Lovász number and related convex functions

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***Abstract*—In this report, we talked about Lovász number. We also wrote about the various related functions.**

***Index Terms*—Lovász number, Shannon capacity,**

I. Introduction

The Lovász number, also called the Lovász theta function, was introduced by László Lovász in his 1979 paper [1]. It is defined as the upper bound of the Shannon capacity of a graph. Its value lies between the clique number and the chromatic number of a graph. Since finding the values of clique number and chromatic number are NP-hard problems, the Lovász theta function can be used to find these values when they are equal, i.e. in perfect graphs.

II. Definitions

A. Strong product of graphs

Let G and H be the given graphs. Let $V(G)$ and $V(H)$ be the vertex sets of graphs G and H respectively. We define the graph strong product of graphs G and H as $G \boxtimes H$. The vertex set of $G \boxtimes H$ is the Cartesian product of graphs $V(G)$ and $V(H)$.
The vertices in $G \boxtimes H$ are defined as a tuple (u, u'). The distinct vertices (u, u') and (v, v') of the graph $G \boxtimes H$ are adjacent to each other if it meets one of the following conditions:

- $u = v$ and u' is adjacent to v'
- $u' = v'$ and u is adjacent to v
- u is adjacent to v and u' is adjacent to v'.

The k-fold strong product of a graph G with itself is, the strong product of G with itself done k times. It is denoted by G_k.

$$G_k = G \boxtimes G \ldots k \text{ times} \ldots \boxtimes G$$

B. Independence number

An independent set of vertices is a set of vertices in which, no two vertices in the set are adjacent to each other.
The independence number, also known as the stability number, of graph G is the size of the largest independent set of G. It is denoted by $\alpha(G)$.

C. Clique number

A clique is a subset of vertices of a graph G in which every pair of vertices in the set are adjacent to each other. It can also be defined as a complete induced subgraph.
The clique number of a graph G is the size of the largest clique in G. It is denoted by $\omega(G)$.

D. Chromatic Number

The chromatic number of a graph G is the minimum number of colors required to color the vertices of G such that, no two adjacent vertices have the same color. It is denoted by $\chi(G)$.

E. Clique cover number

A clique cover of G is the partition of vertices into vertex disjoint cliques.
Clique cover number is the size of the clique cover which uses as few cliques as possible. It is denoted by $\overline{\chi}(G)$.

F. Shannon capacity

The Shannon capacity of a graph G is defined as the number of independent sets of strong graph products. It can be defined as,

$$\begin{aligned}\Theta(G) &= \sup_k \sqrt[k]{\alpha(G_k)} \\ &= \lim_{k\to\infty} \sqrt[k]{\alpha(G_k)}\end{aligned}$$

Shannon capacity of a graph is upper bounded by the Lovász number.

III. Lovász number

Let $G = (V, E)$ be a graph with n vertices. We assign a unit vector u_i to the i^{th} vertex. If the i^{th} and j^{th} vertices are not adjacent to each other in G, we assign u_i and u_j such that they are orthogonal to each other. We call such an assignment of vectors as an orthonormal representation of G. Let it be denoted by U. We can write it as,

$$u_i^T u_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i, j \notin E \end{cases}$$

The Lovász number $\vartheta(G)$ of graph G is written as,

$$\vartheta(G) = \min_{\substack{c,U \\ \|c\|=1 \\ \|u_i\|=1}} \max_{i\in[1,n]} \frac{1}{(c^T u_i)^2}$$

A. Properties

1) If $G \boxtimes H$ is the strong product of G and H,

$$\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$$

2)

$$\vartheta(G)\vartheta(\overline{G}) \geq n$$

Here $\overline{G}$ is the complement of graph G. The complement of a graph G has vertex set of G and the edge set as,

$$\overline{E} = \{\{i,j\} \subset V : \{i,j\} \notin E\}$$

The above inequality becomes equal if the graph G is vertex-transitive.

3) Lovász Sandwich Theorem: The Lovász Sandwich Theorem states that the Lovász number always lies between the clique number and the chromatic number, i.e.

$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$$

4) The Lovász number is an upper bound of the Shannon capacity and the independence number of G

$$\alpha(G) \leq \Theta(G) \leq \chi(G)$$

5) Lovász numbers for common graphs

Graph G	$\vartheta(G)$
Cocktail Party graph $K_{n\times 2}$	2
Complete graph K_n	1
Empty graph K_n	n
Kneser graph $K(n,r)$	$\binom{n}{x}$
Paley graph	$\sqrt{V(G)}$
Cycle graph C_5	$\sqrt{5}$
Petersen graph P	4

IV. Variants of Theta function

A. Definitions

1) Maximum stable set problem (MSSP): The problem of determining $\alpha(G)$ is called the maximum stable set problem or MSSP.

2) Maximum clique problem (MCP): The problem of determining $\omega(G)$ is called the maximum clique problem or MCP. Since $\alpha(G) = \omega(\overline{G})$, the MSSP and MCP are equivalent. They have a wide range of applications in operational research and computer science.

B. LP-based bounds of MSSP

The standard 0-1 LP formulation of the MSSP is as follows. For each $i \in V$, let x_i be a binary variable, taking the value 1 if and only if i is in the stable set. Then,

$$\begin{array}{lll} \max & e^T x & \\ \text{s.t.} & x_i + x_j \quad \leq 1 & (\{i,j\} \in E) \\ & x_i \in \{0,1\} & (i \in V) \end{array}$$

We can strength the LP relaxation by adding the following inequalities:

- Clique inequalities: They take the form $\sum_{i\in C} x_i \leq 1$, where C is a maximal clique in G
- odd hole inequalities: They take the form $\sum_{i\in H} x_i \leq \left\lfloor \frac{|H|}{2} \right\rfloor$

The upper bound obtained, if one uses all clique (and non-negativity) inequalities is called the fractional clique covering number and denoted by $\chi^f(\overline{G})$. By definition, $\alpha(G) \leq \chi^f(\overline{G}) \leq \chi(\overline{G})$.

C. The theta function

Suppose that a vector $x \in \{0,1\}^n$ is feasible for the 0-1 LP formulation presented in IV-B. We define the following,

- $Z = \frac{xx^T}{e^T x} \in [0,1]^{n\times n}$
- $J \bullet Z = \frac{(e^T x)^2}{e^T x} = e^T x$
- $\text{diag}(Z) = \frac{x}{e^T x} \implies \text{Tr}(Z) = 1$

This leads to the SDP relaxation of the MSSP, which we will call '`SDP1`':

$$\begin{array}{ll} \max & J \bullet Z \\ \text{s.t.} & Z_{ij} = 0 \quad (\{i,j\} \in E) \\ & \text{Tr}(Z) = 1 \\ & Z \in \mathcal{S}^n_+ \end{array}$$

The resulting upper bound is $\vartheta(G)$.
It was shown that SDPs can be solved in polynomial time.
Given a feasible vector $x \in \{0,1\}^n$, consider the matrix $X = xx^T \in \{0,1\}^{n\times n}$. X is positive semi-definite matrix. The augmented matrix X^+ is,

$$X^+ = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

It can be shown that X^+ is also positive semi-definite. This leads to another SDP relaxation, which we will call '`SDP2`':

$$\begin{array}{lll} \max & e^T x & \\ \text{s.t.} & X_{ii} = x_i & (i \in V) \\ & X_{ij} = 0 & (\{i,j\} \in E) \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}^{n+1}_+ & \end{array}$$

The resulting upper bound is again $\vartheta(G)$. Here, '`SDP2`' has $n+m$ linear constraints, while '`SDP1`' has $m+1$.
Let (x^*, X^*) be any feasible solution to '`SDP2`' such that the profit (maximal value) γ is positive. We can obtain a feasible solution to '`SDP1`' whose profit is no smaller than γ by setting Z_{ij} to X^*_{ij}/γ for all i and j. Let Z^* be any feasible solution to `SDP1`, again with positive profit γ. Then there exists a feasible solution (x^*, X^*) to '`SDP1`', again with positive profit γ. Then there exists a feasible solution (x^*, X^*) to '`SDP2`' whose profit is no smaller than γ,

$$x^*_i = \frac{\left(\sum_{j=1}^n Z^*_{ij}\right)^2}{\gamma Z^*_{ii}} \text{ if } Z^*_{ii} > 0 \tag{1}$$

$$= 0 \qquad \text{otherwise} \tag{2}$$

D. Variants of theta function

We can define stronger variants of the theta function by adding cutting planes to the SDPs. Adding a cutting plane to the smaller SDP can lead to less bound improvement than adding the analogous inequality to the larger SDP. But, it will come at the cost of dramatically increased computing times. Schrijver proposed to strengthen SDP1 simply by adding the non-negativity inequalities:

$$\begin{array}{ll} Z_{ij} \geq 0 & (\forall \{i,j\} \in \overline{E}) \\ Z_{ik} + Z_{jk} \leq Z_{kk} & (\forall \{i,j\} \in \overline{E}, k \in V \ \{i,j\}) \\ Z_{ik} + Z_{jk} \leq Z_{ij} + Z_{kk} & (\forall \text{ stable } \{i,j,k\}) \end{array}$$

For '`SDP2`', the following inequalities were added,

$$\begin{array}{ll} X_{ij} \geq 0 & (\forall \{i,j\} \in \overline{E}) \\ X_{ik} + X_{jk} \leq x_k & (\forall \{i,j\} \in \overline{E}, k \neq i,j) \\ x_i + x_j + x_k \leq 1 + X_{ik} + X_{jk} & (k \in V \ \{i,j\}) \\ X_{ik} + X_{jk} \leq x_k + X_{ij} & (\forall \text{ stable } \{i,j,k\}) \\ x_i + x_j + x_k \leq 1 + X_{ij} + X_{ik} + X_{jk} & (\forall \text{ stable } \{i,j,k\}) \end{array}$$

It was proposed to weaken the theta function by taking '`SDP2`' and replace $X_{ij} = 0$ with $\sum_{\{i,j\} \in E} X_{ij} = 0$. It can be computed quicker than $\vartheta(G)$, but, it returns a slightly weaker upper bound.

E. The weighted theta function

We consider the weighted version of the stable set problem. For each vertex $i \in V$ in the graph G, let $w_i > 0$ be the associated weight. For '`SDP2`', we can change the objective function to $w^T x$. For '`SDP1`', we have the objective function as

$$\sum_{i \in V} \sum_{j \in V} \sqrt{w_i w_j} Z_{ij}$$

The resulting bound is called the weighted theta function, denoted by $\vartheta(G, w)$. It can be proved that both the SDP formulations lead to the same upper bound.

Theorem 1. *Let Z^* be a feasible solution to '`SDP1`' and $\gamma = J \bullet Z^* > 0$ be the associated profit. Let x^* be the same as defined in* (2). *If $Z^*_{ij} = 0$, then set X^*_{ij} to 0. Otherwise, set X^*_{ij} to*

$$Z^*_{ij} \sqrt{\frac{x^*_i x^*_j}{Z^*_{ii} Z^*_{jj}}} = Z^*_{ij} \frac{(\sum_{k=1}^{n} Z^*_{ik})(\sum_{k=1}^{n} Z^*_{jk})}{\gamma Z^*_{ii} Z^*_{jj}}$$

Then, the resulting pair (x^*, X^*) forms a feasible solution to '`SDP2`' and its profit, $e^T x^*$ is at least as large as γ.

We will illustrate the above theorem using the following example,

Example 1. *Let G be a graph on 3 nodes with $E = \{\{1,2\},\{2,3\}\}$. It holds that $\alpha(G) = \vartheta(G) = 2$. We get the following matrix as the feasible solution for '`SDP1`':*

$$Z^* = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

We have $\gamma = \frac{5}{3}$. Then, we obtain $x^* = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} & \frac{4}{5} \end{pmatrix}^T$ with profit $\frac{9}{5} > \gamma$. Applying the above theorem's formula, we get,

$$X^* = \begin{pmatrix} \frac{4}{5} & 0 & \frac{4}{5} \\ 0 & \frac{1}{5} & 0 \\ \frac{4}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

Theorem 2. *Let (x^*, X^*) be a feasible solution to the weighted version of '`SDP2`', such that $\gamma = w^T x > 0$. We can obtain a feasible solution Z^* to the weighted version of '`SDP1`', whose profit is at least as large, by setting Z^*_{ij} to $\frac{1}{\gamma}\sqrt{w_i w_j} X^*_{ij}$ for all i and j.*

Theorem 3. *Let Z^* be a feasible solution to the weighted version of '`SDP1`' whose profit $\gamma > 0$. We can obtain a feasible solution (x^*, X^*) to the weighted version of '`SDP2`', whose profit is at least as large, as follows. If $Z^*_{ii} = 0$, set $x^*_i = 0$. Otherwise, set x^*_i to,*

$$\frac{\left(\sum_{j=1}^{n} \sqrt{w_j} Z^*_{ij}\right)^2}{\gamma Z^*_{ii}}$$

If $Z^*_{ij} = 0$, set $X^*_{ij} = 0$. Otherwise, set X^*_{ij} to

$$Z^*_{ij} \sqrt{\frac{x^*_i x^*_j}{Z^*_{ii} Z^*_{jj}}} = Z^*_{ij} \frac{(\sum_{k=1}^{n} \sqrt{w_k} Z^*_{ik})(\sum_{k=1}^{n} \sqrt{w_k} Z^*_{jk})}{\gamma Z^*_{ii} Z^*_{jj}}$$

Then, the resulting pair (x^*, X^*) forms a feasible solution to '`SDP2`', and its profit, $w^T x^*$, is at least as large as γ.

V. Conclusion

The Lovász theta function can be computed by solving either of the two equivalent SDPs, which we labelled as '`SDP1`' and '`SDP2`'. We have shown, however, that the equivalence between '`SDP1`' and '`SDP2`' breaks down when they are either strengthened (via cutting planes) or weakened (via constraint aggregation). Our results have implications for the development of exact algorithms for the maximum stable set problem. If one wishes to use the theta function itself as the upper bound, then '`SDP1`' is to be preferred over '`SDP2`'. If, however, one wishes to use a strengthened or weakened variant of the theta number, then the choice between '`SDP1`' and '`SDP2`' is no longer obvious.

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