Global Algorithm :-

→ If value of the function at datapoints is given, or the derivative of the function is given, then one can solve for the i/p's as it is a linear system of eqns.

→ If the curvature or the knots & weights are given, we have a non-linear system of eqns.

↳ for this case perturbation of the data at a point could affect the entire ~~behaviour~~ shape of the surface. Though further away points should have less effect.

Local Algorithm :-

→ Less computationally expensive

→ perturbations to data ~~have very small effects/p~~ only change the surface locally.

Global Algorithm

We are given $\{\underline{Q}_k\}_0^n$ (n+1) points & want to interpolate w/ a ~~polynomi~~ ~~nth~~ p^{th} degree non-rational B-spline curve.

We assign parameter value $\bar{u}_k$ to each $\underline{Q}_k$ & select appropriate knot vector $U = \{u_0, \dots, u_m\}$

$$Q_k = C(\bar{u}_k) = \sum_{i=0}^{n} N_{i,p}(\bar{u}_k) \underbrace{P_i}$$

↳ (n+1) unknown control points.

We now have to determine a method to choose $\{\bar{u}_k\}_0^n$ & U. The options are:-

1. equally spaced:-

$$\bar{u}_0 = 0, \quad \bar{u}_k = k/n \quad k = 1, \ldots, n-1, \quad u_n = 1$$

NOT RECOMMENDED

2. Chord Length:-

$$d = \sum_1^n |Q_k - Q_{k-1}|$$

$$\bar{u}_0 = 0, \quad \bar{u}_n = 1 \quad \& \quad \bar{u}_k = \bar{u}_{k-1} + \frac{|Q_k - Q_{k-1}|}{d}$$

$$k = 1, \ldots, n-1$$

3. Centripetal Method:-

$$d = \sum_{k=1}^n \sqrt{|Q_k - Q_{k-1}|}$$

$$\bar{u}_0 = 0, \quad \bar{u}_n = 1 \quad \& \quad \bar{u}_k = \bar{u}_{k-1} + \frac{\sqrt{|Q_k - Q_{k-1}|}}{d}$$

$$k = 1, \ldots, n-1$$

$\boxed{\text{knots chosen as equally spaced}} \rightarrow$ NOT RECOMMENDED

$$u_0 = \ldots = u_p = 0 \qquad u_{m-p} = \ldots = u_m = 1$$

$$u_{j+p} = j/[n-p+1] \qquad j = 1, \ldots, n-p$$

the above method is not recommended, cause using with centripetal parameterization or chord length parameterization can result in singular system of equations.

$\boxed{\text{Knots can be chosen w/ } \cancel{\text{Moving}} \text{ Average}}$

→ RECOMMENDED!!!

$$u_0 = \dots = u_p = 0 \qquad u_{m-p} = \dots = u_m = 1$$

$$u_{j+p} = \frac{1}{p} \sum_{i=j}^{j+p-1} \bar{u}_i \qquad j = 1, \dots, n-p.$$

the knots reflect the distribution of $\{\bar{u}_k\}_0^n$.

Global Surface Interpolation :-

We are given $(n+1) \times (m+1)$ data points $\{\underline{Q}_{k,l}\}_{k=0,\,l=0}^{k=n,\,l=m}$ & we want to construct a (p,q)th degree B-spline surface interpolating these points,

$$\underline{Q}_{k,l} = \underline{S}(\bar{u}_k, \bar{v}_l) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}(\bar{u}_k) N_{j,q}(\bar{v}_l) \underline{P}_{i,j}$$

the curve fitting the data $\{\underline{Q}_{\cdot,\cdot}\}$

parametric points

$(n+1) \times (m+1)$ control points, we have to solve for.

We can use "Chord length" or "Centripetal" parameterization to get $\bar{u}_0^l, \bar{u}_1^l, \ldots, \bar{u}_n^l$ for each l & then to obtain $\bar{u}_k$ by averaging over l.

$$\bar{u}_k = \frac{1}{m+1} \sum_{l=0}^{m} \bar{u}_k^l \qquad k = 0, \ldots, n$$

$$\bar{u}_0^l = 0, \quad \bar{u}_n^l = 1$$

for each l.

$$d^l = \sum_{k=1}^{n} |Q_{k,l} - Q_{k-1,l}| \quad \text{or} \quad \sum_{k=1}^{n} \sqrt{|Q_{k,l} - Q_{k-1,l}|}$$

$$\bar{u}_k^l = \bar{u}_{k-1}^l + \frac{|Q_{k,l} - Q_{k-1,l}|}{d^l} \quad \text{or} \quad \bar{u}_{k-1}^l + \frac{\sqrt{|Q_{k,l} - Q_{k-1,l}|}}{d^l}$$

the knot vectors are defined as

$$u_0 = \cdots = u_{\cancel{s}} = 0 \qquad u_{\cancel{p}-\cancel{s}} = \cdots = u_{\cancel{p}} = 1$$

$$u_{j+\cancel{s}} = \frac{1}{\cancel{s}} \sum_{i=j}^{j+\cancel{s}-1} \bar{u}_i \qquad \cancel{v_j} \quad j = 1, \ldots, p-s$$

here s & t are the highest derivatives.

$$v_0 = \cdots = v_{\cancel{t}} = 0 \qquad v_{q-t} = \cdots = v_q = 1$$

$$v_{j+t} = \frac{1}{t} \sum_{i=j}^{j+t-1} \bar{v}_i \qquad j = 1, \ldots, q-t$$

Now to the computation of the control points

It would appear that we have a system of $(n+1)\times(m+1)$ equations.

$$\underline{Q}_{k,l} = \sum_{i=0}^{n} N_{i,p}(\bar{u}_k)\left(\sum_{j=0}^{m} N_{j,q}(\bar{v}_l)\underline{P}_{i,j}\right)$$

$$= \sum_{i=0}^{n} N_{i,p}(\bar{u}_k)\,\underline{R}_{i,l} \longrightarrow (*)$$

where $$\underline{R}_{i,l} = \sum_{j=0}^{m} N_{j,q}(\bar{v}_l)\underline{P}_{i,j} \longrightarrow (*1)$$

notice that $(*)$ is curve interpolation through the points $\{\underline{Q}_{k,l}\}_{k=0}^{n}$.

The $\underline{R}_{i,l}$ are control points for the isoparametric curve $S(u,\bar{v}_l)$.

Fixing 'i' & letting 'l' vary, $(*1)$ is curve interpolation through points $\underline{R}_{i,0}, \ldots, \underline{R}_{i,m}$ with $\underline{P}_{i,0}, \ldots, \underline{P}_{i,m}$ as the computed control points.

Thus the Algorithm to obtain the control points $\{P_{i,j}\}_{0,0}^{n,m}$ are

1. Using the knot vector (U) & the ~~con~~ parametric points $\bar{u}_k$, do '$m+1$' curve interpolations through $Q_{0,l}, \ldots, Q_{n,l}$ for $l = 0, \ldots, m$ this yields the control points $\{R_{i,l}\}$

2. Using the knot vector (V) & the parametric points $\bar{v}_l$, do '$n+1$' curve interpolations through $R_{i,0}, \ldots, R_{i,m}$ (for $i = 0, \ldots, n$); this will yield the final control points $\{P_{i,j}\}_{0,0}^{n,m}$

The Local methods have been ignored since they seem a bit cumbersome.