Assignment 3

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Download all python codes from

https://github.com/Digjoy12/Signal−Processing/blob/main/Assignment_3/Code/construction.py

and latex codes from

https://github.com/Digjoy12/Signal−Processing/blob/main/Assignment_3/main.tex

Problem

(Construction - Q2.15) Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.

Solution

Let the center of the circle be

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.1}$$

and radius = 5units.
Therefore the equation of the circle is given by

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{0.0.2}$$

Let the tangent be drawn from a point $\mathbf{P}\begin{pmatrix} x \\ 0 \end{pmatrix}$ on the x-axis which intersect the circle at point **Q** and **R**.
Since, **OQ** and **PQ** are perpendicular

$$(\mathbf{OQ})^\top(\mathbf{QP}) = 0 \tag{0.0.3}$$
$$\implies (\mathbf{O}-\mathbf{Q})^\top(\mathbf{Q}-\mathbf{P}) = 0 \tag{0.0.4}$$
$$\implies \mathbf{O}^\top\mathbf{Q} - \mathbf{O}^\top\mathbf{P} - \|\mathbf{Q}\|^2 + \mathbf{P}^\top\mathbf{Q} = 0 \tag{0.0.5}$$
$$\implies \mathbf{P}^\top\mathbf{Q} = \|\mathbf{Q}\|^2 \tag{0.0.6}$$
$$\implies \mathbf{P}^\top\mathbf{Q} = 25 \tag{0.0.7}$$

Since, $\|\mathbf{Q}\|^2 = 25$
Now, in triangle $\triangle OPQ, \angle P = 30°$ and $\angle Q = 90°$, therefore $\angle POQ = 60°$
i.e the angle between **OQ** and **OP** is 60°.

$$\cos 30° = \frac{(\mathbf{OQ})^\top(\mathbf{OP})}{\|\mathbf{OQ}\|\,\|\mathbf{OP}\|} \tag{0.0.8}$$
$$\implies \frac{1}{2} = \frac{(\mathbf{O}-\mathbf{Q})^\top(\mathbf{O}-\mathbf{P})}{\|\mathbf{O}-\mathbf{Q}\|\,\|\mathbf{O}-\mathbf{P}\|} \tag{0.0.9}$$
$$\implies \frac{1}{2} = \frac{\mathbf{P}^\top\mathbf{Q}}{\|\mathbf{Q}\|\,\|\mathbf{P}\|} \tag{0.0.10}$$
$$\implies \frac{1}{2} = \frac{25}{5\|\mathbf{P}\|} \tag{0.0.11}$$
$$\implies \|\mathbf{P}\| = 10 \tag{0.0.12}$$

Therefore, the point **P** is $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Now, (0.0.7) can be rewritten as

$$\begin{pmatrix} 10 & 0 \end{pmatrix}\mathbf{Q} = 25 \tag{0.0.13}$$
$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{Q} = 2.5 \tag{0.0.14}$$
$$\implies \mathbf{Q} = \begin{pmatrix} 2.5 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.15}$$
$$\implies \mathbf{Q} = \mathbf{q} + \lambda\mathbf{m} \tag{0.0.16}$$

where, $\mathbf{q} = \begin{pmatrix} 2.5 \\ 0 \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now, we know that

$$\|\mathbf{Q}\|^2 = 25 \tag{0.0.17}$$
$$\implies \|\mathbf{q}+\lambda\mathbf{m}\|^2 = 25 \tag{0.0.18}$$
$$\implies (\mathbf{q}+\lambda\mathbf{m})^\top(\mathbf{q}+\lambda\mathbf{m}) = 25 \tag{0.0.19}$$
$$\implies \|\mathbf{q}\|^2 + 2\mathbf{q}^\top\lambda\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 = 25 \tag{0.0.20}$$

Since, $2\mathbf{q}^\top\lambda\mathbf{m} = 0$

$$\implies \lambda^2 = \frac{25 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{0.0.21}$$
$$\implies \lambda^2 = \frac{25 - (2.5)^2}{1} \tag{0.0.22}$$
$$\implies \lambda = \pm 4.33 \tag{0.0.23}$$

Therefore, $\mathbf{Q} = \begin{pmatrix} 2.5 \\ 4.33 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 2.5 \\ -4.33 \end{pmatrix}$

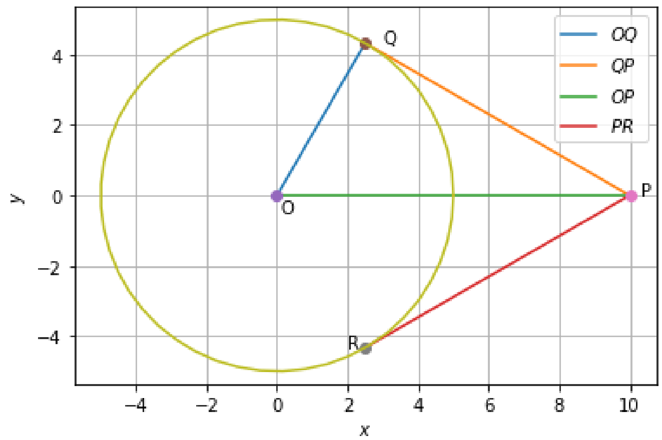

Fig. 0: Plot of the tangents