Empirical Methods

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Problem 1

We first note that the $ARMA(1,1)$ is stationary. The reasoning is as follows. First, note that

$$y_t = 0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t \quad \text{is equivalent to} \quad (1-0.95B)y_t = (1-0.9B)\epsilon_t.$$

Therefore, the zero of the characteric equation of the AR component is

$$\frac{1}{0.95} = \frac{20}{19} > 1.$$

Since we know the series is stationary, we can find its expectation and variance relatively easily. The expected value of y_t is

$$E[y_t] = E\left[0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t\right] = 0.95E[y_{t-1}].$$

It follows that $E[y_t] = 0$. The variance of y_t is

$$\begin{aligned}
Var(y_t) &= Var(0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t) \\
&= 0.95^2 Var(y_{t-1}) + 0.9^2 Var(\epsilon_{t-1}) + Var(\epsilon_t) - 2(0.95)(0.9)Cov(y_{t-1}, \epsilon_{t-1}) \\
&= 0.9025Var(y_{t-1}) + 0.81Var(\epsilon_{t-1}) + Var(\epsilon_t) - 1.71Cov(y_{t-1}, \epsilon_{t-1}) \\
&= 0.9025Var(y_{t-1}) + 0.81Var(\epsilon_{t-1}) + Var(\epsilon_t) - 1.71Cov(0.95y_{t-2} - 0.9\epsilon_{t-2} + \epsilon_{t-1}, \epsilon_{t-1}) \\
&= 0.9025Var(y_{t-1}) + 0.81Var(\epsilon_{t-1}) + Var(\epsilon_t) - 1.71Var(\epsilon_{t-1}) \\
&= 0.9025Var(y_t) + 0.10Var(\epsilon_t) \\
&= 0.9025Var(y_t) + 0.00025.
\end{aligned}$$

In conclusion, we have

$$Var(y_t) = 0.9025Var(y_t) + 0.00025 \quad \text{implies} \quad Var(y_t) \approx 0.00256410.$$

Part 1

To find the first order autocorrelation, we can simply find the covariance between y_t and y_{t+1} and then divide by the variance of y_t, due stationarity. To find the covariance, consider

$$\begin{aligned}
y_t y_{t+1} &= y_t(0.95y_t - 0.9\epsilon_t + \epsilon_{t+1}) \\
&= 0.95y_t^2 - 0.9y_t\epsilon_t + y_t\epsilon_{t+1} \\
&= 0.95y_t^2 - 0.9(0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t)\epsilon_t + y_t\epsilon_{t+1} \\
&= 0.95y_t^2 - 0.855y_{t-1}\epsilon_t + 0.81\epsilon_t\epsilon_{t-1} - 0.9\epsilon_t^2 + y_t\epsilon_{t+1}.
\end{aligned}$$

Since the expectation of y_t is zero, it follows that

$$\begin{aligned}
Cov(y_t, y_{t+1}) &= E[0.95y_t^2 - 0.855y_{t-1}\epsilon_t + 0.81\epsilon_t\epsilon_{t-1} - 0.9\epsilon_t^2 + y_t\epsilon_{t+1}] \\
&= 0.95E[y_t^2] - 0.855E[y_{t-1}\epsilon_t] + 0.81E[\epsilon_t\epsilon_{t-1}] - 0.9E[\epsilon_t^2] + E[y_t\epsilon_{t+1}] \\
&= 0.95E[y_t^2] - 0.9E[\epsilon_t^2] \\
&\approx 0.95(0.0025641) - 0.9(0.05^2) \\
&\approx 0.000185897.
\end{aligned}$$

We conclude that the first order autovariance is

$$\rho_1 = \frac{Cov(y_t, y_{t+1})}{Var(y_t)} \approx \frac{0.000185897}{0.0025641} \approx 0.0725001.$$

Part 2

The caluation for the second order autocorrelation is very similar to the first. Consider

$$y_t y_{t+2} = 0.95 y_t y_{t+1} - 0.9 y_t \epsilon_{t+1} + y_t \epsilon_{t+2}.$$

Once again, we utilize the fact that the expectation of y_t is zero to conclude that

$$Cov(y_t, y_{t+2}) = 0.95E[y_t y_{t+1}] - 0.9E[y_t \epsilon_{t+1}] + E[y_t \epsilon_{t+2}] \approx 0.95(0.000185897) \approx 0.000176602.$$

Dividing by $Var(y_t)$, we have that

$$\rho_2 = \frac{Cov(y_t, y_{t+2})}{Var(y_t)} \approx \frac{0.000176602}{0.00256410} \approx 0.0688749.$$

The ratio

$$\frac{\rho_2}{\rho_1} \approx \frac{0.0688749}{0.0725001} \approx 0.95.$$

As can be seen in ACF plots of $ARMA(1,1)$ models, the first order MA term distorts the natural tapering off of the autocorrelations caused by the autoregressive component. However, this distortion only lasts for the first autocorrelation; after that we expect to see the autocorrelations tapering off like an $AR(1)$ model. The ratio shows exactly that since, $\phi_1 = 0.95$.

Part 3

This is no more than simple computation. We have

$$E_t[y_{t+1}] = 0.95E_t[y_t] - 0.9E_t[\epsilon_t] + E_t[\epsilon_{t+1}] = 0.95(0.6) - 0.9(0.1) = 0.48.$$

Using our result for the expectation of y_{t+1}, we have

$$E_t[y_{t+2}] = 0.95E_t[y_{t+1}] - 0.9E_t[\epsilon_{t+1}] + E_t[\epsilon_{t+2}] = 0.95(0.48) = 0.456.$$

Part 4

We have that

$$\hat{x_t} = E_t[y_{t+1}] = 0.95E_t[y_t] - 0.9E_t[\epsilon_t] + E_t[\epsilon_{t+1}] = 0.95y_t - 0.9\epsilon_t.$$

It follows that

$$E[\hat{x_t}] = 0.95E[y_t] - 0.9E[\epsilon_t] = 0.$$

Calculating the variance is a bit more challenging. We know that

$$\begin{aligned}
\hat{x_t} &= 0.95y_t - 0.9\epsilon_t \\
&= 0.95(0.95y_{t-1} - 0.9\epsilon_{t-1} + \epsilon_t) - 0.9\epsilon_t \\
&= 0.9025y_{t-1} - 0.855\epsilon_{t-1} + 0.95\epsilon_t - 0.9\epsilon_t \\
&= 0.9025y_{t-1} - 0.855\epsilon_{t-1} + 0.05\epsilon_t.
\end{aligned}$$

Furthermore, it is clear

$$\hat{x}_{t-1} = 0.95y_{t-1} - 0.9\epsilon_{t-1} \quad \text{implies} \quad 0.95\hat{x}_{t-1} = 0.9025y_{t-1} - 0.855\epsilon_{t-1}.$$

It follows that

$$\hat{x}_t - 0.95\hat{x}_{t-1} = 0.05\epsilon_t.$$

We conclude that $\hat{x}_t$ is, in fact, a stationary $AR(1)$ model. We are ready to find its variance

$$Var(\hat{x}_t) = 0.95^2 Var(\hat{x}_{t-1}) + 0.05^2 Var(\epsilon_t) \quad \text{implies} \quad Var(\hat{x}_t) = \frac{0.05^4}{1 - 0.95^2} \approx 0.00800641^2.$$

Hence, its standard deviation is

$$\sigma(\hat{x}_t) \approx 0.00800641.$$

All that is left is to find the autocorrelations. Using techniques identical to those in Problem 1.2, we see that

$$Cov(\hat{x}_t, \hat{x}_{t+1}) = E[\hat{x}_t \hat{x}_{t+1}] = 0.95 Var(\hat{x}_t).$$

Therefore, the first order autocorrelation is $\rho_1 = 0.95$.

Problem 2

Observe that

$$y_t = e_t - e_{t-4} = e_{t-1} + \epsilon_t - (e_{t-5} + \epsilon_{t-4}) = e_{t-1} - e_{t-5} + \epsilon_t - \epsilon_{t-4} = y_{t-1} + \epsilon_t - \epsilon_{t-4}.$$

It follows that

$$(1-B)(y_t - \mu_y) = (1 - B^4)\epsilon_t \quad \text{implies} \quad y_t = \mu_y + \frac{1-B^4}{1-B}\epsilon_t = \mu_y + (1 + B + B^2 + B^3)\epsilon_t.$$

In otherwords,

$$y_t = \mu_y + \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + + \epsilon_{t-3}.$$

Since the ϵ-values are independent and identically distributed, it follows that

$$Var(y_t) = Var(\epsilon_t) + Var(\epsilon_{t-1}) + Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) = 4,$$

when $t \geq 4$. As can be seen from the characteristic equation, this series is *not* stationary; for values of t less than four the variance changes. In particular, $Var(y_1) = 1$, $Var(y_2) = 2$, and $Var(y_3) = 3$.

Part 1

Let us find the autocovariances when $t \geq 3$:

$$\begin{aligned}
Cov(y_t, y_t) &= Var(y_t) \\
Cov(y_t, y_{t+1}) &= Cov(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}, \epsilon_{t+1} + \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2}) \\
&= 3, \\
Cov(y_t, y_{t+2}) &= Cov(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}, \epsilon_{t+2} + \epsilon_{t+1} + \epsilon_t + \epsilon_{t-}) \\
&= 2, \\
Cov(y_t, y_{t+3}) &= Cov(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}, \epsilon_{t+3} + \epsilon_{t+2} + \epsilon_{t+1} + \epsilon_t) \\
&= 1, \\
Cov(y_t, y_{t+4}) &= Cov(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}, \epsilon_{t+4} + \epsilon_{t+3} + \epsilon_{t+2} + \epsilon_{t+1}) \\
&= 0,
\end{aligned}$$

and

$$Cov(y_t, y_{t+5}) = Cov(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}, \epsilon_{t+5} + \epsilon_{t+4} + \epsilon_{t+3} + \epsilon_{t+2}) = 0.$$

Part 2

We already completed the reasoning for this step in the preamble for Problem 3. Our conclusion was that

$$y_t = y_{t-1} + \epsilon_t - \epsilon_{t-4}.$$

This is an $ARMA(1,4)$ model of the form

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1} - \ldots - \theta_4 \epsilon_{t-4},$$

where

$$\phi_0 = 0, \quad \phi_1 = 1, \quad \theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = 0, \quad \text{and} \quad \theta_4 = 1.$$

Problem 3

Part 1

Due to the fact the ϵ_{t+1} is independent of x_t, we have

$$Var(R^e_{t+1}) = Var(x_t) + Var(\epsilon_{t+1}) = 0.05^2 + 0.15^2 = 0.025.$$

It follows that the standard devation of excess returns is

$$\sqrt{0.025} \approx 0.158114.$$

Part 2

We know that

$$\beta = \rho \frac{\sigma(R^e_{t+1})}{\sigma(x_t)} \approx 1 \quad \text{implies} \quad R^2 \approx \frac{\sigma^2(x_t)}{\sigma^2(R^e_{t+1})} = 0.1.$$

Part 3

The expected value of excess returns is

$$E[R^2_{t+1}] = E[x_t] = 0.05.$$

Therefore, the Sharpe ratio based on the information given is

$$\frac{0.05}{\sqrt{0.025}} \approx 0.316228.$$

Part 4

Let us find α_t. We know

$$E_t[R^e_{t+1}] = E[x_t] + E_t[\epsilon_{t+1}] = x_t$$

and

$$Var_t(R^e_{t+1}) = Var_t(\epsilon_{t+1}) = 0.15^2.$$

It follows that

$$\alpha_t = \frac{x_t}{0.15^2\gamma} = 10x_t.$$

Hence, if x_t is equal to 0 or 0.10, then

$$\alpha_t = 0 \quad \text{or} \quad 1,$$

respectively.

Let us find the Sharpe ratio under these two scenarios. If $x_t = 0$, then the Sharpe ratio of the portfolio is 0. If $x_t = 0.10$, then the Sharpe ratio is

$$\frac{0.10}{0.15} = \frac{2}{3} \approx 0.666667.$$

Part 5

(a) Let us find the unconditional expectation of excess returns under this strategy. Using the law of iterated expectations, we have

$$E[\alpha_t R^e_{t+1}] = E[E_t[\alpha_t R^e_{t+1}]] = E[\alpha_t x_t] = \frac{1}{0.15^2\gamma} E[x_t^2].$$

Since

$$E[x_t^2] = \frac{1}{2}(0)^2 + \frac{1}{2}(0.10)^2 = 0.005,$$

we conclude that

$$E[\alpha_t R^e_{t+1}] = \frac{0.005}{0.15^2\gamma} = 0.05.$$

(b) We are given that the variance

$$Var(\alpha_t R^e_{t+1}) = E\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - E[\alpha_t x_t]^2.$$

We know that

$$\sigma_t^2(R^e_{t+1}) = \sigma_t^2(\epsilon_{t+1}) = 0.15^2 \quad \text{and} \quad E_t[R^e_{t+1}] = x_t.$$

Then

$$\begin{aligned}
E\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - E[\alpha_t x_t]^2 &= E\left[\left(\frac{x_t}{0.15^2\gamma}\right)^2(x_t^2 + 0.15^2)\right] - E\left[\left(\frac{x_t}{0.15^2\gamma}\right)x_t\right]^2 \\
&= \frac{1}{0.15^4\gamma^2}E\left[x_t^4 + 0.15^2 x_t^2\right] - \frac{1}{0.15^4\gamma^2}E\left[x_t^2\right]^2 \\
&= \frac{1}{0.15^4\gamma^2}\left(E[x_t^4] + 0.15^2 E[x_t^2] - E[x_t^2]^2\right) \\
&= \frac{1}{0.15^4\gamma^2}\left(\frac{0.10^4}{2} + \frac{0.15^2}{2}(0.10)^2 - \frac{1}{4}(0.10)^4\right) \\
&= 0.0137500.
\end{aligned}$$

We conclude that the standard devation is

$$\sqrt{0.01375} \approx 0.117260.$$

(c) Therefore, the unconditional Sharpe ratio of this strategy is

$$\frac{0.05}{0.117260} \approx 0.426403.$$

(d) Let us find the exected value and variance of x_t in this situation. The expected value is

$$E[x_t] = -\frac{0.05}{2} + \frac{0.15}{2} = 0.05$$

and the variance is

$$Var(x_t) = \frac{1}{2}(-0.10)^2 + \frac{1}{2}(0.10)^2 = 0.01.$$

(e) We will proceed to find the R^2 value like we did before. We know

$$\beta = \rho\frac{\sigma(R^e_{t+1})}{\sigma(x_t)} \approx 1 \quad \text{implies} \quad R^2 \approx \frac{\sigma^2(x_t)}{\sigma^2(R^e_{t+1})} = \frac{0.01}{0.01 + 0.15^2} \approx 0.308.$$

(ii) To find the Sharpe ratio, we need to find the expectation and variance. The expectation is

$$\begin{aligned}
E[\alpha_t R^e_{t+1}] &= E\left[\frac{E_t[R^e_{t+1}]}{\gamma\sigma_t^2(R^e_{t+1})}E_t[R^e_{t+1}]\right] \\
&= E\left[\frac{x_t^2}{0.15^2\gamma}\right] \\
&= \frac{1}{0.15^2\gamma}\left[\frac{1}{2}(-0.05)^2 + \frac{1}{2}(0.15)^2\right] \\
&= 0.125.
\end{aligned}$$

The variance

$$\begin{aligned}
E\left[\alpha_t^2\Big(x_t^2+\sigma_t^2(\epsilon_{t+1})\Big)\right]-E[\alpha_t x_t]^2 &= E\left[\left(\frac{x_t}{0.15^2\gamma}\right)^2(x_t^2+0.15^2)\right]-E\left[\left(\frac{x_t}{0.15^2\gamma}\right)x_t\right]^2 \\
&= \frac{1}{0.15^4\gamma^2}E\left[x_t^4+0.15^2x_t^2\right]-\frac{1}{0.15^4\gamma^2}E\left[x_t^2\right]^2 \\
&= \frac{1}{0.15^4\gamma^2}\left(E[x_t^4]+0.15^2E[x_t^2]-E[x_t^2]^2\right) \\
&= \frac{1}{0.15^4\gamma^2}\left(0.000256+0.15^2(0.0125)-0.0125^2\right) \\
&\approx 0.038125.
\end{aligned}$$

We conclude that the standard devation is

$$\sqrt{0.038125}\approx 0.195256.$$

Hence, the Sharpe ratio is

$$\frac{0.125}{0.195256}\approx 0.640184.$$