Empirical Methods in Finance Homework 3

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Problem 1

1

What is the First-order autocorrelation of yt?

Recall from the slides that the first-order autocoreelation of ARMA(1,1) is given by:

$$\rho_1 = \phi_1 - \theta_1 \frac{\sigma_\epsilon^2}{\gamma_0}$$

where

$$\gamma_0 = Var(Y_t) = \sigma_\epsilon^2 \frac{1 + \theta_1^2 - 2\phi_1\theta_1}{1 - \phi_1^2}$$

Put into code, we have $\gamma_0 =$

```
sigma_sq_epsilon=0.05^2
phi_1=0.95
theta_1=0.9
gamma_0=sigma_sq_epsilon*((1+theta_1^2-2*phi_1*theta_1)/(1-phi_1^2))
gamma_0
```

```
## [1] 0.002564103
```

and $\rho_1 =$

```
rho_1=phi_1-theta_1*((sigma_sq_epsilon)/(gamma_0))
rho_1
```

```
## [1] 0.0725
```

2

Recall from slides for higher order autocorrelation for ARMA(1,1):

$$\rho_j = \phi_1 \rho_{j-1}$$

put into codde, $\rho_2 = \rho_1\phi_1 =$

```
rho_2=rho_1*phi_1
rho_2
```

```
## [1] 0.068875
```

The ratio $\rho_2/\rho_1 = \phi_1 = 0.95$

```
rho_2/rho_1
```

```
## [1] 0.95
```

The intuition is very simple. Compared with AR(1) model, the only difference with ARMA(1,1) Model is the effect of residual term one period before.

AR(1):

$$R_t = \phi_0 + \phi_1 R_{t-1} + \epsilon_t, \epsilon_t \sim^{iid} WN(0, \sigma_\epsilon^2)$$

ARMA(1):

$$R_t = \phi_0 + \phi_1 R_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1}, \epsilon_t \sim^{iid} WN(0, \sigma_\epsilon^2)$$

and For AR(1): $\rho_j = \phi_1^j$. Because the last period residual has impacts on current process and the residuals further before have no impacts, the coefficient of the last period residual θ only influence first order autocorrelation ρ_1. Afterward, the influence of the last period residual die out, and so the higher order autocorrelation ρ_j is a product of AR coefficient ϕ_1

Confirm with fuction generating theoretical autocorrelation.

```
ARMAacf(ar = 0.95,ma = -0.9,lag.max = 2,pacf = F)
```

```
##        0        1        2
## 1.000000 0.072500 0.068875
```

3

Given the model, we know:

$$Y_{t+1} = 0.95Y_t - 0.9\epsilon_t + \epsilon_{t+1}$$

Take conditional expectation of Y_t:

$$E(Y_{t+1}|Y_t) = 0.95Y_t - 0.9\epsilon_t = 0.95 * 0.6 - 0.9 * 0.1 = 0.48$$

$$Y_{t+2} = 0.95Y_{t+1} - 0.9\epsilon_{t+1} + \epsilon_{t+2}$$

Take conditional expectation of Y_t

$$E(Y_{t+2}|Y_t) = 0.95E(Y_{t+1}|Y_t) = 0.95 * 0.48 = 0.456$$

#4 Recall from last question, we know

$$\hat{X}_t = 0.95Y_t - 0.9\epsilon_t$$

For the mean:

$$E(\hat{X}_t) = 0.95E(Y_t) = 0$$

$$Var(\hat{X}_t) = 0.95^2 Var(Y_t) + 0.9^2 Var(\epsilon_t) - 2 * 0.95 * 0.9Cov(Y_t, \epsilon_t)$$

where

$$Cov(Y_t, \epsilon_t) = E(Y_t \epsilon_t) = E(\epsilon_t^2) = Var(\epsilon_t) = 0.05^2$$

(ϵ_t is independent of Y_{t-1} and ϵ_{t-1})

Therefore,$Var(\hat{X}_t) =$

```
Var_Xt=0.95^2*gamma_0+0.9^2*sigma_sq_epsilon-2*0.95*0.9*sigma_sq_epsilon
Var_Xt
```

```
## [1] 6.410256e-05
```

and the standard deviation should be

```
Var_Xt^0.5
```

```
## [1] 0.008006408
```

we know $Corr(\hat{X}_t, \hat{X_{t-1}}) = Cov(\hat{X}_t, \hat{X_{t-1}})/Var(\hat{X}_t)$

Recall,

$$\hat{X_{t+1}} = E(Y_{t+2}|Y_t) = 0.95E(Y_{t+1}|Y_t) = 0.95\hat{X}_t$$

So, we want

$$Cov(X_t \hat{+} 1, \hat{X}_t) = E(\hat{X}_t, \hat{X_{t-1}}) = 0.95Var(\hat{X}_t)$$

Therefore:$Corr(\hat{X}_t, \hat{X_{t-1}}) = 0.95$

Problem 2

1

we know that

$$E_t = E_{t-1} + \epsilon_t, \epsilon_t \sim^{iid} WN(0,1)$$

if we assume $\phi = 0$. This implies E_t is a random walk without drift term. Using moving average representation,

$$E_t = \sum_{i=1}^{t} \epsilon_t$$

(Assume $E_0 = 0$)

Therefore,

$$Y_t = E_t - E_{t-4} = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}$$

Therefore, the autocovariances are: (note ϵ 's are iid)

$$\gamma_0 = Var(Y_t) = 4\sigma_\epsilon^2 = 4$$

$$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}, +\epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3} + \epsilon_{t-4}) = 3\sigma_\epsilon^2 = 3$$

By the same argument, $\gamma_2 = 2, \gamma_3 = 1, \gamma_4 = 0, \gamma_5 = 0$

2

Recall ARMA(p,q):

$$Y_t = \phi_0 + \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t$$

and

$$Y_t = E_t - E_{t-4} = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}$$

This can be represented by a MA(3) Model with $\theta_1 = \theta_2 = \theta_3 = 1, \mu_t = 0$

Alternatively,

$$Y_t = Y_{t-1} + \epsilon_t - \epsilon_{t-4}$$

So, it can also be represented by a ARMA(1,4) Model, with $\phi_1 = 1, \theta_4 = -1$ and all other coefficients be 0.