Finite Mixture Model: Gaussian

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RECAP

SESSION III:

- Representing knowledge through graphical models
- Mixture Model:
  - Introduction
  - Multinomial Mixture Model
    - Known parameters
    - Unknown parameters
      - Posterior
      - Fully Collapsed

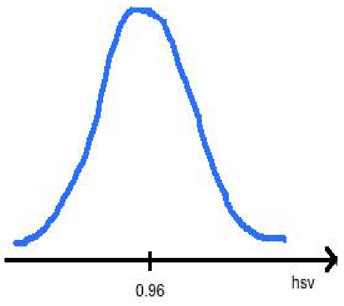
OUTLINE:

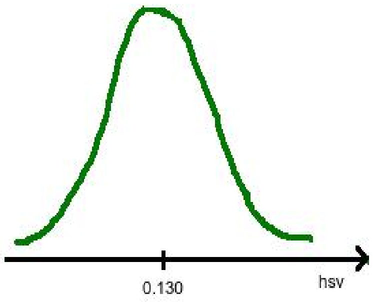
- Gaussian Mixture Model
    - Introduction
    - Known parameters
    - Unknown parameters
        - MLE
        - Posterior
        - Fully Collapsed

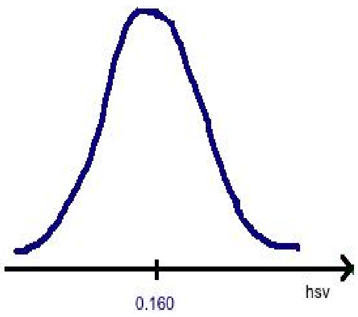
What is the picture representing (sky, river or forest)?

Picture from: Unsplash

HSV color distribution

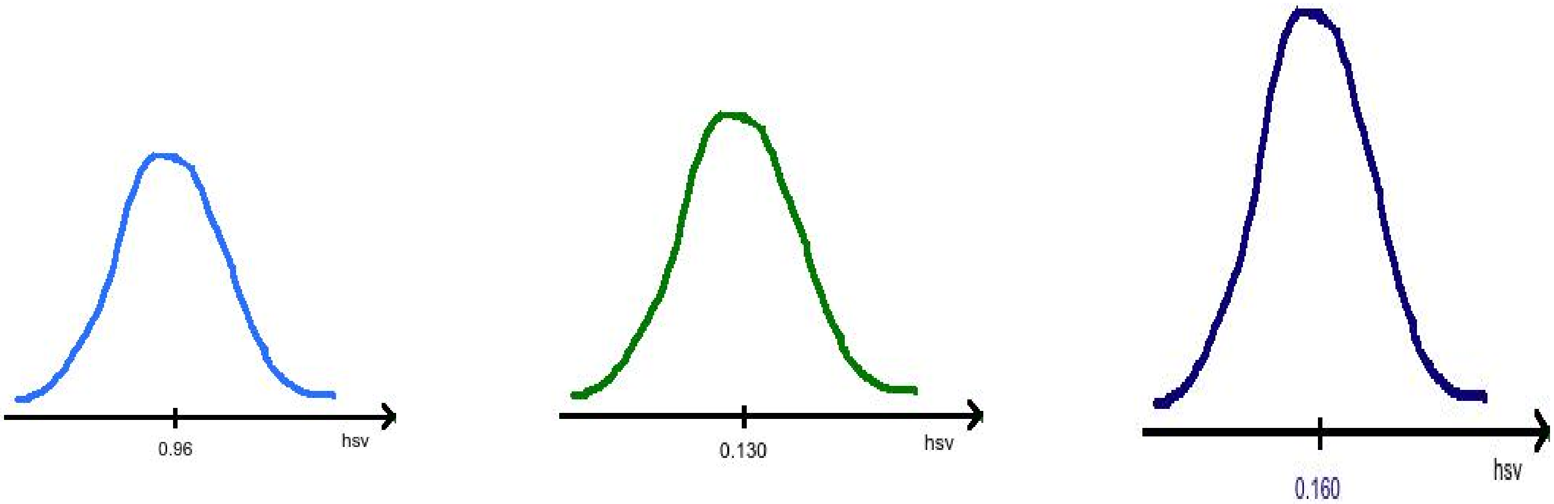






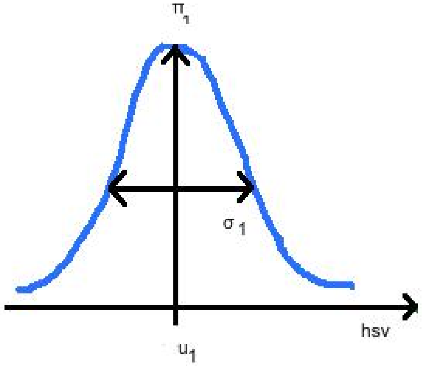
Weight assignment

$\pi_k = [0.2, 0.3, 0.5]$

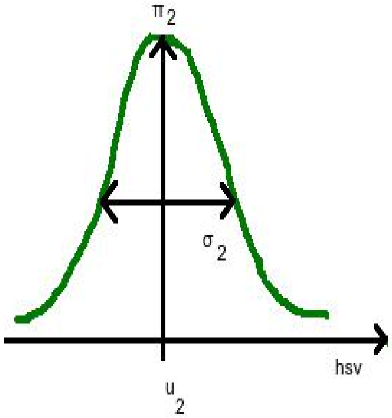


- Each value of π_k lies between 0 and 1.
- The sum of all the values of π_k must be equal to 1.

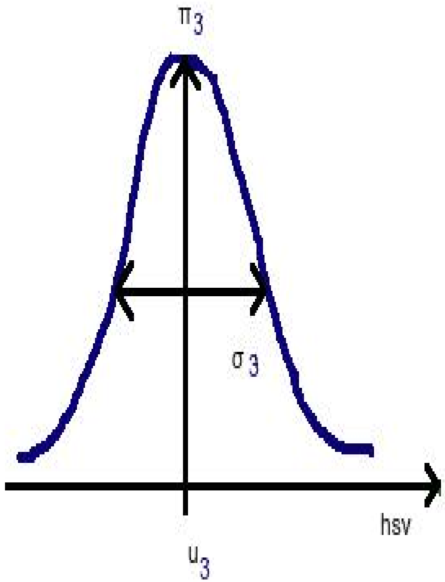
Weight assignment:



N(x| μ_1, σ_1)



N(x| μ_2, σ_2)



N(x| μ_3, σ_3)

Univariate Gaussian Mixture Models

Combining all the distributions, we get:

$$p(x|\theta) = \pi_1 N(x|\,\mu_1, \sigma_1) + \pi_2 N(x|\,\mu_2, \sigma_2) + \pi_3 N(x|\,\mu_3, \sigma_3)$$

Here, $\theta = \{\pi_k,\ \mu_k,\ \sigma_k\}$ are the parameters.

Therefore, $p(x|\theta) = \sum_{k=1}^{K} \pi_k N(x|\,\mu_k,\ \sigma_k)$

Code

Gaussian Mixture Distribution

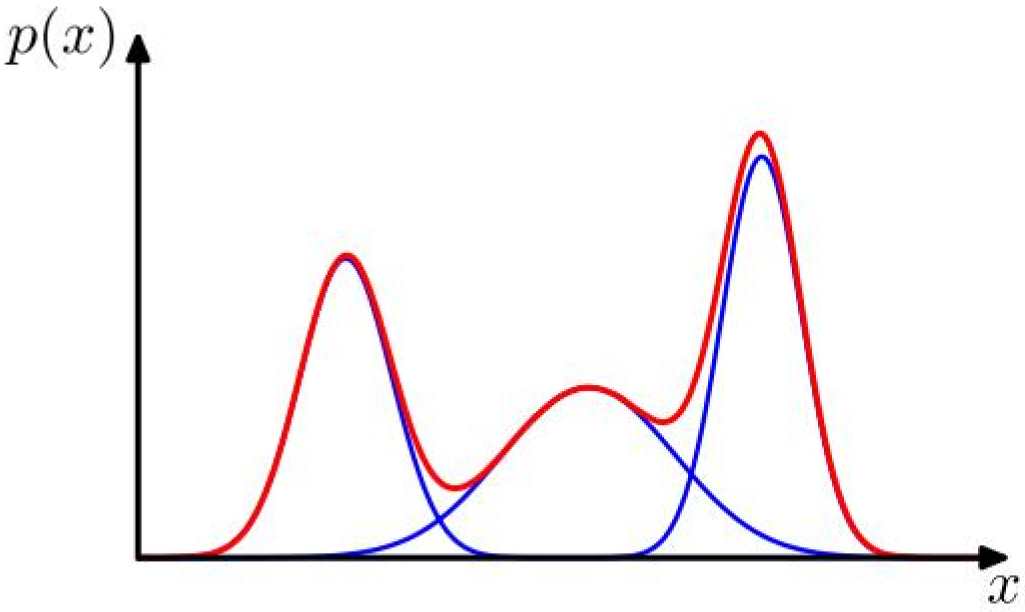


Figure 8: Illustration of Gaussian mixture distribution in one dimension showing three Gaussians (each scaled by a coefficient) in blue and their sum in red

Bishop's PRML (Mixture of Gaussians)

Gaussian mixture models (2D)

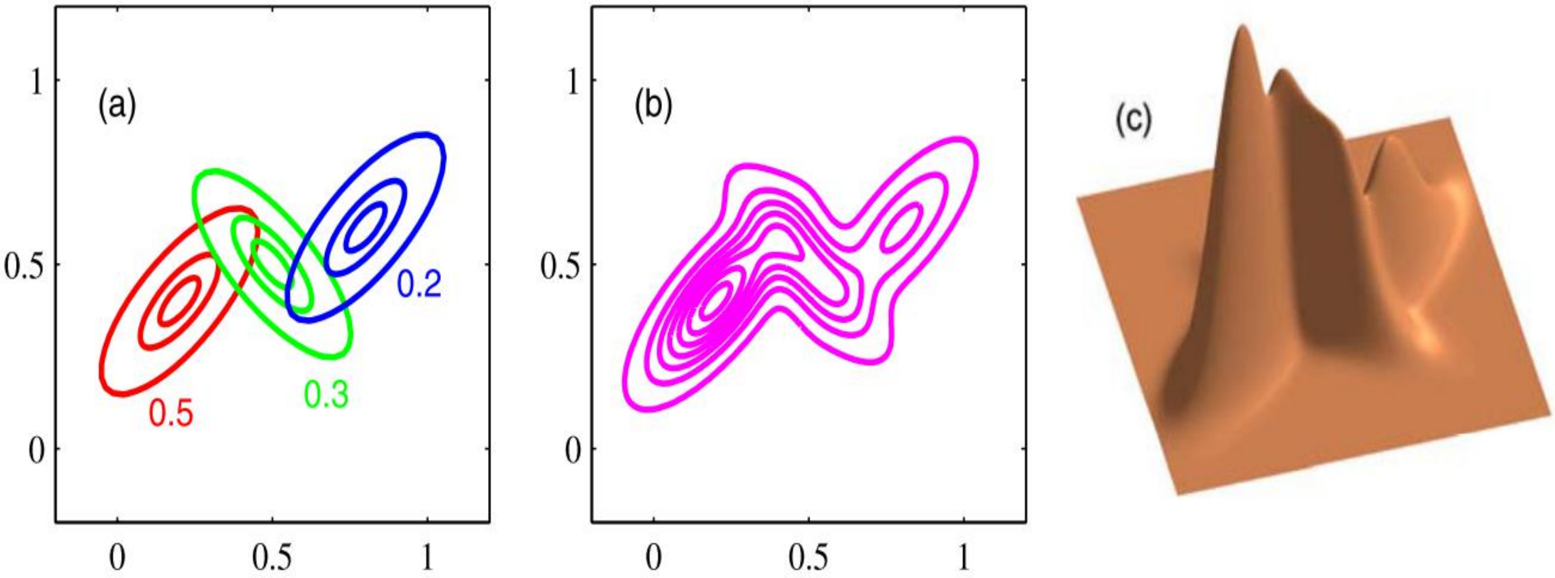


Figure 9: Illustration of a mixture of 3 Gaussians in a two-dimensional space. (a) Contours of constant density for each of the mixture components, in which the 3 components are denoted red, blue and green, and the values of the mixing coefficients are shown below each component. (b) Contours of the marginal probability density p(x) of the mixture distribution. (c) A surface plot of the distribution p(x).

Bishop's PRML (Mixture of Gaussians)

Problem Statement:

Suppose there are three normally distributed mixture components with the following parameters.

```
params = {0: {π1: 0.3, μ1:12, σ1:2},
       1: {π2: 0.2, μ2:0, σ2:1},
       2: {π3: 0.5, μ3:20 , σ3:2}}
```

Cases

1. We know the value of ‘μ’ and ‘σ’.
2. We do not know the value of ‘μ’ and ‘σ’.

Case 1 (‘θ’ known)

- The task is to determine which mixture component was used for generating certain data x_i.

Algorithm:

1. Randomly assign values to cluster, i.e, $z_i = k$ *with uniform probability.*

2. For each i: i) Remove data point (x_i, z_i)

ii) Compute normal pdf , $p_k(x_i|\theta_k)$*, for each cluster k.*

$$p_k(x_i|u_k, \sigma_k) = \frac{1}{\sigma_k\sqrt{2\pi}} e^{-\frac{(x_i-\mu_k)^2}{2\sigma_k{}^2}}$$

iii) Count the number of data points, c_k*, in class k, given by:*

$$c_k = |\{z_i = k| z_i \in Z\}|$$

iv) Estimate Gaussian parameters $\theta_k = (\mu_k, \sigma_k)$ *for each cluster using MLE:*

$$\mu_k = \frac{1}{c_k}\sum_{x\in X_k} x \text{ , where } |X_k| = c_k$$

$$\sigma_k^2 = \frac{1}{c_k}\sum_{x\in X_k} (x-\mu_k)^2$$

Alternatively, use the posterior distribution to sample $\theta, i.e,$

$$p(\theta_k|X, Z, \beta) = p(\theta_k|\{x_i|z_i = k\epsilon Z\}, \beta_k)$$

v) Use dirichlet categorical conditional distribution to compute the probability of observing class k having already observed counts as in equation (1),

$$p(z = k|c, \alpha) = \frac{c_k + \alpha_k}{C+A-1}$$

vi) Add data point (x_i, z_i) back by sampling z_i = k, using,

$$z_i = k \sim p(z = k|c, \alpha)p_k(x_i|\theta_k)$$

where, $p_k(x_i|\theta_k)$ and $p(z = k|c, \alpha)$ are obtained from equation (ii) and (iv) respectively.

Mixture proportions can be estimated at the end.

Case 2 (‘θ’ unknown)(Finding parameters with MLE)

- The task is to determine which mixture component was used for generating certain data x_i.

Algorithm:

1. Randomly assign values to cluster, i.e, $z_i = k$ *with uniform probability.*

2. For each i: i) Remove data point (x_i, z_i)

ii) Count the number of data points, c_k*, in class k, given by:*

$$c_k = |\{z_i \; = \; k| \; z_i \; \in \; Z\}|$$

iii) Compute normal pdf , $p_k(x_i|\theta_k)$, for each cluster k.

$$p_k(x_i|u_k,\, \sigma_k) \;=\; \frac{1}{\sigma_k\sqrt{2\pi}}\; e^{-\frac{(x_i-\mu_k)^2}{2\sigma_k{}^2}}$$

iv) Count the number of data points, c_k, in class k, given by:

$$c_k = |\{z_i \;=\; k|\; z_i \;\in\; Z\}|$$

v) Use Dirichlet categorical conditional distribution to compute the probability of observing class k having already observed counts as in equation (1).

$$p(z = k|c, \alpha) \;=\; \frac{c_k + \alpha_k}{C+A-1}$$

vi) Add data point (x_i, z_i) back by sampling z_i =k, using,

$$z_i = k \sim p(z = k|c, \alpha) p_k(x_i|\theta_k)$$

where, $p_k(x_i|\theta_k)$ and $p(z = k|c, \alpha)$ are obtained from equation (iii) and (v) respectively.

Mixture proportions can be estimated at the end.

Case 2 (‘θ’ unknown)(Finding parameters with posterior Normal Gamma Distribution)

- The task is to determine which mixture component was used for generating certain data x_i.

Algorithm:

1. Randomly assign values to cluster, i.e, $z_i = k$ *with uniform probability.*

2. For each i: i) Remove data point $(x_i,\ z_i)$

ii) Count the number of data points, c_k, in class k, given by:

$$c_k = |\{z_i \ = \ k|\ z_i \ \in \ Z\}|$$

iii) Sample 'μ' and 'σ' with the posterior Normal Gamma Distribution:

$$\mu_{zk}, \sigma_{zk} \sim \mathit{NormalGamma}(\mu, \sigma \mid M, C, A, B)$$

$$M = \frac{c_m + \sum_{i=1}^{n} x_i}{c+n}, \; C = c+n, \; A = a+(n/2),$$

$$B = b+(1/2) * (cm^2 - CM^2 + \sum_{i}^{n} x_i^2)$$

Here, M, C, A, B are the parameters of Normal Gamma and n is the total datapoints in the cluster

iv) Compute normal pdf , $p_k(x_i|\theta_k)$, for each cluster k.

$$p_k(x_i|u_k, \sigma_k) = \frac{1}{\sigma_k\sqrt{2\pi}} \, e^{-\frac{(x_i-\mu_k)^2}{2\sigma_k^2}}$$

v) Use Dirichlet categorical conditional distribution to compute the probability of observing class k having already observed counts as given in equation (2).

$$p(z = k|c, \alpha) = \frac{c_k + \alpha_k}{C+A-1}$$

vi) Add data point (x_i, z_i) *back by sampling* z_i *=k, using,*

$$z_i = k \sim p(z = k|c, \alpha) p_k(x_i|\theta_k)$$

where, $p_k(x_i|\theta_k)$ *and* $p(z = k|c, \alpha)$ *are obtained from equation (iii) and (v) respectively.*

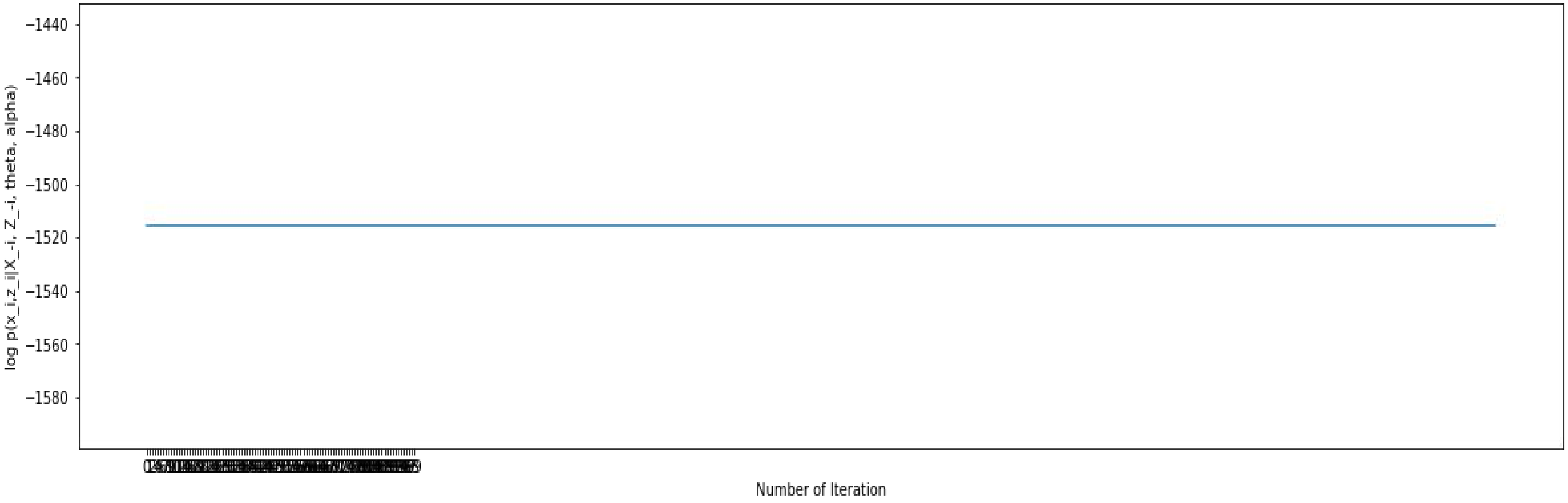
Mixture proportions can be estimated at the end.

- Let us consider the 3 mixture components with the following parameters:

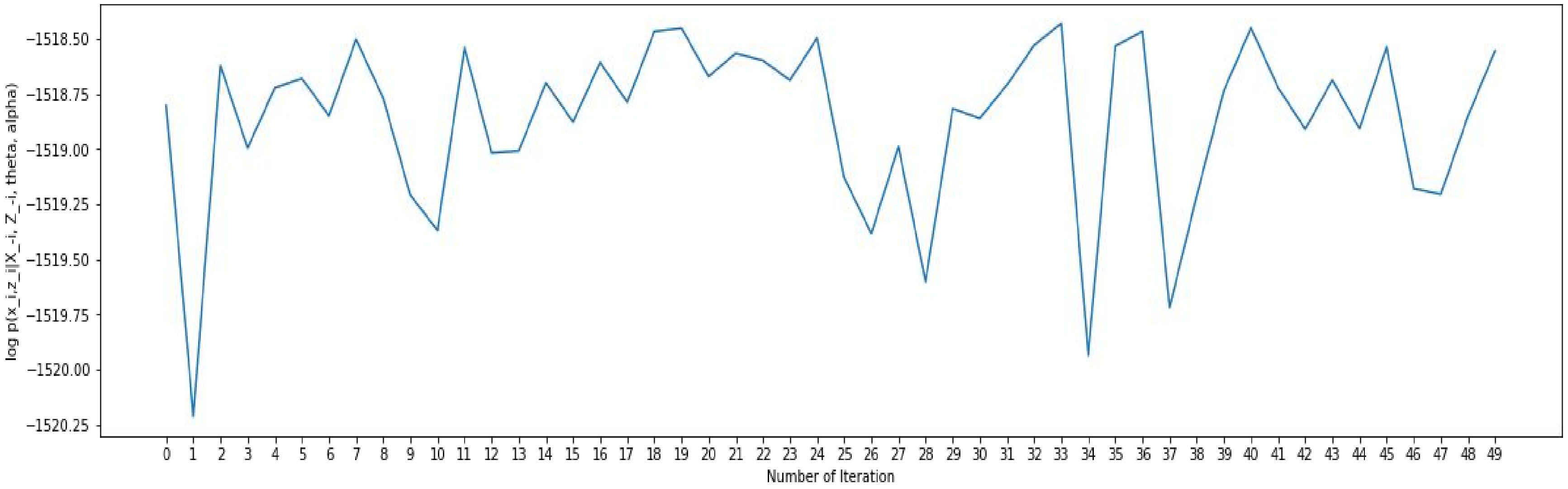
```
params = {
            0: {π1: 0.3, μ1:12, σ1: 2},
            1: {π2: 0.2, μ2:0, σ2: 1},
            2: {π3: 0.5, μ3:20, σ3: 2}
}
```

After running the above algorithms in both Case 1 and Case 2 for 50 iterations, the convergence graph looked as:

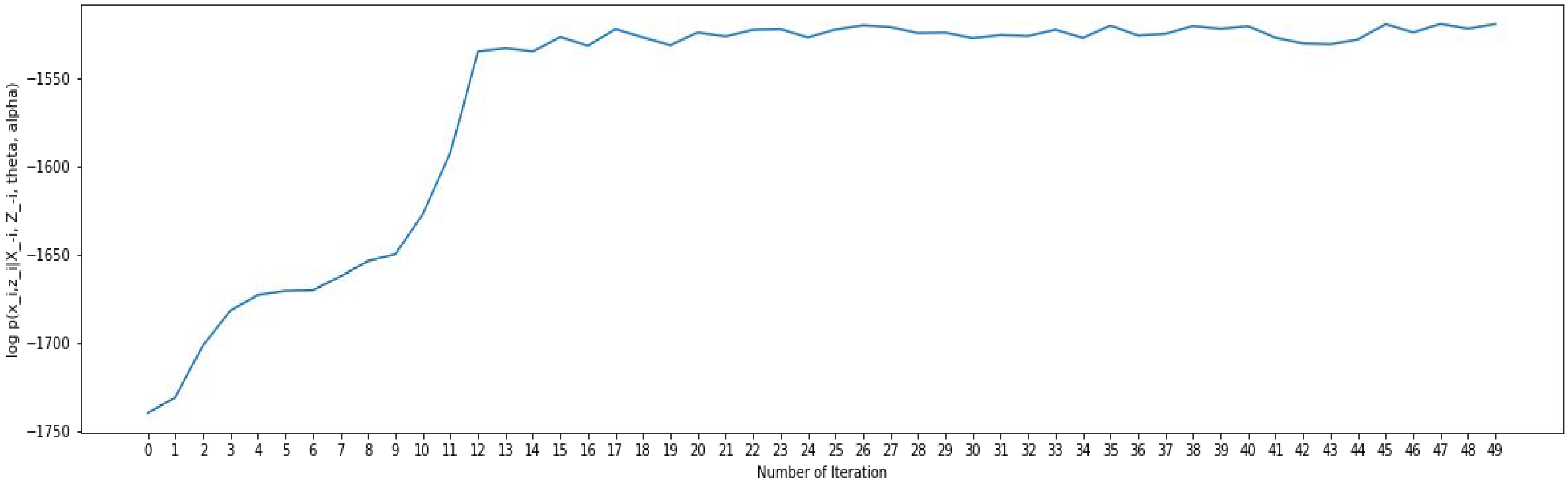
1. Known parameters (after 50 iterations)



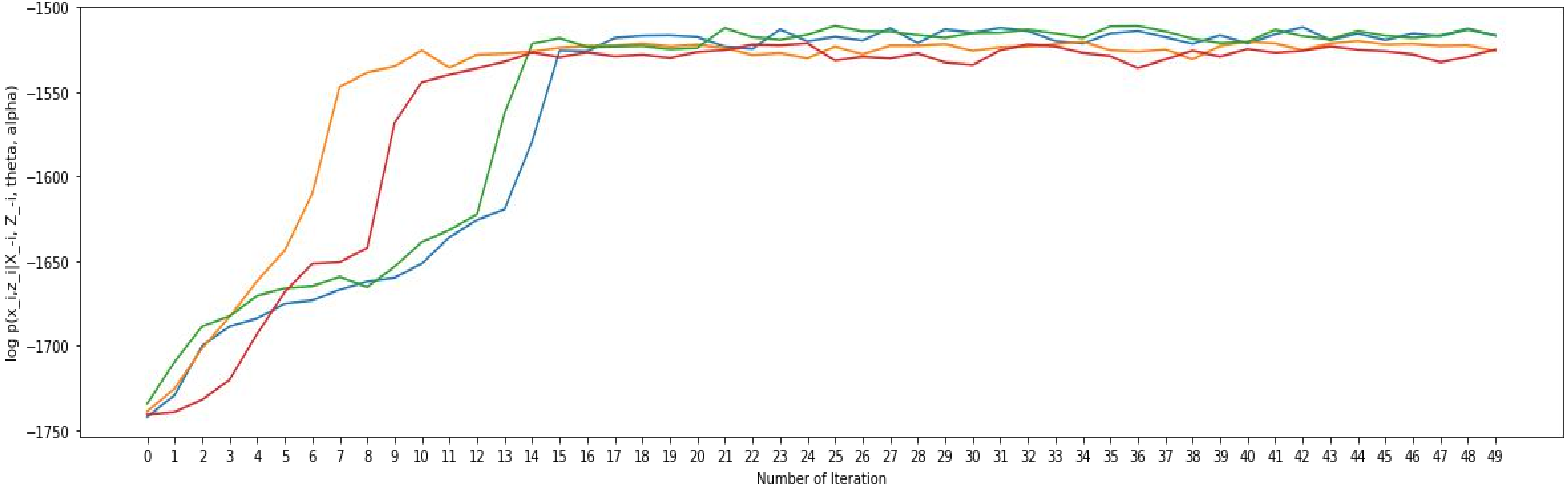
2. Finding parameters with MLE (after 50 iterations)



3. Finding parameters with normal gamma (after 50 iterations)



3. Finding parameters with normal gamma (after 50 iterations) [Multiple dataset]



Sample code:

- Finite mixtures with gaussian prior: (Known and unknown parameters) (Unknown 'μ' and 'σ' are found by taking average and finding variance respectively.)

  https://colab.research.google.com/drive/17xHqCrkIyWhAZ-1_tK_9IEotoZRtfmJh

- Finite mixtures with gaussian prior: (Unknown parameters are found using samples from normal gamma)

  https://colab.research.google.com/drive/1iStXsgTKQfrHTn9kfAIKGfLJz1ZCneSL

Fully Collapsed Gibbs Sampling

- **Gaussian Mixture Model**

The within class gaussian parameters μ_k, λ_k can be integrated out using normal gamma distribution.

Thus, instead of

$$p(x_i|\ \mu_i,\ \lambda_i)$$

we need to compute

$$\frac{p(X_k \mid m,\ c,\ a,\ b)}{p(X_{k,-i}|\ m,\ c,\ a,\ b)}$$

Fully Collapsed Gibbs Sampling

- **Gaussian Mixture Model**

Using NormalGamma as conjugate prior for Normal distribution

$$\frac{p(X_k \mid m, c, a, b)}{p(X_{k,-i} \mid m, c, a, b)} = \frac{\frac{1}{\frac{B'^{A}}{\tau(A')}\sqrt{\frac{C'}{2\pi}}}}{\frac{1}{\frac{B^{A}}{\tau(A)}\sqrt{\frac{C}{2\pi}}}} = \sqrt{\frac{C}{C'}}\,\frac{B^{A}}{B'^{A'}}\,\frac{\tau(A')}{\tau(A)}$$

Here, $\frac{\tau(A')}{\tau(A)} = \frac{\tau(a+\frac{n+1}{2})}{\tau(a+\frac{n}{2})}$

$$= \frac{\tau(A+\frac{1}{2})}{\tau(A)}$$

Fully Collapsed Gibbs Sampling

- **Gaussian Mixture Model**

Multiplying both sides of equation (3) by $\tau(1/2)$,

$$\frac{\tau(A')}{\tau(A)} = \frac{\tau(a+\frac{n+1}{2})}{\tau(a+\frac{n}{2})} * \frac{\tau(1/2)}{\tau(1/2)}$$

As Beta(x, y) = $\frac{\tau(x)\tau(y)}{\tau(x+y)}$, we get,

$$\frac{\tau(A')}{\tau(A)} = \frac{\tau(1/2)}{Beta(x,y)}$$

Fully Collapsed Sampler

- **Algorithm: Gaussian Mixture Model**

1. Randomly assign values to cluster, i.e, $z_i = k$ with uniform probability.

2. For each cluster k, compute $\sqrt{\frac{C}{C'}} \frac{B^A}{B'^{A'}} \frac{\tau(1/2)}{Beta(x,y)}$

3. Add data point (x_i, z_i) back by sampling $z_i = k$, using:

$$z_i = k \sim \frac{c_k + A/k}{N+A-1} \left(\sqrt{\frac{C}{C'}} \frac{B^A}{B'^{A'}} \frac{\tau(1/2)}{Beta(x,y)}\right)$$

Mixture proportions can be estimated at the end.

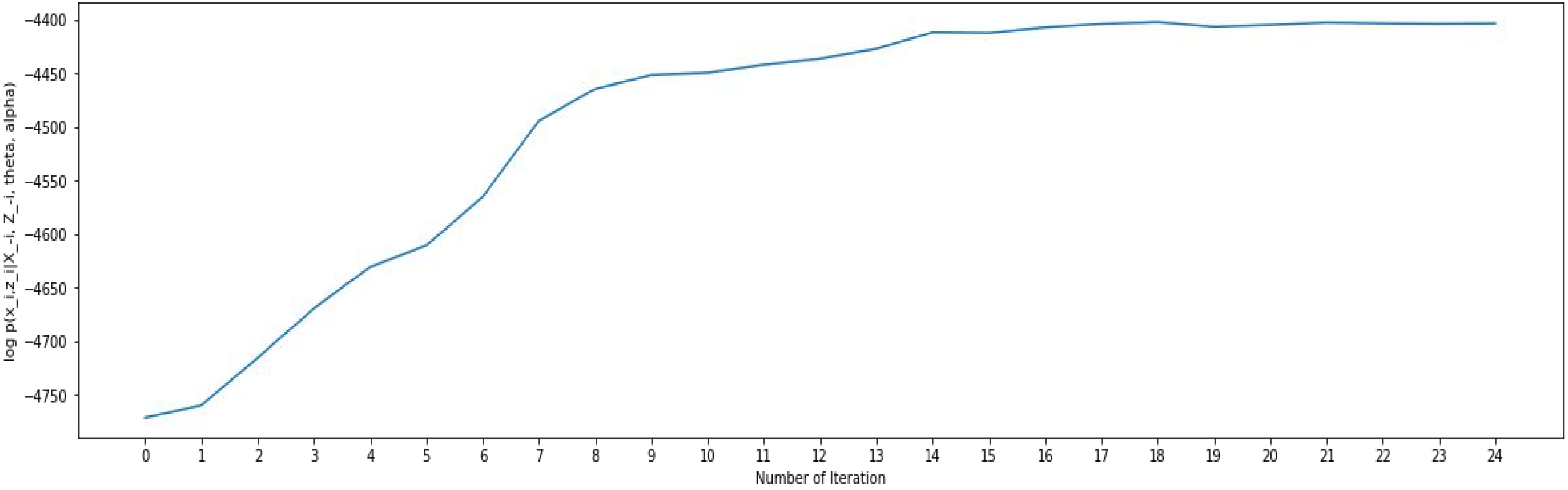
- Suppose we have three mixture components with the following parameters:

```
param = {0: {'pi': 0.3, 'mu':12, 'sigma':2},
         1: {'pi': 0.2, 'mu':5, 'sigma':5},
         2: {'pi': 0.5, 'mu':20 , 'sigma':2}
}

total_iter = 1000
```

Fully collapsed sampler (after 50 iterations)

(As algorithm converge at 24 iterations, iterations after 24 are skipped)



Sample code

- Fully collapsed sampler (Gaussian Mixture Model):

https://colab.research.google.com/drive/1Kcp0ZrL-XCpOAGFVoseObFTe6QpX48dq#scrollTo=cs653ksvm9Gz

Takeaways:

- We learned about Gaussian Mixture Model.

- We learned the way to model gaussian mixture model using gibbs sampling in the cases when parameters are known and unknown.

References:

- Suresh Manandhar. *Bayesian ML : Posterior Distributions and Mixture Models Continuous Probability Density Function*
- Christopher Bishop. *Pattern Recognition and Machine Learning*

Thank you