GATE Assignment

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Download all python codes from

```
https://github.com/Digjoy12/Signal−Processing/
    blob/main/Assignment%202−GATE/Codes/
    code.py
```

and latex codes from

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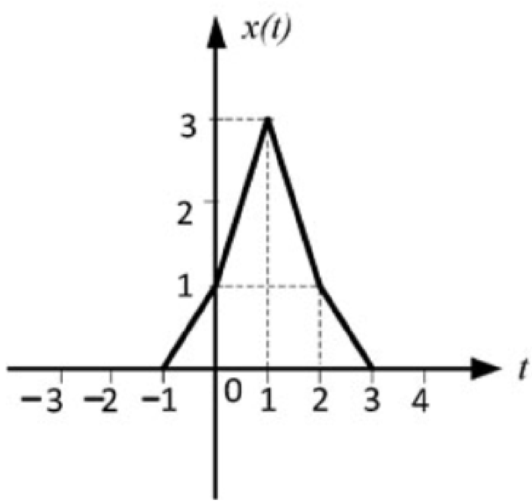
https://github.com/Digjoy12/Signal−Processing/
    blob/main/Assignment%202−GATE/main.tex
```

Problem

(GATE EC 2020 - Q52) X(ω) is the Fourier Transform of x(t) shown below. The value of $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$ (**rounded off to two decimal places**) is ....



Solution

Theorem 1 (Parseval's energy theorem)**.**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{0.0.1}$$

where $X(\omega) = F_{(\omega)}\{x(t)\}$ represents the continuous Fourier transform of x(t) and $\omega = 2\pi f$ is frequency in radians per second.

Proof. The inverse Fourier Transform of x(t) is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(\omega t) d\omega \tag{0.0.2}$$

Taking the conjugate of x(t), we get

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\exp(-\omega t) d\omega \tag{0.0.3}$$

We know that, total energy of signal x(t) is

$$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{0.0.4}$$

$$= \int_{-\infty}^{\infty} x(t)x^*(t) dt \tag{0.0.5}$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\exp(-\omega t) d\omega\right] dt \tag{0.0.6}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\int_{-\infty}^{\infty} x(t)\exp(\omega t) d\omega \tag{0.0.7}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)X(\omega) \tag{0.0.8}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{0.0.9}$$

$\square$

Now, by Parseval's theorem, we know that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad (0.0.10)$$

$$\implies 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad (0.0.11)$$

$$\implies 2\pi \int_{-\infty}^{\infty} x(t)^2 dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad (0.0.12)$$

[*Since, x(t) is real*]

Since, shifting of a signal does not change the energy of the signal, therefore left shifting the signal by 1, we get the signal as y(t) = x(t+1)

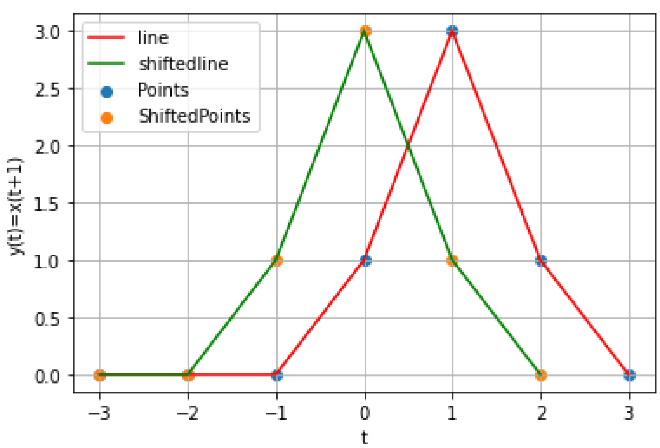


Fig. 0: Graphical Transformation

Now, y(t) is an even function since it is symmetric around the origin.
Therefore $y^2(t)$ is also an even signal.

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} y(t)^2 dt \quad (0.0.13)$$

$$= 4\pi \int_{0}^{\infty} y(t)^2 dt \quad (0.0.14)$$

$$= 4\pi \left[\int_{0}^{1} y(t)^2 dt + \int_{1}^{2} y(t)^2 dt\right] \quad (0.0.15)$$

For t=0 to t=1,

$$y(t) = x(t+1) = -2t+3$$

For t=1 to t=2,

$$y(t) = x(t+1) = -t+2$$

Therefore,

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 4\pi \left[\int_{0}^{1} y(t)^2 dt + \int_{1}^{2} y(t)^2 dt\right] \quad (0.0.16)$$

$$= 4\pi \left[\int_{0}^{1} (-2t+3)^2 dt + \int_{1}^{2} (-t+2)^2 dt\right] \quad (0.0.17)$$

$$= 4\pi\Bigg[\int_{0}^{1} (4t^2 - 12t + 9)dt +$$

$$\int_{1}^{2} (t^2 - 4t + 4)dt\Bigg] \quad (0.0.18)$$

$$= 4\pi\left[\left\{\frac{4t^3}{3} - 6t^2 + 9t\right\}_0^1 + \right. \quad (0.0.19)$$

$$\left.\left\{\frac{t^3}{3} - 2t^2 + 4t\right\}_1^2\right. \quad (0.0.20)$$

$$= 4\pi\left[\frac{13}{3} + \frac{1}{3}\right] \quad (0.0.21)$$

$$= 4\pi\left[\frac{14}{3}\right] \quad (0.0.22)$$

$$= 58.61\% \quad (0.0.23)$$

Hence,

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \mathbf{58.61\%}$$