Empirical Methods in Finance
Homework 3: Solution

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Problem 1: ARMA basics

Consider the ARMA(1,1):

$$y_t = 0.95 \times y_{t-1} - 0.9 \times \varepsilon_{t-1} + \varepsilon_t, \tag{1}$$

where ε_t is i.i.d. Normal with mean zero and variance $\sigma^2 = 0.05^2$. In the below you need to show your work in order to get full credit.

1. What is the first-order autocorrelation of y_t?
2. What is the second-order autocorrelation of y_t? Also, what is the ratio of the second-order to first-order autocorrelation equal to? Give some intuition for this result.
3. If $y_t = 0.6$ and $\varepsilon_t = 0.1$, what is (i) $E_t[y_{t+1}]$, (ii) $E_t[y_{t+2}]$ given the ARMA model?
4. Let $\hat{x}_t = E_t[y_{t+1}]$ where the expectation is taken using the ARMA model. What is the unconditional mean, standard deviation, and first-order auto-correlation of $\hat{x}_t$?
   Suggested Solution:
   1 and 2.

$$V(y_t) = 0.95^2 V(y_{t-1}) + 0.9^2\sigma^2 + \sigma^2 - 2 \times 0.95 \times 0.9\sigma^2$$
$$V(y_t) = 2.56e-3$$
$$C(y_t, y_{t-1}) = C(0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t, y_{t-1})$$
$$= 0.95V(y_{t-1}) - 0.9\sigma^2$$
$$\rho_1 = \frac{C(y_t, y_{t-1})}{V(y_t)} = 0.071$$

$$C(y_t, y_{t-2}) = C(0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t, y_{t-2})$$
$$= 0.95C(y_{t-1}, y_{t-2})$$
$$\rho_2 = \frac{C(y_t, y_{t-2})}{V(y_t)} = 0.068$$

If we calculate the ratio of the second-order to first-order auto-correlation, the ratio is equal to 0.95, which is exactly the coefficient of y_{t-1}. For a ARMA(1,1),the auto-correlation function decays like the corresponding AR(1) model.

3.

$$E_t(y_{t+1}) = 0.95y_t - 0.9\epsilon_t = 0.48$$
$$E_t(y_{t+2}) = 0.95E_t(y_{t+1}) = 0.46$$

4.

$$\hat{x}_t = E_t\left[y_{t+1}\right]$$
$$\hat{x}_t = 0.95y_t - 0.9\epsilon_t$$
$$E(\hat{x}_t) = 0.95E(y_t) - 0.9E(\epsilon_t) = 0$$
$$V(\hat{x}_t) = 0.95^2V(y_t) + 0.9^2V(\epsilon_t) - 2 \times 0.95 \times 0.9\sigma^2 = 6.41e - 05$$
$$std(\hat{x}_t) = 8.00e - 3$$
$$\rho_1(\hat{x}_t) = \frac{C(\hat{x}_t, \hat{x_{t-1}})}{V(\hat{x}_t)} = \frac{C(0.95y_t - 0.9\epsilon_t, 0.95y_{t-1} - 0.9\epsilon_{t-1})}{V(\hat{x}_t)}$$
$$= \frac{0.95^2C(y_t, y_{t-1}) - 0.95 \times 0.9C(y_t, \epsilon_{t-1})}{V(\hat{x}_t)} = 0.95$$

The final answer might be a little different if you put the numbers into the equations at different steps of algebra, but it is better to use the number at the final step.

Problem 2: Year-on-year quarterly data and ARMA dynamics

Assume the true quarterly log market earnings follow:

$$e_t = e_{t-1} + x_t,$$
$$x_t = \phi\, x_{t-1} + \varepsilon_t,$$

where $\mathbb{V}\left(\varepsilon_t\right) = \sigma_\varepsilon^2 = 1$ and ε_t is i.i.d. over time t.

The earnings data you are given is year-on-year earnings growth, with in logs is:

$$y_t \equiv e_t - e_{t-4}.$$

1. Assume $\phi = 0$. Derive auto-covariances of order 0 through 5 for y_t.
2. Assume $\phi = 0$. Determine the number of AR lags and MA lags you need in the ARMA(p, q) process for y_t. Give the associated AR and MA coefficients.
3. Optional: Assume $0 < \phi < 1$. Repeat 1 and 2 under this assumption.

   Suggested Solution: In the general case of $0 < \phi < 1$, we have:

$$\mathbb{V}(x_t) = \phi^2 \mathbb{V}(x_t) + \sigma_\varepsilon^2$$
$$\mathbb{V}(x_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2}$$

and y_t can be written as

$$\begin{aligned} y_t &= e_t - e_{t-4} \\ &= x_{t-3} + x_{t-2} + x_{t-1} + x_t \end{aligned}$$

Recalling that the autocovariances of the AR(1) process are given by

$$\mathbb{C}(x_t, x_{t-j}) = \frac{\phi^j}{1-\phi^2}\sigma_\varepsilon^2$$

We can now calculate the autocovariances:

$$\begin{aligned} \mathbb{C}(y_t, y_t) &= \sum_{i=0}^{3}\sum_{\ell=0}^{3} \mathbb{C}(x_{t-i}, x_{t-\ell}) \\ &= \left(4\frac{1}{1-\phi^2} + 6\frac{\phi}{1-\phi^2} + 4\frac{\phi^2}{1-\phi^2} + 2\frac{\phi^3}{1-\phi^2}\right)\sigma_\varepsilon^2 \\ &= 2 \cdot \frac{2+\phi+\phi^2}{1-\phi}\sigma_\varepsilon^2 \\ \mathbb{C}(y_t, y_{t-1}) &= \sum_{i=0}^{3}\sum_{\ell=1}^{4} \mathbb{C}(x_{t-i}, x_{t-\ell}) \\ &= \frac{(1+\phi)(3+\phi^2)}{1-\phi}\sigma_\varepsilon^2 \\ \mathbb{C}(y_t, y_{t-2}) &= \sum_{i=0}^{3}\sum_{\ell=2}^{5} \mathbb{C}(x_{t-i}, x_{t-\ell}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2 + 2\phi + 2\phi^2 + \phi^3 + \phi^4}{1 - \phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-3}\right) &= \sum_{i=0}^{3}\sum_{\ell=3}^{6}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\left(1+\phi\right)\left(1+\phi^2\right)^2}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-4}\right) &= \sum_{i=0}^{3}\sum_{\ell=4}^{7}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\phi\left(1+\phi\right)\left(1+\phi^2\right)^2}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-5}\right) &= \sum_{i=0}^{3}\sum_{\ell=5}^{8}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\phi^2\left(1+\phi\right)\left(1+\phi^2\right)^2}{1-\phi}\sigma_\varepsilon^2
\end{aligned}
$$

From the above, we see that for $j > 3$, the autocovariances behave like those of an AR(1) process with parameter ϕ. Thus, we know that the ARMA representation is of the form

$$\left(1 - \phi\ B\right) y_t = \theta\left(B\right)\varepsilon_t$$

Expanding the left hand side, we have

$$
\begin{aligned}
\left(1 - \phi\ B\right) y_t &= x_t + \left(1-\phi\right)x_{t-1} + \left(1-\phi\right)x_{t-2} + \left(1-\phi\right)x_{t-3} - \phi x_{t-4} \\
&= \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}
\end{aligned}
$$

Thus, we see that the process is captured by an ARMA(1, 3) with coefficients

$$
\begin{aligned}
\phi\left(B\right) &= 1 - \phi\ B \\
\theta\left(B\right) &= 1 + B + B^2 + B^3
\end{aligned}
$$

Plugging in $\phi = 0$ and $\sigma_\varepsilon^2 = 1$ give the results in the simplified case.

Although not required, it is helpful to note the Wold Decomposition of this process is given by

$$y_t = \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}$$

$$\psi_j = \begin{cases} \sum_{\ell=0}^{j} \phi^j & j \leq 3 \\ \phi\, \psi_{j-1} & j > 3 \end{cases}$$

which allows us to verify the autocovariances using the relation

$$\gamma(h) = \sum_{j=0}^{\infty} \psi_{j+|h|} \psi_j$$

Problem 3: Market-timing and Sharpe ratios

Assume you have an estimate of expected annual excess market returns for each time t, called x_t. You estimate the regression

$$R^e_{t+1} = \alpha + \beta x_t + \epsilon_{t+1} \tag{2}$$

and obtain $\hat{\alpha} = 0$, $\hat{\beta} = 1$, and $\sigma(\hat{\epsilon}_{t+1}) = 15\%$. Further, the sample mean and standard deviation of x_t are both 5%.

1. Calculate the standard deviation of excess returns based on the information given.

   Suggested Solution:

$$\begin{aligned} Var(R^e_{t+1}) &= \beta^2 Var(x_t) + Var(\epsilon_{t+1}) \\ &= 0.05^2 + 0.15^2 \\ &= 0.025 \end{aligned}$$

   Thus, the standard deviation would be 15.81%.

2. Calculate the R^2 of the regression based on the information given.

   Suggested Solution:

$$R^2 = \frac{\beta^2\, Var(x_t)}{Var\left(R^e_{t+1}\right)} = 0.1$$

3. Calculate the sample Sharpe ratio of excess market returns based on the information given.

Suggested Solution:

$$
\begin{aligned}
SR &= \frac{\mathbb{E}(R^e_{t+1})}{\sigma(R^e_{t+1})} \\
&= \frac{0.05}{0.1581} = 0.3163
\end{aligned}
$$

4. Recall from investments that a myopic investors chooses a fraction of wealth

$$
\alpha_t = \frac{\mathbb{E}_t(R^e_{t+1})}{\gamma \sigma^2_t (R^e_{t+1})}
$$

in the risky asset (the market) at each time t, where we assume the risk aversion coefficient, γ, equals 40/9. Further, assume that the residuals ϵ_{t+1} are i.i.d., so $\sigma_t(\epsilon_{t+1}) = 15\%$ for all t. Given this, calculate the weight the investor chooses to hold in the risky asset if $x_t = 0$ and if $x_t = 10\%$. What is conditional Sharpe ratio in each of these cases?

Suggested Solution: The weight the investor chooses to hold in the risky asset is

$$
\alpha_t = \begin{cases} 0 & \text{if } x_t = 0 \\ \\ \dfrac{0.1}{\frac{40}{9} \times 0.15^2} = 1 & \text{if } x_t = 0.1 \end{cases},
$$

The conditional Sharpe ratio is

$$
SR = \begin{cases} 0 & \text{if } x_t = 0 \\ \\ \dfrac{0.1}{0.15} = 0.667 & \text{if } x_t = 0.1 \end{cases}
$$

5. Assume T is large (i.e., $T \to \infty$) and that x_t is either 0% or 10% at each time t, with equal probability (0.5).

   (a) What is the unconditional average excess return for an investor that holds α_t each period?

   Suggested Solution:

$$
\mathbb{E}(\alpha_t R^e_{t+1}) = 0.5 \times 10\% = 5\%
$$

(b) What is the unconditional standard deviation? The following may be helpful for calculating the unconditional variance. You could also simulate a very long series to check your math.

Suggested Solution: Using the law of iterated expectation, we can write

$$\begin{aligned}
Var(\alpha_t R^e_{t+1}) &= \mathbb{E}\left[\mathbb{E}_t\left[(\alpha_t R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\mathbb{E}_t\left[\alpha_t R^e_{t+1}\right]\right]^2 \\
&= \mathbb{E}\left[\alpha_t^2 \mathbb{E}_t\left[(R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\alpha_t \mathbb{E}_t\left[R^e_{t+1}\right]\right]^2 \\
&= \mathbb{E}\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - \mathbb{E}\left[\alpha_t x_t\right]^2 \\
&= \frac{1}{2}\left(0.1^2 + 0.15^2\right) - \left(\frac{1}{2} \times 0.1\right)^2 \\
&= 0.01375
\end{aligned}$$

(c) Finally, what is the unconditional Sharpe ratio of this strategy?

Suggested Solution:

$$SR = \frac{0.05}{\sqrt{0.01375}} = 0.4264$$

The unconditional Sharpe Ratio considering market timing is about 10% higher than the Sharpe Ratio of the simple buy and hold case.

(d) Now, assume the volatility of x_t is higher; x_t it can take the values -5% and $+15\%$ with equal probability.

i. What is the implied R^2 of a forecasting regression of future excess returns on x_t assuming again that $\hat{\alpha} = 0$ and $\hat{\beta} = 1$?

Suggested Solution: The unconditional mean of x_t is $\mathbb{E}[x_t] = \frac{1}{2}(-5\% + 15\%) = 5\%$. The unconditional variance of x_t is $Var(x_t) = E(x^2) - Ex^2 = 0.01$.

$$\begin{aligned}
Var(R^e_{t+1}) &= \beta^2 Var(x_t) + Var(\epsilon_{t+1}) \\
&= 0.01 + 0.15^2 \\
&= 0.0325
\end{aligned}$$

So the R^2 of the forecasting regression is three times higher:

$$R^2 = \frac{\beta^2 Var(x_t)}{Var\left(R^e_{t+1}\right)} = 0.3077$$

ii. What is the unconditional Sharpe ratio the investor that follows the risky asset share rule given above in (4)? Note that a higher R^2 implies a higher Sharpe ratio.

Suggested Solution: The weight the investor chooses to hold in the risky asset

$$\alpha_t = \begin{cases} \dfrac{-0.05}{\frac{40}{9} \times 0.15^2} = -0.5 & \text{if } x_t = -5\% \\ \\ \dfrac{0.15}{\frac{40}{9} \times 0.15^2} = 1.5 & \text{if } x_t = 15\% \end{cases}$$

The unconditional average excess return for an investor that holds α_t each period is

$$\mathbb{E}(\alpha_t R^e_{t+1}) = \frac{1}{2} \left(-0.5 \times -5\% + 1.5 \times 15\%\right) = 12.5\%$$

The unconditional standard deviation is

$$\begin{aligned} Var(\alpha_t R^e_{t+1}) &= \mathbb{E}\left[\mathbb{E}_t\left[(\alpha_t R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\mathbb{E}_t\left[\alpha_t R^e_{t+1}\right]\right]^2 \\ &= \mathbb{E}\left[\alpha_t^2 \mathbb{E}_t\left[(R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\alpha_t \mathbb{E}_t\left[R^e_{t+1}\right]\right]^2 \\ &= \mathbb{E}\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - \mathbb{E}\left[\alpha_t x_t\right]^2 \\ &= \frac{1}{2}\left[(-0.5 \times -0.05)^2 + (1.5 \times 0.15)^2 + (-0.5^2 + 1.5^2) \times 0.15^2\right] \\ &- \left(\frac{1}{2}(-0.5 \times -0.05 + 1.5 \times 0.15)\right)^2 \\ &= 0.038125 \end{aligned}$$

The unconditional Sharpe ratio of this strategy is

$$SR = \frac{0.125}{\sqrt{0.038125}} = 0.6402$$

We see the Sharpe ratio is now much higher with higher R^2.