Assignment 5

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Download all python codes from

https://github.com/Digjoy12/probability/tree/main/Assignment%205/code

and latex codes from

https://github.com/Digjoy12/probability/blob/main/Assignment%205/main.tex

Problem

(CSIR UGC NET Maths June 2018-Q103)-Let X and Y be two random variables with joint probability density function

$$f(x.y) = \begin{cases} \frac{1}{\pi} & 0 \leq x^2 + y^2 \leq 1 \\ 0 & otherwise \end{cases}$$

Which of the following statements are correct?

1) X and Y are independent.
2) $\Pr(X > 0)=\frac{1}{2}$
3) E(Y)=0
4) Cov(X,Y)=0

Solution

The marginal PDF of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy \tag{0.0.1}$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{0.0.2}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{0.0.3}$$

The marginal PDF of Y is given by

$$f_Y(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx \tag{0.0.4}$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{0.0.5}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{0.0.6}$$

Now,

$$f_X(x) \times f_Y(x) = \frac{2\sqrt{1-x^2}}{\pi} \times \frac{2\sqrt{1-y^2}}{\pi} \tag{0.0.7}$$

$$= \frac{4(1-x^2)(1-y^2)}{\pi^2} \tag{0.0.8}$$

$$\neq \frac{1}{\pi} \tag{0.0.9}$$

$$\neq f_{XY}(x, y) \tag{0.0.10}$$

Therefore, X and Y are not independent.

Now,

$$\Pr(X > 0) = \int_0^{\infty} f_X(x)dx \tag{0.0.11}$$

$$= \int_0^1 \frac{2\sqrt{1-x^2}}{\pi}dx \tag{0.0.12}$$

$$= \left(\frac{\arcsin(x) + x\sqrt{1-x^2}}{\pi}\right)_0^1 \tag{0.0.13}$$

$$= \frac{1}{2} \tag{0.0.14}$$

Therefore, option(2) is correct.

Now,

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy \tag{0.0.15}$$
$$= \int_{-1}^{1} \frac{2y\sqrt{1-y^2}}{\pi} dy \tag{0.0.16}$$
$$= \left(\frac{-2(1-y^2)^{\frac{3}{2}}}{3\pi} \right)_{-1}^{1} \tag{0.0.17}$$
$$= 0 \tag{0.0.18}$$

Therefore, option(3) is also correct.

Now,

$$E[XY] = \int_x \int_y xy f_{XY}(x,y) dy dx \tag{0.0.19}$$
$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{xy}{\pi} dy dx \tag{0.0.20}$$
$$= \frac{x}{\pi} \int_{-1}^{1} \left(\frac{y^2}{2} \right)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \tag{0.0.21}$$
$$= 0 \tag{0.0.22}$$

Now,

$$Cov(X,Y) = E[XY] - E[X] E[Y] \tag{0.0.23}$$
$$= 0 - E[X] \times 0 \tag{0.0.24}$$
$$= 0 \tag{0.0.25}$$

Therefore, option(4) is also correct.

The correct options are (2),(3) and (4).