Assignment 3

Digjoy Nandi - AI20BTECH11007

Download all python codes from

```
https://github.com/Digjoy12/probability/tree/main/
    Assignment%203/code
```

and latex codes from

```
https://github.com/Digjoy12/probability/blob/main/
    Assignment%203/main.tex
```

Problem

(GATE MA - 2016 Q.47) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\Pr(X_1 < X_2)$ is equal to ....

Solution

We know that,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{\lambda x} & 0 \leq x < \infty \end{cases} \tag{0.0.1}$$

Given,

$$E(X_1) = \frac{1}{\lambda} = 1 \tag{0.0.2}$$

$$\implies \lambda = 1 \tag{0.0.3}$$

Therefore,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & 0 \leq x < \infty \end{cases} \tag{0.0.4}$$

We know that,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^{\alpha-1} e^{\left(\frac{-x}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)} & 0 \leq x < \infty \end{cases} \tag{0.0.5}$$

Given,

$$E(X_2) = \alpha\beta = 2 \tag{0.0.6}$$

$$V(X_2) = \alpha\beta^2 = 2 \tag{0.0.7}$$

Solving 0.0.6 and 0.0.7, we get, $\alpha = 2$, $\beta = 1$ and $\Gamma(2) = 1$
Therefore,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ xe^{-x} & 0 \leq x < \infty \end{cases} \tag{0.0.8}$$

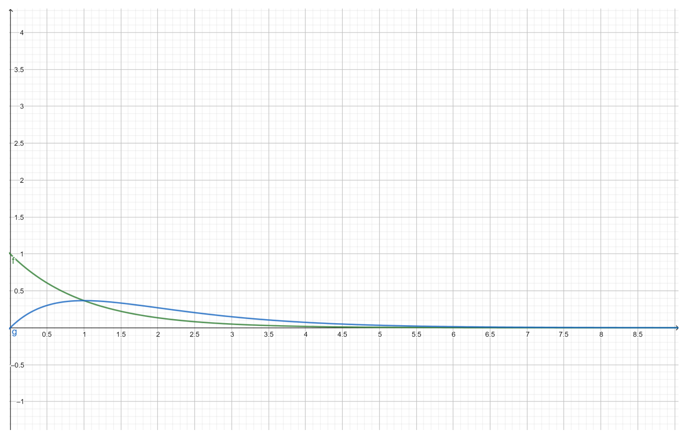

Fig. 0: Probability Distribution of (X_1, X_2)

Calculating the CDF of $f_{X_2}(x)$,
When $x < 0$,

$$F_{X_2}(x) = \int_{-\infty}^{0} 0dx \tag{0.0.9}$$

$$F_{X_2}(x) = 0 \tag{0.0.10}$$

When $x \geq 0$,

$$F_{X_2}(x) = \int_{0}^{\infty} xe^{-x}dx \tag{0.0.11}$$

$$F_{X_2}(x) = -(x+1)e^{-x} \tag{0.0.12}$$

$$\tag{0.0.13}$$

Since,CDF is never negative

$$F_{X_2}(x) = 1 - (x+1)e^{-x} \tag{0.0.14}$$

Alternately, we have CDF of X_1 and X_2 given by

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \leq x < \infty \end{cases} \tag{0.0.15}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - (x+1)e^{-x} & 0 \leq x < \infty \end{cases} \tag{0.0.16}$$

Thus,

$$\Pr(X_1 \leq X_2) = \int_{-\infty}^{\infty} F_{X_1}(x) f_{X_2}(x) dx \tag{0.0.17}$$

$$= \int_{0}^{\infty} (1 - e^{-x})(xe^{-x}) dx \tag{0.0.18}$$

$$= \frac{3}{4} \tag{0.0.19}$$

$$= 0.75 \tag{0.0.20}$$