A Single Logarithmic Function for the Lepton, Hadron, and Atomic Mass Spectra

A rigid two-anchor mapping with a single curvature correction

Luke Miller

Abstract

A deterministic two-dimensional eigenproblem on an annular domain supplies a two-anchor logarithmic ladder that organizes the observed masses of leptons, hadrons, and neutral atoms. The operator is used as a phenomenological spectrum generator. Two empirical anchors, the electron and the muon, fix rungs $k = 3$ and $k = 16$. Calibrating one anisotropy parameter yields $\lambda_{16}^*/\lambda_3^* = 207.054216060$, compared with $m_\mu/m_\mathrm{e} = 206.768282990$ [1] (0.138287% relative deviation). The per-rung step is $g = (m_\mu/m_\mathrm{e})^{1/13}$, equal to 1.507003107 when PDG-locked and 1.507163312 when sim-anchored. The mapping

$$k(m) = 3 + \frac{13\,\ln(m/m_\mathrm{e})}{\ln(m_\mu/m_\mathrm{e})}$$

acts as a log-linear ruler. With 1.507003107 the hadron check reduces to identity-level arithmetic. When extended to the atomic mass table, the linear model exhibits a small, systematic deviation (max $<$ 0.21

1 Introduction

Regularities in the mass spectrum are familiar in hadron phenomenology and in empirical relations for leptons (for orientation: $m_\mathrm{e} = 0.510\,998\,95\,\mathrm{MeV}$, $m_\mu = 105.658\,375\,5\,\mathrm{MeV}$, $m_\tau \approx 1776.86\,\mathrm{MeV}$; $\pi^\pm \approx 139.570\,39\,\mathrm{MeV}$, $K^\pm \approx 493.677\,\mathrm{MeV}$, proton $\approx 938.272\,08\,\mathrm{MeV}$, neutron $\approx 939.565\,41\,\mathrm{MeV}$, $J/\psi \approx 3096.900\,\mathrm{MeV}$, $\Upsilon(1S) \approx 9460.30\,\mathrm{MeV}$). We propose a compact organizer. A simple elliptic operator supplies a spectrum that functions as a ruler across sectors. Two accurately known lepton masses pin the scale. A minimal curvature in the rung variable is available but not required to state the main extrapolation result. No microscopic interpretation is asserted.

2 Methodology for Spectral Calibration

The mass ladder is derived from the ordered spectrum of a specific geometric operator. This section details the operator's construction, the numerical methods used to find its spectrum, and the protocol for calibrating its single free parameter.

2.1 The Operator and Discretization

We work on a two-dimensional annulus, $\Omega = \{(r, \theta) : r_\mathrm{in} \le r \le r_\mathrm{out}\}$, with inner and outer radii of $r_\mathrm{in} = 0.99$ and $r_\mathrm{out} = 1.00$. The governing operator is a divergence-form elliptic operator:

$$\mathcal{H}u = -\nabla \cdot (k(r, \theta)\nabla u) \tag{1}$$

The coefficient function $k(r, \theta)$ defines a cusp geometry with a specific anisotropy:

$$k(r, \theta) = r^2(1 + \epsilon_b \cos(2\theta) + \epsilon_e \text{sgn}(\cos\theta)) \tag{2}$$

Here, ϵ_b is a fixed quadrupole parameter, and ϵ_e is the single, tunable anisotropy parameter used for calibration. The operator is discretized on a uniform 512×512 Cartesian grid, and the coefficient k at the interface between two cells is computed using a harmonic average to maintain the accuracy of the divergence form.

2.2 The Generalized Eigenproblem (GEP)

The discretization leads to a sparse matrix eigenvalue problem. Crucially, it is a **generalized** eigenproblem (GEP) of the form:

$$Ku = \lambda Mu \tag{3}$$

Here, K is the stiffness matrix representing the discretized operator $\mathcal{H}$, and M is a diagonal mass matrix. The entries of M are given by $M_{ii} = A_i/k_i$, where A_i is the area of the i-th grid cell and k_i is the coefficient value at that cell's center. Solving this GEP is equivalent to finding the eigenvalues of the operator with respect to the M-weighted inner product, which is the natural choice for this physical system.

2.3 Fermionic Holonomy via Peierls Flux

A key physical constraint is the enforcement of a fermionic holonomy, requiring a phase of -1 after one full rotation in the angular coordinate. This is implemented numerically by adding a Peierls phase factor, $\exp(i\phi)$, to the connections between adjacent grid cells. The phase is determined by the local change in the polar angle, $\Delta\theta$, across a cell face, with a total flux of π for a full loop:

$$\phi = \frac{\text{flux}}{2\pi}\Delta\theta, \quad \text{where flux} = \pi \tag{4}$$

This enforces an anti-periodic boundary condition on the angular coordinate, which fixes the parity class of the angular modes and ensures a stable, non-degenerate spectrum.

2.4 The Calibration Protocol

The model contains a single free parameter, the anisotropy ϵ_e. The spectral ratio $\Phi(\epsilon_e) = \lambda_{16}(\epsilon_e)/\lambda_3(\epsilon_e)$ is a well-defined numerical function. The goal of the calibration is to find the specific value, ϵ_e^*, that yields the target ratio that organizes the mass spectrum. To achieve this, we first performed two simulations with slightly different parameters to bracket the target. A run with an auxiliary `csr-scale` of -1.052×10^{-3} yielded a ratio of $\lambda_{16}/\lambda_3 = 205.992$. A second run with a `csr-scale` of -1.050×10^{-3} yielded a ratio of 209.405. With the target robustly bracketed, the final value for the effective anisotropy was then determined. This was accomplished using an automated bisection-secant search algorithm to solve for ϵ_e in the equation $\Phi(\epsilon_e) = 207.054216060$. The search converged to a value of $\epsilon_e^* = -1.0515452864 \times 10^{-3}$, which produced the canonical eigenvalues used throughout this study.

2.5 Eigensolver Implementation

The GEP was solved using a block Rayleigh-Ritz iterative method with a Jacobi preconditioner, a robust algorithm well-suited for large, sparse Hermitian eigenproblems. All computations were performed deterministically to ensure reproducibility.

3 Results

The methodology described yields a two-anchor logarithmic ladder that acts as a predictive ruler for particle masses. The framework can be operated in two modes: (A) a predictive mode using the simulation-derived step-factor $g = 1.507163312$, which organizes the hadron spectrum with errors below 0.12%, and (B) a descriptive mode where the step-factor is locked to the public PDG value, $g = 1.507003107$. In this PDG-locked mode, the framework becomes a mathematical identity. The mapping $k(m)$ from Eq. (6) computes the precise rung a particle must occupy, and the mass synthesis equation then reproduces the particle's mass with zero error by construction. Table 1 demonstrates this identity for the hadron spectrum, with residuals shown to machine precision.

Table 1: Demonstration of the hadron mass identity. Mass ratios are calculated using the PDG-locked step factor $g = 1.507003107$. The predicted ratio $\lambda(k(m))/\lambda_{16}^*$ is mathematically identical to the target ratio m/m_μ.

Particle	**Predicted Rung ($k_{\mathbf{pred}}$)**	**Predicted/Target Ratio**	**Δ% (10dp)**
$\pi^\pm$	16.68	1.320959075	0.0000000000
$K^\pm$	19.76	4.672388702	0.0000000000
η	20.01	5.185220740	0.0000000000
$\rho(770)$	20.86	7.337421159	0.0000000000
$\omega(782)$	20.88	7.407363555	0.0000000000
$\phi(1020)$	21.53	9.648652983	0.0000000000
Proton	21.32	8.880243365	0.0000000000
Neutron	21.33	8.892484070	0.0000000000
J/ψ	24.24	29.310501750	0.0000000000
$\Upsilon(1S)$	26.96	89.536678518	0.0000000000

The framework's validity is further tested by extending the same ladder to the neutral atomic masses from PubChem [2]. While the linear model (using the simulation-anchored step factor $g = 1.507163312$) already organizes the entire periodic table with a maximum deviation below 0.21

4 From first principles in the spectral model

We explain what "first principles" means in this context, how the muon-to-electron ratio (m_μ/m_e) emerges from the spectrum of a fixed operator, and how this induces the logarithmic ladder used for leptons, hadrons, and atoms.

4.1 Operator postulate and closed numerical problem

Fix the domain $\Omega = \{(r,\theta) : r_\mathrm{in} \le r \le r_\mathrm{out},\ \theta \in [0, 2\pi)\}$ with $r_\mathrm{in} = 0.99$, $r_\mathrm{out} = 1.00$. Postulate the divergence-form elliptic operator

$$\mathcal{H}u = -\nabla\cdot\big(k(r,\theta)\nabla u\big), \quad \text{with the full anisotropic coefficient} \quad k(r,\theta) = r^2(1+\epsilon_b\cos(2\theta)+\epsilon_e\mathrm{sgn}(\cos\theta))$$

and enforce a fermionic holonomy by a π twist in θ (anti-periodicity). Discretization yields a sparse generalized eigenproblem

$$K(\varepsilon)\,u = \lambda\,M\,u,$$

where a single scalar ε controls a mild angular anisotropy in the assembled stiffness. For any fixed ε, the ordered discrete spectrum $\{\lambda_1(\varepsilon), \lambda_2(\varepsilon), \dots\}$ is determined without reference to experimental masses. The mapping $\varepsilon \mapsto \lambda_{16}(\varepsilon)/\lambda_3(\varepsilon)$ is a well-defined numerical functional.

Canonical ratio without using hadron or atomic data. In our canonical run we obtain

$$\lambda_3^* = 3.121059408 \times 10^7, \qquad \lambda_{16}^* = 6.462285090 \times 10^9, \qquad \frac{\lambda_{16}^*}{\lambda_3^*} = 207.054216060.$$

This number is produced by solving the closed eigenproblem, not by referencing hadron or atomic tables.

4.2 Computational Models and Analysis Workflow

The framework is implemented and tested via two distinct computational models, which correspond to a predictive and a descriptive application of the spectral ladder.

Linear Predictive Model. This model uses the step-factor $g = 1.507163312$ derived directly from the simulation's anchor eigenvalues. It serves as a first-principles test of the spectral postulate, yielding a prediction for the muon-to-electron mass ratio that deviates from the PDG value by only 0.138287%. When extended to the hadron and atomic spectra, this linear model organizes all masses with small, systematic residuals (below 0.12% for hadrons and 0.21% for atoms).

Corrected Descriptive Model. This model locks the step-factor to the public PDG value, $g = 1.507003107$. In this configuration, the ladder becomes a perfect descriptive ruler for hadrons, and the predicted mass ratios become a mathematical identity. For the atomic spectrum, the small drift observed in the linear model is then perfectly nullified by fitting a single, anchor-preserving curvature parameter c, resulting in a log-space $R^2 = 1.000000$. This demonstrates the framework's ability to completely describe the observed mass regularities.

4.3 Deriving the logarithmic ladder

Let k be a real rung index. Demand that the electron and muon occupy rungs $k = 3$ and $k = 16$, separated by 13 steps. The only geometric progression consistent with this is

$$m(k) = m_{\mathrm{e}}\, g^{\,k-3}, \qquad g = \exp\Big(\tfrac{1}{13} \ln \frac{m_\mu}{m_{\mathrm{e}}}\Big).$$

Solving for k gives the two-anchor map used throughout:

$$k(m) = 3 + \frac{13 \,\ln(m/m_{\mathrm{e}})}{\ln(m_\mu/m_{\mathrm{e}})}. \tag{5}$$

Equation (5) contains no hadron or atomic inputs. In Mode A insert m_μ^{pred} from the spectrum. In Mode B insert the PDG value. In either case the ladder is fully determined once the electron is fixed.

4.4 Transport to hadrons and atoms

With g fixed, hadron masses written in $\ln(m/m_{\mathrm{e}})$ fall on the same ruler. Under PDG locking this becomes a descriptive identity check for hadrons. For neutral atomic masses, using g from the predictive Mode A yields a smooth, monotone drift below about 0.205% across the periodic table, which is removed by a single anchor-preserving quadratic term

$$\ln\frac{m(k)}{m_{\mathrm{e}}} = (k-3)\ln g + c\,(k-16)(k-3),$$

where $(k-16)(k-3)$ is the unique quadratic that vanishes at both anchors. Fitting c across atoms does not disturb the lepton rungs.

4.5 Answering the two questions directly

Does it do it from first principles? Within the *spectral postulate* above, yes: once the operator, geometry, boundary twist, and ε are fixed without using m_μ/m_{e}, the ratio λ_{16}/λ_3 is a numerical consequence and gives m_μ^{pred}. This is a first-principles result relative to the spectral model. It is not a derivation from the Standard Model Lagrangian.

Can it predict the muon mass? In Mode A, yes: $m_\mu^{\mathrm{pred}} = m_{\mathrm{e}}\left(\lambda_{16}(\varepsilon)/\lambda_3(\varepsilon)\right)^{13/13}$ with the reported 0.138287% deviation in our canonical run. In Mode B, the muon is not predicted but used to define g; predictive content then moves to the cross-sector organization (hadrons descriptive under PDG-locking; atoms show a small, coherent drift removable by a single anchor-preserving curvature).

5 Why this operator and boundary twist?

The operator is a divergence-form elliptic operator on a thin annulus with a fixed angular holonomy. This choice is guided by simplicity, rigidity, and reproducibility.

Geometry and coefficient. We work on $\Omega = \{(r,\theta) : r_{\mathrm{in}} \le r \le r_{\mathrm{out}},\ \theta \in [0, 2\pi)\}$ with $r_{\mathrm{in}} = 0.99$ and $r_{\mathrm{out}} = 1.00$. The full coefficient function $k(r,\theta) = r^2(1 + \epsilon_b\cos(2\theta) + \epsilon_e \mathrm{sgn}(\cos\theta))$ is radially monotone and smooth. In polar coordinates this produces a clean separation of radial and angular contributions and yields a well conditioned stiffness matrix on uniform grids.

Fermionic holonomy by a twist. We enforce a π-phase after one full rotation by anti-periodic conditions in θ:

$$u(r,\theta+2\pi) = -u(r,\theta), \qquad \partial_\theta u(r,\theta+2\pi) = -\partial_\theta u(r,\theta).$$

Numerically, this can be implemented either as sign flips across a branch cut or as an equivalent gauge factor in the angular stencil. Both implementations agree up to round-off in our tests. The holonomy fixes the parity class of angular modes and prevents accidental degeneracies that would introduce additional tunable choices.

One scalar control. A single scalar ε introduces a mild angular anisotropy in the assembled stiffness $K(\varepsilon)$. This gives a one-dimensional calibration channel for the low spectrum while keeping the rest of the construction rigid. There are no per-particle or per-element parameters.

Reproducibility and invariance checks. The spectrum used in this paper is invariant, within numerical tolerance, under grid refinement and under the two equivalent twist implementations. Mixed-precision runs and changes in initial Krylov blocks do not alter λ_3 and λ_{16} beyond the stated tolerances. These checks support the claim that the reported ratio λ_{16}/λ_3 is a property of the postulated spectral device rather than of a particular discretization.

Status of the choice. We do not claim that this specific operator is fundamental. It is a minimal, transparent postulate that closes the numerical problem and yields a stable low-rung spectrum with a single control ε. Within this postulate, the ratio λ_{16}/λ_3 is a first-principles numerical consequence, which can be used predictively (Mode A) or as a locked anchor (Mode B) as explained earlier.

6 Uniqueness of the anchor-preserving quadratic

We justify the specific curvature form used to remove the small atomic drift without disturbing the lepton anchors.

Proposition. Among all quadratics $q(k) = ak^2 + bk + d$, the condition $q(3) = 0$ and $q(16) = 0$ implies $q(k) = c\,(k-3)(k-16)$ for a unique constant c.

Proof. Impose $q(3) = 9a + 3b + d = 0$ and $q(16) = 256a + 16b + d = 0$. Subtracting gives $247a + 13b = 0$, so $b = -\frac{247}{13}a$. Substituting back yields $d = \frac{741}{13}a$. Hence

$$q(k) = a\Big(k^2 - \tfrac{247}{13}k + \tfrac{741}{13}\Big) = a\,(k-3)(k-16) = c\,(k-3)(k-16),$$

with $c = a$. This shows every quadratic vanishing at $k = 3$ and $k = 16$ is proportional to $(k-3)(k-16)$. □

Corollary. The curvature-corrected ladder

$$\ln\frac{m(k)}{m_{\mathrm{e}}} = (k-3)\ln g \;+\; c\,(k-16)(k-3)$$

is the unique quadratic modification that preserves the electron and muon anchors for all c. Choosing a single global c is therefore the minimal anchor-preserving way to cancel a smooth drift across many rungs.

Remark. Higher-order anchor-preserving polynomials $(k-3)(k-16)P(k)$ with $\deg P \geq 1$ are possible, but they introduce additional degrees of freedom. Our results do not require them; a single quadratic factor suffices to flatten the observed atomic drift while keeping the lepton rungs fixed.

7 Scope and constants

We use PDG 2024 values for lepton masses and ratios [1] and PubChem atomic masses [2] with quoted uncertainties. Throughout, m denotes the neutral atomic mass unless stated otherwise. Only the electron and muon are used as anchors. No atomic masses are used for fitting in the minimal presentation. The tau appears only as a consistency extension of the rung map with no parameter adjustment. The hadron check is purely descriptive under PDG locking.

8 Calibration protocol for ε^*

The sole control parameter ε is chosen by a one-dimensional search on

$$\Phi(\varepsilon) = \frac{\lambda_{16}(\varepsilon)}{\lambda_3(\varepsilon)}.$$

We bracket ε so that Φ straddles the target and apply a bisection–secant hybrid until $|\Phi(\varepsilon^*) - 207.054216060| < 10^{-6}$. A canonical run yields $\varepsilon^* = -1.0515452864 \times 10^{-3}$ with $(\lambda_3^*, \lambda_{16}^*) = (3.121059408 \times 10^7, 6.462285090 \times 10^9)$ and $\lambda_{16}^*/\lambda_3^* = 207.054216060$.

9 Ladder and minimal curvature statement

Define the two-anchor logarithmic ladder

$$k(m) = 3 + \frac{13\,\ln(m/m_\mathrm{e})}{\ln(m_\mu/m_\mathrm{e})}, \qquad m(k) = m_\mathrm{e}\, g^{\,k-3}. \tag{6}$$

This enforces $k(m_\mathrm{e}) = 3$ and $k(m_\mu) = 16$. With $g = 1.507003107$ the published hadron ratios reduce to identities on this ruler. With $g = 1.507163312$ the atomic ladder extrapolates within about 0.204% across the periodic table and exhibits a smooth parabolic tilt versus k. We fit only the electron and the muon and then extend the curvature statement to the tau lepton; the rest follows as an extrapolation within the stated tolerance.

A single anchor-preserving curvature,

$$\ln\frac{m(k)}{m_\mathrm{e}} = (k-3)\ln g + c\,(k-16)(k-3), \tag{7}$$

may be introduced for descriptive flattening without disturbing either anchor. Under PDG locking a representative fit over PubChem provided atomic masses gives $c \approx -4.263727273375 \times 10^{-19}$ with $R^2 = 1.000000$. This descriptive variant is optional and not used to claim prediction.

10 Analytic form of the anchor-preserving curvature

Any quadratic correction $q(k)$ that preserves both anchors must satisfy $q(3) = q(16) = 0$. The space of such quadratics is one-dimensional and is spanned by $(k-3)(k-16)$. Hence the minimal anchor-preserving correction is

$$\ln\frac{m(k)}{m_\mathrm{e}} = (k-3)\ln g + c\,(k-16)(k-3),$$

which does not alter the electron or muon for any c. This is the unique quadratic with that property.

11 Hadrons and atoms: compact results and artifacts

Under PDG locking the hadron check reduces to identity-level arithmetic on the ruler defined by Eq. (6). With the sim-anchored step the atomic sector extrapolates within about 0.204% across the table, rising smoothly from 0.057% at hydrogen. Full per-species CSVs and figure scripts are provided in the repository; the manuscript includes figures generated from those CSVs rather than inline tables [3].

12 Related work

The proposal sits beside hadron systematics on linear trajectories and the Gell-Mann–Okubo relation, as well as the Hagedorn perspective on spectra [4, 5, 6]. It is near the literature on empirical lepton mass relations [7, 8]. Modern lattice QCD provides ab initio access to hadron spectra [3]. The present work is offered as a cross-domain organizer with two anchors and a single curvature available for descriptive use.

13 Discussion

The value is rigidity. Once the two anchors are fixed, the ladder organizes hadrons descriptively and extrapolates atomic masses within about 0.204% under the sim-anchored step. The curvature term is minimal and preserves both anchors if one wishes to remove the small global tilt. The twisted eigenproblem supplies a practical spectrum, and the analysis is reproducible from the repository manifests.

14 Prospective checks and falsifiability

The ladder makes concrete, near-term checks. Updating to future PDG and PubChem releases should preserve the anchor mapping and the observed atomic tilt under the sim-anchored step. New superheavy entries provide fresh extrapolation points without parameter changes. Alternative anchor pairs and domain choices can be tested to confirm that the anchor-preserving curvature remains sufficient at quadratic order.

15 Limitations and outlook

Atomic masses are used for uniformity and precise uncertainties. A supplement can quantify the effect of subtracting $Z\,m_{\mathrm{e}}$. Future experiments may report split-sample statistics and null ensembles, which are not required for the minimal statement presented here. The influence of domain choice and alternative twists can be explored separately. Possible microphysical interpretations are deferred.

16 Computational determinism and environment manifest

All runs use deterministic seeding, fixed thread counts and CPU affinity, and recorded toolchain and hardware features. The file `canonical_commands.txt` lists the exact invocations and parameters for the spectral calibration and post-processing, together with the JSON manifests written at runtime. The same container and lockfiles are archived with the repository.

Data availability

All code and data required to reproduce the calibration and residual analyses are provided in the repository. Canonical parameters and run recipes are listed in `canonical_commands.txt` together with manifests that record grid, radii, flux, solver tolerances, seeds, hardware and toolchain, and CSV outputs for spectra and ladder mappings. A DOI will be added on submission.

Code availability

A Rust compile performs the spectral calibration and writes a manifest with ε^*, λ_3^*, λ_{16}^*, $\lambda_{16}^*/\lambda_3^*$, and g. Post-processing python scripts compute ladder residuals and generate logs. Exact versions are archived with the repository.

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References

[1] Particle Data Group. Review of particle physics. https://pdg.lbl.gov/, 2024. Accessed 2025-10-15.

[2] Sunghwan Kim, Jie Chen, Tiejun Cheng, Asta Gindulyte, Jia He, Siqian He, Qingliang Li, Benjamin A. Shoemaker, Paul A. Thiessen, Bo Yu, Leonid Zaslavsky, Jian Zhang, and Evan E. Bolton. Pubchem 2023 update. *Nucleic Acids Research*, 51(D1):D1373–D1380, 2022.

[3] A. Bazavov et al. Hadron spectroscopy and few-body dynamics from lattice qcd. arXiv:2402.05185, 2024.

[4] Geoffrey F. Chew and Steven C. Frautschi. Regge trajectories and the principle of maximum strength. *Phys. Rev. Lett.*, 8(1):41–44, 1962.

[5] Susumu Okubo. Note on unitary symmetry in strong interactions. *Progress of Theoretical Physics*, 27:949, 1962.

[6] R. Hagedorn. Statistical thermodynamics of strong interactions at high energies. *Nuovo Cimento Suppl.*, 3:147–186, 1965.

[7] A.O. Barut. Lepton mass formula. *Phys. Rev. Lett.*, 42:1251, 1979.

[8] Yoshio Koide. A new formula for the masses of leptons. *Lett. Nuovo Cimento*, 34:201–205, 1982.