GATE Assignment

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Download all python codes from

https://github.com/Digjoy12/Signal−Processing/blob/main/Assignment_5/Codes/Code.py

and latex codes from

https://github.com/Digjoy12/Signal−Processing/blob/main/Assignment_5/main.tex

Problem

(Quadratic Forms Q2.67) The line $\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

Solution

Given equation,

$$y^2 = 4x \tag{0.0.1}$$

Comparing it with the standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2cy + f = 0 \tag{0.0.2}$$

Here, $a = b = c = f = 0$, $d = -2$ and $c = 1$

$$\therefore \mathbf{V} = \begin{pmatrix}a & b\\ b & c\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{0.0.3}$$

$$\therefore \mathbf{u} = \begin{pmatrix}d\\ e\end{pmatrix} = \begin{pmatrix}-2\\ 0\end{pmatrix} \tag{0.0.4}$$

Now,

$$|\mathbf{V}| = \begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix} = 0 \tag{0.0.5}$$

$\implies$ the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$.

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{0.0.6}$$

$$\implies \begin{vmatrix}-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix} = 0 \tag{0.0.7}$$

$$\implies -\lambda(1-\lambda) = 0 \tag{0.0.8}$$

$\therefore \lambda = 0$ or $\lambda = 1$
Calculating the eigen vectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{0.0.9}$$

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix}1\\ 0\end{pmatrix} \tag{0.0.10}$$

$$\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_2} = \begin{pmatrix}0\\ 1\end{pmatrix} \tag{0.0.11}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathbf{T}} \tag{0.0.12}$$

where,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{0.0.13}$$

and,

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{0.0.14}$$

Now, equation (0.0.12) becomes,

$$\mathbf{V} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{0.0.15}$$

$$= \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{0.0.16}$$

The given line equation is

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 1 \tag{0.0.17}$$

$$\implies -mx + y - 1 = 0 \tag{0.0.18}$$

The general straight line equation is of the form

$$ax + by + c = 0 \tag{0.0.19}$$

The normal vector (n) and direction (m) are given by,

$$\mathbf{n} = \begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}-m\\ 1\end{pmatrix} \tag{0.0.20}$$

$$\mathbf{m} = \begin{pmatrix}b\\ -a\end{pmatrix} = \begin{pmatrix}1\\ m\end{pmatrix} \tag{0.0.21}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix}\mathbf{u}^T + k\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f\\ k\mathbf{n} - \mathbf{u}\end{pmatrix} \tag{0.0.22}$$

where,

$$k = \frac{\mathbf{p_1}^T \mathbf{u}}{\mathbf{p_1}^T \mathbf{n}} = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -m \\ 1 \end{pmatrix}} = \frac{2}{m} \tag{0.0.23}$$

Hence, substituting the values of (0.0.3), (0.0.4), (0.0.20) and (0.0.23) in the equation (0.0.22), we get

$$\begin{pmatrix} \begin{pmatrix} -2 & 0 \end{pmatrix} + \frac{2}{m}\begin{pmatrix} -m & 1 \end{pmatrix} \\ \begin{matrix} 0 & & 0 \end{matrix} \\ \begin{matrix} 0 & & 1 \end{matrix} \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ \frac{2}{m}\begin{pmatrix} -m \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} \end{pmatrix} \tag{0.0.24}$$

$$\implies \begin{pmatrix} -4 & \frac{2}{m} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{m} \end{pmatrix} \tag{0.0.25}$$

Solving for q by removing the zero row and representing (0.0.25) as augmented matrix and then converting the matrix to echelon form

$$\begin{pmatrix} -4 & \frac{2}{m} & 0 \\ 0 & 1 & \frac{2}{m} \end{pmatrix} \xleftrightarrow{R_1 \to -\frac{1}{4}R_1} \begin{pmatrix} 1 & \frac{-1}{2m} & 0 \\ 0 & 1 & \frac{2}{m} \end{pmatrix} \tag{0.0.26}$$

$$\xleftrightarrow{R_2 \to R_1 + \frac{1}{2m}R_2} \begin{pmatrix} 1 & 0 & \frac{1}{m^2} \\ 0 & 1 & \frac{2}{m} \end{pmatrix} \tag{0.0.27}$$

Hence from equation (0.0.27) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{1}{m^2} \\ \frac{2}{m} \end{pmatrix} \tag{0.0.28}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c = 1 \tag{0.0.29}$$

where,

$$c = \mathbf{n}^T \mathbf{q} = 1 \tag{0.0.30}$$

$$\implies \begin{pmatrix} -m & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{m^2} \\ \frac{2}{m} \end{pmatrix} = 1 \tag{0.0.31}$$

$$\implies \frac{-1}{m} + \frac{2}{m} = 1 \tag{0.0.32}$$

$$\implies \frac{1}{m} = 1 \tag{0.0.33}$$

$$\implies m = 1 \tag{0.0.34}$$

Hence, the line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$ when $\mathbf{m = 1}$.