

Constructions using Python

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Abstract—This book introduces constructions in high school geometry using Python. The content and exercises are based on NCERT textbooks from Class 6-12. A simple introduction to Python and \LaTeX figures is provided in the process.

Download all python codes from

```
svn co https://github.com/gadepall/school/trunk/ncert/constructions/codes
```

and latex-tikz codes from

```
svn co https://github.com/gadepall/school/trunk/ncert/constructions/figs
```

1 EXAMPLES

1.1. Draw Fig. 1.1.1 for $a = 4, c = 3$.

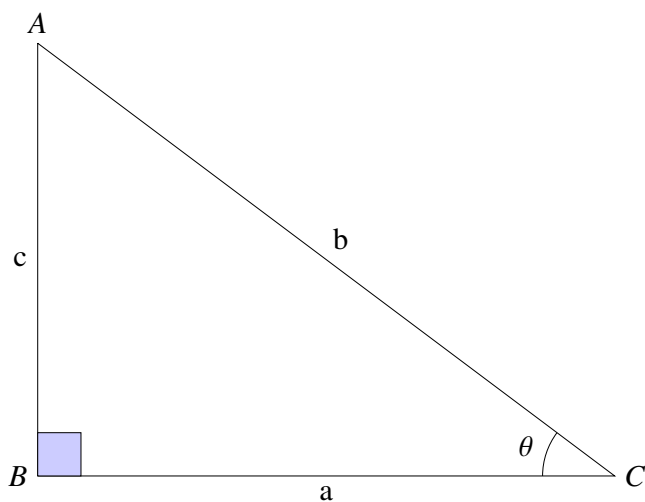


Fig. 1.1.1: Right Angled Triangle

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Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.1.1)$$

The python code for Fig. 1.1.1 is

```
codes/triangle/tri_right_angle.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_right_angle.tex
```

The above latex code can be compiled as a standalone document as

```
figs/triangle/tri_right_angle_alone.tex
```

1.2. Draw Fig. 1.2.1 for $a = 4, c = 3$.

Solution: The vertex \mathbf{A} can be expressed in polar coordinate form as

$$\mathbf{A} = b \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (1.2.1)$$

where

$$b = \sqrt{a^2 + c^2} = 5, \tan \theta = \frac{3}{4} \quad (1.2.2)$$

The python code for Fig. 1.2.1 is

```
codes/triangle/tri_polar.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_polar.tex
```

1.3. Draw Fig. 1.3.1 with $a = 6, b = 5$ and $c = 4$.

Solution: Let the vertices of $\triangle ABC$ and \mathbf{D} be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 0 \end{pmatrix} \quad (1.3.1)$$

Then

$$AB = \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (1.3.2)$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (1.3.3)$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \quad (1.3.4)$$

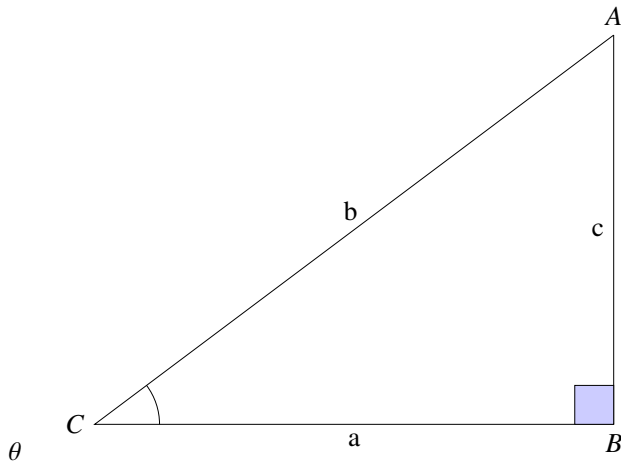


Fig. 1.2.1: Right Angled Triangle

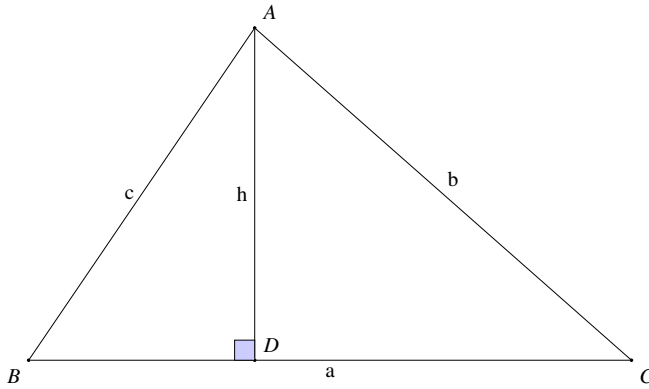


Fig. 1.3.1

From (1.3.4),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \quad (1.3.5)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{C}^T \mathbf{C} - \mathbf{A}^T \mathbf{C} - \mathbf{C}^T \mathbf{A} \quad (1.3.6)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T \mathbf{C} \quad (\because \mathbf{A}^T \mathbf{C} = \mathbf{C}^T \mathbf{A}) \quad (1.3.7)$$

$$= a^2 + c^2 - 2ap \quad (1.3.8)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (1.3.9)$$

From (1.3.2),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \quad (1.3.10)$$

$$\Rightarrow q = \pm \sqrt{c^2 - p^2} \quad (1.3.11)$$

The python code for Fig. 1.3.1 is

```
codes/triangle/tri_sss.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_sss.tex
```

- 1.4. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.

Solution:

The vertex \mathbf{A} can be expressed in *polar coordinate form* as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} \quad (1.4.1)$$

From $\triangle ABC$, we use the law of cosines:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad (1.4.2)$$

$$= 0.5625 \quad (1.4.3)$$

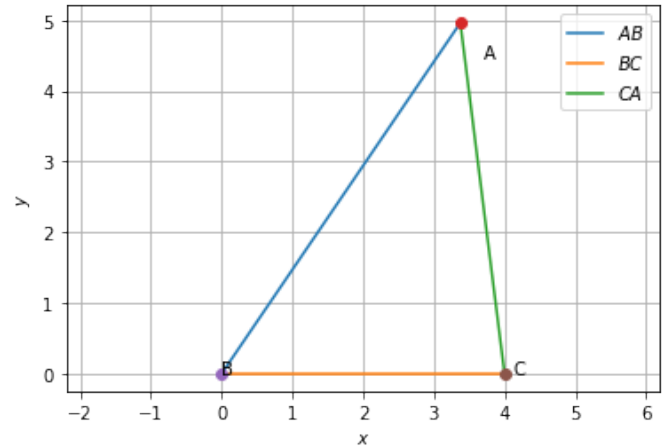
$$\Rightarrow B = 55.771^\circ \quad (1.4.4)$$

Thus,

$$\mathbf{A} = 6 \begin{pmatrix} \cos 55.771 \\ \sin 55.771 \end{pmatrix} \quad (1.4.5)$$

$$\mathbf{A} = \begin{pmatrix} 3.375 \\ 4.960 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \quad (1.4.6)$$

which are plotted in Fig. 1.4.1

Fig. 1.4.1: $\triangle ABC$

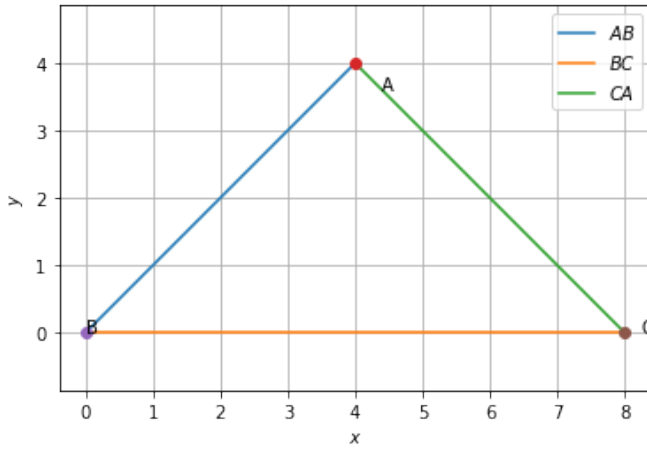
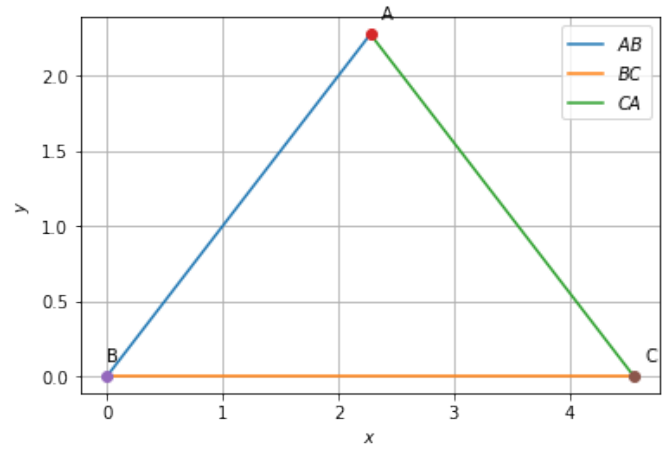
- 1.5. Construct an isosceles triangle whose base is $a = 8\text{cm}$ and altitude $AD = h = 4\text{cm}$

Solution: From the given information,

$$\mathbf{A} = \begin{pmatrix} a/2 \\ h \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \quad (1.5.1)$$

which are used to plot the triangle in Fig. 1.5.1

- 1.6. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.

Fig. 1.5.1: isosceles triangle $\triangle ABC$ Fig. 1.6.1: $\triangle ABC$

Solution: Use sine formula,

$$b \sin 45 = c \sin 45 \quad (1.6.1)$$

$$\Rightarrow b = c \quad (1.6.2)$$

$$a \sin 45 = b \sin 90 \quad (1.6.3)$$

$$\Rightarrow a = \sqrt{2}b \quad (1.6.4)$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -\sqrt{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (1.6.5)$$

solving which yields

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3.22 \end{pmatrix} \quad (1.6.6)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.6.7)$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4.55 \\ 0 \end{pmatrix} \quad (1.6.8)$$

resulting in $\triangle ABC$ plotted in Fig. 1.6.1.

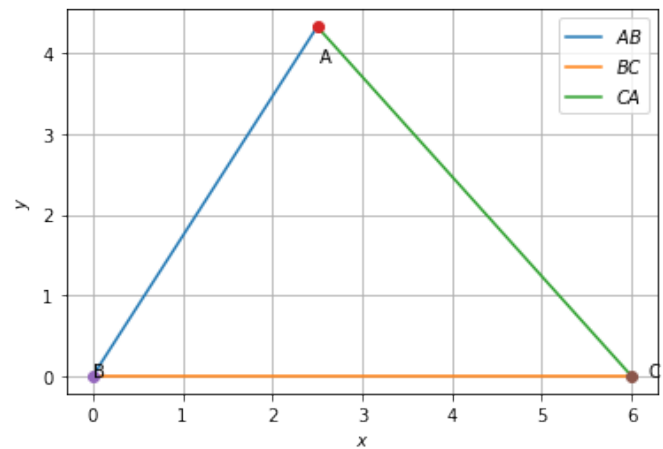
1.7. Draw $\triangle ABC$ with $a = 6$, $c = 5$ and $\angle B = 60^\circ$.

Solution: The vertex \mathbf{A} can be expressed in polar coordinate form as

$$\mathbf{A} = c \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad (1.7.1)$$

$$\Rightarrow \mathbf{A} = 5 \begin{pmatrix} \cos 60 \\ \sin 60 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5\sqrt{3} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.7.2)$$

upon substituting the given values. The triangle is plotted in Fig. 1.7.1.

Fig. 1.7.1: $\triangle ABC$

1.8. Draw $\triangle ABC$ with $a = 7$, $\angle B = 45^\circ$ and $\angle A = 105^\circ$.

Solution: Let

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.8.1)$$

$$\therefore \angle C = 30^\circ, \quad (1.8.2)$$

By law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.8.3)$$

$$\Rightarrow c = \frac{7 \sin 30^\circ}{\sin 105^\circ} \quad (1.8.4)$$

$$c = 3.62 \quad (1.8.5)$$

and

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} \quad (1.8.6)$$

$$= \begin{pmatrix} 2.55 \\ 2.55 \end{pmatrix} \quad (1.8.7)$$

Thus, the vertices of given $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 2.55 \\ 2.55 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \quad (1.8.8)$$

and $\triangle ABC$ is plotted in Fig. 1.8.1.

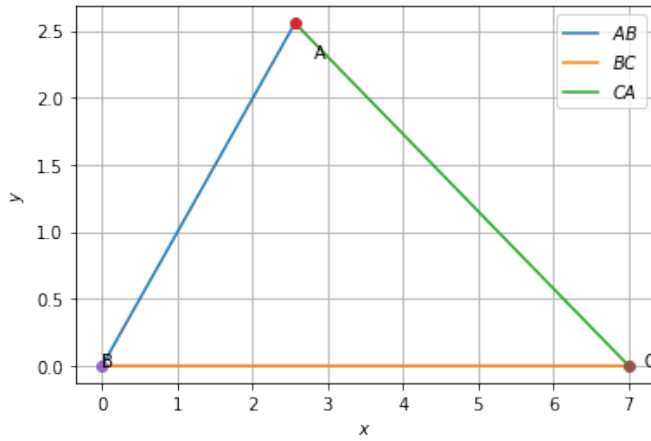


Fig. 1.8.1: $\triangle ABC$

1.9. $\triangle ABC$ is right angled at \mathbf{B} . If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.

Solution: Let,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.9.1)$$

Given,

$$a = 12, b + c = 18 \quad (1.9.2)$$

From $\triangle ABC$, using the Baudhayana sutra,

$$b^2 = c^2 + a^2 \quad (1.9.3)$$

$$\Rightarrow b - c = 8 \quad (\because b + c = 18) \quad (1.9.4)$$

Now we have,

$$b + c = 18 \quad (1.9.5)$$

$$b - c = 8 \quad (1.9.6)$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix} \quad (1.9.7)$$

Applying row reduction,

$$\begin{pmatrix} 1 & 1 & 18 \\ 1 & -1 & 8 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 18 \\ 0 & -2 & -10 \end{pmatrix} \quad (1.9.8)$$

$$\xrightarrow{R_1 \rightarrow 2R_1 + R_2} \begin{pmatrix} 2 & 0 & 26 \\ 0 & -2 & -10 \end{pmatrix} \xrightarrow{\begin{matrix} R_1 \rightarrow \frac{R_1}{2} \\ R_2 \rightarrow -\frac{R_2}{2} \end{matrix}} \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.9.9)$$

Therefore,

$$\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix} \quad (1.9.10)$$

Thus,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (1.9.11)$$

and $\triangle ABC$ is plotted in Fig. 1.9.1

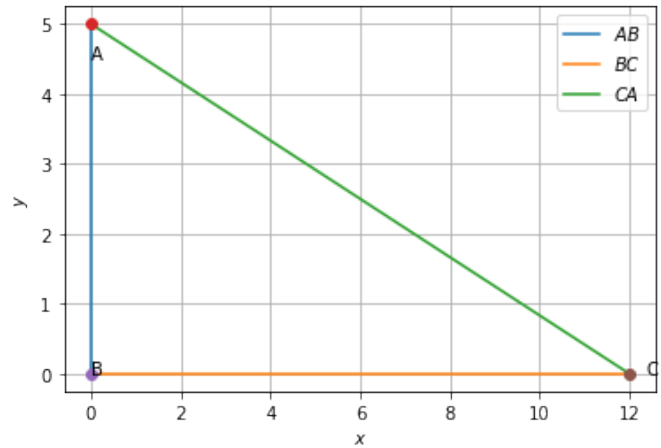


Fig. 1.9.1: Right Angle $\triangle ABC$

1.10. In $\triangle ABC$, $a = 8$, $\angle B = 45^\circ$ and $c - b = 3.5$. Sketch $\triangle ABC$.

Solution: Let

$$\mathbf{A} = c \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad (1.10.1)$$

Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac \cos B \quad (1.10.2)$$

$$\Rightarrow (c+b)(c-b) + 8^2 - 2 \times 8 \times \left(\frac{1}{\sqrt{2}}\right)c = 0 \quad (1.10.3)$$

$$\Rightarrow (7 - 16\sqrt{2})c + 7b = -128 \quad (1.10.4)$$

upon simplification. From the given information,

$$c - b = \frac{7}{2}, \quad (1.10.5)$$

and the above equations can be expressed as the matrix equation

$$\begin{pmatrix} 7 - 16\sqrt{2} & 7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -128 \\ \frac{7}{2} \end{pmatrix} \quad (1.10.6)$$

yielding

$$\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} 11.99 \\ 8.49 \end{pmatrix} \quad (1.10.7)$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = 11.99 \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (1.10.8)$$

which are used to plot Fig. 1.10.1.

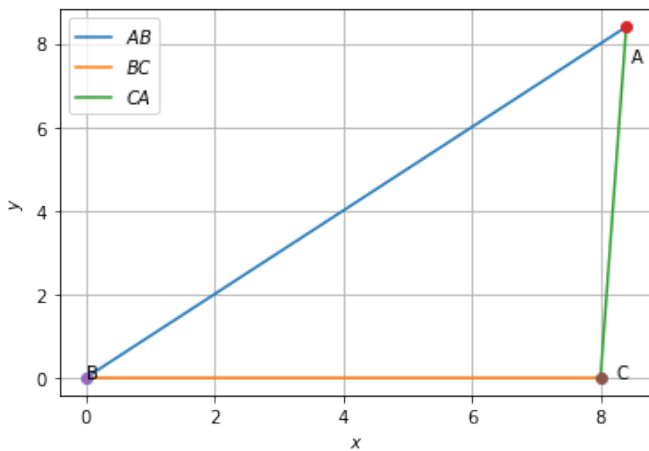


Fig. 1.10.1: $\triangle ABC$

1.11. In $\triangle ABC$, $a = 6$, $\angle B = 60^\circ$ and $b - c = 2$. Sketch $\triangle ABC$.

Let

$$\mathbf{A} = b \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad (1.11.1)$$

Using the cosine formula,

$$b^2 = a^2 + c^2 - 2ac \cos B \quad (1.11.2)$$

$$\Rightarrow (b+c)(b-c) = 6^2 - 2(6)\frac{1}{2}c \quad (\because \angle B = 60^\circ) \quad (1.11.3)$$

$$\Rightarrow (b+c)(2) = 36 - 6c \quad (\because b - c = 2) \quad (1.11.4)$$

$$\text{or, } b + 4c = 18 \quad (1.11.5)$$

From the above, we obtain the matrix equation

$$\begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 18 \\ 2 \end{pmatrix} \quad (1.11.6)$$

By applying row reduction:

$$\begin{pmatrix} 1 & 4 & 18 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 4 & 18 \\ 0 & -5 & -16 \end{pmatrix} \quad (1.11.7)$$

$$\xrightarrow{R_1 \rightarrow 5R_1 + 4R_2} \begin{pmatrix} 5 & 0 & 26 \\ 0 & -5 & -16 \end{pmatrix} \quad (1.11.8)$$

$$\xrightarrow{\substack{R_1 \rightarrow \frac{R_1}{5} \\ R_2 \rightarrow -\frac{R_2}{5}}} \begin{pmatrix} 1 & 0 & \frac{26}{5} \\ 0 & 1 & \frac{16}{5} \end{pmatrix} \quad (1.11.9)$$

$$\therefore \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{26}{5} \\ \frac{16}{5} \end{pmatrix} \quad (1.11.10)$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = \frac{26}{5} \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} = \begin{pmatrix} 2.6 \\ 4.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.11.11)$$

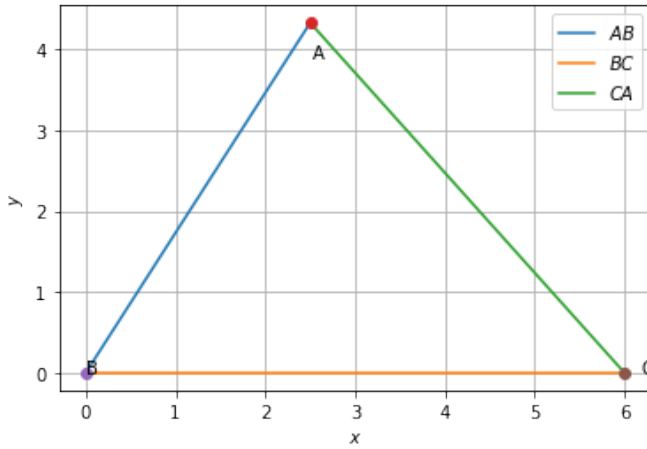
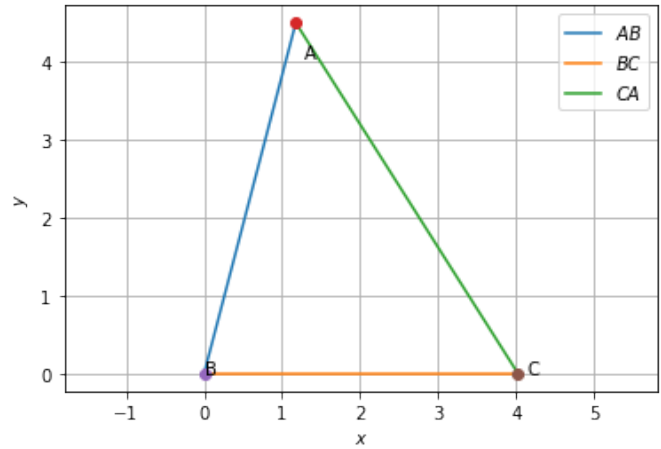
and the plot of $\triangle ABC$ is obtained in Fig. 1.11.1
1.12. Draw $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 30^\circ$ and $\angle C = 90^\circ$.

Solution: Using the sine formula,

$$b \sin C = c \sin B \quad (1.12.1)$$

$$\Rightarrow b \sin 90^\circ = c \sin 30^\circ \quad (1.12.2)$$

$$\text{or, } c = 2b \quad (1.12.3)$$

Fig. 1.11.1: $\triangle ABC$ Fig. 1.12.1: $\triangle ABC$

Similarly,

$$a \sin B = b \sin A \quad (1.12.4)$$

$$\Rightarrow a = \sqrt{3}b \quad (1.12.5)$$

Formulating the above as a matrix equation

$$\begin{pmatrix} 0 & -2 & 1 \\ 1 & -\sqrt{3} & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (1.12.6)$$

Solving the above,

$$a = 4.026, b = 2.32, c = 4.64 \quad (1.12.7)$$

which are used to obtain the vertices of $\triangle ABC$ using Problem 1.3.

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 4.64 \end{pmatrix} \quad (1.12.8)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.12.9)$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4.02 \\ 0 \end{pmatrix} \quad (1.12.10)$$

The desired triangle is plotted in Fig. 1.12.1.

1.13. Construct $\triangle xyz$ where $xy = 4.5$, $yz = 5$ and $zx = 6$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix} \quad (1.13.1)$$

The vertex C can be expressed in polar coordinate form as

$$\mathbf{C} = b \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad (1.13.2)$$

Using the cosine formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (1.13.3)$$

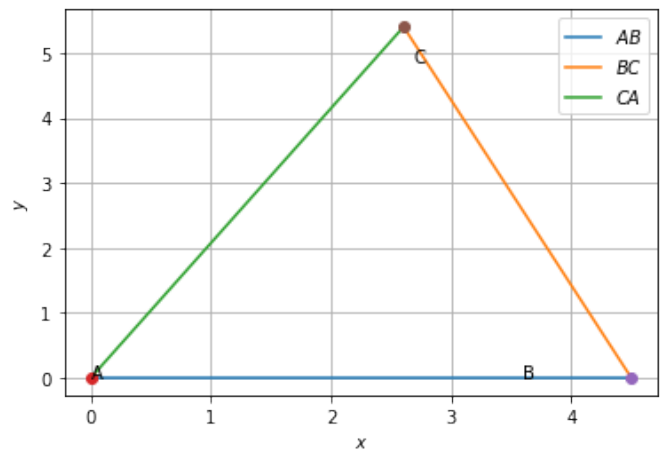
$$\Rightarrow A = 54.640^\circ \quad (1.13.4)$$

Hence,

$$\mathbf{C} = 6 \begin{pmatrix} \cos 54.640^\circ \\ \sin 54.640^\circ \end{pmatrix} = \mathbf{C} = \begin{pmatrix} 3.472 \\ 3.990 \end{pmatrix}, \quad (1.13.5)$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix} \quad (1.13.6)$$

which are plotted in Fig. 1.13.1

Fig. 1.13.1: $\triangle ABC$

1.14. Draw an equilateral triangle of side 5.5.

Solution:

Let,

$$\mathbf{A} = a \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.14.1)$$

$$= 5.5 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5.5 \\ 0 \end{pmatrix} \quad (1.14.2)$$

after substituting $\theta = 60^\circ$ and $a = 5.5$. The triangle is then plotted in Fig. 1.14.1

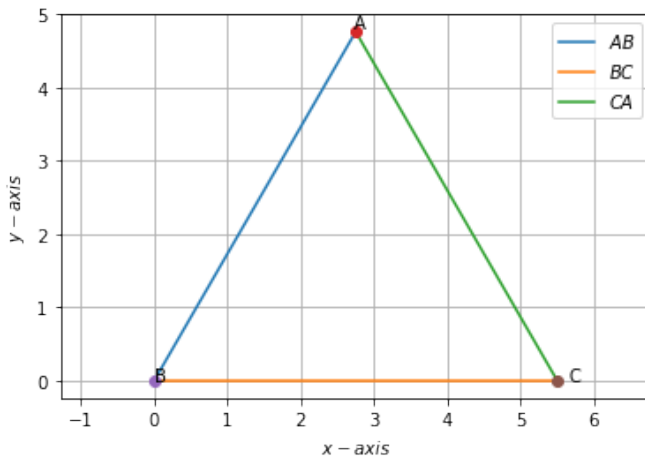


Fig. 1.14.1: $\triangle ABC$

1.15. Draw $\triangle PQR$ with $PQ = 4$, $QR = 3.5$ and $PR = 4$. What type of triangle is this?

Solution: Let

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{R} = PR \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (1.15.1)$$

where,

$$PR \left(\frac{\sin \theta}{2} \right) = \frac{QR}{2} \quad (1.15.2)$$

$$\Rightarrow \theta = 2 \sin^{-1} \left(\frac{QR}{2PR} \right) \quad (1.15.3)$$

$$= 51.88 \quad (1.15.4)$$

Thus, the vertices of $\triangle PQR$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 2.47 \\ 3.15 \end{pmatrix} \quad (1.15.5)$$

which are used to plot $\triangle PQR$ in Fig. 1.15.1.

1.16. Construct $\triangle ABC$ such that $AB = 2.5$, $BC = 6$ and $AC = 6.5$. Find $\angle B$.

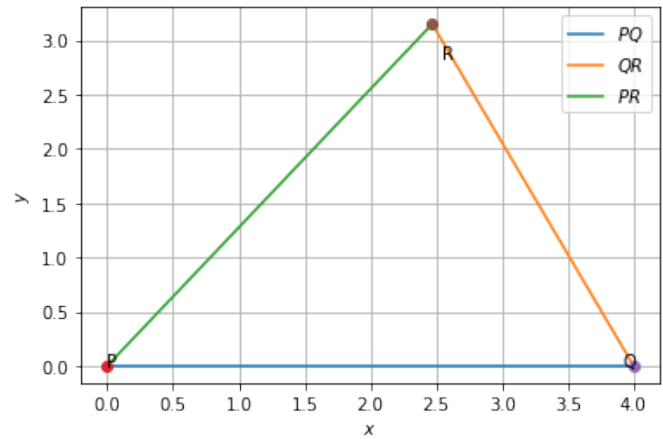


Fig. 1.15.1: isosceles $\triangle PQR$

Solution: From the given information,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad (1.16.1)$$

$$\Rightarrow \cos B = 0 \quad (1.16.2)$$

$$\text{or, } \angle B = 90^\circ \quad (1.16.3)$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.16.4)$$

and plotted in Fig. 1.16.1.

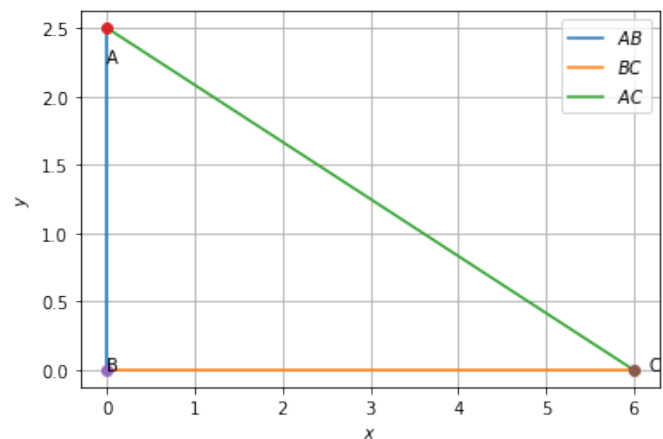


Fig. 1.16.1: $\triangle ABC$

1.17. Construct $\triangle DEF$ such that $DE = 5$, $DF = 3$ and $\angle D = 90^\circ$.

Solution: From the given information, the vertices of $\triangle DEF$ are

$$\mathbf{E} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.17.1)$$

which are used to plot Fig. 1.17.1.

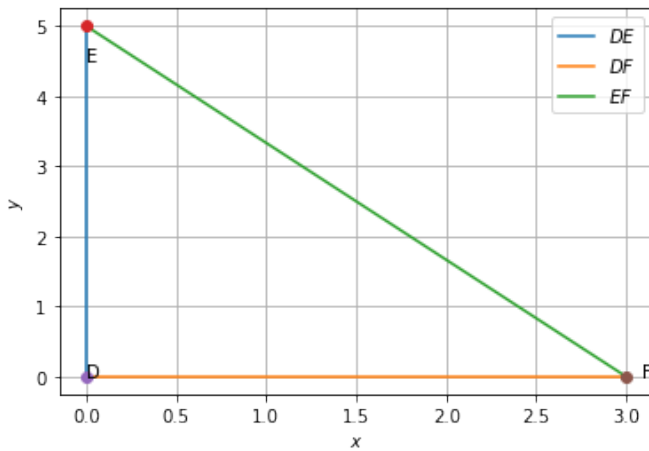


Fig. 1.17.1

- 1.18. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110° .

Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \mathbf{B} = c \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad (1.18.1)$$

Then, the vertices of isosceles $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6.5 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2.22313 \\ 6.10798 \end{pmatrix} \quad (1.18.2)$$

which are plotted in Fig. 1.18.1.

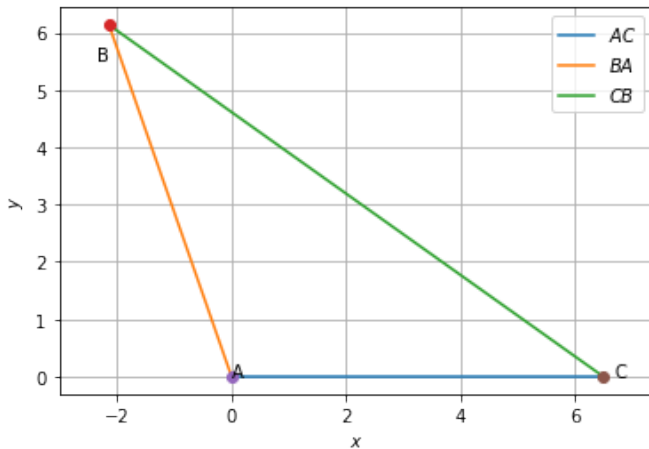


Fig. 1.18.1: Isosceles $\triangle ABC$

- 1.19. Construct $\triangle ABC$ given that $\angle A = 60^\circ$, $\angle B = 30^\circ$ and $AB = 5.8$.

Solution: From the given information,

$$\angle C = 90^\circ \quad (1.19.1)$$

Hence,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \sin B \end{pmatrix} \quad (1.19.2)$$

$$= \begin{pmatrix} 0 \\ 2.9 \end{pmatrix} \quad (1.19.3)$$

$$\mathbf{B} = \begin{pmatrix} c \cos B \\ 0 \end{pmatrix} \quad (1.19.4)$$

$$= \begin{pmatrix} 5.02294 \\ 0 \end{pmatrix} \quad (1.19.5)$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.19.6)$$

which are used to draw $\triangle ABC$ in Fig. 1.19.1.

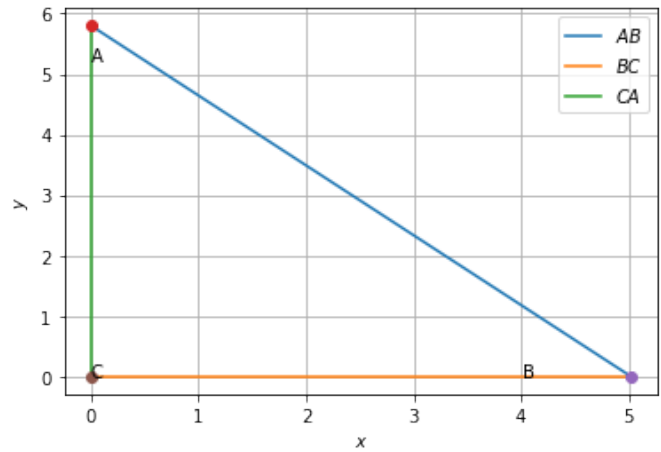


Fig. 1.19.1: $\triangle ABC$

- 1.20. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.

Solution:

Let

$$\mathbf{L} = \begin{pmatrix} 0 \\ l \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.20.1)$$

From the given information,

$$\|\mathbf{N} - \mathbf{M}\|^2 = \|\mathbf{N}\|^2 = 3^2 = 9 \quad (1.20.2)$$

$$\|\mathbf{L} - \mathbf{M}\|^2 = \|\mathbf{L}\|^2 = l^2 \quad (1.20.3)$$

$$\|\mathbf{L} - \mathbf{N}\|^2 = 5^2 = 25 \quad (1.20.4)$$

which can be expressed as

$$\|\mathbf{L} - \mathbf{N}\|^2 = (\mathbf{L} - \mathbf{N})^T (\mathbf{L} - \mathbf{N}) \quad (1.20.5)$$

$$= \|\mathbf{L}\|^2 + \|\mathbf{N}\|^2 - 2\mathbf{L}^T \mathbf{N} \quad (1.20.6)$$

$$\Rightarrow l^2 + 9 = 25 \quad (1.20.7)$$

$$\text{or, } l = \pm 4 \quad (1.20.8)$$

For $l=4$, $\triangle LMN$ is plotted in the first quadrant in Fig. 1.20.1.

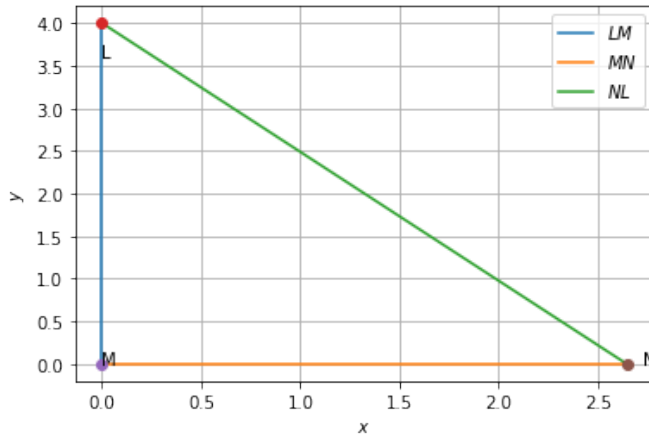


Fig. 1.20.1

- 1.21. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.

Solution: Let

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (1.21.1)$$

Then,

$$\|\mathbf{P} - \mathbf{R}\|^2 = (\mathbf{P} - \mathbf{R})^T (\mathbf{P} - \mathbf{R}) \quad (1.21.2)$$

$$= \|\mathbf{P}\|^2 + \|\mathbf{R}\|^2 \quad (1.21.3)$$

$$\therefore \mathbf{P}^T \mathbf{R} = \mathbf{R}^T \mathbf{P}, \mathbf{R}^T \mathbf{P} = 0 \quad (1.21.4)$$

$$= p^2 + 64 = 10^2 \quad (1.21.5)$$

$$\Rightarrow p = \pm 6 \quad (1.21.6)$$

Since positive area is considered here, only $p = 6$ is taken into consideration. Thus,

$$\mathbf{P} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (1.21.7)$$

and the desired triangle is plotted in Fig. 1.21.1

- 1.22. Construct right angled \triangle whose hypotenuse is 6 and one of the legs is 4.

Solution: Let us consider $\triangle PQR$ right angled at Q and assume that we are restricted to first quadrant such that

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (1.22.1)$$

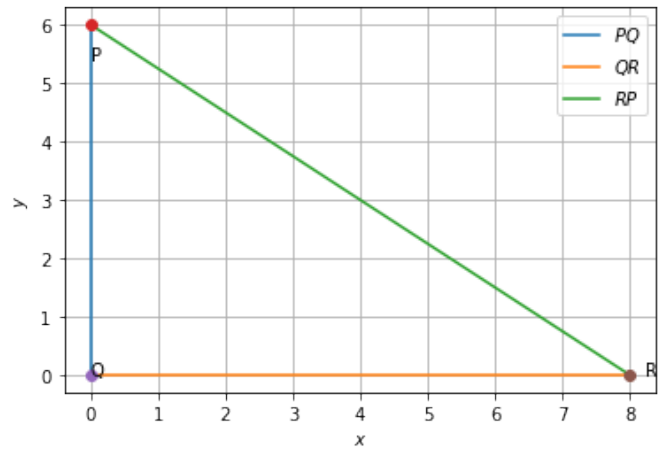


Fig. 1.21.1: Right Angle $\triangle PQR$

Then,

$$\|\mathbf{P} - \mathbf{R}\|^2 = 36 \quad (1.22.2)$$

$$\Rightarrow p^2 + 16 = 36 \quad (1.22.3)$$

$$\Rightarrow p = \pm 2\sqrt{5} \quad (1.22.4)$$

Since first quadrant was assumed here, only $p = +2\sqrt{5}$ is taken into consideration. So, the vertices of $\triangle PQR$ in Fig. 1.22.1 are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 2\sqrt{5} \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.22.5)$$

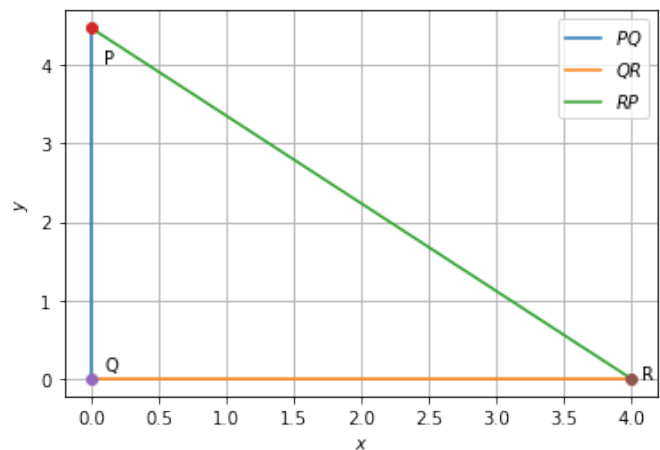


Fig. 1.22.1: Right Angled $\triangle PQR$

- 1.23. Construct an isosceles right angled $\triangle ABC$ right angled at C such that $AC = 6$.

Solution:

$\therefore \triangle ABC$ is isosceles, its vertices are

$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (1.23.1)$$

which are used to plot the desired triangle in Fig. 1.23.1.

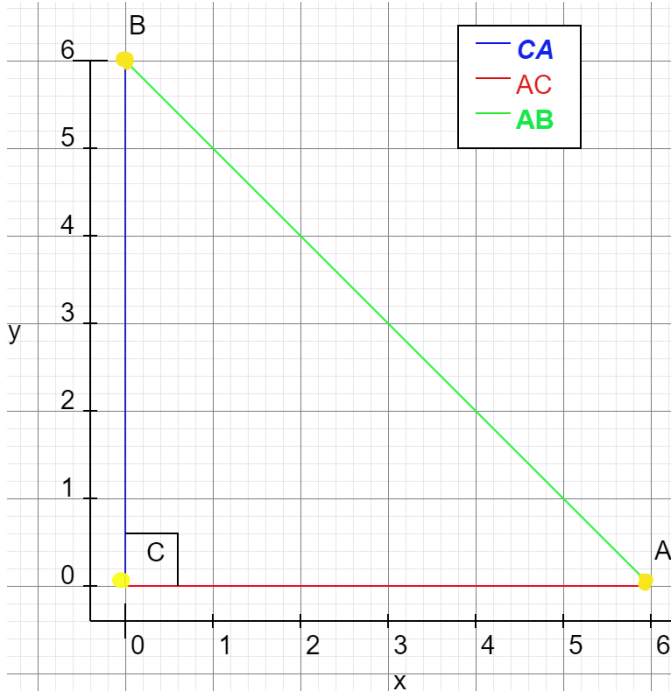


Fig. 1.23.1: Isosceles Right Angle $\triangle ABC$

- 1.24. Construct $\triangle PQR$, given that $PQ = 3$, $QR = 5.5$ and $\angle PQR = 60^\circ$.

Solution:

$$\therefore \mathbf{P} = r \begin{pmatrix} \cos Q \\ \sin Q \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} p \\ 0 \end{pmatrix} \quad (1.24.1)$$

from the given information,

$$\mathbf{P} = 3 \begin{pmatrix} \cos 60 \\ \sin 60 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 5.5 \\ 0 \end{pmatrix} \quad (1.24.2)$$

and plotted in Fig. 1.24.1.

- 1.25. Construct $\triangle ABC$ with $BC = 7.5$, $AC = 5$ and $\angle C = 60^\circ$.

Solution:

$$\therefore \mathbf{A} = b \begin{pmatrix} \cos C \\ \sin C \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (1.25.1)$$

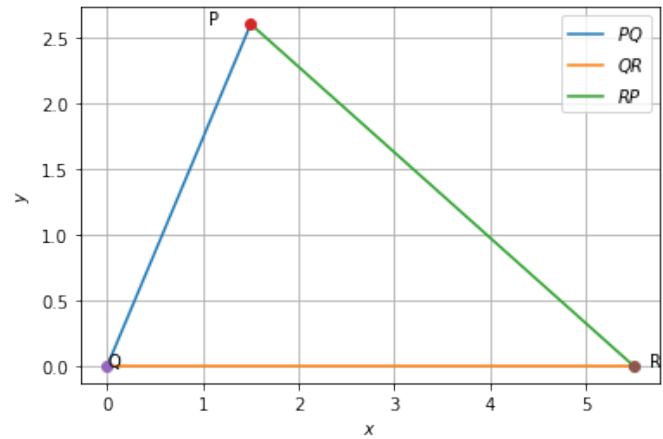


Fig. 1.24.1: The Constructed triangle

substituting the given values,

$$\mathbf{A} = 5 \begin{pmatrix} \cos 60 \\ \sin 60 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5\sqrt{3} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.25.2)$$

which are plotted in Fig. 1.25.1.

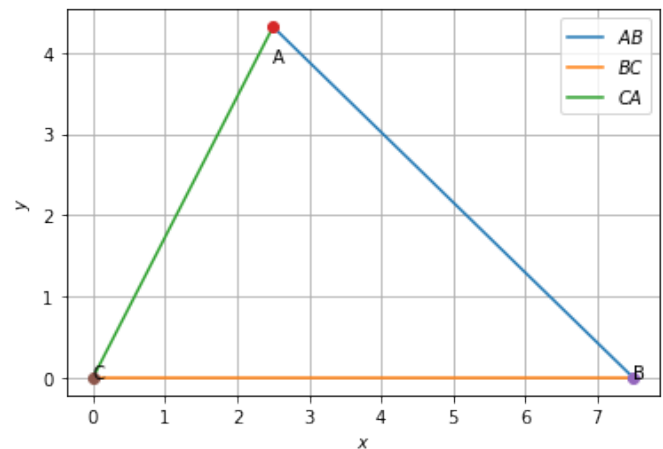


Fig. 1.25.1: The Constructed triangle

- 1.26. Construct $\triangle XYZ$ if $XY = 6$, $\angle X = 30^\circ$ and $\angle Y = 100^\circ$.

Solution: Let

$$XY = z, YZ = x, XZ = y \quad (1.26.1)$$

From the given information,

$$y = z \left(\frac{\sin Y}{\sin Z} \right) = 6 \left(\frac{\sin 100^\circ}{\sin 50^\circ} \right) = 7.7134 \quad (1.26.2)$$

and the vertices are

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Y} = z \begin{pmatrix} \cos X^\circ \\ \sin X^\circ \end{pmatrix}, \mathbf{Z} = \begin{pmatrix} y \\ 0 \end{pmatrix} \quad (1.26.3)$$

$$(1.26.4)$$

$$\Rightarrow \mathbf{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Y} = 6 \begin{pmatrix} \cos 30^\circ \\ \sin 30^\circ \end{pmatrix} = \begin{pmatrix} 3\sqrt{3} \\ 3 \end{pmatrix}, \mathbf{Z} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \quad (1.26.5)$$

and plotted in Fig. 1.26.1.

- 1.27. If $AC = 7$, $\angle A = 60^\circ$ and $\angle B = 50^\circ$, can you draw the triangle?

Solution: From the given information,

$$\angle C = 70^\circ \quad (1.27.1)$$

and

$$a = b \left(\frac{\sin A}{\sin B} \right) \quad (1.27.2)$$

$$= 7.913611 \quad (1.27.3)$$

Thus, the coordinates are

$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = a \begin{pmatrix} \cos 70^\circ \\ \sin 70^\circ \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \quad (1.27.4)$$

and plotted in Fig. 1.27.1.

- 1.28. Construct $\triangle PQR$ if $PQ = 5$, $\angle Q = 105^\circ$ and $\angle R = 40^\circ$.

From the given information,

$$\angle P = 35^\circ \quad (1.28.1)$$

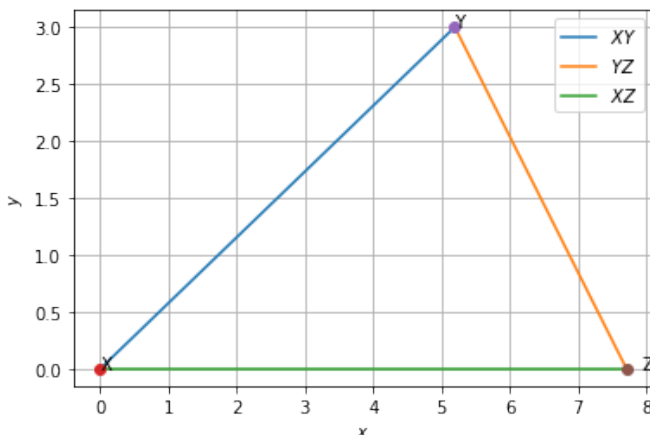


Fig. 1.26.1: Constructed Triangle

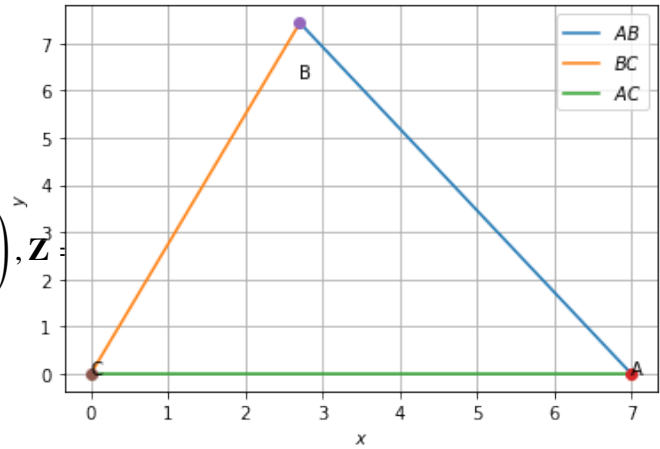


Fig. 1.27.1: $\triangle ABC$

and

$$q = r \left(\frac{\sin Q}{\sin R} \right) \quad (1.28.2)$$

$$= 7.5135 \quad (1.28.3)$$

Thus, the coordinates are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{R} = q \begin{pmatrix} \cos 35^\circ \\ \sin 35^\circ \end{pmatrix} \quad (1.28.4)$$

See Fig. 1.28.1.

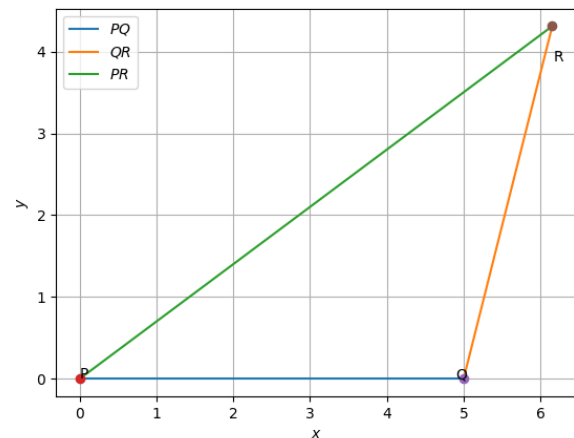


Fig. 1.28.1: $\triangle PQR$

- 1.29. Can you construct $\triangle DEF$ such that $EF = 7.2$, $\angle E = 110^\circ$ and $\angle F = 180^\circ$?

- 1.30. Construct parallelogram $ABCD$ in Fig. 1.30.1 given that $BC = 5$, $AB = 6$, $\angle C = 85^\circ$.

Solution: BD is found using the cosine formula and $\triangle BDC$ is drawn using the approach in

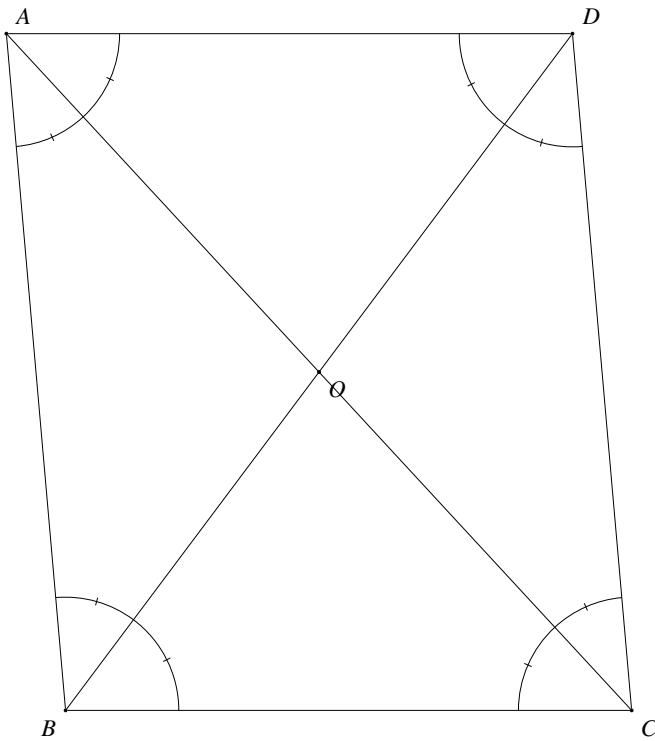


Fig. 1.30.1: Parallelogram Properties

Construction 1.3 with

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad (1.30.1)$$

Since the diagonals bisect each other,

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{D}}{2} \quad (1.30.2)$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{C}. \quad (1.30.3)$$

AB and AD are then joined to complete the ||gm. The python code for Fig. 1.30.1 is

```
codes/quad/pgm_sas.py
```

and The equivalent latex-tikz code is

```
figs/quad/pgm_sas.tex
```

- 1.31. Draw the ||gm $ABCD$ in Fig. 1.31.1 with $BC = 6$, $CD = 4.5$ and $BD = 7.5$. Show that it is a rectangle.

Solution: It is easy to verify that

$$BD^2 = BC^2 + C^2 \quad (1.31.1)$$

Hence, using Baudhayana theorem,

$$\angle BCD = 90^\circ \quad (1.31.2)$$

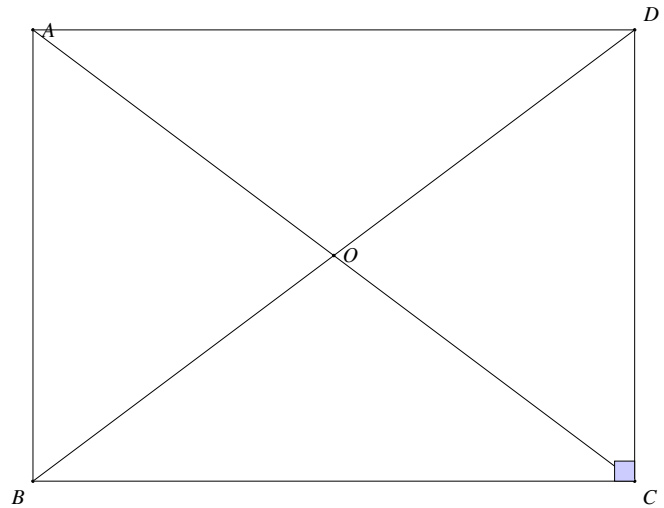


Fig. 1.31.1: Rectangle

and $ABCD$ is a rectangle.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \quad (1.31.3)$$

The python code for Fig. 1.31.1 is

```
codes/quad/pgm_sss.py
```

and the equivalent latex-tikz code is

```
figs/quad/pgm_sss.tex
```

- 1.32. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.

Solution: The coordinates of the various points in Fig. 1.32.1 are obtained as

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4.5 \end{pmatrix} \quad (1.32.1)$$

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad (1.32.2)$$

- 1.33. A square is a rectangle whose sides are equal. Draw a square of side 4.5.

Solution: The coordinates of the various points in Fig. 1.33.1 are obtained as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \quad (1.33.1)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix}, \mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (1.33.2)$$

- 1.34. With the same centre \mathbf{O} , draw two circles of

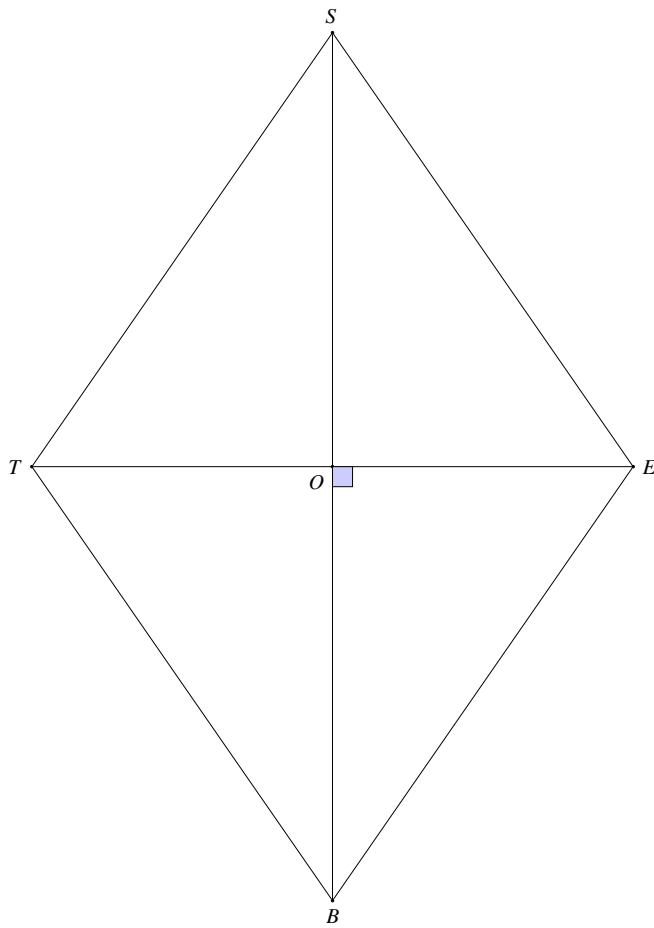


Fig. 1.32.1: Rhombus

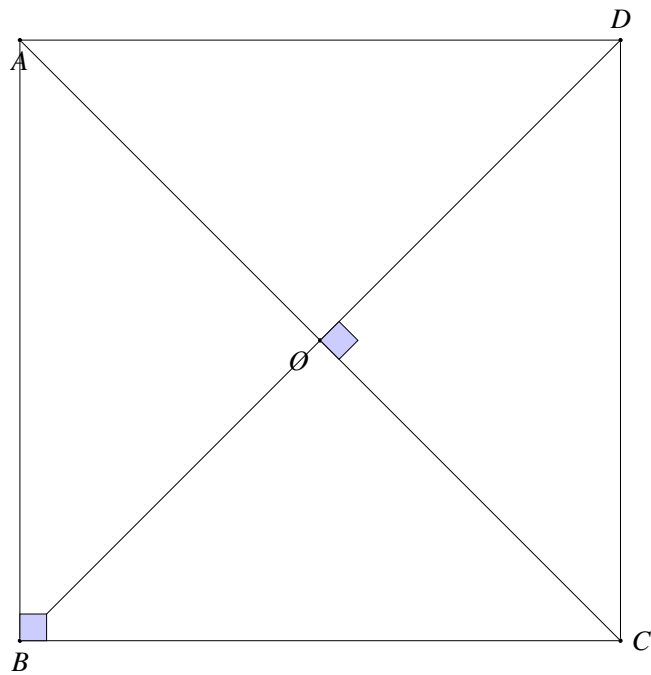


Fig. 1.33.1: Square

radii 4 and 2.5

Solution:

All input values required to plot Fig. 1.34.1 are given in Table 1.34.1 as shown below

	Symbols	Circle1	Circle2
Centre	\mathbf{O}	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Radius	r_1, r_2	2.5	4
Polar coordinate	$\mathbf{C}_1, \mathbf{C}_2$	$2.5 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$	$4 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$
Angle	θ	$0-2\pi$	$0-2\pi$

TABLE 1.34.1: Input values

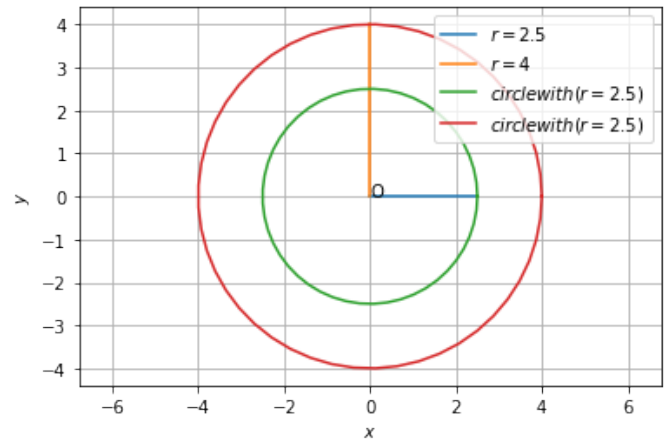


Fig. 1.34.1: Concentric circles with centre as origin and radii 2.5 and 4 respectively

- 1.35. Let \mathbf{A} and \mathbf{B} be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at \mathbf{C} and \mathbf{D} . Is $AB \perp CD$?

Solution: The centers and radii of the two circles without any loss of generality are given in Table 1.35.1

	Circle 1	Circle 2
Centre	$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbf{B} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$
Radius	$r_1 = r_2 = 3$	

TABLE 1.35.1: Input values

Let

$$\mathbf{u} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \theta \in [0, 2\pi]. \quad (1.35.1)$$

Then on Circle 1 and Circle 2 are given by

$$\mathbf{x} = \mathbf{A} + r\mathbf{u} \quad (1.35.2)$$

$$\mathbf{x} = \mathbf{B} + r\mathbf{u} \quad (1.35.3)$$

Fig. 1.35.1 is plotted using the above equations.
Fig. 1.35.1

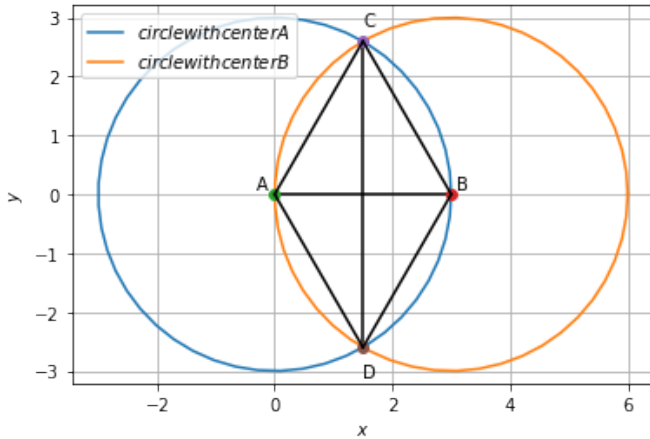


Fig. 1.35.1: Circles with their points of intersection

The general equation of Circle 1 is given by

$$\|\mathbf{x} - \mathbf{A}\|^2 = r^2 \quad (1.35.4)$$

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{A}^T \mathbf{x} + \|\mathbf{A}\|^2 - r^2 = 0 \quad (1.35.5)$$

Similarly, for Circle 2,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{B}^T \mathbf{x} + \|\mathbf{B}\|^2 - r_2^2 = 0 \quad (1.35.6)$$

Subtracting (1.35.6) from (1.35.5),

$$2\mathbf{B}^T \mathbf{x} = \|\mathbf{B}\|^2 \quad (1.35.7)$$

$$(1 \ 0) \mathbf{x} = \frac{3}{2} \quad (1.35.8)$$

which can be expressed as

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.35.9)$$

$$= \mathbf{q} + \lambda \mathbf{m} \text{ where} \quad (1.35.10)$$

$$\mathbf{q} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.35.11)$$

Substituting (1.35.10) in (1.35.5)

$$\|\mathbf{x}\|^2 = r^2 \quad (\because \mathbf{A} = 0) \quad (1.35.12)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 = r^2 \quad (1.35.13)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) = r^2 \quad (1.35.14)$$

$$\Rightarrow \mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) = r^2 \quad (1.35.15)$$

$$\Rightarrow \|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 = r^2 \quad (1.35.16)$$

$$\Rightarrow \|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 = r^2 \quad (1.35.17)$$

$$\Rightarrow \lambda = \pm \sqrt{\frac{9 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2}} \quad \because \mathbf{q}^T \mathbf{m} = 0 \quad (1.35.18)$$

Substituting the value of λ in (1.35.10),

$$\mathbf{C} = \mathbf{q} + \lambda \mathbf{m} \quad (1.35.19)$$

$$\mathbf{D} = \mathbf{q} - \lambda \mathbf{m} \quad (1.35.20)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^T (\mathbf{C} - \mathbf{D}) = 2 \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{6.75} \end{pmatrix} \quad (1.35.21)$$

$$= 0 \quad (1.35.22)$$

$$\Rightarrow AB \perp CD \quad (1.35.23)$$

1.36. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.

Solution: The given information is summarised in Table 1.36.1. See Fig. 1.36.1. Let P be a

	Symbols	Circle1	Circle2
Centre	O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Radius	r_1, r_2	4	6

TABLE 1.36.1

point on Circle 2 with radius 6. Then

$$\mathbf{P} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.36.1)$$

Let PQ and PR be tangents from point P on circle with radius 6 to the points Q and R on

circle with radius 4 . Now,

$$(\mathbf{O} - \mathbf{Q})^T(\mathbf{Q} - \mathbf{P}) = 0 \quad (\because OQ \perp QP) \quad (1.36.2)$$

$$\Rightarrow \mathbf{P}^T \mathbf{Q} = 16 \quad (\because \|\mathbf{Q}\|^2 = 16) \quad (1.36.3)$$

$$\text{or, } \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Q} = \frac{8}{3} \quad (1.36.4)$$

$$\Rightarrow \mathbf{Q} = \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.36.5)$$

$$= \mathbf{q} + \lambda \mathbf{m} \quad (1.36.6)$$

$$\text{where } \mathbf{q} = \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.36.7)$$

We know,

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 = r_1^2 \quad (1.36.8)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T(\mathbf{q} + \lambda \mathbf{m}) = r_1^2 \quad (1.36.9)$$

$$\lambda^2 = \frac{r_1^2 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.36.10)$$

$$\lambda = \pm 2.98 \quad (1.36.11)$$

Substituting the above in (1.36.5),

$$\mathbf{Q} = \begin{pmatrix} \frac{8}{3} \\ 2.98 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \frac{8}{3} \\ -2.98 \end{pmatrix} \quad (1.36.12)$$

The circles as well as the tangents are plotted in Fig. 1.36.1

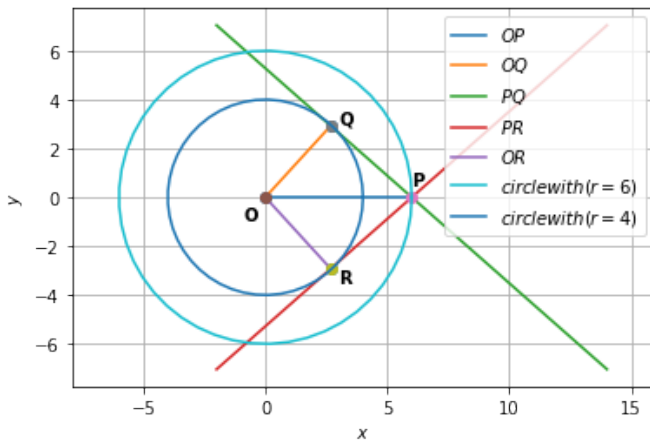


Fig. 1.36.1: Tangent lines to circle of radius 4 units.

- 1.37. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**.

Solution: Data from the given question is

available in Table 1.37.1: Let

	Symbols	Circle1
Centre	C	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Radius	r	3.4

TABLE 1.37.1: Input values

$$\mathbf{P} = r \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} = \begin{pmatrix} 1.7 \\ 2.9 \end{pmatrix} \quad (1.37.1)$$

$$\mathbf{Q} = r \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 2.4 \end{pmatrix} \quad (1.37.2)$$

Then the perpendicular bisector of PQ passes through

$$\mathbf{M} = \frac{\mathbf{P} + \mathbf{Q}}{2} \quad (1.37.3)$$

and has normal vector

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \quad (1.37.4)$$

resulting in the equation

$$(\mathbf{P} - \mathbf{Q})^T \left(\mathbf{x} - \frac{\mathbf{P} + \mathbf{Q}}{2} \right) = 0 \Rightarrow (\mathbf{P} - \mathbf{Q})^T \mathbf{x} = 0 \quad (1.37.5)$$

after simplification. It is obvious that **O** satisfies the above equation as can be verified in Fig. 1.37.1.

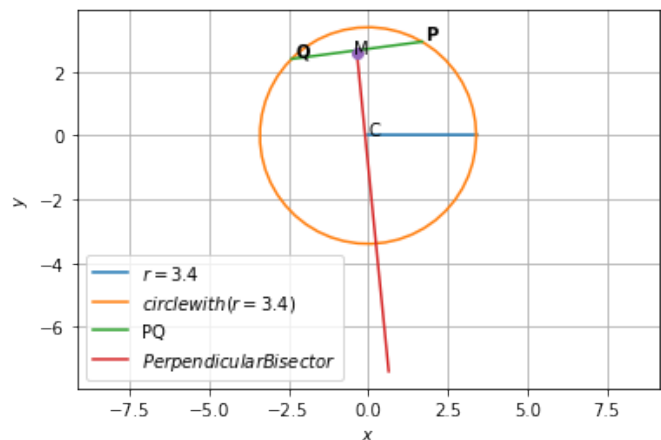


Fig. 1.37.1: perpendicular bisector of the chord passes through the center

- 1.38. Construct a quadrilateral $ABCD$ such that $AB = 5$, $\angle A = 50^\circ$, $AC = 4$, $BD = 5$ and $AD = 6$.

Solution:

The rough figure of the expected quadrilateral ABCD is given in Fig. 1.38.1

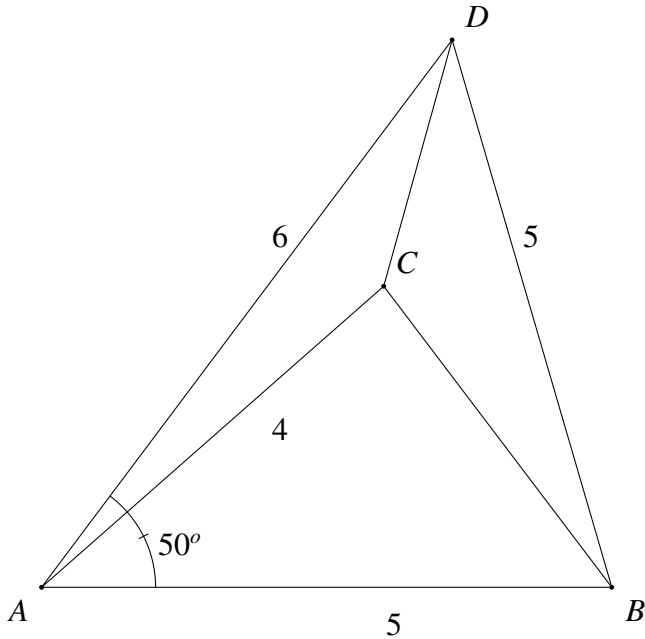


Fig. 1.38.1: Rough Figure

From the given information, in $\triangle ABD$,

$$\cos A = \frac{\|B - A\|^2 + \|D - A\|^2 - \|D - B\|^2}{2 \|B - A\| \|D - A\|} \quad (1.38.1)$$

$$\Rightarrow \angle A = \cos^{-1}(0.6) \approx 53.13^\circ \quad (1.38.2)$$

$$\neq 50^\circ \quad (1.38.3)$$

resulting in a contradiction. Therefore construction of quadrilateral with given measurements is not possible.

- 1.39. Can you construct a quadrilateral $PQRS$ with $PQ = 3, RS = 3, PS = 7.5, PR = 8$ and $SQ = 4$?

Solution: From the given information,

$$\|P - Q\| = 3 \quad (1.39.1)$$

$$\|R - S\| = 3 \quad (1.39.2)$$

$$\|P - S\| = 7.5 \quad (1.39.3)$$

$$\|P - R\| = 8 \quad (1.39.4)$$

$$\|S - Q\| = 4 \quad (1.39.5)$$

Let quadrilateral $PQRS$ be made up of two triangles $\triangle PSQ$ and $\triangle PSR$ on base PS .

- a) In $\triangle PSR$,

$$\begin{aligned} \|P - S\| + \|R - S\| &= 7.5 + 3 = 10.5 \\ &> \|P - R\| \end{aligned} \quad (1.39.6)$$

$$\|P - R\| + \|R - S\| = 8 + 3 = 11 > \|P - S\| \quad (1.39.7)$$

$$\begin{aligned} \|P - S\| + \|P - R\| &= 7.5 + 8 = 15.5 \\ &> \|R - S\| \end{aligned} \quad (1.39.8)$$

\therefore using triangle inequality, construction of $\triangle PSR$ is possible.

- b) In $\triangle PSQ$,

$$\begin{aligned} \|P - S\| + \|S - Q\| &= 7.5 + 4 = 11.5 \\ &> \|P - Q\| \end{aligned} \quad (1.39.9)$$

$$\begin{aligned} \|P - S\| + \|P - Q\| &= 7.5 + 3 = 10.5 \\ &> \|S - Q\| \end{aligned} \quad (1.39.10)$$

$$\|P - Q\| + \|S - Q\| = 3 + 4 = 7 < \|P - S\| \quad (1.39.11)$$

which violates triangle inequality. \therefore construction of $\triangle PSQ$ is not possible.

Fig. 1.39.1 highlights this.

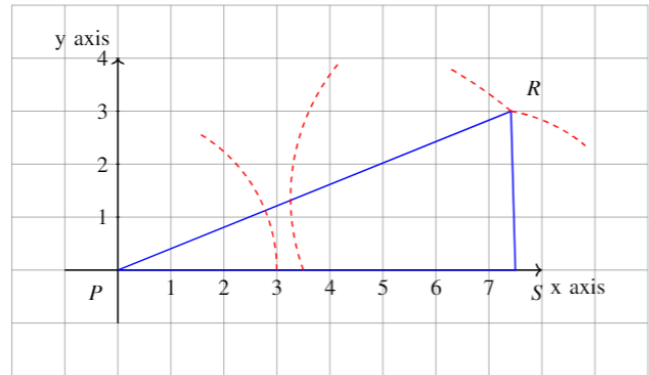


Fig. 1.39.1: Construction of quadrilateral $PQRS$

- 1.40. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.

Solution: In $\triangle LDO$

$$\|O - L\| + \|O - D\| = 17.5 > \|L - D\| \quad (1.40.1)$$

$$\|O - D\| + \|L - D\| = 15 > \|O - L\| \quad (1.40.2)$$

$$\|O - L\| + \|L - D\| = 12.5 > \|O - D\| \quad (1.40.3)$$

and triangle inequality is satisfied. Similarly, in

$\triangle LDG$

$$\|\mathbf{L} - \mathbf{D}\| + \|\mathbf{G} - \mathbf{L}\| = 11 > \|\mathbf{G} - \mathbf{D}\| \quad (1.40.4)$$

$$\|\mathbf{G} - \mathbf{L}\| + \|\mathbf{G} - \mathbf{D}\| = 12 > \|\mathbf{L} - \mathbf{D}\| \quad (1.40.5)$$

$$\|\mathbf{L} - \mathbf{D}\| + \|\mathbf{G} - \mathbf{D}\| = 11 > \|\mathbf{G} - \mathbf{L}\| \quad (1.40.6)$$

and triangle inequality is satisfied. \therefore the given sides form a quadrilateral which can be constructed by using the approach in Problem 1.3 to obtain the vertices of $\triangle LDO$ and $\triangle LDG$ as

$$\mathbf{L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} -1.875 \\ 7.26 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 2.5 \\ 5.5 \end{pmatrix} \quad (1.40.7)$$

and plotting the quadrilateral GOLD in Fig. 1.40.1

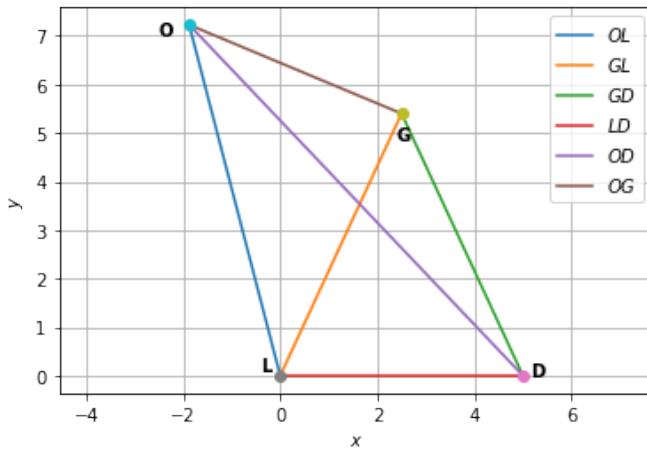


Fig. 1.40.1: Quadrilateral GOLD

- 1.41. Construct $ABCD$, where $AB = 4$, $BC = 5$, $CD = 6.5$, $\angle B = 105^\circ$ and $\angle C = 80^\circ$.

Solution:

Let

$$\angle B = 105^\circ = \theta \quad (1.41.1)$$

$$\angle C = 80^\circ = \alpha \quad (1.41.2)$$

$$\|\mathbf{A} - \mathbf{B}\| = 4 = p \quad (1.41.3)$$

$$\|\mathbf{C} - \mathbf{B}\| = 5 = q \quad (1.41.4)$$

$$\|\mathbf{D} - \mathbf{C}\| = 6.5 = r \quad (1.41.5)$$

and

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.41.6)$$

Lemma 1.1.

$$\mathbf{A} = p\mathbf{b} \quad \left(\because \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad (1.41.7)$$

$$\mathbf{D} = \mathbf{C} + r\mathbf{c} \quad (1.41.8)$$

where

$$\mathbf{b} = \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \cos C \\ \sin C \end{pmatrix} \quad (1.41.9)$$

Thus,

$$\mathbf{A} = 4 \begin{pmatrix} \cos 105^\circ \\ \sin 105^\circ \end{pmatrix} \quad (1.41.10)$$

$$= \begin{pmatrix} -1.03 \\ 3.86 \end{pmatrix} \quad (1.41.11)$$

and

$$\mathbf{D} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + 6.5 \begin{pmatrix} \cos 80^\circ \\ \sin 80^\circ \end{pmatrix} \quad (1.41.12)$$

$$= \begin{pmatrix} 6.12 \\ 6.39 \end{pmatrix} \quad (1.41.13)$$

which are then used to plot Fig. 1.41.1

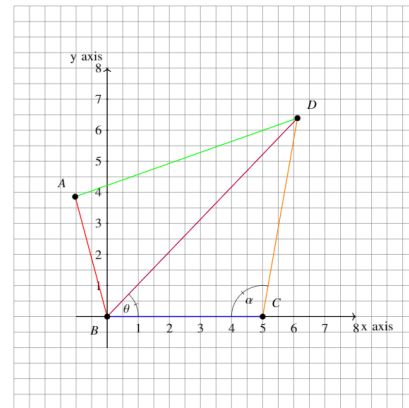


Fig. 1.41.1: Quadrilateral ABCD

1.42. Construct $DEAR$ with $DE = 4$, $EA = 5$, $AR = 4.5$, $\angle E = 60^\circ$ and $\angle A = 90^\circ$.

Solution: The given information can be expressed as

$$\angle E = 60^\circ = \theta \quad (1.42.1)$$

$$\angle A = 90^\circ = \alpha \quad (1.42.2)$$

$$\|\mathbf{D} - \mathbf{E}\| = 4 = a \quad (1.42.3)$$

$$\|\mathbf{E} - \mathbf{A}\| = 5 = b \quad (1.42.4)$$

$$\|\mathbf{A} - \mathbf{R}\| = 4.5 = c \quad (1.42.5)$$

Let,

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.42.6)$$

Lemma 1.2.

$$\mathbf{D} = a\mathbf{e} \quad \left(\because \mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad (1.42.7)$$

$$\mathbf{R} = \mathbf{A} + c\mathbf{a} \quad (1.42.8)$$

where

$$\mathbf{e} = \begin{pmatrix} \cos E \\ \sin E \end{pmatrix}, \mathbf{a} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad (1.42.9)$$

Thus, from (1.42.1) and (1.42.3) in (1.42.7),

$$\mathbf{D} = 4 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} \quad (1.42.10)$$

$$= \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \quad (1.42.11)$$

and from (1.42.2) and (1.42.5) in (1.42.8),

$$\mathbf{R} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + 4.5 \begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} \quad (1.42.12)$$

$$= \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad (1.42.13)$$

Thus

$$\mathbf{D} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad (1.42.14)$$

and the quadrilateral $DEAR$ is plotted in Fig. 1.42.1.

1.43. Construct $TRUE$ with $TR = 3.5$, $RU = 3$, $UE = 4$, $\angle R = 75^\circ$ and $\angle U = 120^\circ$.

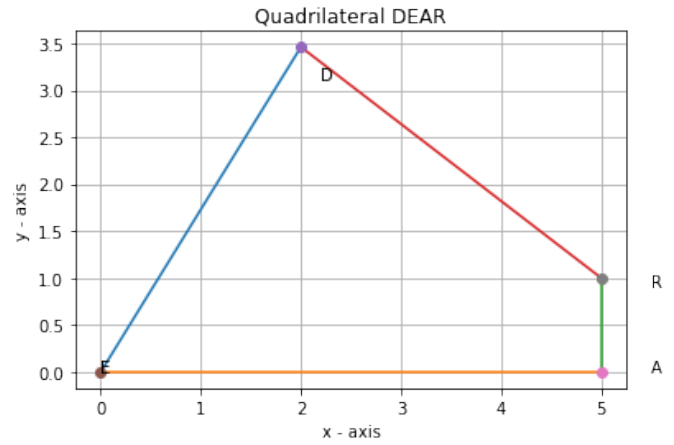


Fig. 1.42.1: Quadrilateral $DEAR$

Solution: From the given information,

$$\angle R = 75^\circ = \theta \quad (1.43.1)$$

$$\angle U = 120^\circ = \alpha \quad (1.43.2)$$

$$\|\mathbf{T} - \mathbf{R}\| = 3.5 = a \quad (1.43.3)$$

$$\|\mathbf{U} - \mathbf{R}\| = 3 = b \quad (1.43.4)$$

$$\|\mathbf{E} - \mathbf{U}\| = 4 = c \quad (1.43.5)$$

Let,

$$\mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.43.6)$$

Lemma 1.3.

$$\mathbf{T} = C\mathbf{u} \quad \left(\because \mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad (1.43.7)$$

$$\mathbf{E} = \mathbf{U} + a\mathbf{r} \quad (1.43.8)$$

where

$$\mathbf{r} = \begin{pmatrix} \cos R \\ \sin R \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \cos U \\ \sin U \end{pmatrix} \quad (1.43.9)$$

Thus,

$$\mathbf{T} = 4 \begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix} \quad (1.43.10)$$

$$= \begin{pmatrix} -2 \\ 3.46 \end{pmatrix} \quad (1.43.11)$$

and

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 3.5 \begin{pmatrix} \cos 75^\circ \\ \sin 75^\circ \end{pmatrix} \quad (1.43.12)$$

$$= \begin{pmatrix} 3.39 \\ 3.38 \end{pmatrix} \quad (1.43.13)$$

The vertices of given quadrilateral TRUE can be written as,

$$\mathbf{T} = \begin{pmatrix} -2 \\ 3.46 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 3.39 \\ 3.38 \end{pmatrix} \quad (1.43.14)$$

which are plotted in Fig. 1.43.1.

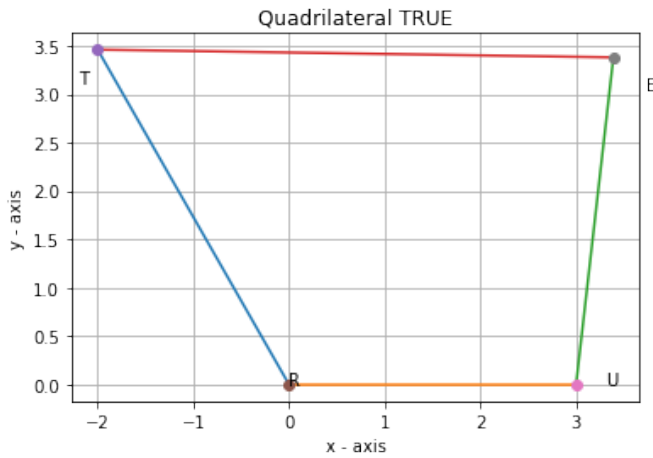


Fig. 1.43.1: Quadrilateral TRUE

1.44. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?

Solution: We obtain the vertices of the rhombus as follows

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 3.5 \end{pmatrix} \quad (1.44.1)$$

which are plotted in Fig. 1.44.1.

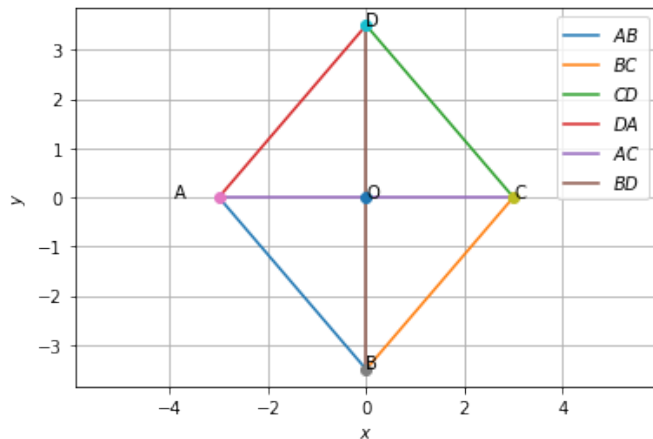


Fig. 1.44.1: Rhombus ABCD

1.45. Draw a square $READ$ with $RE = 5.1$.

Solution: The vertices are given by

$$\mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 5.1 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5.1 \\ 5.1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 5.1 \end{pmatrix} \quad (1.45.1)$$

The desired square is plotted in Fig. 1.45.1

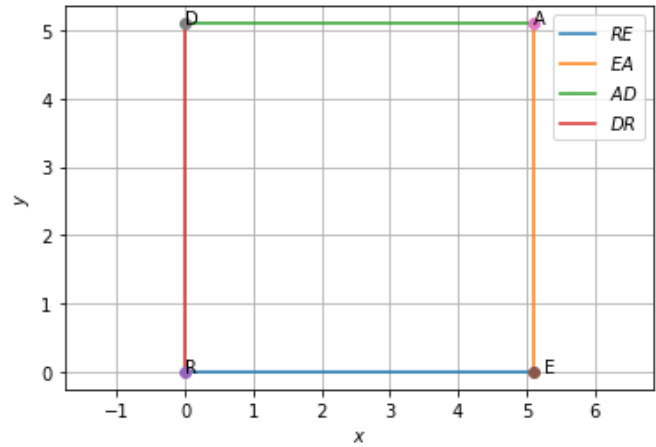


Fig. 1.45.1: Square $READ$

1.46. Draw a rhombus whose diagonals are 5.2 and 6.4.

Solution: We obtain the vertices of the rhombus as

$$\mathbf{A} = \begin{pmatrix} -2.6 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3.2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2.6 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 3.2 \end{pmatrix} \quad (1.46.1)$$

which are plotted in Fig. 1.46.1

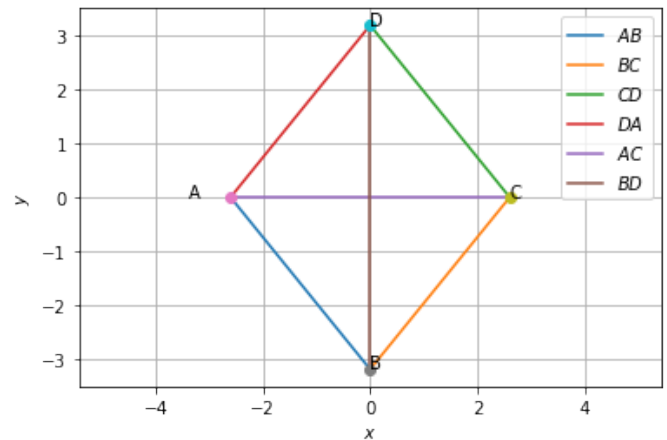


Fig. 1.46.1: Rhombus ABCD

1.47. Draw a rectangle with adjacent sides 5 and 4.

Solution: The vertices of rectangle $ABCD$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{D} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (1.47.1)$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad (1.47.2)$$

where $a = 5$ and $c = 4$. The rectangle $ABCD$ is plotted in Fig. 1.47.1

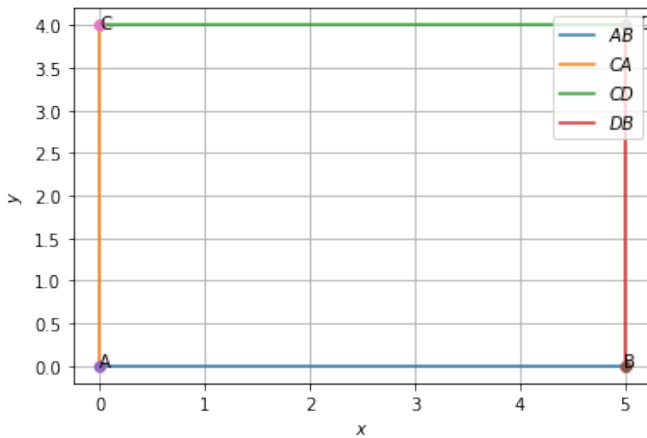


Fig. 1.47.1: Rectangle $ABCD$

1.48. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

Solution: There are infinite number of parallelograms that can be drawn. For a unique parallelogram, one angle needs to be specified.

2 EXERCISES

2.1. Construct the triangles in Table 2.1.1. **Solu-**

S.No	Triangle	Given Measurements		
1	$\triangle ABC$	$\angle A = 85^\circ$	$\angle B = 115^\circ$	$AB = 5$
2	$\triangle PQR$	$\angle Q = 30^\circ$	$\angle R = 60^\circ$	$QR = 4.7$
3	$\triangle ABC$	$\angle A = 70^\circ$	$\angle B = 50^\circ$	$AC = 3$
4	$\triangle LMN$	$\angle L = 60^\circ$	$\angle N = 120^\circ$	$LM = 5$
5	$\triangle ABC$	$BC = 2$	$AB = 4$	$AC = 2$
6	$\triangle PQR$	$PQ = 2.5$	$QR = 4$	$PR = 3.5$
7	$\triangle XYZ$	$XY = 3$	$YZ = 4$	$XZ = 5$
8	$\triangle DEF$	$DE = 4.5$	$EF = 5.5$	$DF = 4$

TABLE 2.1.1

tion:

a)

b) **Solution:** From the given information, $\triangle PQR$ is a right angled triangle. Let $QR = p$ and $\theta = 30^\circ$. Then the vertices of the triangle are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.1)$$

$$\mathbf{Q} = \begin{pmatrix} 0 \\ p \cos \theta \end{pmatrix} \quad (2.1.2)$$

$$= \begin{pmatrix} 0 \\ 4.07 \end{pmatrix} \quad (2.1.3)$$

$$\mathbf{R} = \begin{pmatrix} p \sin \theta \\ 0 \end{pmatrix} \quad (2.1.4)$$

$$= \begin{pmatrix} 2.35 \\ 0 \end{pmatrix} \quad (2.1.5)$$

The triangle is plotted in Fig. 2.1.1

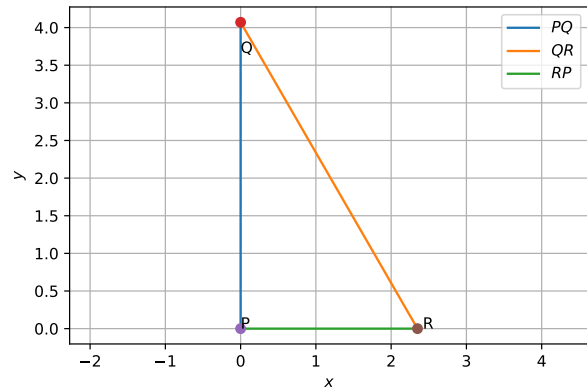


Fig. 2.1.1: $\triangle PQR$ constructed using python

c) **Solution:** From the given information,

$$\angle C = 60^\circ \quad (2.1.6)$$

Using the sine formula,

$$c = b \left(\frac{\sin C}{\sin B} \right) \quad (2.1.7)$$

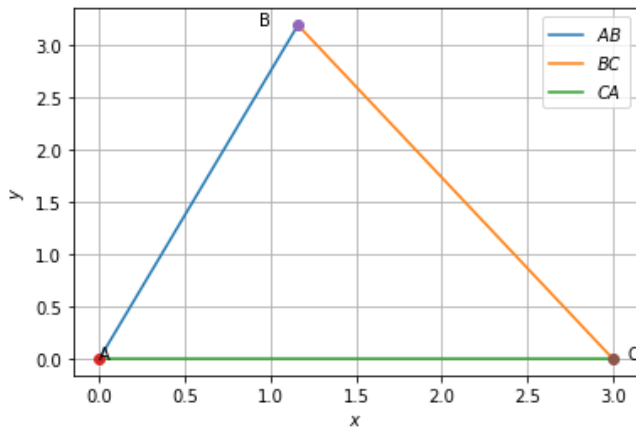
$$= 3.3915 \quad (2.1.8)$$

the vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = c \begin{pmatrix} \cos 70^\circ \\ \sin 70^\circ \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.1.9)$$

and plotted in Fig. 2.1.2.

2.2. Construct $PQRS$ where $PQ = 4$, $QR = 6$, $RS = 5$, $PS = 5.5$ and $PR = 7$.

Fig. 2.1.2: Plot of $\triangle ABC$

- 2.3. Draw *JUMP* with $JU = 3.5$, $UM = 4$, $MP = 5$, $PJ = 4.5$ and $PU = 6.5$
- 2.4. Construct a quadrilateral *ABCD* such that $BC = 4.5$, $AC = 5.5$, $CD = 5$, $BD = 7$ and $AD = 5.5$.
- 2.5. Construct *LIFT* such that $LI = 4$, $IF = 3$, $TL = 2.5$, $LF = 4.5$, $IT = 4$.
- 2.6. DRAW rhombus *BEND* such that $BN = 5.6$, $DE = 6.5$.
- 2.7. construct a quadrilateral *MIST* where $MI = 3.5$, $IS = 6.5$, $\angle M = 75^\circ$, $\angle I = 105^\circ$ and $\angle S = 120^\circ$.
- 2.8. Can you construct the above quadrilateral *MIST* if $\angle M = 100^\circ$ instead of 75° .
- 2.9. Can you construct the quadrilateral *PLAN* if $PL = 6$, $LA = 9.5$, $\angle P = 75^\circ$, $\angle L = 150^\circ$ and $\angle A = 140^\circ$?
- 2.10. Construct *MORE* where $MO = 6$, $OR = 4.5$, $\angle M = 60^\circ$, $\angle O = 105^\circ$, $\angle R = 105^\circ$.
- 2.11. Construct *PLAN* where $PL = 4$, $LA = 6.5$, $\angle P = 90^\circ$, $\angle A = 110^\circ$ and $\angle N = 85^\circ$.
- 2.12. Draw rectangle *OKAY* with $OK = 7$ and $KA = 5$.
- 2.13. Construct a kite *EASY* if $AY = 8$, $EY = 4$ and $SY = 6$.
- 2.14. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
- 2.15. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60° .
- 2.16. Let *ABC* be a right triangle in which $a = 8$, $c = 6$ and $\angle B = 90^\circ$. *BD* is the perpendicular from **B** on *AC* (altitude). The circle through **B**, **C**, **D** (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
- 2.17. Draw a circle of diameter 6.1
- 2.18. Draw a circle with centre **B** and radius 6. If **C** be a point 10 units away from its centre, construct the pair of tangents *AC* and *CD* to the circle.
- 2.19. Draw a circle of radius 3 and any two of its diameters. Draw the ends of these diameters. What figure do you get?
- 2.20. Draw a line segment *AB* of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.