Assignment 3

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Download all python codes from

https://github.com/Digjoy12/probability/tree/main/Assignment%203/code

and latex codes from

https://github.com/Digjoy12/probability/blob/main/Assignment%203/main.tex

Problem

(GATE MA - 2016 Q.47) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\Pr(X_1 < X_2)$ is equal to ....

Solution

We know that,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{\lambda x} & 0 \le x < \infty \end{cases} \tag{0.0.1}$$

Given,

$$E(X_1) = \frac{1}{\lambda} = 1 \tag{0.0.2}$$

$$\implies \lambda = 1 \tag{0.0.3}$$

Therefore,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & 0 \le x < \infty \end{cases} \tag{0.0.4}$$

We know that,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^{\alpha-1} e^{\left(\frac{-x}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)} & 0 \le x < \infty \end{cases} \tag{0.0.5}$$

Given,

$$E(X_2) = \alpha\beta = 2 \tag{0.0.6}$$

$$V(X_2) = \alpha\beta^2 = 2 \tag{0.0.7}$$

Solving 0.0.6 and 0.0.7, we get, $\alpha = 2$, $\beta = 1$ and $\Gamma(2) = 1$
Therefore,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ xe^{-x} & 0 \le x < \infty \end{cases} \tag{0.0.8}$$

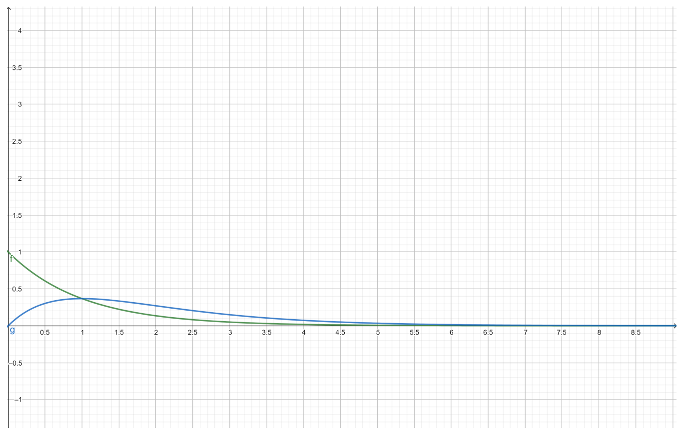

Fig. 0: Probability Distribution of (X_1, X_2)

Calculating the CDF of $f_{X_2}(x)$,

$$F_{X_2}(x) = \int_0^x f_{X_2}(x) \tag{0.0.9}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - \sum_{k=0}^{\alpha-1} \dfrac{\left(\frac{t}{\beta}\right)^k}{k!} e^{\left(\frac{-t}{\beta}\right)} & 0 \le x < \infty \end{cases} \tag{0.0.10}$$

Let $\alpha = n$ and $\lambda = \frac{1}{\beta}$
Alternately, we have CDF of X_1 and X_2 given by

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \le x < \infty \end{cases} \tag{0.0.11}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - \sum_{k=0}^{n-1} \dfrac{(\lambda t)^k}{k!} e^{-\lambda t} & 0 \le x < \infty \end{cases} \tag{0.0.12}$$

Thus,

$$\begin{aligned}
\Pr\left(X_1 \le X_2\right) &= \int_{-\infty}^{\infty} F_{X_1}(x) f_{X_2}(x) dx && (0.0.13)\\
&= \int_{0}^{\infty} (1 - e^{-x})(xe^{-x}) dx && (0.0.14)\\
&= \frac{3}{4} && (0.0.15)\\
&= 0.75 && (0.0.16)
\end{aligned}$$