Empirical Methods in Finance
Homework 3

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Problem 1: ARMA basics

Consider the ARMA(1,1):

$$y_t = 0.95 \times y_{t-1} - 0.9 \times \varepsilon_{t-1} + \varepsilon_t, \tag{1}$$

where ε_t is i.i.d. Normal with mean zero and variance $\sigma^2 = 0.05^2$. In the below you need to show your work in order to get full credit.

1. What is the first-order autocorrelation of y_t?
2. What is the second-order autocorrelation of y_t? Also, what is the ratio of the second-order to first-order autocorrelation equal to? Give some intuition for this result.
3. If $y_t = 0.6$ and $\varepsilon_t = 0.1$, what is (i) $E_t[y_{t+1}]$, (ii) $E_t[y_{t+2}]$ given the ARMA model?
4. Let $\hat{x}_t = E_t[y_{t+1}]$ where the expectation is taken using the ARMA model. What is the unconditional mean, standard deviation, and first-order autocorrelation of $\hat{x}_t$?

Problem 2: Year-on-year quarterly data and ARMA dynamics

A substantial amount of quantity data, such as earnings, exhibit seasonalities. These can be hard to model. It is therefore common to use so-called Year-on-Year data (e.g., Q1 earnings vs Q1 earnings a year ago, Q2 earnings vs Q2 earnings a year ago, etc). In this problem we will see that such a practice can induce MA-terms due to the overlap in the quarterly year-on-year observations.

Assume the true quarterly log market earnings follow:

$$
\begin{aligned}
e_t &= e_{t-1} + x_t, \\
x_t &= \phi x_{t-1} + \varepsilon_t,
\end{aligned}
$$

where $var(\varepsilon_t) = \sigma_\varepsilon^2 = 1$ and ε_t is i.i.d. over time t.

The earnings data you are given is year-on-year earnings growth, which in logs is:

$$
y_t \equiv e_t - e_{t-4}.
$$

1. Assume $\phi = 0$. Derive autocovariances of order 0 through 5 for y_t. I.e., $cov(y_t, y_{t-j})$ for $j = 0, ..., 5$.

2. Assume $\phi = 0$. Determine the number of AR lags and MA lags you need in the ARMA(p,q) process for y_t. Give the associated AR and MA coefficients.

3. Optional: assume $1 > \phi > 0$. Repeat 1. and 2. under this assumption.

Problem 3: Market-timing and Sharpe ratios (a little harder)

Much of this class is about prediction. In this problem you will derive how market timing can improve the unconditional Sharpe ratio of a fund. The market timing is based on "forecasting regressions" akin to those we undertake in a VAR. However, we are only forecasting one period ahead here.

Assume you have an estimate of expected annual excess market returns for each time t, called x_t. You estimate the regression

$$
R^e_{t+1} = \alpha + \beta x_t + \varepsilon_{t+1},
$$

and obtain $\hat{\alpha} = 0$, $\hat{\beta} = 1$, and $\sigma(\hat{\varepsilon}_{t+1}) = 15\%$. Further, the sample mean and standard deviation of x_t are both 5%.

1. Calculate the standard deviation of excess returns based on the information given.

2. Calculate the R^2 of the regression based on the information given.

3. Calculate the sample Sharpe ratio of excess market returns based on the information given.

4. Recall from investments that myopic investors chooses a fraction of wealth

$$\alpha_t = \frac{E_t\left[R^e_{t+1}\right]}{\gamma\sigma^2_t\left[R^e_{t+1}\right]}$$

in the risky asset (the market) at each time t, where we assume risk aversion coefficient, γ, equals 40/9. Further, assume that the residuals ε_{t+1} are i.i.d., so $\sigma_t(\varepsilon_{t+1}) = 15\%$ for all t. Given this, calculate the weight the investor chooses to hold in the risky asset if $x_t = 0\%$ and if $x_t = 10\%$. What is conditional Sharpe ratio in each of these cases?

5. Assume T is large (i.e., $T \to \infty$) and that x_t is either 0% or 10% at each time t, with equal probability (0.5).

   (a) What is the unconditional average excess return for an investor that holds α_t each period?

   (b) What is the unconditional standard deviation? The following may be helpful for calculating the unconditional variance. You could also simulate a very long series to check your math.

$$\begin{aligned} Var\left(\alpha_t R^e_{t+1}\right) &= E\left[E_t\left[\left(\alpha_t R^e_{t+1}\right)^2\right]\right] - E\left[E_t\left[\alpha_t R^e_{t+1}\right]\right]^2 \\ &= E\left[\alpha_t^2 E_t\left[\left(R^e_{t+1}\right)^2\right]\right] - E\left[\alpha_t E_t\left[R^e_{t+1}\right]\right]^2 \\ &= E\left[\alpha_t^2\left(x_t^2 + \sigma_t^2\left(\varepsilon_{t+1}\right)\right)\right] - E\left[\alpha_t x_t\right]^2. \end{aligned}$$

   (c) Finally, what is the unconditional Sharpe ratio of this strategy?

   (d) Now, assume the volatility of x_t is higher: it can take the values -5% and $+15\%$ with equal probability.

      i. What is the implied R^2 of a forecasting regression of future excess returns on x_t assuming again that $\hat{\alpha} = 0$ and $\hat{\beta} = 1$?

      ii. What is the unconditional Sharpe ratio the investor that follows the risky asset share rule given above in (4)? Note that a higher R^2 implies a higher Sharpe ratio.