

Assignment 3

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Download all python codes from

<https://github.com/Digjoy12/probability/tree/main/Assignment%203/code>

and latex codes from

<https://github.com/Digjoy12/probability/blob/main/Assignment%203/main.tex>

PROBLEM

(GATE MA - 2016 Q.47) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\Pr(X_1 < X_2)$ is equal to

SOLUTION

We know that,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & 0 \leq x < \infty \end{cases} \quad (0.0.1)$$

Given,

$$E(X_1) = \frac{1}{\lambda} = 1 \quad (0.0.2)$$

$$\Rightarrow \lambda = 1 \quad (0.0.3)$$

Therefore,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & 0 \leq x < \infty \end{cases} \quad (0.0.4)$$

We know that,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)} & 0 \leq x < \infty \end{cases} \quad (0.0.5)$$

Given,

$$E(X_2) = \alpha\beta = 2 \quad (0.0.6)$$

$$V(X_2) = \alpha\beta^2 = 2 \quad (0.0.7)$$

Solving 0.0.6 and 0.0.7, we get, $\alpha = 2, \beta = 1$ and $\Gamma(2) = 1$

Therefore,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ x e^{-x} & 0 \leq x < \infty \end{cases} \quad (0.0.8)$$

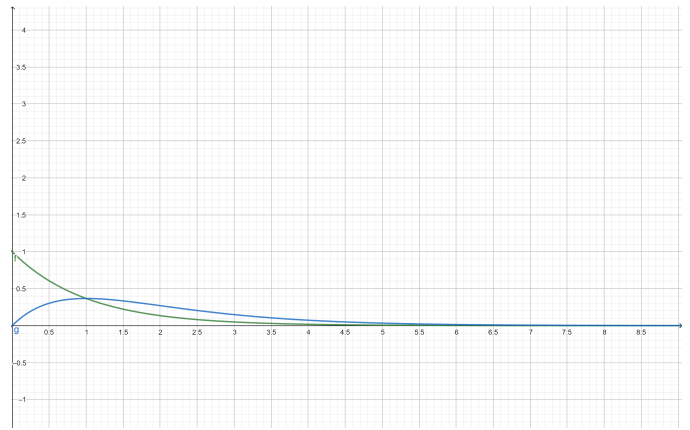


Fig. 0: Probability Distribution of (X_1, X_2)

Calculating the CDF of $f_{X_2}(x)$,

$$F_{X_2}(x) = \int_0^x f_{X_2}(x) \quad (0.0.9)$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - \sum_{k=0}^{\alpha-1} \frac{\left(\frac{t}{\beta}\right)^k}{k!} e^{\left(\frac{-t}{\beta}\right)} & 0 \leq x < \infty \end{cases} \quad (0.0.10)$$

For $\alpha = 2$ and $\beta = 1$

Alternately, we have CDF of X_1 and X_2 given by

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \leq x < \infty \end{cases} \quad (0.0.11)$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - \sum_{k=0}^1 \frac{(t)^k}{k!} e^{-t} & 0 \leq x < \infty \end{cases} \quad (0.0.12)$$

Thus,

$$\Pr(X_1 \leq X_2) = \int_{-\infty}^{\infty} F_{X_1}(x)f_{X_2}(x)dx \quad (0.0.13)$$

$$= \int_0^{\infty} (1 - e^{-x})(xe^{-x})dx \quad (0.0.14)$$

$$= \frac{3}{4} \quad (0.0.15)$$

$$= 0.75 \quad (0.0.16)$$