Lovász Number and Related convex function

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Introduction

The Lovász number, also called the Lovász theta function, was introduced by László Lovász in his 1979 paper [1]. It is defined as the upper bound of the Shannon capacity of a graph. Its value lies between the clique number and the chromatic number of a graph. Since finding the values of clique number and chromatic number are NP-hard problems, the Lovász theta function can be used to find these values when they are equal, i.e. in perfect graphs.

Definitions

- **Strong product of graphs**
  Let G and H be the given graphs.We define the graph strong product of graphs G and H as $G \boxtimes H$.

The vertices in $G \boxtimes H$ are defined as a tuple (u, u'). The distinct vertices (u, u') and (v, v') of the graph $G \boxtimes H$ are adjacent to each other if it meets one of the following conditions:

- $u = v$ and u' is adjacent to v'
- $u' = v'$ and u is adjacent to v
- u is adjacent to v and u' is adjacent to v'.

The k-fold strong product of a graph G is denoted by G_k

$$G_k = G \boxtimes G \ldots k \text{ times} \ldots \boxtimes G$$

- **Independence number**
  An independent set of vertices is a set of vertices in which, no two vertices in the set are adjacent to each other.
  The independence number, also known as the stability number, of graph G is the size of the largest independent set of G. It is denoted by $\alpha(G)$.
- **Clique number**
  A clique is a subset of vertices of a graph G in which every pair of vertices in the set are adjacent to each other. It can also be defined as a complete induced subgraph.
  The clique number of a graph G is the size of the largest clique in G. It is denoted by $\omega(G)$.
- **Chromatic number**
  The chromatic number of a graph G is the minimum number of colors required to color the vertices of G such that, no two adjacent vertices have the same color. It is denoted by $\chi(G)$.

- **Clique cover number**
  A clique cover of G is the partition of vertices into vertex disjoint cliques.
  Clique cover number is the size of the clique cover which uses as few cliques as possible. It is denoted by $\overline{\chi}(G)$.
- **Shannon capacity**
  The Shannon capacity of a graph G is defined as the number of independent sets of strong graph products.

$$\begin{aligned}\Theta(G) &= \sup_k \sqrt[k]{\alpha(G_k)} \\ &= \lim_{k\to\infty} \sqrt[k]{\alpha(G_k)}\end{aligned}$$

Lovász number

Let $G = (V, E)$ be a graph with n vertices. We assign a unit vector u_i to the i^{th} vertex. If the i^{th} and j^{th} vertices are not adjacent to each other in G, we assign u_i and u_j such that they are orthogonal to each other. We call such an assignment of vectors as an orthonormal representation of G.

$$u_i^T u_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i, j \notin E \end{cases}$$

The Lovász number $\vartheta(G)$ of graph G is written as,

$$\vartheta(G) = \min_{\substack{c, U \\ \|c\|=1 \\ \|u_i\|=1}} \max_{i \in [1,n]} \frac{1}{(c^T u_i)^2}$$

Properties of Lovász number

- If $G \boxtimes H$ is the strong product of G and H,

$$\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$$

- Let $\overline{G}$ be the complement of graph G. The complement of a graph G has vertex set of G and the edge set as,

$$\overline{E} = \{\{i, j\} \subset V : \{i, j\} \notin E\}$$

$$\vartheta(G)\vartheta(\overline{G}) \geq n$$

  The above inequality becomes equal if the graph G is vertex-transitive.

- Lovász Sandwich Theorem: The Lovász Sandwich Theorem states that the Lovász number always lies between the clique number and the chromatic number, i.e.

$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$$

- The Lovász number is an upper bound of the Shannon capacity and the independence number of G

Vraiants of Theta function

- Maximum stable set problem (MSSP)
  The problem of determining $\alpha(G)$ is called the maximum stable set problem or MSSP.
- Maximum clique problem (MCP)
  The problem of determining $\omega(G)$ is called the maximum clique problem or MCP.
  Since $\alpha(G) = \omega(\overline{G})$, the MSSP and MCP are equivalent. They have a wide range of applications in operational research and computer science.

Variants of Theta function

LP-based bounds of MSSP

The standard 0-1 LP formulation of the MSSP is as follows. For each $i \in V$, let x_i be a binary variable, taking the value 1 if and only if i is in the stable set.

$$\begin{array}{lllr} \max & e^T x & & \\ \text{s.t.} & x_i + x_j & \leq 1 & (\{i,j\} \in E) \\ & x_i \in \{0,1\} & & (i \in V) \end{array}$$

Variants of Theta function

The theta function

Suppose that a vector $x \in \{0, 1\}^n$ is feasible for the 0-1 LP formulation presented in **??**. We define the following,

- $Z = \frac{xx^T}{e^T x} \in [0, 1]^{n \times n}$
- $JZ = \frac{(e^T x)^2}{e^T x} = e^T x$
- $\mathrm{diag}(Z) = \frac{x}{e^T x} \implies \mathrm{Tr}(Z) = 1$

This leads to the SDP relaxation of the MSSP, which we will call '`SDP1`':

$$\begin{array}{ll} \max & JZ \\ \text{s.t.} & Z_{ij} = 0 \quad (\{i, j\} \in E) \\ & \mathrm{Tr}(Z) = 1 \\ & Z \in \mathcal{S}^n_+ \end{array}$$

The resulting upper bound is $\vartheta(G)$.
It was shown that SDPs can be solved in polynomial time.

Variants of Theta function

The theta function derived by *Grotschel et al*

Given a feasible vector $x \in \{0,1\}^n$, consider the matrix $X = xx^T \in \{0,1\}^{n \times n}$. X is positive semi-definite matrix. The augmented matrix X^+ is,

$$X^+ = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

It can be shown that X^+ is also positive semi-definite. This leads to another SDP relaxation, which we will call '`SDP2`':

$$\begin{array}{lll} \max & e^T x & \\ \text{s.t.} & X_{ii} = x_i & (i \in V) \\ & X_{ij} = 0 & (\{i,j\} \in E) \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}_+^{n+1} & \end{array}$$

Variants of Theta function

Relating SDP1 to SDP2 and vice-versa

Let (x^*, X^*) be any feasible solution to '`SDP2`' such that the profit (maximal value) γ is positive. We can obtain a feasible solution to '`SDP1`' whose profit is no smaller than γ by setting Z_{ij} to X^*_{ij}/γ for all i and j. Let Z^* be any feasible solution to `SDP1`, again with positive profit γ. Then there exists a feasible solution (x^*, X^*) to '`SDP1`', again with positive profit γ. Then there exists a feasible solution (x^*, X^*) to '`SDP2`' whose profit is no smaller than γ,

$$\begin{aligned} x^*_i &= \frac{\left(\sum_{j=1}^{n} Z^*_{ij}\right)^2}{\gamma Z^*_{ii}} \text{ if } Z^*_{ii} > 0 && (1) \\ &= 0 \qquad\qquad \text{otherwise} && (2) \end{aligned}$$

The weighted theta function

We consider the weighted version of the stable set problem. For each vertex $i \in V$ in the graph G, let $w_i > 0$ be the associated weight. For 'SDP2', we can change the objective function to $w^T x$. For 'SDP1', we have the objective function as

$$\sum_{i \in V} \sum_{j \in V} \sqrt{w_i w_j} Z_{ij}$$

The resulting bound is called the weighted theta function, denoted by $\vartheta(G, w)$. It can be proved that both the SDP formulations lead to the same upper bound.

The weighted theta function

Theorem

Let Z^* be a feasible solution to 'SDP1' and $\gamma = JZ^* > 0$ be the associated profit. Let x^* be the same as defined in (2). If $Z^*_{ij} = 0$, then set X^*_{ij} to 0. Otherwise, set X^*_{ij} to

$$Z^*_{ij}\sqrt{\frac{x^*_i x^*_j}{Z^*_{ii} Z^*_{jj}}} = Z^*_{ij}\frac{(\sum_{k=1}^{n} Z^*_{ik})(\sum_{k=1}^{n} Z^*_{jk})}{\gamma Z^*_{ii} Z^*_{jj}}$$

Then, the resulting pair (x^*, X^*) forms a feasible solution to 'SDP2' and its profit, $e^T x^*$ is at least as large as γ.

The weighted theta function

We will illustrate the above theorem using the following example,

Example

Let G be a graph on 3 nodes with $E = \{\{1,2\},\{2,3\}\}$. It holds that $\alpha(G) = \vartheta(G) = 2$. We get the following matrix as the feasible solution for '`SDP1`':

$$Z^* = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

We have $\gamma = \frac{5}{3}$. Then, we obtain $x^* = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} & \frac{4}{5} \end{pmatrix}^T$ with profit $\frac{9}{5} > \gamma$. Applying the above theorem's formula, we get,

$$X^* = \begin{pmatrix} \frac{4}{5} & 0 & \frac{4}{5} \\ 0 & \frac{1}{5} & 0 \\ \frac{4}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

The weighted theta function

Theorem

Let (x^*, X^*) be a feasible solution to the weighted version of 'SDP2', such that $\gamma = w^T x > 0$. We can obtain a feasible solution Z^* to the weighted version of 'SDP1', whose profit is at least as large, by setting Z^*_{ij} to $\frac{1}{\gamma}\sqrt{w_i w_j} X^*_{ij}$ for all i and j.

The weighted theta function

Theorem

Let Z^* be a feasible solution to the weighted version of 'SDP1' whose profit $\gamma > 0$. We can obtain a feasible solution (x^*, X^*) to the weighted version of 'SDP2', whose profit is at least as large, as follows. If $Z^*_{ii} = 0$, set $x^*_i = 0$. Otherwise, set x^*_i to,

$$\frac{\left(\sum_{j=1}^{n} \sqrt{w_j} Z^*_{ij}\right)^2}{\gamma Z^*_{ii}}$$

If $Z^*_{ij} = 0$, set $X^*_{ij} = 0$. Otherwise, set X^*_{ij} to

$$Z^*_{ij} \sqrt{\frac{x^*_i x^*_j}{Z^*_{ii} Z^*_{jj}}} = Z^*_{ij} \frac{(\sum_{k=1}^{n} \sqrt{w_k} Z^*_{ik})(\sum_{k=1}^{n} \sqrt{w_k} Z^*_{jk})}{\gamma Z^*_{ii} Z^*_{jj}}$$

Then, the resulting pair (x^*, X^*) forms a feasible solution to 'SDP2', and its profit, $w^T x^*$, is at least as large as γ.

Conclusion

The Lovász theta function can be computed by solving either of the two equivalent SDPs, which we labelled as SDP1 and SDP2. Our results have implications for the development of exact algorithms for the maximum stable set problem. If one wishes to use the theta function itself as the upper bound, then SDP1 is to be preferred to SDP2. If, however, one wishes to use a strengthened or weakened variant of the theta number, then the choice between SDP1 and SDP2 is no longer obvious.

References

- László Lovász, "On the Shannon capacity of a graph", 1979
- Laura Galli, Adam N. Letchford, "On the Lovász Theta Function and Some Variants"
- Carlos J. Luz, "An upper bound on the independence number of a graph computable in polynomial-time", 1995
- Carlos J. Luz, Alexander Schrijver, "A Convex Quadratic Characterization of the Lovász Theta Number", 2005

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