

# Lecture Note 10

## The Cross-Section of Stock Returns: Factor Models

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# Overview of Lecture Note 10: The Cross-Section

## 1 The single-factor model

- ▶ Portfolio choice and the curse of dimensionality
- ▶ The simplest covariance matrix: A single factor
- ▶ Single-factor model math
- ▶ Portfolios or single stocks: estimation error and beta stability

## 2 Multi-factor models

- ▶ Math
- ▶ Properties
- ▶ Estimation

## 3 Applications

- ▶ Firm characteristics-based factors (the Fama-French model)

# Portfolio Problem

Portfolio of  $N$  securities with  $N \times N$  variance-covariance matrix  $\Sigma$

Vector of  $N \times 1$  portfolio weights:  $\mathbf{w}$

Variance of portfolio return is given by:

$$\begin{aligned}\sigma_P^2 &= \mathbf{w}'\Sigma\mathbf{w} \\ &= \sum_{i,j=1}^N w_i w_j \sigma_{ij} \\ &= \sum_{i=1}^N w_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{j=1, j \neq i}^N w_i w_j \sigma_i \sigma_j \rho_{ij}\end{aligned}$$

Markowitz (1952): choose  $\mathbf{w}$  to minimize the variance of the portfolio for a given mean  $\mu_P$ .

# Inputs to Portfolio Analysis

**Standard deviation** of portfolio return:

$$\begin{aligned}\sigma_P &= \left[ \sum_{i,j=1}^N w_i w_j \sigma_{ij} \right]^{1/2} \\ &= \left[ \sum_{i=1}^N w_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{j=1, j \neq i}^N w_i w_j \sigma_i \sigma_j \rho_{ij} \right]^{1/2} \\ &= [\mathbf{w}' \mathbf{\Sigma} \mathbf{w}]^{1/2}\end{aligned}$$

**Mean** of portfolio return:

$$E[R_P] = \sum_{i=1}^N w_i E[R_i] = \sum_{i=1}^N w_i \mu_i = \mu_P$$

# Inputs to Portfolio Analysis

① Expected returns  $E[R_i] = \mu_i$  and risk  $\Sigma$ :

- ▶ we could use historical data to estimate expected returns
- ▶ we could use equilibrium models to estimate expected returns (see Black and Litterman (1992))

② Pairwise correlations:

- ▶ hard to estimate
- ▶ unstable through time

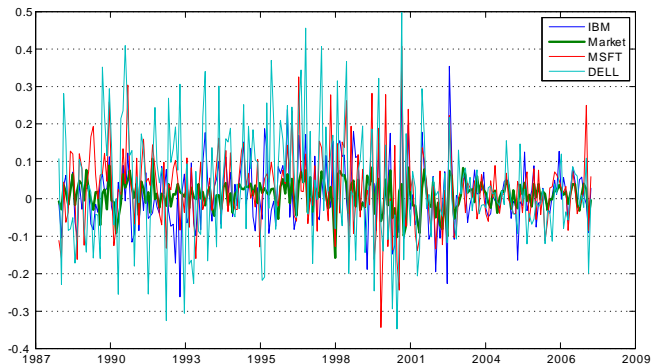
# Curse of Dimensionality

- number of parameters to estimate necessary to carry out portfolio analysis is staggering.
- suppose you follow between 150 and 250 stocks
- we need all correlation coefficients  $\rho_{ij}$  for stocks  $i$  and  $j$
- total number of free correlations in an  $N \times N$  covariance matrix  $\Sigma$  is  $\frac{N(N-1)}{2}$  (11175 in the case of  $N=150$  stocks)
- in the case of a large number of assets: traditional mean-variance analysis tends to perform very poorly out of sample; see, e.g. DeMiguel, Garlappi, and Uppal (2009)
- For more on the impact of the curse of dimensionality in statistics, see Chapter 2 of Hastie, Tibshirani, and Friedman (2009)

# Single Factor Model

- single factor models: co-movement is due to a single factor.
- **economic factor models**: use economic theory or economic intuition to determine the factor.
- **statistical factor models**: use the data to determine the factor.
- some textbooks call this a **single index model**. The factor is often a stock market index.
- Capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965), and Mossin (1966) is the most famous single-factor model.
  - ▶ The 'market return' is the factor

# Stock Returns



Monthly data from CRSP: The Market Return is the CRSP Value-Weighted Return on NYSE-AMEX-NASDAQ. Sample 1987:1-2007.12



# Single Factor Model: The Market Model

- stocks move in tandem due to a common factor  $f_t$
- the return on a stock  $i$  can be written as:

$$R_{it} = \alpha_i + \beta_i f_t + \varepsilon_{it}$$

- let  $R_{mt}$  denote the market return.
- suppose the common factor is the market. Then, we have

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$$

- ▶  $\varepsilon_{it}$  is the component of the return that is independent of the market return  $R_{mt}$ .
- ▶ assume  $E[\varepsilon_{it}] = 0$ .
- ▶  $\beta_i$  is a constant.

# Single Factor Model

Two key (simplifying) assumptions:

- ① disturbances are independent of the market return:

$$\text{Cov}[\varepsilon_{it}, R_{mt}] = 0$$

- ②  $\varepsilon_{it}$  is independent of  $\varepsilon_{jt}$ :

$$\text{Cov}[\varepsilon_{it}, \varepsilon_{jt}] = 0$$

$\Rightarrow$  no effects beyond the market that account for covariation (e.g. industry effects)

# Moments of the Single Factor Model

- This model then implies:

$$E[R_{it}] = \alpha_i + \beta_i E[R_{mt}], \quad i = 1, \dots, N$$

$$\mu_i = \alpha_i + \beta_i \mu_m, \quad i = 1, \dots, N$$

$$V[R_{it}] = \beta_i^2 V[R_{mt}] + V[\varepsilon_{it}], \quad i = 1, \dots, N$$

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2, \quad i = 1, \dots, N$$

$$\text{Cov}[R_{it}, R_{jt}] = \beta_i \beta_j \sigma_m^2, \quad i = 1, \dots, N$$

# Single Factor Model Portfolio Math

The expected return on a portfolio is:

$$\begin{aligned}\mu_p &= \sum_{i=1}^N w_i \mu_i \\ &= \sum_{i=1}^N w_i \alpha_i + \mu_m \sum_{i=1}^N w_i \beta_i\end{aligned}$$

Recall that the variance of the portfolio return is:

$$V[R_P] = \left[ \sum_{i=1}^N w_i^2 (\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2) + \sum_{i=1}^N \sum_{j=1}^N w_i w_j \beta_i \beta_j \sigma_m^2 \right]$$

Note the dramatic decrease in number of parameters to be estimated

- $3N + 2$  estimates:  $\alpha_i, \beta_i, \sigma_{\varepsilon_i}^2, i = 1, \dots, N$  and  $\sigma_m^2, E[R_m]$

# Single Factor Model Portfolio Math (cont'd)

- The beta of a portfolio  $\beta_p$  is:

$$\beta_p = \sum_{i=1}^N w_i \beta_i$$

where  $w_i$  are the portfolio weights

- The alpha of a portfolio  $\alpha_p$  is:

$$\alpha_p = \sum_{i=1}^N w_i \alpha_i$$

where  $w_i$  are the portfolio weights

- then, given the factor model, the expected return on the portfolio is:

$$\mu_p = \alpha_p + \beta_p \mu_m$$

# Singel Factor Model: Market Model

- if the factor  $f_t$  is the market return  $R_{mt}$ , then the factor is a traded asset.
- the model should apply to the market return as well.
- then the expected return on the market portfolio is:

$$\mu_m = \alpha_m + \beta_m \mu_m$$

- this immediately implies that:
  - ▶  $\alpha_m = 0$
  - ▶  $\beta_m = 1$
- the market portfolio has a beta of one and an alpha of zero.

# Single Factor Model Parameters

- to use the single-factor model, we need *estimates* of  $\alpha_i$  and  $\beta_i$
- what you need, is the betas going forward (i.e. the beta for a security that applies over the next say 5 years)
- we typically use historical betas as estimates of future betas
  - ▶ these are **backward** looking
  - ▶ but they are informative about **future** betas

# Single Factor Model Estimation

- define

$$\mathbf{X} = \begin{bmatrix} 1 & R_{m,1} \\ 1 & R_{m,2} \\ \vdots & \vdots \\ 1 & R_{m,T} \end{bmatrix}, \mathbf{y}_i = \begin{bmatrix} R_{i,1} \\ R_{i,2} \\ \vdots \\ R_{i,T} \end{bmatrix}$$

- write the single factor model as:

$$\mathbf{y}_i = \mathbf{X}\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i$$

with  $\boldsymbol{\beta}_i = [\alpha_i \quad \beta_i]'$

- Use least squares to estimate  $\boldsymbol{\beta}_i$  for each asset  $i$



# Sampling Error

- consider the case of individual securities
- the estimator of the slope coefficient is :

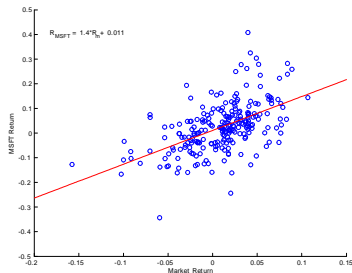
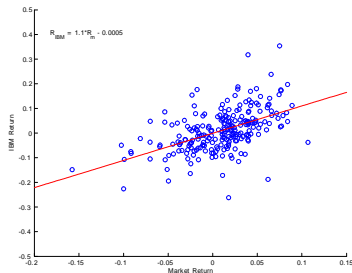
$$\hat{\beta}_i = \frac{\sum_{t=1}^T (R_{mt} - \bar{R}_m)(R_{it} - \bar{R}_i)}{\sum_{t=1}^T (R_{mt} - \bar{R}_m)^2}$$

- the variance of the estimator of the slope coefficient is :

$$V(\hat{\beta}_i - \beta_i) = \frac{1}{T} \frac{\text{var}(\hat{\varepsilon}_{it})}{\text{var}(R_{mt})}$$

- measure of sampling error
- sampling error is influenced by  $\sigma_{\varepsilon_i}^2$ , which can be large for individual stocks

# IBM and MSFT

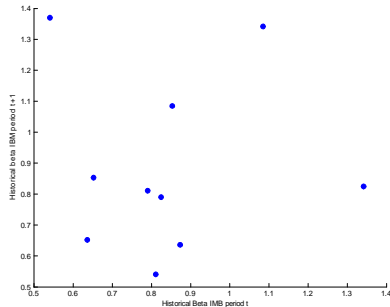
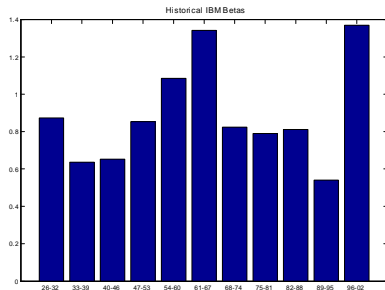


Monthly data from CRSP. Sample 1987.1-2007.12. IBM is left and MSFT is right.

# Time-Variation in Betas

- empirical evidence suggests there is quite a lot of time variation in historical estimates of  $\beta$ 's over different samples
  - ▶ could be because true betas are varying over time
  - ▶ could be sampling error
- for individual securities, little relation between estimated betas in subsequent time samples
- for portfolios of securities sorted on certain characteristics  $\Rightarrow$  more stability
- for portfolios of securities  $\Rightarrow$  less sampling error

# Instability in IBM Betas



Monthly data from CRSP. Sample 1926.1-2002.12. Estimates over 7-year non-overlapping intervals.

# Portfolio sorts

- the academic finance literature sorts stocks into portfolios
- statistically, this reduces idiosyncratic volatility of the 'asset'
  - ▶ more noisy measurement of the true beta at the individual security level (larger 'sampling error')
  - ▶ less at the portfolio level
- economically, risk characteristics of individual securities might change over time
  - ▶ **key idea:** sort individual securities into portfolios based on characteristics
  - ▶ **examples:**
    - 1 size (market cap)
    - 2 book-to-market
    - 3 industries

# Portfolio Sampling Error

- consider the case of a portfolio with  $N$  assets.
- the estimator of the slope coefficient is :

$$\hat{\beta}_p = \frac{\sum_{t=1}^T (R_{mt} - \bar{R}_m)(R_{pt} - \bar{R}_p)}{\sum_{t=1}^T (R_{mt} - \bar{R}_m)^2}$$

- the variance of the estimator of the slope coefficient is :

$$V(\hat{\beta}_p - \beta_p) = \frac{1}{T} \frac{\text{var}(\hat{\varepsilon}_{pt})}{\text{var}(R_{mt})}$$

where

$$\text{var}(\hat{\varepsilon}_{pt}) = \sum_{i=1}^N w_i^2 \text{var}(\hat{\varepsilon}_{it})$$

## Sampling Error (cont'd)

- originally, we have  $N$  total firms.
- separate  $N$  firms into a smaller set of  $j = 1, \dots, J$  portfolios. Each portfolio has  $N^{(j)}$  assets in it. Assume no firm shows up in more than 1 portfolio at a given time.
- consider an equally-weighted portfolio with  $w_i = \frac{1}{N^{(j)}}$ ,  $i = 1, \dots, N^{(j)}$
- by building portfolios we reduce sampling error, by pushing this term

$$\sigma_{\varepsilon_p}^2 = \frac{1}{(N^{(j)})^2} \sum_{i=1}^{N^{(j)}} \sigma_{\varepsilon_i}^2$$

to zero as  $N^{(j)}$  increases.

- moreover, we might create more stable betas

# Size and Book-to-Market

Fama and French (1992, 1993) argue that *size* and *book/market* are key risk characteristics

They construct portfolios by sorting stocks on these characteristics

- The portfolios, which are constructed at the end of each June, are the intersections of 2 portfolios formed on size (market equity, ME) and 3 portfolios formed on the ratio of book equity to market equity (BE/ME).
  - ▶ The size breakpoint for year  $t$  is the median NYSE market equity at the end of June of year  $t$ .
  - ▶ BE/ME for June of year  $t$  is the book equity for the last fiscal year end in  $t-1$  divided by ME for December of  $t-1$ .
  - ▶ The BE/ME breakpoints are the 30th and 70th NYSE percentiles.



## Size and Book-to-Market (cont'd)

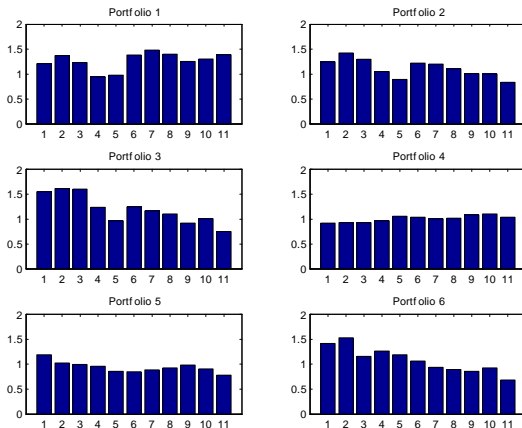
The six portfolios are

size vs. B/M	low	medium	high
small	1	2	3
big	4	5	6

We can give them nice labels

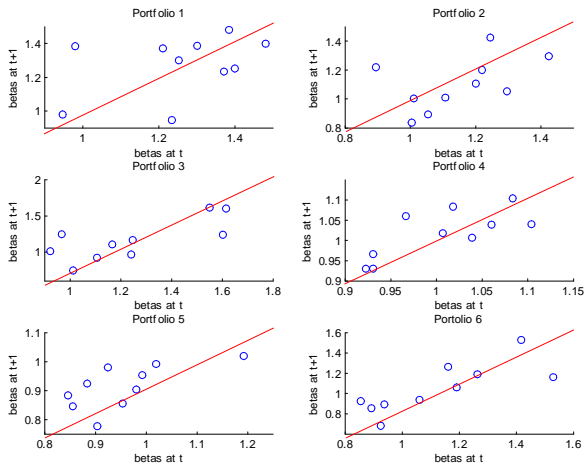
size vs. B/M	low	medium	high
small	Small Growth	Small Neutral	Small Value
big	Big Growth	Big Neutral	Big Value

# Portfolio Betas



6 Fama-French portfolios sorted on size and book-to-market. Monthly data from CRSP. Sample 1926.1-2002.12. Estimates over 11 non-overlapping 7-year intervals.

# Stability in Portfolio Betas



6 Fama-French portfolios sorted on size and book-to-market. Monthly data from CRSP. Sample 1926.1-2002.12. Estimates over 11 non-overlapping 7-year intervals.

# Accuracy in Historical Betas

- empirical evidence suggests historical portfolio betas are better predictors of future portfolio betas
- substantial gain in stability
- sorting stocks into portfolios is one way to produce stability.
- there are other ways to produce stability. In statistics, these are called 'shrinkage estimators' or Bayesian estimators
  - ▶ the basic idea is to push our parameter estimates to a prior value that is economically plausible.
  - ▶ Vasicek (1976) and Blume (1975) are early applications of this idea

# Multi-factor Models

- so far, we have assumed that all of the covariation among securities is due to a single common factor  $f_t$
- the common factor  $f_t$  is typically taken to be the market return  $R_{mt}$
- however, there might be other reasons security prices co-move  $\Rightarrow$  multiple factors
- what should the extra factors be?
- some examples:
  - ① industry factors
  - ② macro-economic factors
  - ③ factors built from accounting variables
  - ④ statistical (& latent) factors

# Multi-factor Models (cont'd)

- stocks move in tandem
- the return on a stock  $i$  can be written as:

$$R_{it} = \alpha_i + \beta_{i1}f_{1t} + \beta_{i2}f_{2t} + \dots + \beta_{iK}f_{Kt} + \varepsilon_{it}, \quad i = 1, \dots, N$$

- $f_{it}$  are correlated with each other.
- for example:
  - ▶  $f_{1t}$  is a market index
  - ▶  $f_{2t}$  is an interest rate index

# Multi-factor Models in Matrix Notation

- let  $\mathbf{f}_t$  denote a  $K \times 1$  vector of factors
- let  $\boldsymbol{\beta}_i$  denote a  $K \times 1$  vector of factor loadings
- we can write the model in matrix notation as

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}, \quad i = 1, \dots, N$$

- we can then stack all  $N$  assets together

$$\mathbf{R}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{f}_t + \boldsymbol{\varepsilon}_t$$

- $\mathbf{R}_t$  is a  $N \times 1$  vector of returns,  $\boldsymbol{\beta}$  is a  $N \times K$  matrix of factor loadings
- let  $\boldsymbol{\Sigma}_f$  denote the  $K \times K$  covariance matrix of the factors.
- let  $\boldsymbol{\Sigma}_\varepsilon$  denote the  $N \times N$  covariance matrix of the idiosyncratic errors

# Multi-factor Model: Assumptions

- 1 disturbances are independent of the factor:

$$\text{Cov}[\varepsilon_{it}, f_{jt}] = 0$$

- 2  $\varepsilon_{it}$  is independent of  $\varepsilon_{jt}$ :

$$\text{Cov}[\varepsilon_{it}, \varepsilon_{jt}] = 0$$

no effects beyond the factors that we have included that account for covariation. This means that  $\Sigma_{\varepsilon}$  is diagonal.



# Multi-factor Model: Moments

Mean:

$$\begin{aligned} E[\mathbf{R}_t] &= \alpha + \beta E[\mathbf{f}_t] \\ &= \alpha + \beta \mu_f \end{aligned}$$

Covariance matrix:

$$\begin{aligned} \text{Var}[\mathbf{R}_t] &= \beta \text{Var}[\mathbf{f}_t] \beta' + \text{Var}[\varepsilon_t] \\ &= \beta \Sigma_f \beta' + \Sigma_\varepsilon \end{aligned}$$

# Number of Parameters to be Estimated

- suppose the factors  $f_t$  are observable
- total number of parameters to be estimated:

$$K + 2N + NK + \frac{K(K+1)}{2}$$

- ▶ we need estimates of the  $K \times 1$  vector  $\mu_f$
- ▶ we need estimates of  $\alpha$  and the diagonal elements of  $\Sigma_\varepsilon$
- ▶ we need estimates of the  $N \times K$  matrix of factor loadings  $\beta$
- ▶ we need estimates of the covariance matrix  $\Sigma_f$ . It has  $\frac{K(K+1)}{2}$  free parameters.

# Industry Factor Model

- common co-movement between stocks not driven by the market
- some of this was related to industry effects
  - ▶ 30 to 50 % of variation in stock returns is due to the market
  - ▶ 10 % of variation in stock returns is due to industry effects

# Industry Factor Model: General

Seems most appropriate for firms that operate in multiple industries

The return on a stock  $i$  can be written as:

$$R_{it} = \alpha_i + \beta_{im} R_{mt} + \beta_{i1} R_{1t} + \dots + \beta_{iK} R_{Kt} + \varepsilon_{it}, \quad i = 1, \dots, N$$

- the  $R_{jt}, j = 1, \dots, K$  are returns on industries
- $R_{mt}$  is the market return

# Industry Factor Model: Simple

Seems more appropriate for firms that operate in a single industry

The return on a stock  $i$  can be written as:

$$R_{it} = \alpha_i + \beta_{im} R_{mt} + \beta_{ij} R_{jt} + \varepsilon_{it}, \quad i = 1, \dots, N$$

- the  $R_{jt}$  is the return on an industry based index in which firm  $i$  operates
- $R_{mt}$  is the market index

# Fama and French (1993) Model

The Fama/French factors are constructed using the 6 value-weighted portfolios formed on size and book-to-market.

- SMB (Small Minus Big)

- ▶  $SMB = 1/3 (\text{Small Value} + \text{Small Neutral} + \text{Small Growth}) - 1/3 (\text{Big Value} + \text{Big Neutral} + \text{Big Growth})$ .

- HML (High Minus Low)

- ▶  $HML = 1/2 (\text{Small Value} + \text{Big Value}) - 1/2 (\text{Small Growth} + \text{Big Growth})$ .

- $R_{mt} - R_{ft}$ , the excess return on the market, is the value-weight return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates).

## Fama and French (1993) Model (cont'd)

- multi-factor model for monthly returns:

$$R_{it} = \alpha_i + \beta_{i,M} R_t^m + \beta_{i,SMB} R_t^{SMB} + \beta_{i,HML} R_t^{HML} + \varepsilon_{it}$$

- we will use  $N = 10$  test assets formed on industries

# Fama and French (1993): Industry Loadings

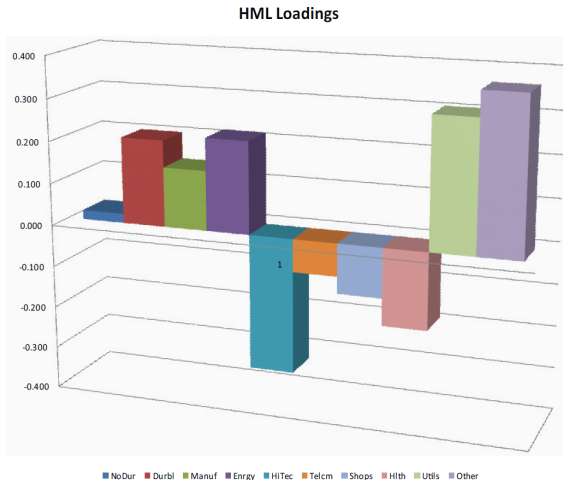
Column1	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
<b>HML</b>	0.022	0.205	0.142	0.219	-0.321	-0.078	-0.118	-0.182	0.308	0.368
se	0.021	0.034	0.016	0.034	0.027	0.027	0.025	0.031	0.032	0.018
<b>SMB</b>	-0.031	0.007	-0.013	-0.219	0.080	-0.133	0.078	-0.093	-0.160	0.064
se	0.023	0.038	0.017	0.037	0.029	0.030	0.028	0.034	0.035	0.020
<b>Market</b>	0.760	1.192	1.105	0.862	1.274	0.683	0.954	0.896	0.785	1.049
se	0.014	0.023	0.011	0.023	0.018	0.019	0.017	0.021	0.022	0.012
<b>R^2</b>	0.769	0.764	0.928	0.621	0.848	0.587	0.780	0.660	0.622	0.909

Sample: 1926.7 – 2008.12



# HML Loadings

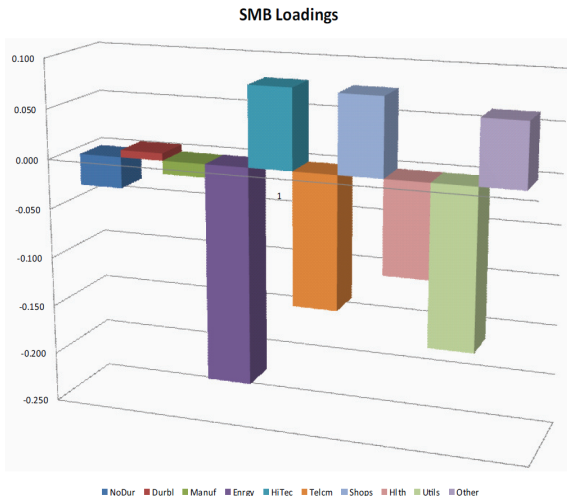
- $b_{HML}$  for each industry:



Sample: 1926.7 – 2008.12

# SMB Loadings

- $b_{SMB}$  for each industry:



Sample: 1926.7 – 2008.12