Volume =  $32cc$ .

$$\Rightarrow xyz = 32, x, y, z > 0. \quad (1)$$

We have to minimize the amount of material used for the construction of the box.

Let  $S$  be the surface area of the box whose top is open

$$\therefore S = xy + 2xz + 2yz \quad (2)$$

By Lagrange's method

Let  $F = s + \lambda\phi = xy + 2yz + 2xz + \lambda(xyz - 32)$  where  $\lambda$  is the Lagrange's multiplier.

$F_x = y + 2z + \lambda yz, F_y = x + 2z + \lambda xz, F_z = 2y + 2x + \lambda xy$ .

To find the stationary points, solve

$$F_x = 0, F_y = 0, F_z = 0, \phi = 0$$

$$F_x = 0 \Rightarrow y + 2z + \lambda yz = 0$$

$$\Rightarrow y + 2z = -\lambda yz$$

$$\begin{aligned}\Rightarrow \frac{y}{yz} + \frac{2z}{yz} &= -\lambda \\ \Rightarrow \frac{1}{z} + \frac{2}{y} &= -\lambda\end{aligned}\tag{3}$$

$$\begin{aligned}F_y = 0 \Rightarrow x + 2z + \lambda xz &= 0 \\ \Rightarrow x + 2z &= -\lambda xz\end{aligned}$$

$$\begin{aligned}\text{i.e., } xy + 2yz &= -\lambda xyz \\ \Rightarrow \frac{xy}{xyz} + \frac{2yz}{xyz} &= -\lambda \\ \Rightarrow \frac{1}{z} + \frac{2}{x} &= -\lambda\end{aligned}\tag{4}$$

$$\begin{aligned}F_z = 0 \Rightarrow 2y + 2x + \lambda xy &= 0 \\ \Rightarrow 2x + 2y &= -\lambda xy \\ 2xz + 2yz &= -\lambda xyz \\ \Rightarrow \frac{2xz}{xyz} + \frac{2yz}{xyz} &= -\lambda \\ \Rightarrow \frac{2}{y} + \frac{2}{x} &= -\lambda\end{aligned}\tag{5}$$

$$\begin{aligned}(3) - (4) \Rightarrow \quad \frac{2}{y} - \frac{2}{x} &= 0 \\ \frac{1}{y} &= \frac{1}{x} \\ \Rightarrow x &= y\end{aligned}\tag{6}$$

$$\begin{aligned}(3) - (5) \Rightarrow \quad \frac{1}{z} - \frac{2}{x} &= 0 \\ \frac{1}{z} &= \frac{2}{x} \\ \Rightarrow x &= 2z\end{aligned}\tag{7}$$

From (6)and(7) we obtain  $x = y = 2z$ .

$$(1) \Rightarrow 2z \cdot 2z \cdot z = 32 \Rightarrow 4z^3 = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2.$$

$$\therefore x = 4, y = 4.$$

The stationary point is  $(4, 4, 2)$ .

The dimensions are  $4\text{cm}, 4\text{cm}, 2\text{cm}$ .

**Example 2.89.** Find the dimensions of the rectangular box open at the top of maximum capacity whose surface area is  $432 \text{ sq.m}$ . [Jun 2013]

**Solution.** Let the dimensions of the box be  $x, y, z$ .

Given, surface area = 432.

$$xy + 2xz + 2yz = 432 \quad (1)$$

Let  $V$  be the volume of the box.

We have to maximize  $V$ .

$$V = xyz \quad (2)$$

By lagrange's method

$$F = V + \lambda\phi, \text{ where } \lambda \text{ is the lagrange multiplier.}$$

$$F = xyz + \lambda(xy + 2xz + 2yz - 432).$$

$$F_x = yz + \lambda(y + 2z).$$

$$F_y = xz + \lambda(x + 2z).$$

$$F_z = xy + \lambda(2x + 2y).$$

$$F_\lambda = xy + 2xz + 2yz - 432.$$

For stationary points,  $F_x = 0, F_y = 0, F_z = 0, F_\lambda = 0$ .

$$\begin{aligned} F_x = 0 &\Rightarrow yz + \lambda(y + 2z) = 0 \\ &\Rightarrow xyz + \lambda(xy + 2xz) = 0. \end{aligned} \quad (1)$$

$$\begin{aligned} F_y = 0 &\Rightarrow xz + \lambda(x + 2z) = 0 \\ &\Rightarrow xyz + \lambda(xy + 2yz) = 0. \end{aligned} \quad (2)$$

$$F_z = 0 \Rightarrow xy + \lambda(2x + 2y) = 0$$

$$xyz + \lambda(2xz + 2yz) = 0. \quad (3)$$

$$(1) + (2) + (3) \Rightarrow$$

$$3xyz + \lambda(2xy + 4xz + 4yz) = 0$$

$$3xy + 2\lambda(xy + 2xz + 2yz) = 0$$

$$3xyz + 2\lambda \times 432 = 0$$

$$3xyz = -864\lambda$$

$$\lambda = -\frac{3xyz}{864} = -\frac{xyz}{288}$$

Substituting the value of  $\lambda$  in  $F_x = 0$  we get

$$yz - \frac{xyz}{288}(y + 2z) = 0$$

$$1 - \frac{x}{288}(y + 2z) = 0$$

$$1 = \frac{x}{288}(y + 2z)$$

$$xy + 2xz = 288 \quad (4)$$

$$F_y = 0 \Rightarrow xz - \frac{xyz}{288}(x + 2z) = 0$$

$$1 - \frac{y}{288}(x + 2z) = 0$$

$$1 - \frac{y}{288}(x + 2z) = 0, 1 = \frac{y}{288}(x + 2z)$$

$$xy + 2yz = 288 \quad (5)$$

$$F_z = 0 \Rightarrow xy - \frac{xyz}{288}(2x + 2y) = 0$$

$$1 - \frac{z}{288}(2x + 2y) = 0, 1 = \frac{z}{288}(2x + 2y)$$

$$2xz + 2yz = 288 \quad (6)$$