Lecture 7
Vector Autoregressive Models

Lars A. Lochstoer

UCLA Anderson School of Management

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Outline

1. Vector Autoregressive Models (VARs)
2. Worked example of VAR analysis
   - Out-of-sample performance
   - **Optional**: Impulse-Response plots and Sims' orthogonalization

Vector Autoregressive Models (VARs)

Vector Autoregressive Models

Definition

A multivariate time series $\mathbf{r}_t$ is a VAR process of order 1

$$\mathbf{r}_t = \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{r}_{t-1} + \boldsymbol{\varepsilon}_t, \qquad \varepsilon_t \sim \mathrm{WN}(0, \boldsymbol{\Sigma})$$

where $\boldsymbol{\Phi}_0$ is $N \times 1$ and $\boldsymbol{\Phi}_1$ is a $N \times N$ matrix.

The covariance matrix $\boldsymbol{\Sigma}$ is required to be positive definite.

Bivariate Case

- stack two AR(1)'s on top of each other...
- as an example, we consider the bivariate case:

$$\begin{aligned} r_{1t} &= \phi_{10} + \phi_{11} r_{1,t-1} + \phi_{12} r_{2,t-1} + \varepsilon_{1t} \\ r_{2t} &= \phi_{20} + \phi_{21} r_{1,t-1} + \phi_{22} r_{2,t-1} + \varepsilon_{2t} \end{aligned}$$

- ϕ_{12} measures the conditional effect of $r_{2,t-1}$ on r_{1t} given $r_{1,t-1}$
- ϕ_{12} measures the linear dependence of $r_{1,t}$ on $r_{2,t-1}$ given $r_{1,t-1}$

Writing AR(p) models as VAR(1)

- Consider the following univariate AR(p) model

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \ldots \phi_p r_{t-p} + \sigma \varepsilon_t$$

- We can write this as a VAR(1) as

$$\mathbf{r}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{r}_{t-1} + \mathbf{R}\varepsilon_t \qquad \varepsilon_t \sim \mathrm{WN}(0,1)$$

$$\begin{bmatrix} r_t \\ r_{t-1} \\ r_{t-2} \\ \vdots \\ r_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_p \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{t-1} \\ r_{t-2} \\ r_{t-3} \\ \vdots \\ r_{t-p} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \varepsilon_t$$

- We can now analyze the AR(p) as a VAR(1), which is simple.

MA Representation

- condition for stationarity

- compute the mean

$$E[\mathbf{r}_t] = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 E[\mathbf{r}_{t-1}]$$

which implies that

$$\boldsymbol{\mu} = (I_N - \mathbf{\Phi}_1)^{-1}\mathbf{\Phi}_0$$

- rewrite this system in deviations from the mean:

$$\widetilde{\mathbf{r}}_t = \mathbf{\Phi}_1 \widetilde{\mathbf{r}}_{t-1} + \boldsymbol{\varepsilon}_t, \qquad \widetilde{\mathbf{r}}_t \equiv \mathbf{r}_t - \boldsymbol{\mu}$$

- by backward substitution:

$$\widetilde{\mathbf{r}}_t = \sum_{j=0}^{\infty} \mathbf{\Phi}_1^j \boldsymbol{\varepsilon}_{t-j}$$

Stationarity

- the MA(∞) expression

$$\mathbf{r}_t \; = \; \boldsymbol{\mu} + \sum_{j=0}^{\infty} \boldsymbol{\Phi}_1^j \boldsymbol{\varepsilon}_{t-j}$$

  implies we need $\boldsymbol{\Phi}_1^j \to 0$ as $j \to \infty$ to get stationarity

- the N eigenvalues $\boldsymbol{\lambda}$ need to be less than one in modulus:

$$|\boldsymbol{\lambda} \mathbf{I}_N - \boldsymbol{\Phi}_1| = 0$$

- for a complex number $z = x + iy$ the **modulus** is defined as

$$|z| \; = \; \sqrt{x^2 + y^2}$$

VEC and Kronecker products

- The VEC operator stacks the columns of an $m \times n$ matrix A and turns it into a $mn \times 1$ vector.

- Consider a matrix A and the vec operator vec(A) given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{vec}(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{pmatrix}$$

- If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the **Kronecker product** defined by $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$$

VEC and Kronecker products

- Some useful properties of these operations:

$$
\begin{aligned}
\text{vec}(A+B) &= \text{vec}(A)+\text{vec}(B) \\
\text{vec}(ABC) &= (C'\otimes A)\text{vec}(B) \\
(A\otimes B)^{-1} &= A^{-1}\otimes B^{-1} \\
(A\otimes B)' &= A'\otimes B'
\end{aligned}
$$

- More properties can be found in any matrix algebra book, see e.g. Abadir and Magnus (2005) (or Google it).

(Unconditional) variances

- compute the unconditional variance:

$$E[\widetilde{\mathbf{r}}_t \widetilde{\mathbf{r}}_t'] \;=\; \mathbf{\Phi}_1 E[\widetilde{\mathbf{r}}_t \widetilde{\mathbf{r}}_t'] \mathbf{\Phi}_1' + \mathbf{\Sigma},$$

  or equivalently:

$$\mathbf{\Gamma}_0 = \mathbf{\Phi}_1 \mathbf{\Gamma}_0 \mathbf{\Phi}_1' + \mathbf{\Sigma},$$

- Solve this equation:
  - take the 'vec' of both sides

$$\mathrm{vec}(\mathbf{\Gamma}_0) \;=\; \mathrm{vec}(\mathbf{\Phi}_1 \mathbf{\Gamma}_0 \mathbf{\Phi}_1') + \mathrm{vec}(\mathbf{\Sigma}),$$

  - use the Kronecker product

$$\mathrm{vec}(\mathbf{\Gamma}_0) \;=\; [\mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1]\mathrm{vec}(\mathbf{\Gamma}_0) + \mathrm{vec}(\mathbf{\Sigma}),$$

  - solve

$$\mathrm{vec}(\mathbf{\Gamma}_0) \;=\; [I_{N^2} - \mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1]^{-1}\mathrm{vec}(\mathbf{\Sigma}),$$

Autocovariances

- compute the auto- covariances by taking expectations on the lhs and rhs:

$$E[\widetilde{\mathbf{r}}_t \widetilde{\mathbf{r}}'_{t-k}] = \mathbf{\Phi}_1 E[\widetilde{\mathbf{r}}_{t-k} \widetilde{\mathbf{r}}'_{t-k}]$$

  which implies that:

$$\mathbf{\Gamma}_k = \mathbf{\Phi}_1 \mathbf{\Gamma}_{k-1}, k > 0$$

  where $\mathbf{\Gamma}_j$ is the lag j cross-covariance matrix

- by repeated substitution, we obtain the following autocovariogram:

$$\mathbf{\Gamma}_k = \mathbf{\Phi}_1^k \mathbf{\Gamma}_0, k > 0$$

Forecasting

- forecasting is very easy. Take the 1-step ahead conditional expectation:

$$E_t[\mathbf{r}_{t+1}] \ = \ (I_N - \mathbf{\Phi}_1^1)\boldsymbol{\mu} + \mathbf{\Phi}_1^1 \mathbf{r}_t$$

- doing this h steps ahead:

$$E_t[\mathbf{r}_{t+h}] = (I_N - \mathbf{\Phi}_1^h)\boldsymbol{\mu} + \mathbf{\Phi}_1^h \mathbf{r}_t$$

- Notice that if we take the limit as $h \longrightarrow \infty$, we get

$$\lim_{h \to \infty} E_t[\mathbf{r}_{t+h}] = \boldsymbol{\mu}$$

Autocorrelations

- auto-covariances:

$$\mathbf{\Gamma}_k = \mathbf{\Phi}_1 \mathbf{\Gamma}_{k-1}, k > 0$$

- let $\mathbf{D} = \text{diag}[\sqrt{\Gamma_{11}(0)} \ldots \sqrt{\Gamma_{NN}(0)}]$.

- by pre-and post-multiplication with $\mathbf{D}$:

$$\begin{aligned} \boldsymbol{\rho}_k &= \mathbf{D}^{-1} \mathbf{\Phi}_1 \mathbf{\Gamma}_{k-1} \mathbf{D}^{-1} \\ &= \mathbf{D}^{-1} \mathbf{\Phi}_1 \mathbf{D} \mathbf{D}^{-1} \mathbf{\Gamma}_{k-1} \mathbf{D}^{-1} \\ &= \mathbf{\Upsilon} \boldsymbol{\rho}_{k-1}, \qquad k > 0 \end{aligned}$$

with $\mathbf{\Upsilon} = \mathbf{D}^{-1} \mathbf{\Phi}_1 \mathbf{D}$

Vector Autoregressive Models of order p

Definition

A multivariate time series $\mathbf{r}_t$ is a VAR process of order p

$$\mathbf{r}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{r}_{t-1} + \ldots + \mathbf{\Phi}_p \mathbf{r}_{t-p} + \boldsymbol{\varepsilon}_t, \qquad \varepsilon_t \sim \mathrm{WN}(0, \mathbf{\Sigma})$$

where $\mathbf{\Phi}_0$ is $N \times 1$, $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_p$ are $N \times N$ matrix and $\boldsymbol{\varepsilon}_t$ is a sequence of white noise random vectors.

The covariance matrix $\mathbf{\Sigma}$ is required to be positive definite.

VAR(p) Model

- condition for stationarity
- compute the mean

$$E[\mathbf{r}_t] = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 E[\mathbf{r}_{t-1}] + \ldots + \mathbf{\Phi}_p E[\mathbf{r}_{t-p}]$$

  which (under stationarity) implies that:

$$(I_N - \mathbf{\Phi}_1 - \ldots - \mathbf{\Phi}_p)\,\boldsymbol{\mu} = \mathbf{\Phi}_0$$

  and we can back out the mean provided that the inverse exists

- the autocovariances satisfy:

$$\mathbf{\Gamma}_k = \mathbf{\Phi}_1 \Gamma_{k-1} + \ldots + \mathbf{\Phi}_p \Gamma_{k-p}, \qquad k > 0$$

Vector Autoregressive Models of order 4

Definition

A multivariate time series $\mathbf{r}_t$ is a VAR process of order $p = 4$ if

$$\mathbf{r}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{r}_{t-1} + \ldots + \mathbf{\Phi}_4 \mathbf{r}_{t-4} + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim \mathrm{WN}(0, \mathbf{\Sigma})$$

where $\mathbf{\Phi}_0$ is a N-dimensional vector, $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_4$ are $N \times N$ matrices.

The variance covariance matrix is required to be positive definite.

Vector Autoregressive Models of order 4

- we can represent this VAR(4) as a VAR(1):

$$\begin{bmatrix} \mathbf{r}_{t+1} \\ \mathbf{r}_t \\ \mathbf{r}_{t-1} \\ \mathbf{r}_{t-2} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \mathbf{\Phi}_3 & \mathbf{\Phi}_4 \\ I_N & 0 & 0 & 0 \\ 0 & I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_t \\ \mathbf{r}_{t-1} \\ \mathbf{r}_{t-2} \\ \mathbf{r}_{t-3} \end{bmatrix} + \begin{bmatrix} I_N \\ 0 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{\varepsilon}_{t+1}$$

- much easier to handle!

- this is related to linear, state space models (get to this at end of class)

Estimation

- in this system
$$\mathbf{r}_t = \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{r}_{t-1} + \varepsilon_t$$
- each equation can be estimated separately using OLS
  - errors are serially uncorrelated with constant variances
  - right hand side variables are predetermined (not exogenous)

Specification Test

- the lag length p is an important part of VAR(p) modeling
- we can use the AIC to choose the lag length
- the AIC of a VAR(p) model is:

$$AIC(i) = \ln(|\Sigma_p|) + \frac{2N^2 p}{T}$$

  where $\Sigma_p = \frac{1}{T-1} \sum_{t=p+1}^{T} \varepsilon_t \varepsilon_t'$
- the lag length p should be chosen to minimize the AIC
- run a multi-variate Q-test on residuals

VARs: A worked example

Impulse-Responses and Sims shock orthogonalization

The data

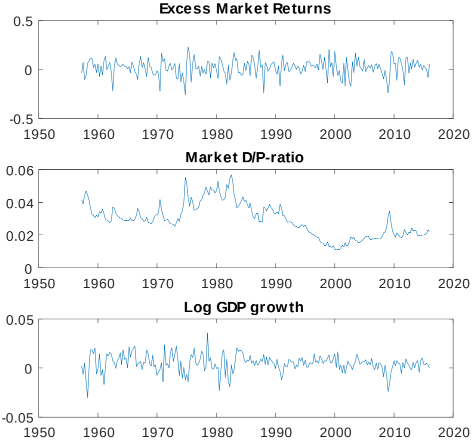
We will fit a higher order VAR to the following quarterly data:

1. Excess stock market returns (CRSP)
2. The market Price-Dividend ratio (CRSP)
   - Market value divided by sum of last 12 months of dividends (to avoid seasonalities)
3. Real, per capita quarterly log GDP growth

The sample is 1957Q1 through 2015Q4.

Step 1: Plot the data

- Always, if you can, plot the data you are using.
  - non-stationarities, outliers, data-errors, etc.?
- Note that the DP ratio is *very* persistent. Otherwise, returns and gdp growth both are pretty clearly stationary. No obvious data-errors.



MatLab code for previous slide

```
load VAR_data

subplot(3,1,1)
plot(date,ExcessMktRet)
title('Excess Market Returns')

subplot(3,1,2)
plot(date,DP)
title('Market D/P-ratio')

subplot(3,1,3)
plot(date,D_gdp)
title('Log GDP growth')
```

Model specification

We are interested in understanding how the equity risk premium varies with business conditions (GDP growth) and how GDP growth links to valuation ratios and stock returns

The choice of the number of lags in the VAR is a tricky one.

1. Use as few lags as you can while still capturing the main effects in the data (AIC, BIC). VAR(1) is a good start.
2. Overall, model specification choice is a bit of an art

Having looked a bit at the return forecasting equation, and to illustrate some of the topics from Lecture 5, I have chosen a restricted VAR(2)

- See next slide for specification

Model specification

Here are the forecasting equations I want embedded in the VAR:

$$\begin{aligned}
R^e_{t+1} &= \phi_0^{(1)} + \phi_{11} R^e_t + \phi_{12} DP_t + \phi_{13} \Delta gdp_t + \phi_{14} \Delta gdp_{t-1} + \sigma_R \varepsilon_{R,t+1}, \\
DP_{t+1} &= \phi_0^{(2)} + \phi_{21} R^e_t + \phi_{22} DP_t + \phi_{23} \Delta gdp_t + \phi_{24} \Delta gdp_{t-1} + \sigma_{DP} \varepsilon_{DP,t+1}, \\
\Delta gdp_{t+1} &= \phi_0^{(3)} + \phi_{31} R^e_t + \phi_{32} DP_t + \phi_{33} \Delta gdp_t + \phi_{34} \Delta gdp_{t-1} + \sigma_{gdp} \varepsilon_{gdp,t+1}.
\end{aligned}$$

How do I map this into a VAR(1)?

$$z_{t+1} = \phi_0 + \phi_1 z_t + error$$

- What is the z_t vector? What is ϕ_0? What is ϕ_1?

Model specification

So:

$$z_t = \left[R_t^e \quad DP_t \quad \Delta gdp_t \quad \Delta gdp_{t-1}\right]'$$

and

$$\phi_0 = \left[\phi_0^{(1)} \quad \phi_0^{(2)} \quad \phi_0^{(3)} \quad 0\right]'$$

and

$$\phi_1 = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Model estimation

Simply estimate the parameters in each of the below using standard OLS regressions:

$$
\begin{aligned}
R_{t+1}^{e} &= \phi_0^{(1)} + \phi_{11} R_t^e + \phi_{12} DP_t + \phi_{13} \Delta gdp_t + \phi_{14} \Delta gdp_{t-1} + \sigma_R \varepsilon_{R,t+1}, \\
DP_{t+1} &= \phi_0^{(2)} + \phi_{21} R_t^e + \phi_{22} DP_t + \phi_{23} \Delta gdp_t + \phi_{24} \Delta gdp_{t-1} + \sigma_{DP} \varepsilon_{DP,t+1}, \\
\Delta gdp_{t+1} &= \phi_0^{(3)} + \phi_{31} R_t^e + \phi_{32} DP_t + \phi_{33} \Delta gdp_t + \phi_{34} \Delta gdp_{t-1} + \sigma_{gdp} \varepsilon_{gdp,t+1}.
\end{aligned}
$$

Save the residuals so you can estimate their variance-covariance matrix later

MatLab code for estimation

```
% estimate the restricted VAR(2), set it up as a VAR(1)
phi0 = zeros(4,1);
phi1 = zeros(4,4);

out = LL_olsNW(ExcessMktRet(3:end),cat(2,ones(234,1),ExcessMktRet(2:end-
1),DP(2:end-1),D_gdp(2:end-1),D_gdp(1:end-2)),0);
phi0(1) = out.beta(1);
phi1(1,:) = out.beta(2:end)';
eps1 = out.res;
```

MatLab code for estimation

```
out = LL_olsNW(DP(3:end),cat(2,ones(234,1),ExcessMktRet(2:end-
1),DP(2:end-1),D_gdp(2:end-1),D_gdp(1:end-2)),0);
phi0(2) = out.beta(1);
phi1(2,:) = out.beta(2:end)';
eps2 = out.res;

out = LL_olsNW(D_gdp(3:end),cat(2,ones(234,1),ExcessMktRet(2:end-
1),DP(2:end-1),D_gdp(2:end-1),D_gdp(1:end-2)),0);
phi0(3) = out.beta(1);
phi1(3,:) = out.beta(2:end)';
eps3 = out.res;

% add element that corresponds to second lag of GDP
phi1(4,3) = 1;
display(phi0);
display(phi1);
```

MatLab code for estimation

- In general, you may want to display R^2 values and t-stat's, but we are keeping it simple here:

```
phi0 =

  -0.0071
   0.0008
   0.0033
        0


phi1 =

   0.0813    0.9028    0.1656   -1.1466
  -0.0034    0.9624   -0.0078    0.0609
   0.0247   -0.0251    0.2744    0.1163
        0         0    1.0000         0
```

Check for stationarity

How? Recall that all we need when we have a VAR(1) is that the Eigenvalues are less than one in modulus:

```
% check that the system is stationary
eigenvalues = eig(phi1);
modulus = sqrt(real(eigenvalues).^2 + imag(eigenvalues).^2)

modulus =
0.3230
0.3506
0.3506
0.9590
```

Demean system for easy manipulation

Recall that the unconditional mean of a VAR(1) is:

$$\boldsymbol{\mu} = \left(\mathbf{I}_N - \boldsymbol{\phi}_1\right)^{-1} \boldsymbol{\phi}_0$$

So the MatLab code is:

```
% Calculate unconditional mean vector:
mu = (eye(4) - phi1)\phi0;

% demean all variables for simplicity
z_t = cat(2,ExcessMktRet(2:end),DP(2:end),D_gdp(2:end),D_gdp(1:end-1)) -
kron(ones(length(DP)-1,1),mu');
```

Expected value plot: 1 quarter ahead

The VAR gives you the expected value at any future date: $E_t\left[\tilde{z}_{t+k}\right] = \boldsymbol{\phi}_1^k z_t$

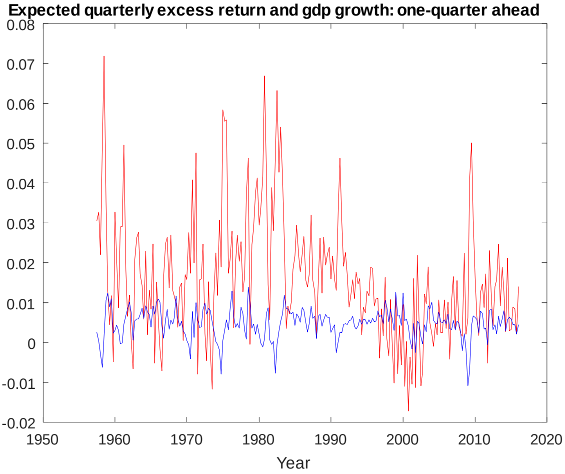
```
% plot the expected returns and the expected GDP growth series
% one-period (a quarter) ahead expectations
Et1 = (phi1 * z_t')' + kron(ones(length(DP)-1,1),mu');

plot(date(2:end),Et1(:,1),'r')
hold on
plot(date(2:end),Et1(:,3),'b')
xlabel('Year')
title('Expected quarterly excess return and gdp growth: one-quarter ahead');
```

Expected value plot: 1 quarter ahead

Notice that the 1-period ahead forecast is quite volatile

- GDP is in blue (less volatile), return is in red (more volatile)
- Note the negative correlation between expected GDP growth and expected returns: The risk premium appears to be *counter-cyclical*!



Expected value plot: 4 quarters ahead

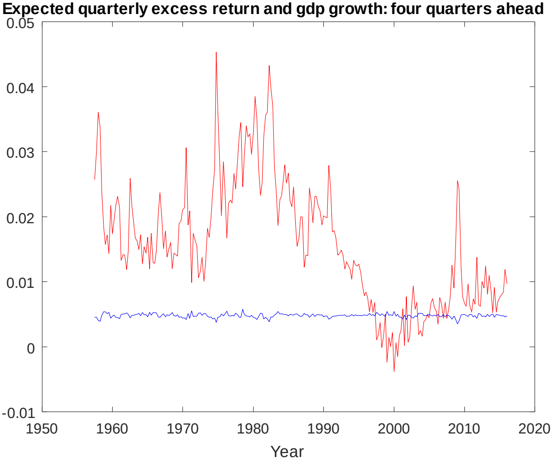
```
% one year (4 quarters) ahead expectations

Et4 = (phi1^4 * z_t')' + kron(ones(length(DP)-1,1),mu');
plot(date(2:end),Et4(:,1),'r')
hold on
plot(date(2:end),Et4(:,3),'b')
xlabel('Year')
title('Expected quarterly excess return and gdp growth: four quarters ahead');
```

Expected value plot: 4 quarters ahead

Notice that the 4-period ahead forecast is less volatile

- GDP is in blue (less volatile), return is in red (more volatile)
- Expected returns still vary a great deal, due to effect of persistent PD-ratio
  - Almost no forecasting power for GDP 4 quartes out...



Expected value plot: 20 quarters ahead

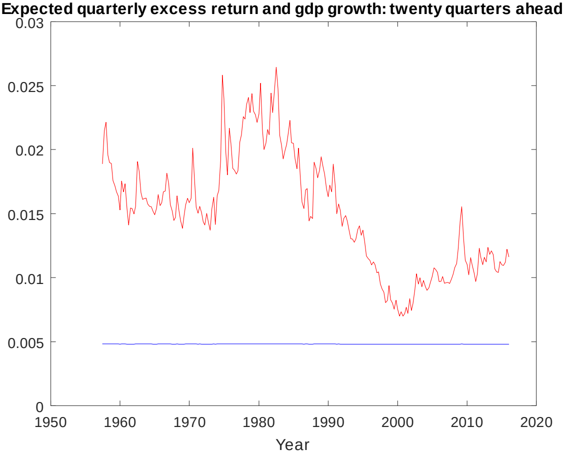
```
% 5 years (20 quarters) ahead expectations

Et20 = (phi1^20 * z_t')' + kron(ones(length(DP)-1,1),mu');
plot(date(2:end),Et20(:,1),'r')
hold on
plot(date(2:end),Et20(:,3),'b')
xlabel('Year')
title('Expected quarterly excess return and gdp growth: twenty quarters ahead');
```

Expected value plot: 20 quarters ahead

Notice that the 20-period ahead forecast is even less volatile

- GDP is in blue (less volatile), return is in red (more volatile)
- Expected returns still vary a great deal, again due to effect of highly persistent PD-ratio
  - No forecasting power for GDP 20 quartes out, simply unconditional average



VARs: Out-of-sample test

Out-of-sample test

Overfitting means we get 'too high' R^2's in sample as we explain some of the true variation in actual residuals

- Out-of-sample performance can then be bad

Related, parameters may be unstable, varying over time in reality. In such cases, the out-of-sample performance will likely suffer.

Here, present a natural out-of-sample test

- Focus only on the 1-period ahead return forecasting equation

$$R^e_{t+1} = \phi_0^{(1)} + \phi_{11} R^e_t + \phi_{12} DP_t + \phi_{13} \Delta gdp_t + \phi_{14} \Delta gdp_{t-1} + \sigma_R \varepsilon_{R,t+1}$$

Out-of-sample test

1. Choose an initial training sample, for instance covering the first 80% of the sample. This depends on overall sample length
   - Too short initial sample means the parameters are estimated with a lot of noise, which of course leads to low out-of-sample performance.

2. Using data up to an including time τ to estimate the model, get the model prediction for time $\tau + 1$ and compare with actual outcome at time $\tau + 1$

3. Natural metrics of fit (expectations are averages over the out-of-sample period used (e.g, the last 20% of sample)

$$MSPE = E\left[(predicted - actual)^2\right]$$

$$R^2_{out-of-sample} = 1 - \frac{E\left[(predicted - actual)^2\right]}{E\left[(actual - E\left[actual\right])^2\right]}$$

MatLab code for out-of-sample tests

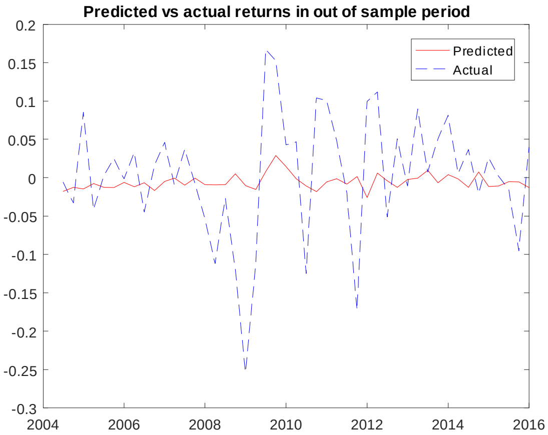
```
% first, set an initial training sample.
T = length(z_t(:,1));
train = round(T*4/5);
pred = zeros(T-train-1,1);
actual = z_t(train+1:T,1);

% do just return equation
for tt = train:T-1
out = LL_olsNW(z_t(2:tt,1),cat(2,ones(tt-1,1),z_t(1:tt-1,1), z_t(1:tt-1,2),
z_t(1:tt-1,3), z_t(1:tt-1,4)),0);
pred(1+tt-train) = out.beta' * cat(2,1,z_t(tt,:))';
end

MSE = mean((pred - actual).^2);
RMSE = sqrt(MSE);
R2_outofsample = 1 - MSE / mean((actual - mean(actual)).^2);
```

Predicted vs Actual in out of sample period

- The out of sample R^2 is 1.71% for quarterly excess return forecasts
- The in-sample R^2 for the full sample is 1.61%
- While these numbers seem small, the predicted *quarterly* risk premium varies from about -2% to 3%!



Impulse-Response Plots

(Advanced: Optional Material)

Impulse-Response Plots

Recall that the $MA\left(\infty\right)$ representation of a demeaned $VAR\left(1\right)$ is:

$$z_t = \sum_{j=0}^{\infty} \boldsymbol{\phi}_1^j \boldsymbol{\varepsilon}_{t-j},$$

where $\boldsymbol{\varepsilon}_{t-j}$ is the, in our case, 4×1 vector of shocks at time $t-j$:

$$\boldsymbol{\varepsilon}_{t-j} = \begin{bmatrix} \sigma_R \varepsilon_{R,t-j} \\ \sigma_{DP} \varepsilon_{DP,t-j} \\ \sigma_{gdp} \varepsilon_{gdp,t-j} \\ 0 \end{bmatrix}$$

Note that the last shock is always zero as it is the shock corresponding to the lagged GDP growth equation.

Impulse-Response Plots

The Impulse-Response (IR) function is the dynamic multiplier on a particular shock on a particular outcome at a particular lag

- The Impulse is a one-time one-standard deviation positive shock (typically), *with all other shocks set to zero*
- The Response is how this shock affects future outcomes as a function of the lag j

Example 1: Get the response of returns (the first element of z_t) to a shock to Δgdp (the third element of z_t), 5 lags ago

- Get the (1,3) element of $\boldsymbol{\phi}_1^5$. Multiply this number by the standard deviation of the shock, σ_{gdp}.
- The shock we are talking about is ε_{gdp}

Example 2: Get the response of log GDP growth (the third element of z_t) to a shock to returns (the first element of z_t), 20 lags ago

- Get the (3,1) element of $\boldsymbol{\phi}_1^{20}$. Multiply this number by the standard deviation of the shock, σ_R.
- The shock we are talking about is ε_R

Impulse-Response Plots: MatLab code

```
% create impulse responses (i.e., plot dynamic multipliers for each of the
% three shocks (note, the fourth variable (the VAR(2) gdp term) does not
% have a shock).
Sigma = cov(cat(2,eps1,eps2,eps3));
StDevs = sqrt(diag(Sigma));

% plot impulse-response (dynamic multipliers at different lags up to 20
% lags).
phi1_series = zeros(4,4,20);
for j = 1:21
    phi1_series(:,:,j) = phi1^(j-1);
end
```

Impulse-Response Plots: MatLab code

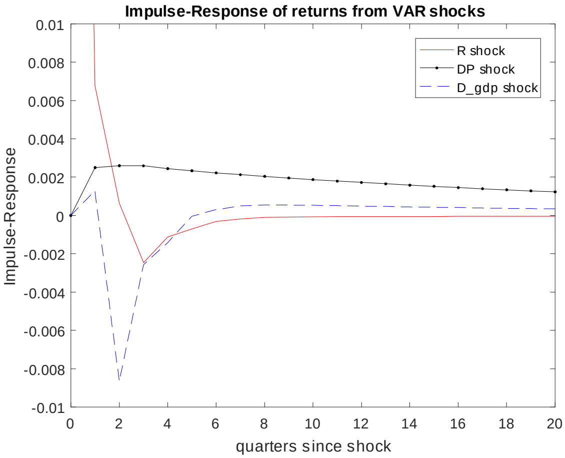
```
% start with the IR for returns of a shock to returns, DP, and GDP.

% first, let the return shock happen at time t, with all other shocks set
% to zero, and trace out the response
plot((0:1:20)',StDevs(1)*squeeze(phi1_series(1,1,:)),'r')
hold on
% next, DP shock effect on returns
plot((0:1:20)',StDevs(2)*squeeze(phi1_series(1,2,:)),'k.-')
% next, GDP shock effect on returns
plot((0:1:20)',StDevs(3)*squeeze(phi1_series(1,3,:)),'b--')
hold off

ylim([-0.01 0.01]);
xlabel('quarters since shock')
ylabel('Impulse-Response')
legend('R shock','DP shock', 'D\_gdp shock')
title('Impulse-Response of returns from VAR shocks')
```

Impulse-Response Plot for Excess Market Returns

- Note the very persistent effect of the valuation ratio (DP-ratio) shock
  - (note: 0-th lag for returns is outside of plot limits, at about 0.08)
- Note that GDP growth has a negative effect after a lag
  - Good times means risk premium goes down



Impulse-Response Plots: MatLab code

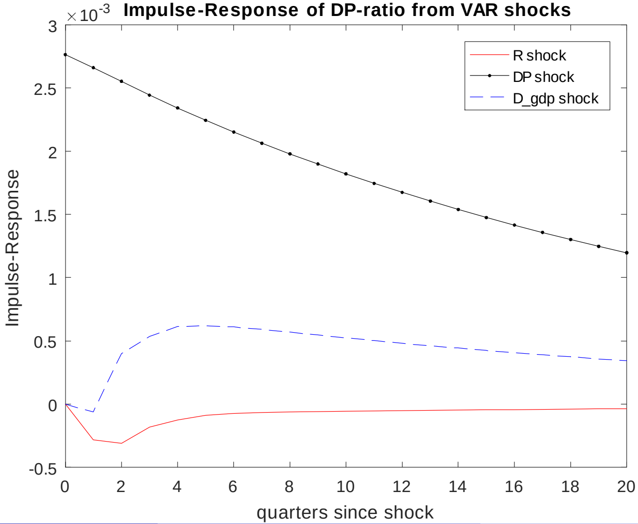
```
% next, the IR for DP of a shock to returns, DP, and GDP.

% first, let the return shock happen at time t, with all other shocks set
% to zero, and trace out the response
plot((0:1:20)',StDevs(1)*squeeze(phi1_series(2,1,:)),'r')
hold on
% next, DP shock effect on returns
plot((0:1:20)',StDevs(2)*squeeze(phi1_series(2,2,:)),'k.-')
% next, GDP shock effect on returns
plot((0:1:20)',StDevs(3)*squeeze(phi1_series(2,3,:)),'b–')
hold off

xlabel('quarters since shock')
ylabel('Impulse-Response')
legend('R shock','DP shock', 'D\_gdp shock')
title('Impulse-Response of DP-ratio from VAR shocks')
```

Impulse-Response Plot for DP ratio

- Note the very persistent effect of the valuation ratio (DP-ratio) shock
- Note that GDP growth increase DP due to higher expected returns (higher discount rates)



Impulse-Response Plots: MatLab code

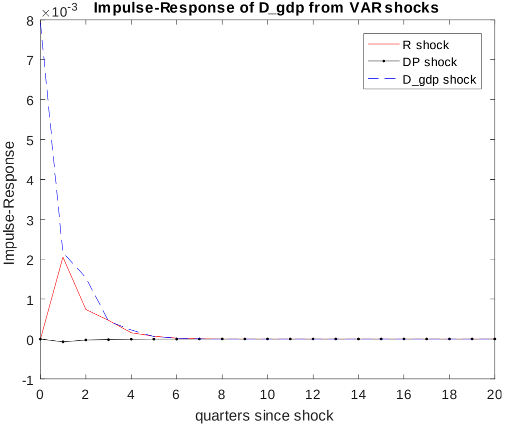
```
% the IR for Delta gdp of a shock to returns, DP, and GDP.

% first, let the return shock happen at time t, with all other shocks set
% to zero, and trace out the response
plot((0:1:20)',StDevs(1)*squeeze(phi1_series(3,1,:)),'r')
hold on
% next, DP shock effect on returns
plot((0:1:20)',StDevs(2)*squeeze(phi1_series(3,2,:)),'k.-')
% next, GDP shock effect on returns
plot((0:1:20)',StDevs(3)*squeeze(phi1_series(3,3,:)),'b–')
hold off

xlabel('quarters since shock')
ylabel('Impulse-Response')
legend('R shock','DP shock', 'D\_gdp shock')
title('Impulse-Response of D\_gdp from VAR shocks')
```

Impulse-Response Plot for D_gdp

- Note the DP-shock has no impact, thus no persistent predictions for GDP growth
- Note that returns positively forecast future GDP growth
- Also, significant positive autocorrelation in GDP growth



Sims VAR shock orthogonalization

The impulse responses just derived are a bit strange if the shocks in the VAR are correlated

- What does it mean to hold the other shocks constant at zero, when typically they would not be constant given non-zero shock correlations...?

A natural (and most common) approach is to write the (demeaned) VAR as:

$$\mathbf{z}_{t+1} = \boldsymbol{\phi}\mathbf{z}_t + \Sigma_C \boldsymbol{\varepsilon}^O_{t+1},$$

where

1. $\boldsymbol{\varepsilon}^{\mathbf{O}}_{\mathbf{t+1}}$ is a 3×1 vector of uncorrelated shocks with mean zero and unit variance.
2. Let the variance covariance matrix of the original shocks be Σ. Then, let Σ_C be the Cholesky decomposition of this matrix
   - The Cholesky decomposition yields a lower triangular matrix with the property that

     $$\Sigma_C \boldsymbol{\varepsilon}^O_{t+1} = \boldsymbol{\varepsilon}_{t+1},$$

     where $\boldsymbol{\varepsilon}_{\mathbf{t+1}}$ is the 3×1 vector of original shocks estimated in the earlier OLS regressions, with mean zero and variance-covariance matrix Σ.

The Cholesky Decomposition

The Cholesky decomposition of the positive-definite variance-covariance matrix Σ satisifes:

$$\Sigma = \Sigma_C \Sigma_C',$$

where Σ_C is a lower-triangular matrix.
Concretely, in our case:

$$\boldsymbol{\varepsilon}_t = \Sigma_C \boldsymbol{\varepsilon}_t^O$$

means (using earlier notation)

$$\boldsymbol{\varepsilon}_t = \begin{bmatrix} \sigma_R \varepsilon_{R,t} \\ \sigma_{DP} \varepsilon_{DP,t} \\ \sigma_{gdp} \varepsilon_{gdp,t} \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma_C^{(1,1)} \varepsilon_{1,t}^O \\ \Sigma_C^{(2,1)} \varepsilon_{1,t}^O + \Sigma_C^{(2,2)} \varepsilon_{2,t}^O \\ \Sigma_C^{(3,1)} \varepsilon_{1,t}^O + \Sigma_C^{(3,2)} \varepsilon_{2,t}^O + \Sigma_C^{(3,3)} \varepsilon_{3,t}^O \\ 0 \end{bmatrix}$$

- Now, each shock can be meaningfully considered in isolation
  - Note, however, that the ordering of variables in the VAR matters!

VAR variable ordering

When applying the Cholesky decomposition to get orthogonalized shocks, the ordering of variables matters.

- The first shock (top row) is the main shock, subsequent rows have additional shocks capturing the part of the other shocks that are orthogonal to the previous shocks

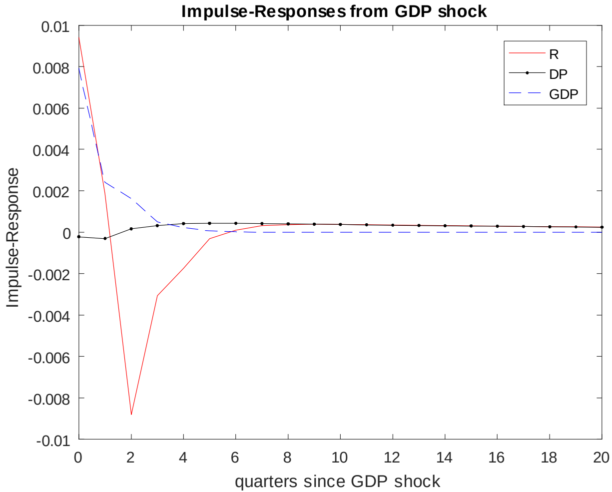
Thus, it is sensible to rank the shocks in terms of their perceived causality

- For instance, most models would imply that the GDP shock is the fundamental cash flow shock driving the economy
- Perhaps preference shocks (e.g., to risk aversion) affects the valuation ratio, so let the DP shock be second and then the return shock last?
  - I'm not so convinced by the ordering of these two...

1. Order the original shock vector accordingly, get variance covariance matrix
2. Do Cholesky decomposition, make sure your routine delivers a lower-triangular matrix (MatLab's chol() commend delivers upper-triangular)
3. Redo impulse-response. Look at effect of GDP shock on all three variables, then the DP shock, then the R shock.

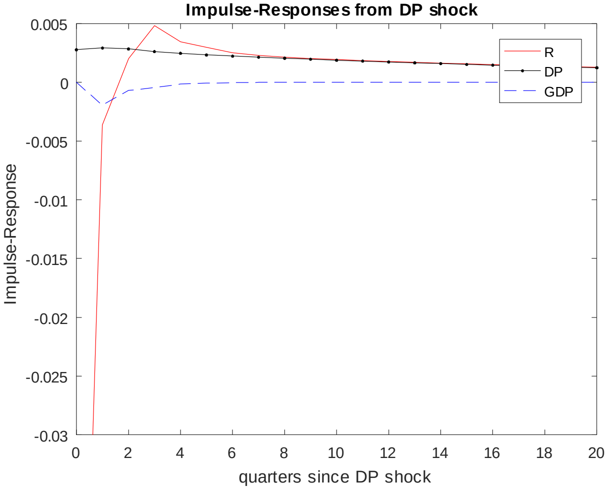
Orthogonal IRs from our example

Effect of 1 st.dev. positive GDP shock on all variables



Orthogonal IRs from our example

Effect of 1 st.dev. positive DP shock (orthogonal to GDP shock) on all variables



Orthogonal IRs from our example

Effect of 1 st.dev. positive R shock (orthogonal to GDP and DP shocks) on all variables