Lecture 8
Models of Volatility Dynamics

Lars A. Lochstoer

UCLA Anderson School of Management

Winter 2020

Outline

1. Stylized facts of volatility clustering
2. Realized Variance
3. ARCH models
4. GARCH models
   - GARCH(1,1)
   - I-GARCH, GARCH-M, EGARCH, GJR
5. Value-at-Risk

ARMA(p,q)

- so far, we have focused on modeling the conditional mean:

$$r_t = E\left[r_t | r_{t-1}, r_{t-2}, ...; \boldsymbol{\theta}\right] + \varepsilon_t$$

- the benchmark models for the conditional mean are ARMA models

$$r_t = \phi_0 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} + \ldots - \theta_q \varepsilon_{t-q}$$

- given estimates, $\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\theta}_{q-1}, \hat{\theta}_q$, we can calculate the residuals $\hat{\varepsilon}_t = v_t(0)$ recursively from the initial condition.

- the key modeling goal is to make sure that the residuals $\{\hat{\varepsilon}_t\}$ are white noise
  - No additional predictable components left

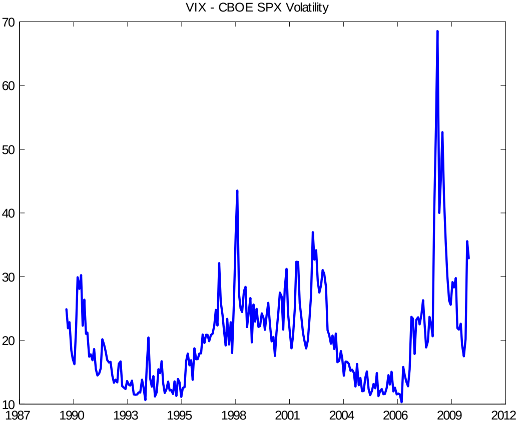
Volatility Clustering

- suppose we have a good model for the conditional mean μ_t (e.g. ARMA(p, q))
- now we look at the squared residuals $\{\hat{\varepsilon}_t^2\}$ to test for conditional heteroskedasticity
- for returns on most financial assets, there is a lot **more autocorrelation** in conditional second moments than in the conditional first moments.
- volatility is predictable!

Volatility?

- In most asset markets, volatility varies dramatically over time.
- episodes of high volatility seem to be clustered.
- we don't simply observe volatility
- how do we measure *vol* (vol is short-hand for volatility)?
  1. implied volatility (back out volatility from option prices)
  2. (non-parametric) realized volatility (e.g., realized volatility of stock returns over one-month using daily data)
  3. (parametric) model volatility

VIX: Option Implied Market Volatility



Annualized measure of 30-day vol. VIX, designed to measure the market's expectation of 30-day volatility implied by at-the-money S&P 100 Index (OEX) option prices. Monthly data. 1990-2009. VIX white paper: https://www.cboe.com/micro/vix/vixwhite.pdf.

Why do we care about Volatility?

- why do we care about modeling vol?
  1. portfolio allocation
  2. risk management
  3. option pricing

ARCH model of Engle (1982)

Definition

The ARCH(m) model is given by:

$$\varepsilon_t = \sigma_t \eta_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_m \varepsilon_{t-m}^2$$

where η_t is i.i.d., mean zero, has variance of one , $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$

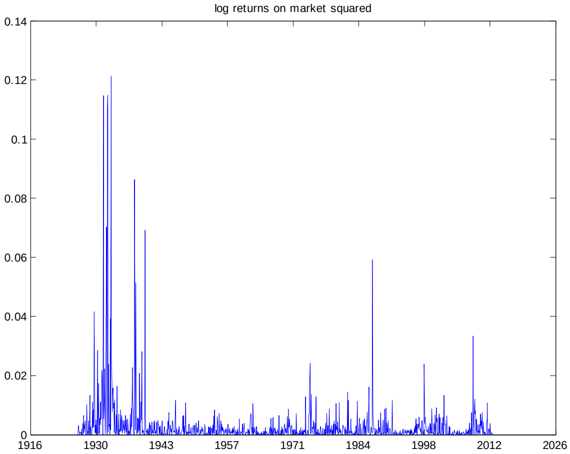
- the standard normal is a common choice for η_t
- the (standardized) Student's t is another option for η_t
- allows for large (small) shocks to be followed by more large (small) shocks: volatility clustering

Building a Volatility Model

1. estimate a model for the conditional mean (e.g. ARMA model)
2. use the residuals $\hat{\varepsilon}_t$ from the mean equation to test for ARCH effects
3. specify a volatility model for ARCH effects
4. check the fitted model

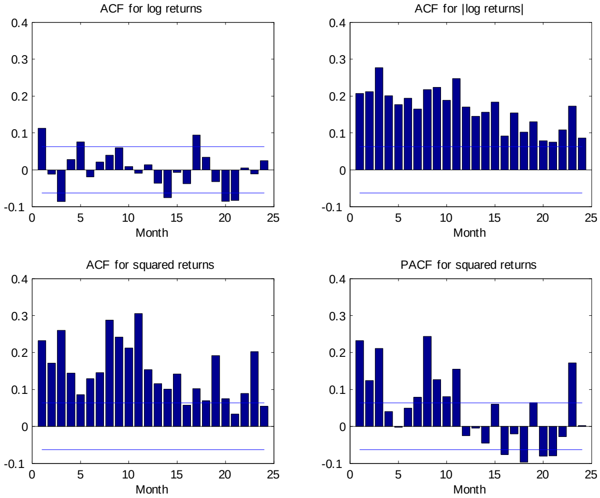
Stylized Facts on Volatility Clustering

Squared Monthly Log Market Returns



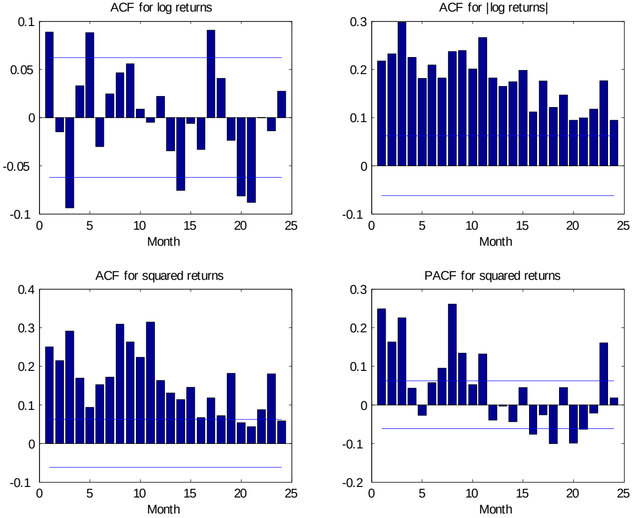
log stock returns on market -squared. Monthly data. 1926-2012.

(P)ACF's of absolute value, squared, and regular market returns; Long sample



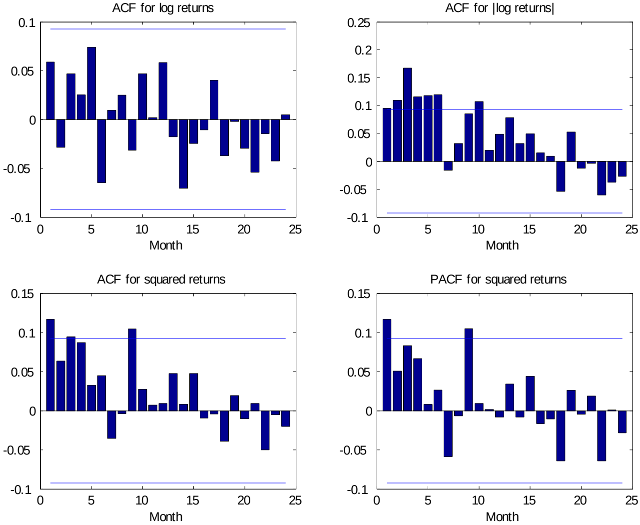
log stock returns on CRSP-VW. Monthly data, 1926-2012.

Dynamics of Variance: Monthly Returns, postwar Sample



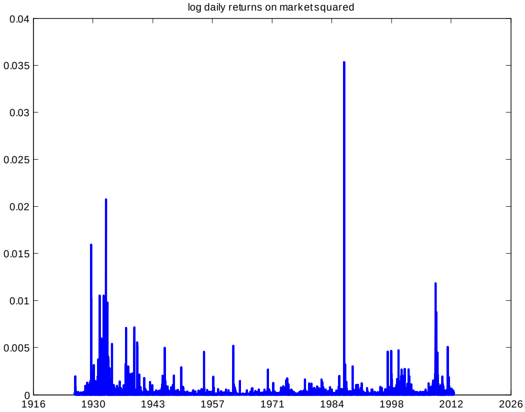
log stock returns on CRSP-VW. Monthly data. 1945-2012.

Dynamics of Variance: Monthly Returns, Short Sample



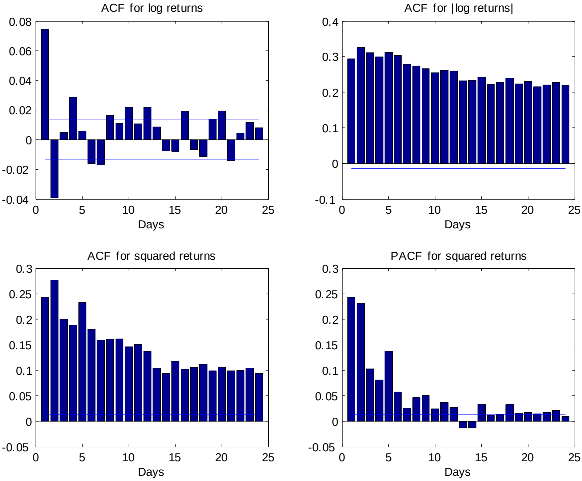
log stock returns on CRSP-VW. Monthly data. 1970-2012.

Daily Market Returns Squared



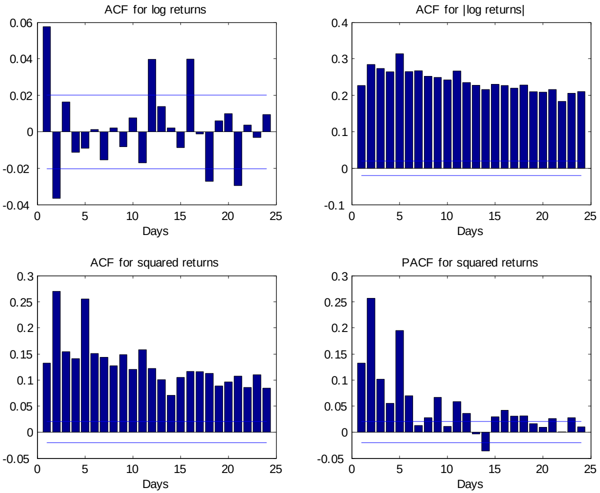
log stock returns on market -squared. Daily data. 1988-2012.

Dynamics of Variance: Daily Returns, Long Sample



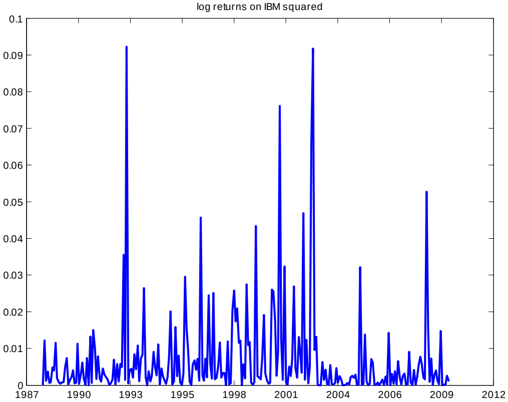
log stock returns on CRSP-VW. Daily data. 1926-2012.

Dynamics of Variance: Daily Returns, Short Sample



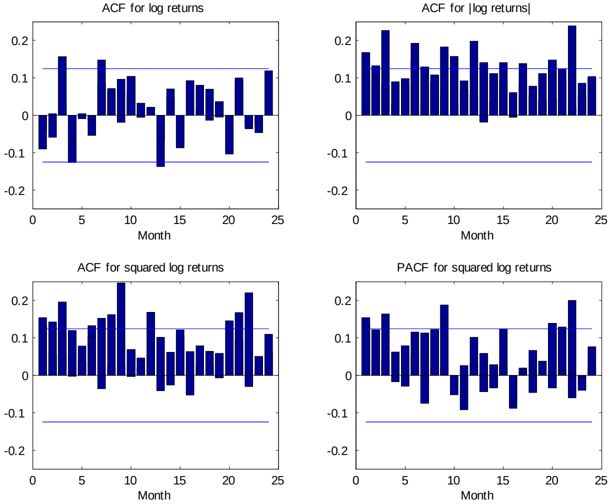
log stock returns on CRSP-VW. Daily data. 1970-2012.

IBM



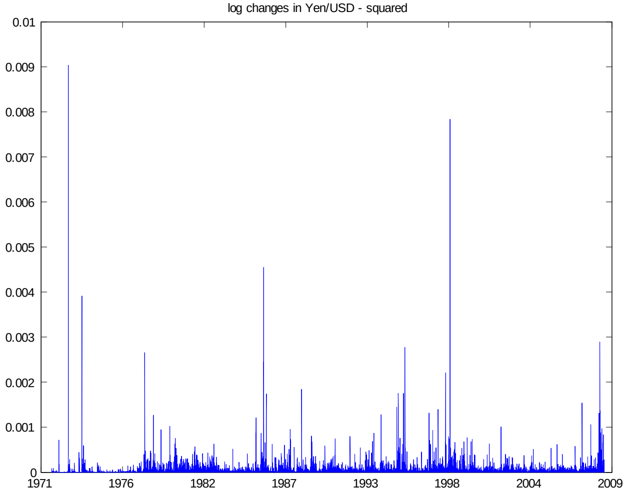
log stock returns on IBM -squared. Monthly data. 1988-2009.

IBM



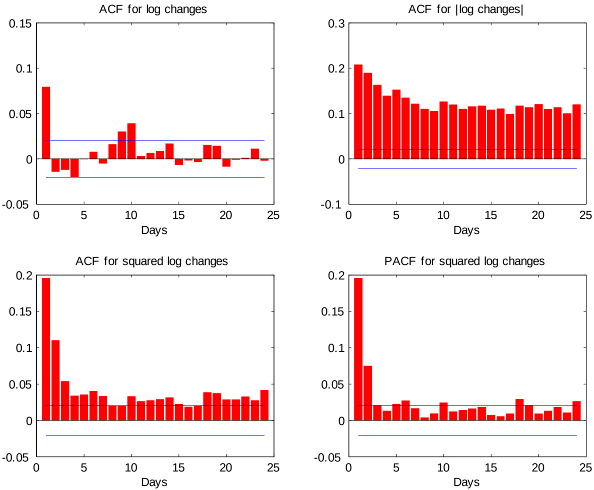
log stock returns on IBM. Monthly data. 1988-2009.

Yen/USD Squared Log Changes: Daily Data



log changes in Yen/USD squared. Daily data. 1971-2009.

Yen/USD (P)ACF's



log changes in Yen/USD. Daily data. 1971-2009.

Testing for ARCH

- estimate a model for the conditional mean: e.g. ARMA(p, q)
- use the squared residuals to test for ARCH
- specify a volatility model if ARCH effects are documented

Testing for ARCH(1)

- we can test for autoregressive conditional heteroscedasticity using a **LB** Q-test:
- suppose the model for the conditional mean is ARMA(p, q)
- estimate the model
- run a Q-test on the estimated squared residuals $\{\hat{\varepsilon}_t^2\}$
  - test the null that:

$$H_0 : \rho_1 = \rho_2 = \ldots = \rho_m = 0$$

    where ρ_i denotes the autocorrelation of the squared residuals
  - if you cannot reject the null, no need for ARCH machinery

Testing for ARCH(2)

- we can test for autoregressive conditional heteroscedasticity using a **LM** test:
- suppose the model is ARMA(p, q)
- estimate the model
- run a LM-test on the estimated squared residuals $\{\varepsilon_t^2\}$
  - test the null that $\alpha_i = 0, i = 1, 2, \ldots, m$:

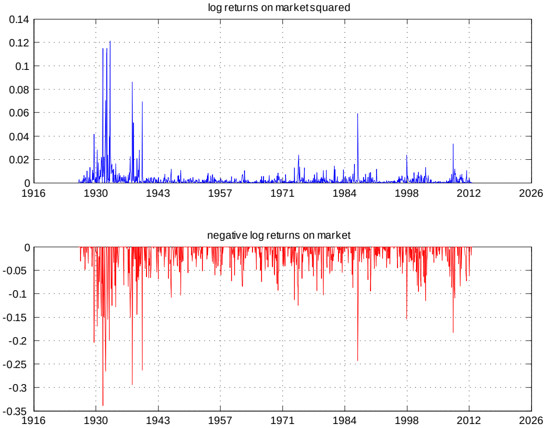
$$\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_m \varepsilon_{t-m}^2 + e_t, t = m+1, \ldots, T$$

  - test the null that:

$$H_0 : \alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$$

  - use the usual F-test, reject the null if F exceeds $\chi^2(\alpha)$ –the $(1-\alpha)$-*th* upper percentile

Leverage Effects: Monthly Returns on Market Squared



log stock returns on market -squared. Monthly data. 1926-2012.

Volatility Summary

- volatility clusters
- volatility is stationary
- leverage effect (asymmetry)
  - negative returns seem to be followed by larger increases in volatility than positive returns
  - first emphasized by Black (1976)
  - Black (1976) suggested that negative returns (decreases in price) change a company's debt/equity ratio, increasing their leverage.

Realized Variance

High-Frequency Data

- let r_t denote log of gross returns over a period of time t
- t may denote 1 day, 1 week, 1 month.
- suppose over the time t, we observe equally spaced log-returns $\{r_{t,i}\}_{i=1}^{n}$ at a higher frequency. Example: t is monthly and i are days in the month

$$r_t = \sum_{i=1}^{n} r_{t,i}$$

- for log returns, the variance of the return over time t is given by:

$$V\left(r_t \middle| F_{t-1}\right) = \sum_{i=1}^{n} V\left(r_{t,i} \middle| F_{t-1}\right) + 2\sum_{i<j} \mathrm{Cov}\left(r_{t,i}, r_{t,j} \middle| F_{t-1}\right)$$

- if we assume that the returns at the higher frequency are i.i.d., then

$$V\left(r_t | F_{t-1}\right) = nV\left(r_{t,1} | F_{t-1}\right)$$

where $V\left(r_{t,1} | F_{t-1}\right)$ can be estimated from:

$$\widehat{\sigma}_t^2 = \frac{\sum_{i=1}^{n}\left(r_{t,i} - \overline{r}_t\right)^2}{n-1}$$

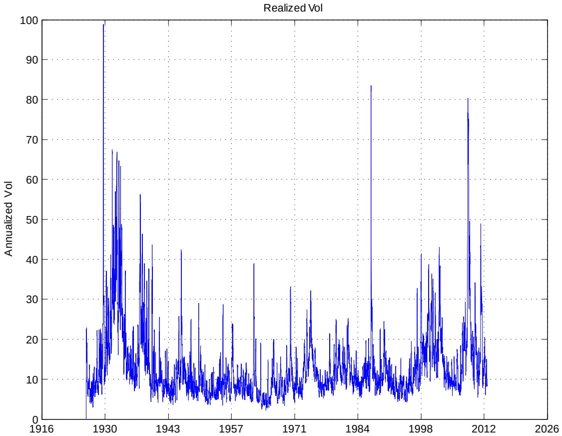
Realized Variance (Volatility)

- the estimate for the variance is given by:

$$\widehat{\sigma}_t^2 = \frac{\sum_{i=1}^{n} \left(r_{t,i} - \overline{r}_t\right)^2}{n-1}$$

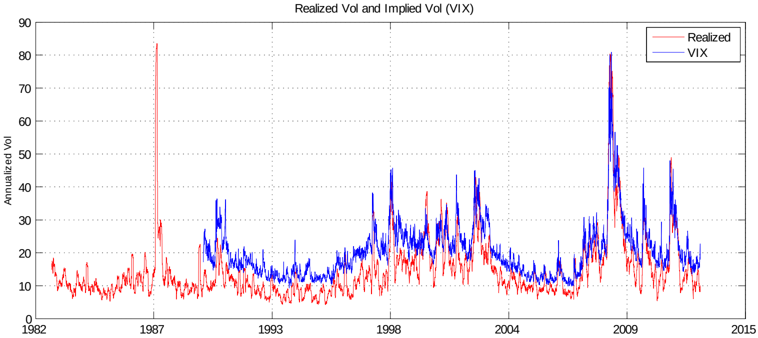
- in practice, t is often 1 day or 1 month.
- assumption: estimator assumes errors are i.i.d. meaning that volatility is roughly constant over t.
- note: the term **realized variance** is not used consistently in the literature.
  - many finance papers will call this **realized volatility**. In econometrics, **realized volatility** is the square root of **realized variance**.

Realized Volatility



Annualized (Monthly) Realized Vol. Daily data. 1926-2012.

Realized Volatility and Implied Vol



Annualized (Monthly) Realized Vol and the VIX. Daily data. 1983-2012.

ARCH models

ARCH(1)

Definition

The ARCH(1) process is given by:

$$\varepsilon_t \;=\; \sigma_t \eta_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

where η_t is i.i.d., mean zero, has variance of one , $\alpha > 0$ and $\alpha_i \geq 0$ for $i > 0$

Variance

- the unconditional mean is zero:

$$E\left[\varepsilon_t\right] = E(E\left[\varepsilon_t | F_{t-1}\right]) = E(\sigma_t E\left[\eta_t | F_{t-1}\right]) = 0$$

- the unconditional variance is:

$$V\left[\varepsilon_t\right] = E\left[\varepsilon_t^2\right] = E(E\left[\varepsilon_t^2 | F_{t-1}\right]) = E(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)$$

- because of stationarity, we get that:

$$V\left[\varepsilon_t\right] = \alpha_0 + \alpha_1 V\left[\varepsilon_t\right]$$

- as a result, the variance is:

$$V\left[\varepsilon_t\right] = \frac{\alpha_0}{1 - \alpha_1}$$

- we require that $0 \leq \alpha_1 < 1$

Fourth Moment

- the unconditional fourth moment is :

$$E\left[\varepsilon_t^4\right] = E(E\left[\varepsilon_t^4 | F_{t-1}\right]) = E\left(3\sigma_t^4\right) = 3\left[E(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)\right]^2$$

- using stationarity, this implies

$$m_4 = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}$$

which means $0 \leq \alpha_1^2 < 1/3$

- the unconditional kurtosis is given by:

$$\frac{E\left[\varepsilon_t^4\right]}{V\left[\varepsilon_t\right]^2} = 3\frac{1-\alpha_1^2}{1-3\alpha_1^2} > 3$$

- **positive kurtosis** even though innovations are conditionally Gaussian

Normality

- ARCH(1) process is conditionally normal if $\eta_t \sim \mathrm{N}(0,1)$
- ARCH(1) process is not unconditionally normal
  - time variation in the variance generates fat tails

1-step ahead forecast

- the ARCH(m) model is given by:

$$\sigma^2_{t+1} = \alpha_0 + \alpha_1 \varepsilon^2_t + \ldots + \alpha_m \varepsilon^2_{t-m+1}$$

- hence the forecast is simply:

$$\sigma^2_t(1) = \alpha_0 + \alpha_1 \varepsilon^2_t + \ldots + \alpha_m \varepsilon^2_{t-m+1}$$

2-step ahead forecast

- the ARCH(m) model is given by:

$$\sigma_{t+2}^2 = \alpha_0 + \alpha_1 \varepsilon_{t+1}^2 + \ldots + \alpha_m \varepsilon_{t-m+2}^2$$

- hence the forecast is simply:

$$\sigma_t^2(2) = \alpha_0 + \alpha_1 \sigma_t^2(1) + \alpha_2 \varepsilon_t^2 \ldots + \alpha_m \varepsilon_{t-m+2}^2$$

- note that we have used the following:

$$E_t[\varepsilon_{t+1}^2 | \mathcal{F}_t] = E_t[\sigma_{t+1}^2 \eta_{t+1}^2 | \mathcal{F}_t] = \sigma_{t+1}^2 E_t[\eta_{t+1}^2 | \mathcal{F}_t]$$

h-step ahead forecast

- the ARCH(m) model is given by:

$$\sigma^2_{t+h} = \alpha_0 + \alpha_1 \varepsilon^2_{t+h} + \ldots + \alpha_m \varepsilon^2_{t-m+h}$$

- General expression:

$$\sigma^2_t(h) = \alpha_0 + \alpha_1 \sigma^2_t(h-1) + \ldots + \alpha_m \sigma^2_t(h-m)$$

where $\sigma^2_t(h-j) = \varepsilon^2_{t+h-j}$ if $h-j<0$

Building an ARCH model

- we can pick the order of an ARCH model by looking at *PACF* at $\{\varepsilon_t^2\}$; you might need a large number of lags
- for a well-specified ARCH model, the standardized residual:

$$\hat{\eta}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

  should be white noise
- this can be tested by examining $\hat{\eta}_t$
  1. check the volatility spec by doing a Q-test on $\hat{\eta}_t^2$
  2. check the mean spec by doing a Q-test on $\hat{\eta}_t$

Weakness of ARCH Models

1. symmetry in effects of positive and negative shocks on vol
2. ARCH model: mechanical description of volatility (no economics)
3. sometimes many lags are needed to describe vol dynamics

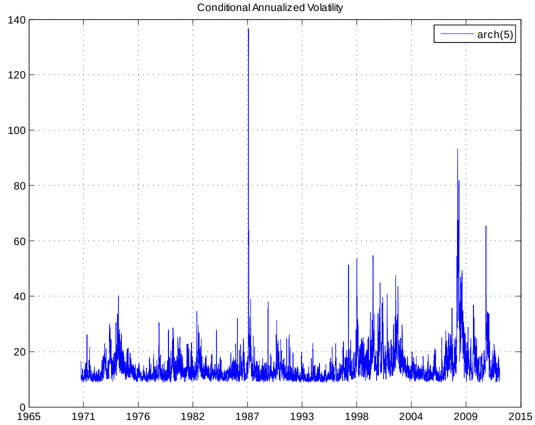
ARCH(5)

Table: ARCH(5)

Parameter	Value	s.e.	t-stat
Constant	3.09E-05	6.65E-07	46.472
ARCH1	0.099307	0.005071	19.5838
ARCH2	0.144208	0.008707	16.5623
ARCH3	0.141901	0.009809	14.4668
ARCH4	0.168116	0.01012	16.6119
ARCH5	0.150002	0.010459	14.3424

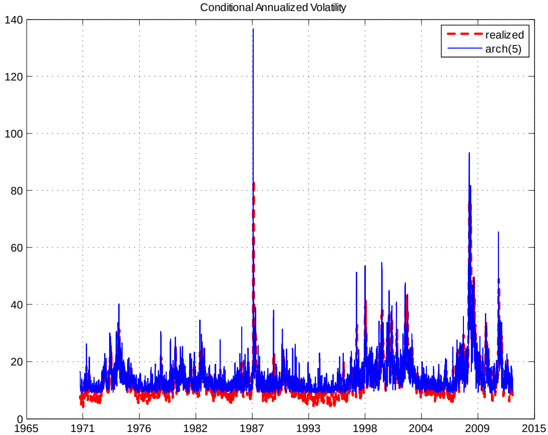
ARCH(5) estimated on daily CRSP-VW stock returns 1971-2012.

Parametric Volatility σ_t



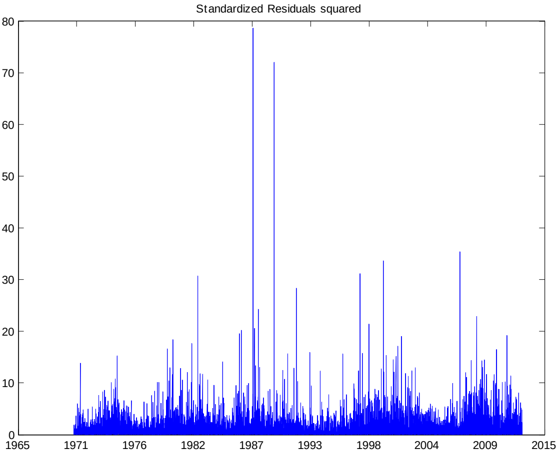
ARCH(5). Daily data. CRSP-VW.1971-2012.

Parametric Volatility against Realized Vol.



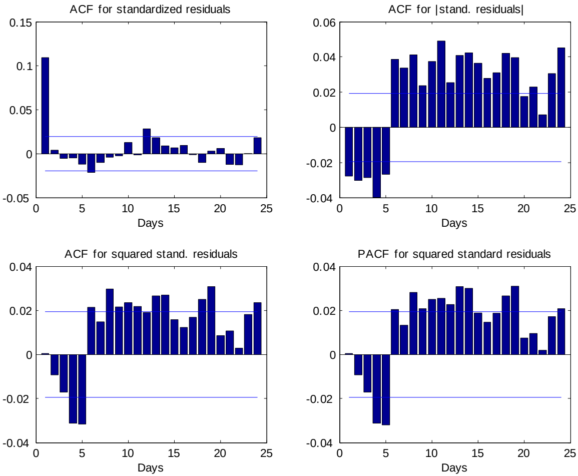
ARCH(5). Daily data. CRSP-VW. 1971-2012.

Standardized Residuals $(\varepsilon_t/\sigma_t)^2$



ARCH(5). Daily data. CRSP-VW.1971-2012.

Standardized Residuals ε_t / σ_t



ARCH(5). Daily data. CRSP-VW.1971-2012.

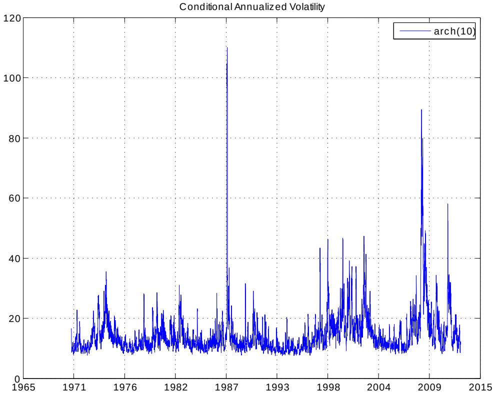
ARCH(10)

Table: ARCH(10)

Parameter	Value	standard error	t-stat
Constant	2.04E-05	7.56E-07	26.927
ARCH1	0.083715	0.004496	18.6207
ARCH2	0.113967	0.008698	13.1022
ARCH3	0.080502	0.009147	8.80109
ARCH4	0.095266	0.00937	10.1667
ARCH5	0.107659	0.011581	9.29659
ARCH6	0.075962	0.009292	8.1747
ARCH7	0.052205	0.008305	6.28631
ARCH8	0.083885	0.00877	9.56551
ARCH9	0.073778	0.008694	8.4864
ARCH10	0.049606	0.008884	5.58371

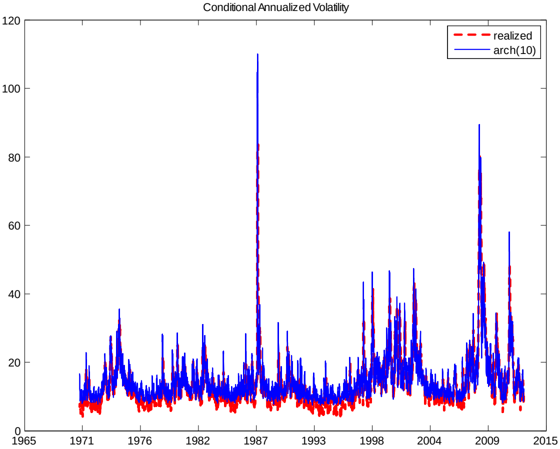
ARCH(10) estimated on daily CRSP-VW stock returns 1971-2012.

Parametric Volatility σ_t



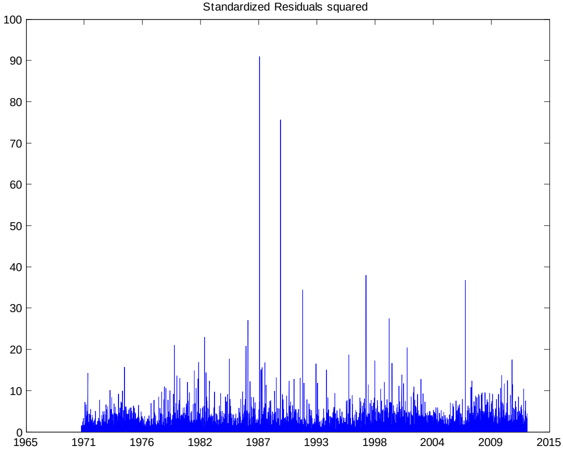
ARCH(10). Daily data. CRSP-VW.1971-2012.

Parametric Volatility against Realized Variance.



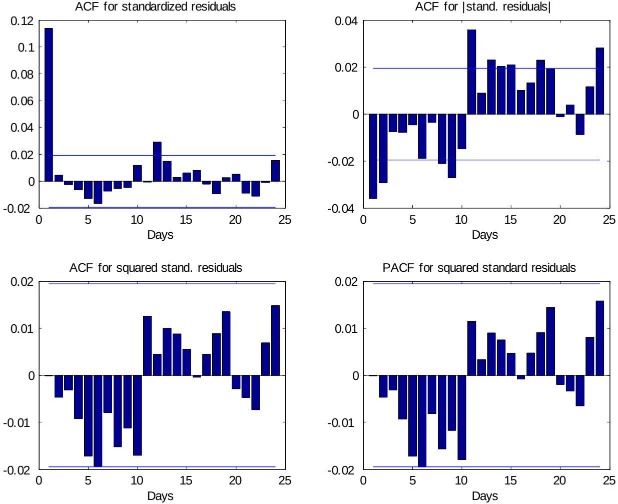
ARCH(10). Daily data. CRSP-VW.1971-2012.

Standardized Residuals $(\varepsilon_t/\sigma_t)^2$



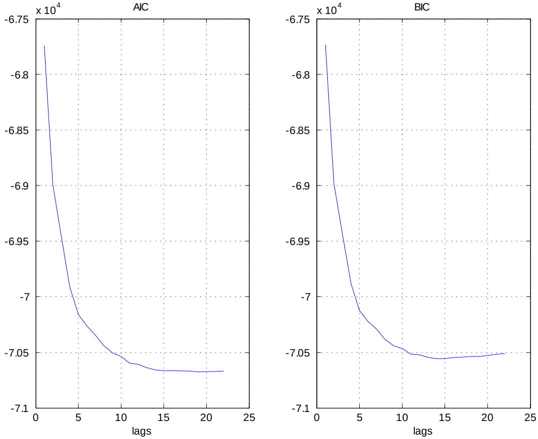
ARCH(10). Daily data. CRSP-VW.1971-2012.

Standardized Residuals ε_t / σ_t



ARCH(10). Daily data. CRSP-VW.1971-2012.

AIC/BIC



ARCH(lags). Daily data. CRSP-VW.1971-2012.

GARCH models

GARCH model of Bollerslev (1986)

Definition

Consider $\varepsilon_t = r_t - \mu_t$. The ε_t follows a GARCH(m, s) model if

$$\varepsilon_t = \sigma_t \eta_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where η_t is i.i.d., mean zero, has variance of one , $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$ and $\beta_j \geq 0$ for $j > 0$, and

$$\sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i) < 1$$

GARCH is ARMA for vol

- define:

$$\kappa_t = \varepsilon_t^2 - \sigma_t^2$$

- hence

$$\sigma_t^2 = \varepsilon_t^2 - \kappa_t$$

- this delivers an ARMA-like representation:

$$\varepsilon_t^2 = \alpha_0 + \sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i)\varepsilon_{t-i}^2 - \sum_{j=1}^{s} \beta_j \kappa_{t-j} + \kappa_t$$

  where $E(\kappa_t) = 0$ and $E(\kappa_t \kappa_{t-j}) = 0$ but is not white noise

- the unconditional variance is simply:

$$E\left(\varepsilon_t^2\right) = \frac{\alpha_0}{1 - \sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i)}$$

GARCH(1,1) of Bollerslev (1986)

- consider GARCH(1,1) model:

$$\begin{aligned} \varepsilon_t &= \sigma_t \eta_t \qquad\qquad \eta_t \sim \mathrm{N}(0,1) \\ \sigma_{t+1}^2 &= \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2 \end{aligned}$$

  - large $\varepsilon_t^2, \sigma_t^2$ leads to large σ_{t+1}^2

- the unconditional kurtosis is given by:

$$\frac{E\left[\varepsilon_t^4\right]}{V\left[\varepsilon_t\right]^2} = 3\frac{1-(\alpha_1+\beta_1)^2}{1-(\alpha_1+\beta_1)^2-2\alpha_1^2} > 3$$

- positive kurtosis even though innovations η_t are Gaussian

Forecasting with GARCH(1,1)

- consider GARCH(1,1) model:

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2$$

- the one-step ahead forecast of volatility:

$$\sigma_t^2(1) = \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2$$

- the two-step ahead forecast of volatility:

$$\sigma_t^2(2) = \alpha_0 + (\alpha_1 + \beta_1)[\sigma_t^2(1)]$$

  where we used the following expression

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2 \left(\eta_t^2 - 1\right)$$

Forecasting with GARCH(1,1)

- consider GARCH(1,1) model:

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2$$

- the one-step ahead forecast of volatility:

$$\sigma_t^2(1) = \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2$$

- the two-step ahead forecast of volatility:

$$\sigma_t^2(2) = \alpha_0 + (\alpha_1 + \beta_1)[\sigma_t^2(1)]$$

- in general:

$$\sigma_t^2(h) = \alpha_0 + (\alpha_1 + \beta_1)[\sigma_t^2(h-1)]$$

Forecasting with GARCH(1,1) at longer horizons

The multi-step volatility forecast $\sigma_t^2(h)$

$$\sigma_t^2(h) = \alpha_0 + (\alpha_1 + \beta_1)[\sigma_t^2(h-1)]$$

converges to the unconditional variance as h increases

Picking Order and Checking Model

- for a well-specified GARCH model, the standardized residual:

$$\hat{\eta}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

  should be white noise

- this can be tested by examining $\hat{\eta}_t$
  1. check the volatility specification by doing a Q-test on $\hat{\eta}_t^2$
  2. check the mean spec. by doing a Q-test on $\hat{\eta}_t$

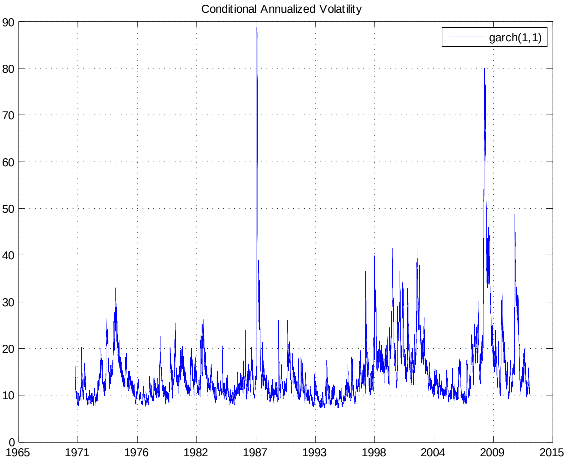
GARCH(1,1)

Table: GARCH(1,1)

Parameter	Value	standard error	t-stat
Constant	1.16E-06	2.23E-07	5.18121
GARCH1	0.909294	0.002906	312.933
ARCH1	0.07998	0.001725	46.3601

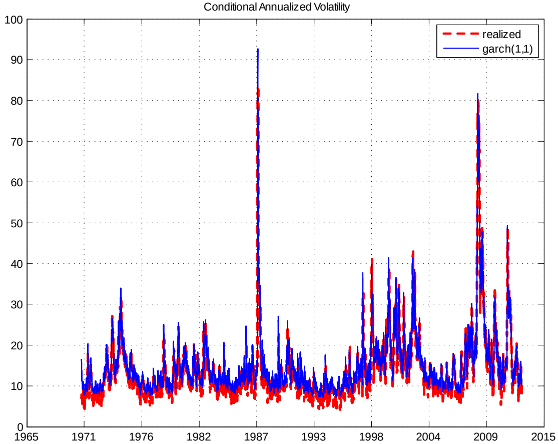
GARCH(1,1). Daily data. CRSP-VW.1971-2012.

Parametric Volatility σ_t



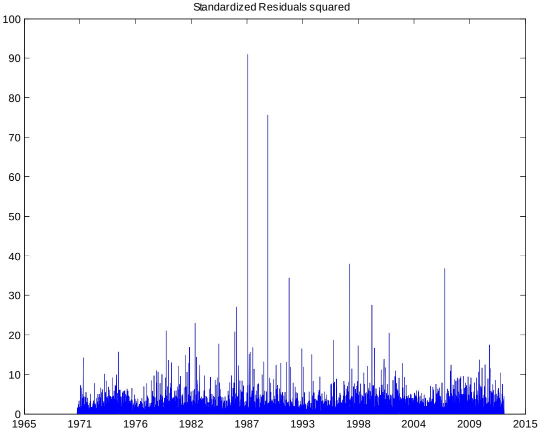
GARCH(1,1). Daily data. CRSP-VW.1971-2012.

Parametric Volatility against Realized Vol.



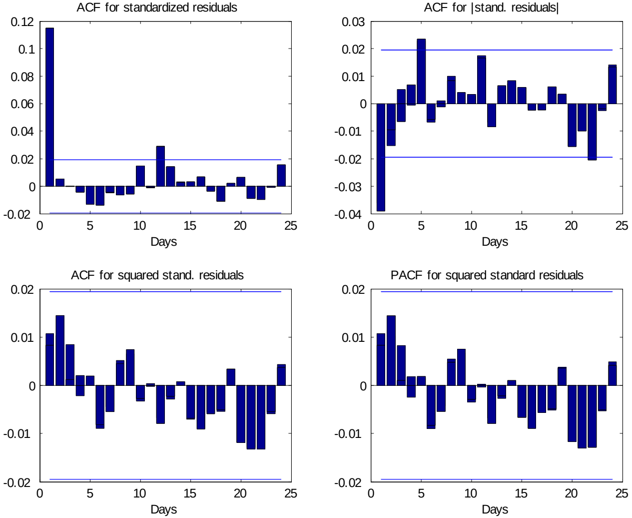
GARCH(1,1). Daily data. CRSP-VW.1971-2012.

Standardized Residuals $(\varepsilon_t/\sigma_t)^2$



GARCH(1,1). Daily data. CRSP-VW.1971-2012.

Standardized Residuals ε_t / σ_t



GARCH(1,1). Daily data. CRSP-VW.1971-2012.

AIC/BIC

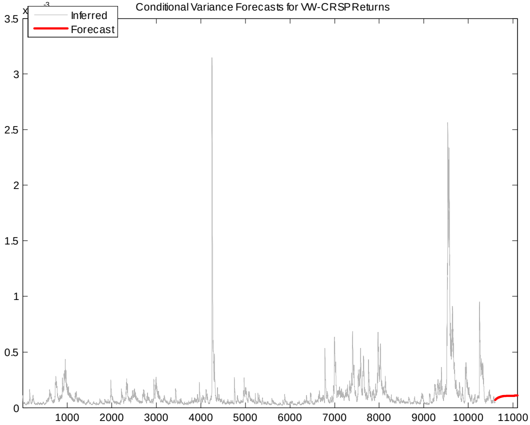
- AIC/BIC for GARCH(1,1)

$$\begin{aligned} aic &= -7.0738e+04 \\ bic &= -7.0665e+04 \end{aligned}$$

- GARCH(1,1) has smaller AIC and BIC than ARCH(10)

- you could use the BIC/AIC criterion to find the best GARCH(p,q)

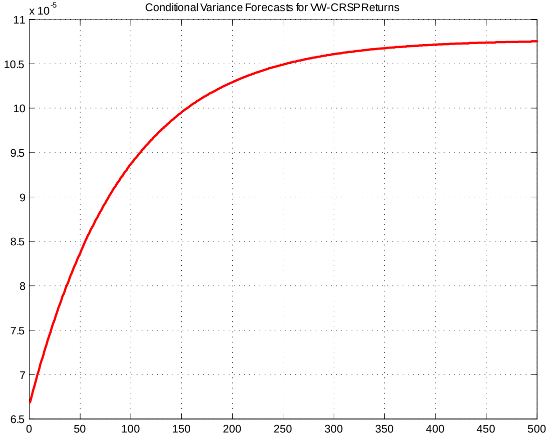
Predicted one-period ahead variance from GARCH(1,1)



GARCH(1,1). Daily data. CRSP-VW.1971-2012.

Prediction variance

- x-axis gives number of periods ahead forecast is



GARCH(1,1). Daily data. CRSP-VW.1971-2012.

I-GARCH, EGARCH, GARCH-M, GJR models

I-GARCH

Definition

Consider $\varepsilon_t = r_t - \mu_t$. The ε_t follows a *I*-GARCH$(1,1)$ model if

$$\varepsilon_t = \sigma_t \eta_t, \qquad \sigma_t^2 = \alpha_0 + (1 - \beta_1)\varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where η_t is i.i.d., mean zero, has variance of one. $0 < \beta < 1$

- the unconditional variance is not defined!
- hard to justify for returns

Forecasting with I-GARCH(1,1)

- in general for GARCH(1,1):

$$\sigma_t^2(h) = \alpha_0 + (\alpha_1 + \beta_1)[\sigma_t^2(h-1)]$$

- now set $\alpha_1 + \beta_1 = 1$
- repeated substitution yields:

$$\sigma_t^2(h) = (h-1)\alpha_0 + [\sigma_t^2(1)]$$

  - effects on future vols are persistent!

GARCH(1,1)-M

Definition

Consider

$$\varepsilon_t = r_t - \mu_t - \gamma\sigma_t^2.$$

The ε_t follows a GARCH$(1,1)$-M model if

$$\varepsilon_t = \sigma_t \eta_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where η_t is i.i.d., mean zero, has variance of one. $0 < \beta < 1$.

M stands for GARCH in the mean.

- the parameter γ measures a variance risk premium
- the risk premium increases when volatility increases

E-GARCH model of Nelson (1991)

Definition

Consider $\varepsilon_t = r_t - \mu_t$. The ε_t follows a E-GARCH$(1,1)$ model if

$$\varepsilon_t = \sigma_t \eta_t, \qquad (1 - \alpha B) \ln \sigma_t^2 = (1 - \alpha)\alpha_0 + g(\eta_{t-1})$$

where η_t is i.i.d., mean zero, has variance of one.

$$g(\eta_t) = (\delta + \gamma)\eta_t - \gamma E(|\eta_t|) \quad \textit{if} \quad \eta_t \geq 0$$

$$g(\eta_t) = (\delta - \gamma)\eta_t - \gamma E(|\eta_t|) \quad \textit{if} \quad \eta_t < 0$$

- built-in asymmetry to capture leverage effect
- we expect $\delta < 0$

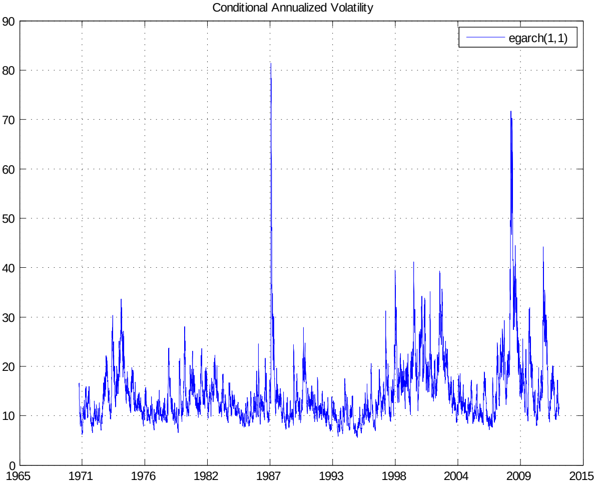
E-GARCH(1,1)

Table: E-GARCH(1,1)

Parameter	Value	standard error	t-stat
———	———	———	———
Constant	-0.17256	0.011062	-15.599
GARCH1	0.980926	0.001135	864.546
ARCH1	0.13937	0.005217	26.7121
Leverage1	-0.07601	0.003161	-24.0473

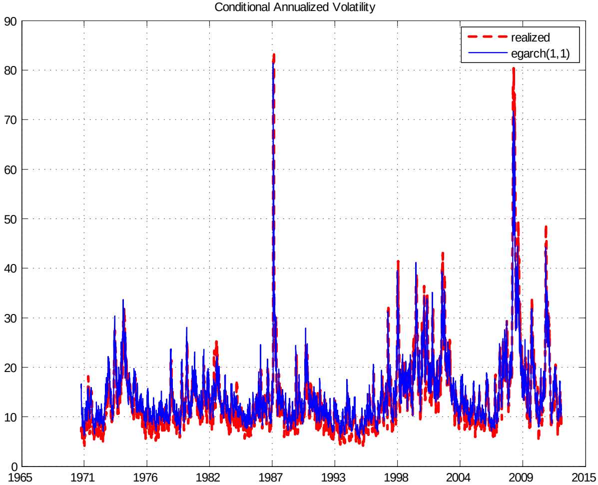
EGARCH(1,1). Daily data. CRSP-VW.1971-2012.

Parametric Volatility σ_t



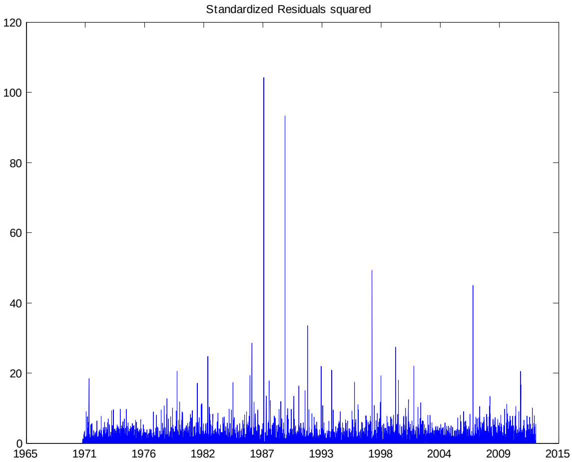
EGARCH(1,1). Daily data. CRSP-VW.1971-2012.

Parametric Volatility against Realized Vol.



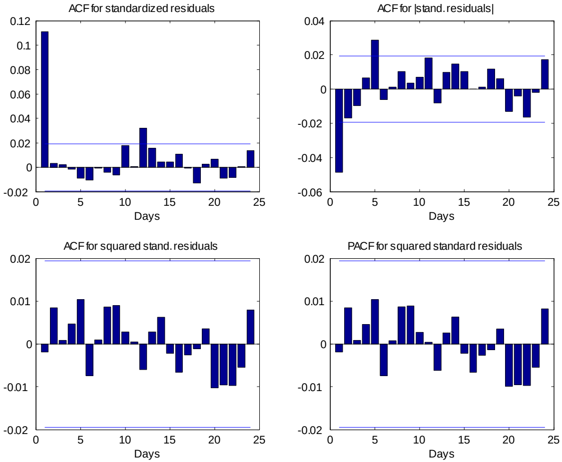
EGARCH(1,1). Daily data. CRSP-VW.1971-2012.

Standardized Residuals $(\varepsilon_t/\sigma_t)^2$



EGARCH(1,1). Daily data. CRSP-VW.1971-2012.

Standardized Residuals ε_t / σ_t



EGARCH(1,1). Daily data. CRSP-VW.1971-2012.

Glosten, Jagannathan, and Runkel (1993)

Definition

Consider $\varepsilon_t = r_t - \mu_t$. The ε_t follows a GJR-GARCH$(1,1)$ model if

$$\varepsilon_t = \sigma_t \eta_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + I_{t-1}\delta$$

where η_t is i.i.d., mean zero, has variance of one.

$$I_t = \begin{cases} 0 & \text{if} \quad \varepsilon_t \geq 0 \\ 1 & \text{if} \quad \varepsilon_t < 0 \end{cases}$$

- simple way to capture leverage effect
- we expect $\delta > 0$

Value-at-Risk

VaR

Definition

Let the r.v. $L_{t,t+h}$ denote the loss of a financial position over a time horizon $t \longrightarrow t+h$. The Value at Risk over the time horizon h is the amount of money you could lose with probability p. This is given by:

$$p = Pr\left[L_{t,t+h} \geq VaR\right] \;=\; 1 - Pr\left[L_{t,t+h} < VaR\right].$$

- For a given probability p, how much could a position lose? This is the amount of money (value) that is at risk.
- in applications, we need
  1. the probability (say $p = .01$)
  2. the time horizon h (say $h = 10$ days)
  3. the frequency of the data
  4. the loss cdf $F_{t,t+h}$
  5. the size of the position

Remarks

- By convention, VaR is reported as a positive number.
- VaR is the $1-p$ quantile function of the loss distribution.
- For a long position, a loss is a negative return.
  - Option #1: we can look at the distribution of $-r_t$ and calculate the $1-p$-th quantile.
  - Option #2: we can look at the distribution of r_t, calculate the p-th quantile, and take the absolute value.
- Warning: be careful when calculating this. The distribution of returns is typically not symmetric.

Simple example

- Suppose you bought \$10000 of MSFT.

- Let r_{t+1} denote the log-return over the next month. Our model is:

$$r_{t+1} \sim \mathrm{N}(0.05, 0.01)$$

- The one-period return CDF is: $F_{t,t+1}(r_{t+1}) = \Phi(r_{t+1}|0.05, 0.01)$

- Let $p = 0.01$. Calculate the quantile $F^{-1}_{t,t+1}(0.01) = -0.1826$. The value at risk is:

$$VaR_{0.01} = |10000 * -0.1826| = 1826$$

- Let $p = 0.05$. Calculate the quantile $F^{-1}_{t,t+1}(0.05) = -0.1145$. The value at risk is:

$$VaR_{0.05} = |10000 * -0.1145| = 1145$$

Comments

- The quantile function is known (or easily calculated) when returns are conditionally normal or conditionally Student's t.
- In most models used for risk management, the conditional distribution of returns is not known. We need to calculate VaR by simulation methods.
- RiskMetrics uses the IGARCH(1,1) model with normally distributed errors η_t and no drift $\mu_t = 0$. This is a special case where returns are conditionally normal at all horizons. (Next example.)

RiskMetrics

Definition

The RiskMetrics Value at Risk assumes the daily log return

$$r_t | F_{t-1} \sim N(\mu_t, \sigma_t^2)$$

where

$$\mu_t = 0, \qquad \sigma_t^2 = \alpha \sigma_{t-1}^2 + (1-\alpha) r_{t-1}^2, 1 > \alpha > 0$$

Hence,

$$r_t = \varepsilon_t = \sigma_t \eta_t \qquad \eta_t \sim N(0,1)$$

is an IGARCH(1,1) process without drift.

- The combination of no drift, normal errors, and IGARCH implies that the conditional distribution is known at all horizons.

VaR in RiskMetrics

- Using the independence assumption for η_t:

$$V\left(r_t[h]|\mathcal{F}_t\right) = \sum_{i=1}^{h} V\left(\varepsilon_{t+i}|\mathcal{F}_t\right)$$

  where $V\left(\varepsilon_{t+i}|\mathcal{F}_t\right) = E\left(\sigma_{t+i}^2|\mathcal{F}_t\right)$ can be computed recursively.
- from the I-GARCH(1,1) specification:

$$\sigma_t^2 = \sigma_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2(\eta_{t-1}^2 - 1) \quad \forall \quad t$$

- this implies that the conditional variance can be stated as:

$$\sigma_{t+i}^2 = \sigma_{t+i-1}^2 + (1-\alpha)\sigma_{t+i-1}^2(\eta_{t+i-1}^2 - 1), \qquad i = 2, \ldots, h$$

  and hence $E\left(\sigma_{t+i}^2|\mathcal{F}_t\right) = \sigma_{t+1}^2, i = 2, \ldots, h$
- hence, the conditional variance of the long-horizon return is given by:

$$V\left(r_t[h]|\,\mathcal{F}_t\right) = hV\left(\varepsilon_{t+1}|\,\mathcal{F}_t\right) = h\sigma_{t+1}^2$$

VaR in RiskMetrics

- For the IGARCH(1,1) model under these assumptions, we have

$$r_t[h] \sim \mathrm{N}(0, h\sigma^2_{t+1})$$

- Important assumptions:
  - normal errors: sum of normal r.v.'s is normal.
  - no drift
  - volatility does not mean-revert (it has no mean)
- suppose the tail probability is 5%
- the daily VaR under RiskMetrics is :

$$VaR = Amount\ of\ \ Position \times 1.65\sigma_{t+1}$$

- the VaR at the h-day horizon under RiskMetrics is :

$$VaR(h) = Amount\ of\ \ Position \times 1.65\sqrt{h}\sigma_{t+1}$$

  - the square root of time rule...

Discussion of RiskMetrics

- simple, makes risk transparent and tangible
- normality assumption is a weakness
- square root of time rule is an artefact of the assumption that $\mu_t = 0$
  - in general:

$$VaR(h) = Amount\ of\ Position \times \left(1.65\sqrt{h}\sigma_{t+1} - h\mu\right)$$

$$VaR(h) \neq \sqrt{k}Var(1)$$

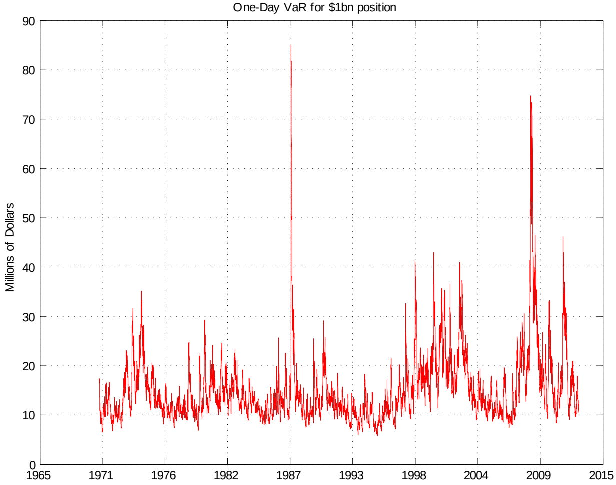
Calculation of VaR from a general GARCH model

- For many models, we do not know the CDF of the losses at horizon h. We must use simulation.
- Start at time t with known estimate σ_t^2 and mean μ_t.
    1. For $i = 1, \ldots, M$, simulate future returns

    $$r_{t+1}^{(i)}, r_{t+2}^{(i)}, \ldots, r_{t+h}^{(i)}$$

    which requires simulating $\mu_{t+1}^{(i)}, \mu_{t+2}^{(i)}, \ldots, \mu_{t+h}^{(i)}$ and $\sigma_{t+1}^{2,(i)}, \sigma_{t+2}^{2,(i)}, \ldots, \sigma_{t+h}^{2,(i)}$
    2. For each return sequence i, calculate the loss $L_{t,t+h}^{(i)}$ at horizon h on the position.
    3. Sort the losses $\left\{L_{t,t+h}^{(i)}\right\}_{i=1}^{M}$ from smallest to largest. This is an approximation to the CDF of the loss distribution.
    4. As your estimator of VaR, take the nearest integer $(1-p) * M$ in the sorted values as the quantile.
- The larger the value of M the more accurate the estimate.

VaR



EGARCH(1,1). Daily data. CRSP-VW.1971-2012. VaR for $1bn position in stocks.

References

Black, F. (1976). Studies of stock market volatility changes. In Proceedings of the American Statistical Association, Business and Economic Statistics Section, Volume 177, pp. 181.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 31(3), 307-327.

Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of the United Kingdom inflation. Econometrica 50(4), 987-1007.

Glosten, L. R., R. Jagannathan, and D. E. Runkle (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. The Journal of Finance 48(5), 1779-1801.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: a new approach. Econometrica 59(2), 347-370.