Assignment 3

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Download all python codes from

https://github.com/Digjoy12/probability/tree/main/Assignment%203/code

and latex codes from

https://github.com/Digjoy12/probability/blob/main/Assignment%203/main.tex

Problem

(GATE MA - 2016 Q.47) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\Pr(X_1 < X_2)$ is equal to ....

Solution

We know that,

$$f(X_1) = \begin{cases} 0 & X_1 < 0 \\ \lambda e^{\lambda x} & 0 \le X_1 < \infty \end{cases} \tag{0.0.1}$$

Given,

$$E(X_1) = \frac{1}{\lambda} = 1 \tag{0.0.2}$$

$$\implies \lambda = 1 \tag{0.0.3}$$

Therefore,

$$f(X_1) = \begin{cases} 0 & X_1 < 0 \\ e^{-X_1} & 0 \le X_1 < \infty \end{cases} \tag{0.0.4}$$

We know that,

$$f(X_2) = \begin{cases} 0 & X_2 < 0 \\ \frac{X_2^{\alpha-1} e^{(\frac{-X_2}{\beta})}}{\beta^\alpha \Gamma(\alpha)} & 0 \le X_2 < \infty \end{cases} \tag{0.0.5}$$

Given,

$$E(X_2) = \alpha\beta = 2 \tag{0.0.6}$$

$$V(X_2) = \alpha\beta^2 = 2 \tag{0.0.7}$$

Solving 0.0.6 and 0.0.7, we get, $\alpha = 2$, $\beta = 1$ and $\Gamma(2) = 1$

Therefore,

$$f(X_2) = \begin{cases} 0 & X_2 < 0 \\ X_2 e^{-X_2} & 0 \le X_2 < \infty \end{cases} \tag{0.0.8}$$

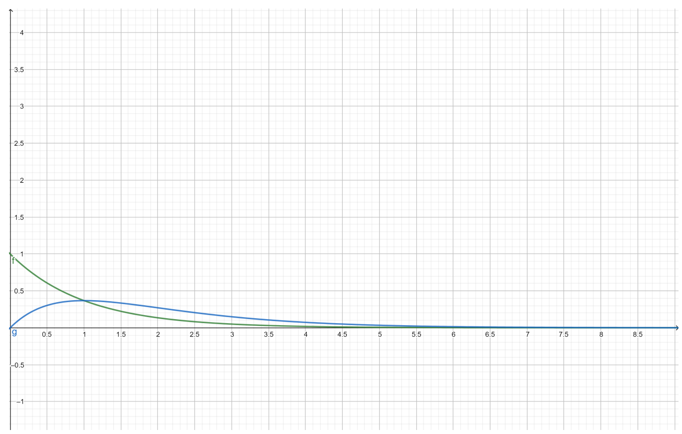

Fig. 0: Probability Distribution of (X_1, X_2)

Alternately, we have CDF of X_1 and X_2 given by

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \le x < \infty \end{cases} \tag{0.0.9}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - (x+1)e^{-x} & 0 \le x < \infty \end{cases} \tag{0.0.10}$$

Thus,

$$\begin{aligned} \Pr(X_1 \le X_2) &= \int_{-\infty}^{\infty} F_{X_1}(x) f(X_2) dX && (0.0.11) \\ &= \int_0^{\infty} (1 - e^{-x})(xe^{-x}) dx && (0.0.12) \\ &= \frac{3}{4} && (0.0.13) \\ &= 0.75 && (0.0.14) \end{aligned}$$