CDC01-REG1152

Global Stabilization of a Flat Underactuated System: the Inertia Wheel Pendulum

Reza Olfati-Saber
California Institute of Technology
Control and Dynamical Systems 107-81
Pasadena, CA 91125
olfati@cds.caltech.edu

Abstract

Inertial Wheel Pendulum (IWP) is a planar pendulum with a revolving wheel (that has a uniform mass distribution) at the end. The pendulum is unactuated and the wheel is actuated. Our main result is to address global asymptotic stabilization of the inertia wheel pendulum around its up-right position. Simulation results are provided for parameters taken from a real-life model of the IWP.

1 Introduction

Inertia Wheel Pendulum, depicted in Fig. 1, is a planar inverted pendulum with a revolving wheel at the end. The wheel is actuated and the joint of the

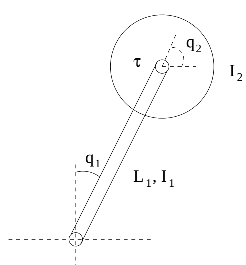

Figure 1: The Inertia Wheel Pendulum

pendulum at the base is unactuated. The inertial wheel pendulum was first introduced by Spong *et al.* in [3] where a supervisory hybrid/switching control strategy is applied to asymptotic stabilization of the IWP around its upright equilibrium point.

Here, we show that based on a recent result of the author in [1], the dynamics of the inertia wheel pendulum can be transformed into a cascade nonlinear system in strict feedback form using a global change of coordinates in an explicit form. Then, global asymptotic stabilization of the up-right equilibrium point can be achieved using backstepping procedure.

2 Dynamics and Control of IWP

The Lagrangian of the inertia wheel pendulum is as the following

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - V(q_1)$$

where $q = (q_1, q_2)^T$ is the configuration vector in $\mathbb{R}^2$, M is a constant inertia matrix with elements

$$\begin{aligned} m_{11} &= m_1 l_1^2 + m_2 L_1^2 + I_1 + I_2 \\ m_{12} &= m_{21} = m_{22} = I_2 \end{aligned}$$

and

$$V(q_1) = (m_1 l_1 + m_2 L_1) g_0 \cos(q_1) =: m_0 \cos(q_1)$$

is the potential energy of the IWP [3]. The Euler-Lagrange equations of motion for the inertia wheel pendulum are as the following

$$\begin{array}{ccccccc} m_{11}\ddot{q}_1 & + & m_{12}\ddot{q}_2 & + & g_1(q_1) & = & 0 \\ m_{21}\ddot{q}_1 & + & m_{22}\ddot{q}_2 & & & = & \tau \end{array} \tag{1}$$

where $g_1(q_1) = -m_0 \sin(q_1)$. Due to the fact that the kinetic energy of the inertia wheel pendulum is invariant with respect to the group action $(q_1, q_2) \to (q_1 + \alpha, q_2 + \beta)$, the system has an apparent kinetic symmetry and according to [1] one can apply the following global change of coordinates

$$\begin{array}{rcl} z_1 & = & \partial\mathcal{L}/\partial\dot{q}_1 = m_{11}\dot{q}_1 + m_{12}\dot{q}_2 \\ z_2 & = & q_1 \\ z_3 & = & \dot{q}_2 \end{array} \quad (2)$$

and change of control

$$\tau = (m_{22} - m_{21}m_{12}/m_{11})u + (m_{21}m_0/m_{11})\sin(q_1)$$

to transform the dynamics of the IWP into a strict feedback system

$$\begin{array}{rcl} \dot{z}_1 & = & m_0 \sin(z_2) \\ \dot{z}_2 & = & (z_1 - m_{12}z_3)/m_{11} \\ \dot{z}_3 & = & u \end{array} \quad (3)$$

Note that q_2 does not play any important role in the dynamics of the IWP and is ignored as a state variable. Here is our main result:

Proposition 1. *There exists a nonlinear state feedback law in the following explicit form*

$$u = \frac{m_{11}}{m_{12}}(c_2\mu_1 + c_3\mu_2 + \frac{m_0}{m_{11}}\sin(z_2) - \ddot{k}_1) \quad (4)$$

that globally asymptotically stabilizes $(z_1, z_2, z_3) = 0$ *for the Inertia Wheel Pendulum in (3) where*

$$\begin{array}{rcl} k_1(z_1) & = & c_0\sigma(c_1z_1) \\ \dot{k}_1(z_1, z_2) & = & c_0c_1m_0\sin(z_2)(1 - \sigma(c_1z_1)^2) \\ \ddot{k}_1(z_1, z_2, z_3) & = & c_0c_1m_0(1 - \sigma(c_1z_1)^2) \\ & \cdot & [\cos(z_2)(z_1/m_{11} - m_{12}z_3/m_{11}) \\ & - & 2c_1m_0\sigma(c_1z_1)\sin(z_2)^2] \\ \mu_1 & = & z_2 - k_1(z_1) \\ \mu_2 & = & (z_1 - m_{12}z_3)/m_{11} - \dot{k}_1 \end{array}$$

with the choice of sigmoidal function $\sigma(s) = \tanh(s)$ *and constants*

$$-\pi/2 < c_0 < 0, \; c_1, c_2, c_3 > 0$$

Proof. The proof relies on global stability of cascade nonlinear systems with a minimum-phase zero-dynamics that are robust w.r.t. L_1-bounded disturbances. See [2] pages 133–134 in [2] for the full proof. □

The state trajectories and the control input of the inertia wheel pendulum starting at the downward position ($q_1 = \pi$) are shown in Fig. 2. For comparison purposes, we used the same numeric values for the parameters of the dynamics of the IWP as in [3]. These parameters are as follows: $m_{11} = 4.83 \times 10^{-3}$, $m_{12} = m_{21} = m_{22} = 32 \times 10^{-6}$, $\bar{m} = 38.7 \times 10^{-3}$, $g_0 = 9.8$, and $m_0 = \bar{m}g_0$. The simulation results demonstrate that the obtained nonlinear controller aggressively stabilizes the pendulum to its upright position. The values of the controller parameters for the simulation are $c_0 = -\pi/10, c_1 = .03, c_2 = 16, c_3 = 8$ and the maximal applied torque was $\tau_{max} = 0.6$ Nm.

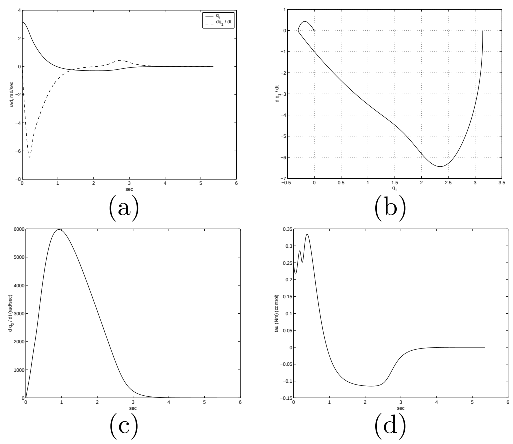

Figure 2: State trajectories and control of IWP: (a) the trajectories of $(q_1, \dot{q}_1)$, (b) the state path in $(q_1, \dot{q}_1)$ plane, (c) $\dot{q}_2$, and (d) control input τ.

3 Conclusion

In this paper, global stabilization of the Inertial Wheel Pendulum to its up-right equilibrium point using a nonlinear state feedback in closed form was achieved based on a key global change of coordinates in [1].

References

[1] R. Olfati-Saber. "Control of underactuated mechanical systems with two degrees of freedom and symmetry". *Proc. of American Control Conference*, 6:4092–4096, Chicago, IL, June 2000.

[2] R. Olfati-Saber. *"Nonlinear Control of Underactuated Mechanical Systems with Application to Robotics and Aerospace Vehicles"*. PhD thesis, Massachusetts Institute of Technology, Department of Electrical Engineering and Computer Science, February 2001.

[3] M. W. Spong, P. Corke, and R. Lozano. "Nonlinear control of the Inertia Wheel Pendulum". *Automatica*, submitted, September, 1999.