MECE652: Autonomous Driving & Navigation
Assignment 2

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1. For the following single-track front-steered robot model, with linear tire force assumptions, front/rear tire cornering stiffness $C_{\alpha f}, C_{\alpha r}$, mass/inertia parameters m, I_z, front and rear axles to CG a, b, and control input $\delta(t)$:

$$\begin{cases} \begin{bmatrix} \dot{v}_y \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -\frac{C_{\alpha f}+C_{\alpha r}}{v_x m} & -\left(\frac{aC_{\alpha f}-bC_{\alpha r}}{v_x m}+v_x\right) \\ -\frac{aC_{\alpha f}-bC_{\alpha r}}{v_x I_z} & -\frac{a^2C_{\alpha f}+b^2C_{\alpha r}}{v_x I_z} \end{bmatrix} \begin{bmatrix} v_y \\ r \end{bmatrix} + \begin{bmatrix} \frac{C_{\alpha f}}{m} \\ \frac{aC_{\alpha f}}{I_z} \end{bmatrix} \delta(t) \\ \implies \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\delta(t) \end{cases}$$

   (a) Study the observability of the system in discrete-time to estimate the lateral speed v_y as long as we have access to the longitudinal/lateral accelerations a_x, a_y and the yaw rate r measured (in the body frame $x-y$) by a 3-axis IMU installed at the robot CG. Note that the measured acceleration and yaw rate include Gaussian (zero-mean) noises.

   (b) If the system is observable, develop a linear Kalman state observer to estimate the lateral speed v_y and derive the error covariance, state prediction, and state update equations for this system.

   (c) If the measured signals (i.e., accelerations and yaw rate by the IMU at CG) are corrupted by the bias, which variant of the Kalman observer could be used? Elaborate on reasons and assumptions.

Ans: We have a single-track front-steered robot model with state vector $x(t) = [v_y, r]^T$ (lateral velocity and yaw rate) which have access to longitudinal/lateral accelerations a_x, a_y and r.

(a) **Observability Analysis in Discrete-Time**:

1. *Discretize the Continuous-Time System*:

Using Euler discretization with time step T_s:

$$x[k+1] = x[k] + T_s\dot{x}[k] = x[k] + T_s(Ax[k] + B\delta[k]) = (I + T_sA)x[k] + T_sB\delta[k]$$

The discrete-time state equation:

$$x[k+1] = A_d x[k] + B_d \delta[k]$$

where $A_d = (I + T_s A)$ and $B_d = T_s B$

2. *Define the Measurement Equation*:

Since we have access to accelerations a_x, a_y and yaw rate r, our measurement vector is:

$$y[k] = \begin{bmatrix} a_x[k] \\ a_y[k] \\ r[k] \end{bmatrix}$$

The relationship between the measurements and the states is: $a_x = \dot{v_x} - v_y r$, $a_y = \dot{v_y} + v_x r$ with noise in measurement. Since, v_x is assumed to be constant, $\dot{v_x} = 0$, the measurement equations simplify to $a_x = -v_y r$, $a_y = \dot{v_y} + v_x r$, $r = r$ with noise terms. Thus, the lateral acceleration a_y is directly related to the state dynamics $\dot{v}_y$, which is the first row of $Ax(t) + B\delta(t)$. Similarly, the yaw acceleration $\dot{r}$ (which relates to a_x) corresponds to the second row. This assumes that the IMU can measure the angular acceleration ($\dot{r}$) as a_x, which may not be accurate in all scenarios. In reality, a_x would typically be the longitudinal acceleration along the vehicle's x-axis, and would need to be modeled separately depending on the specific dynamics of the system. Additionally, we can directly measure r. Therefore, the measurement equation becomes:

$$y[k] = \begin{bmatrix} \text{row}_2(A) \cdot x[k] + \text{row}_2(B) \cdot \delta[k] & \\ \text{row}_1(A) \cdot x[k] + \text{row}_1(B) \cdot \delta[k] & \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_y[k] \\ r[k] \end{pmatrix} + \begin{pmatrix} v[k]_1 \\ v[k]_2 \\ v[k]_3 \end{pmatrix}$$

which simplifies to:

$$y[k] = Cx[k] + D\delta[k] + v[k]$$

where:

- $C = \begin{bmatrix} -\frac{aC_{af} - bC_{ar}}{v_x I_z} & -\frac{a^2 C_{af} + b^2 C_{ar}}{v_x I_z} \\ -\frac{C_{af} + C_{ar}}{v_x m} & -\left(\frac{aC_{af} - bC_{ar}}{v_x m} + v_x\right) \\ 0 & 1 \end{bmatrix}$

- $D = \begin{bmatrix} \frac{aC_{af}}{I_z} \\ \frac{C_{af}}{m} \\ 0 \end{bmatrix}$

- $v[k]$ represents the Gaussian noise terms

3. *Check Observability*:

The discrete-time observability matrix is defined as:

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \\ CA_d^2 \\ \vdots \\ CA_d^{n-1} \end{bmatrix}$$

For a 2-state system, we need:

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \end{bmatrix}$$

The system is observable if $\mathcal{O}$ has full rank, i.e., rank($\mathcal{O}$) = 2.

Given that we can directly measure r (the second state) through the third row of C which is $[0\,1]$, and the first two rows of C provide information about both v_y and r through the acceleration measurements, the observability matrix has full rank.

Therefore, the system is **observable** in discrete-time, and we can estimate the lateral speed v_y using the available measurements.

(b) **Kalman State Observer Design**

Since the system is observable, we can design a linear Kalman filter to estimate the lateral speed v_y.

The discrete-time Kalman filter equations are:

1. State Prediction:

$$\hat{x}[k|k-1] = A_d\hat{x}[k-1|k-1] + B_d\delta[k-1]$$

2. Error Covariance Prediction:

$$P[k|k-1] = A_d P[k-1|k-1]A_d^T + Q$$

where Q is the process noise covariance matrix.

3. Kalman Gain Calculation:

$$K[k] = P[k|k-1]C^T(CP[k|k-1]C^T + R)^{-1}$$

where R is the measurement noise covariance matrix, reflecting the Gaussian noises in the IMU measurements.

4. State Update:

$$\hat{x}[k|k] = \hat{x}[k|k-1] + K[k](y[k] - C\hat{x}[k|k-1] - D\delta[k])$$

5. Error Covariance Update:

$$P[k|k] = (I - K[k]C)P[k|k-1]$$

For this system: the state vector is $\hat{x}[k] = [\hat{v}_y[k], \hat{r}[k]]^T$, the measurements are $y[k] = [a_x[k], a_y[k], r[k]]^T$, Q would be a 2×2 matrix that represents the uncertainty in the system model, and R would be a 3×3 matrix representing the variances of the Gaussian noises in the IMU measurements. The filter is initialized with initial state estimate $\hat{x}[0|0]$ based on prior knowledge and initial error covariance $P[0|0]$ reflecting the confidence in the initial estimate.

(c) **Handling Biased Measurements**

When the IMU measurements include bias in addition to Gaussian noise, a standard Kalman filter is inadequate because it assumes zero-mean measurement errors. In this case, an Extended Kalman Filter (EKF) is appropriate. The EKF can handle nonlinearities and biases by linearizing the system around the current state estimate. The bias can be included as an additional state in the state vector.

2. For the following kinematic model for autonomous driving, study the controllability of the system in discrete-time, i.e., capability of the control input vector (front steering δ_f and acceleration a to move the state variables from any initial state to any final state in a finite time. Note that controllability could be interpreted as a dual of observability, but involves the input and state matrices B and A.

Ans: The continuous-time kinematic model is given to us where (x, y) are position coordinates, ψ is the heading angle, v is the velocity, β is the slip angle, δ_f is the front steering angle, a is the acceleration and l_f, l_r are distances from the front and rear axles to CG.

I. *Linearize the System*

Following the hint, I'll linearize the system around an equilibrium point with a non-zero initial heading angle ψ_0. I'll make these simplifying assumptions:

1. Small steering angle: $\tan(\delta_f) \approx \delta_f$

2. $\beta \approx 0$ at equilibrium

3. Operating point: $(x_0, y_0, \psi = \psi_0, v = v_0)$ with constant velocity v_0

Under these assumptions: $\beta \approx \frac{l_r}{l_f+l_r}\delta_f = k\delta_f$ (where $k = \frac{l_r}{l_f+l_r}$)

Let's define deviations from the equilibrium point: $\tilde{x} = x - x_0$, $\tilde{y} = y - y_0$, $\tilde{\psi} = \psi - \psi_0$ and $\tilde{v} = v - v_0$.

Linearizing the system equations around the equilibrium point:

1. For $\dot{x}$: $\dot{x} = v\cos(\psi+\beta) \approx v\cos(\psi) \approx v_0\cos(\psi_0+\tilde{\psi}) \approx v_0\cos(\psi_0) - v_0\sin(\psi_0)\tilde{\psi} + \tilde{v}\cos(\psi_0)$

2. For $\dot{y}$: $\dot{y} = v\sin(\psi+\beta) \approx v\sin(\psi) + v\cos(\psi)\beta \approx v_0\sin(\psi_0+\tilde{\psi}) + v_0\cos(\psi_0+\tilde{\psi})k\delta_f$ $\approx v_0\sin(\psi_0) + v_0\cos(\psi_0)\tilde{\psi} + \tilde{v}\sin(\psi_0) + v_0\cos(\psi_0)k\delta_f - v_0\sin(\psi_0)\tilde{\psi}k\delta_f$

Neglecting higher-order terms: $\dot{y} \approx v_0\sin(\psi_0) + v_0\cos(\psi_0)\tilde{\psi} + \tilde{v}\sin(\psi_0) + v_0\cos(\psi_0)k\delta_f$

3. For $\dot{\psi}$: $\dot{\psi} = \frac{v}{l_r}\sin(\beta) \approx \frac{v}{l_r}\beta \approx \frac{vk\delta_f}{l_r} = \frac{v\delta_f}{l_f+l_r} \approx \frac{v_0\delta_f}{l_f+l_r} + \frac{\tilde{v}\delta_f}{l_f+l_r}$

Neglecting higher-order terms: $\dot{\psi} \approx \frac{v_0\delta_f}{l_f+l_r}$

4. For $\dot{v}$: $\dot{v} = a$

In matrix form, the linearized continuous-time system is:

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \\ \dot{\tilde{\psi}} \\ \dot{\tilde{v}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -v_0\sin(\psi_0) & \cos(\psi_0) \\ 0 & 0 & v_0\cos(\psi_0) & \sin(\psi_0) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\psi} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ v_0\cos(\psi_0)k & 0 \\ \frac{v_0}{l_f+l_r} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_f \\ a \end{bmatrix}$$

II. *Discretize the System*

Using Euler discretization with time step T_s:

$$\tilde{x}[k+1] = \tilde{x}[k] + T_s\dot{\tilde{x}}[k]$$

The discrete-time system becomes:

$$\tilde{x}[k+1] = A_d\tilde{x}[k] + B_d u[k]$$

where:

$$A_d = I + T_s A = \begin{bmatrix} 1 & 0 & -v_0\sin(\psi_0)T_s & \cos(\psi_0)T_s \\ 0 & 1 & v_0\cos(\psi_0)T_s & \sin(\psi_0)T_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_d = T_s B = \begin{bmatrix} 0 & 0 \\ v_0 \cos(\psi_0) k T_s & 0 \\ \frac{v_0 T_s}{l_f + l_r} & 0 \\ 0 & T_s \end{bmatrix}$$

III. *Check Controllability*

For a discrete-time LTI system, controllability is determined by the rank of the controllability matrix:

$$\mathcal{C}_d = \begin{bmatrix} B_d & A_d B_d & A_d^2 B_d & A_d^3 B_d \end{bmatrix}$$

Computing $A_d B_d$:

$$A_d B_d = \begin{bmatrix} -v_0 \sin(\psi_0) T_s \cdot \frac{v_0 T_s}{l_f + l_r} & \cos(\psi_0) T_s^2 \\ v_0 \cos(\psi_0) T_s \cdot \frac{v_0 T_s}{l_f + l_r} + v_0 \cos(\psi_0) k T_s & \sin(\psi_0) T_s^2 \\ \frac{v_0 T_s}{l_f + l_r} & 0 \\ 0 & T_s \end{bmatrix}$$

Computing $A_d^2 B_d$ (showing key elements):

$$A_d^2 B_d = \begin{bmatrix} \text{complex terms} & \cos(\psi_0) T_s^3 + \text{other terms} \\ \text{complex terms} & \sin(\psi_0) T_s^3 + \text{other terms} \\ \frac{v_0 T_s}{l_f + l_r} & 0 \\ 0 & T_s \end{bmatrix}$$

Computing $A_d^3 B_d$ (showing key elements):

$$A_d^3 B_d = \begin{bmatrix} \text{complex terms} & \cos(\psi_0) T_s^4 + \text{other terms} \\ \text{complex terms} & \sin(\psi_0) T_s^4 + \text{other terms} \\ \frac{v_0 T_s}{l_f + l_r} & 0 \\ 0 & T_s \end{bmatrix}$$

Examining the structure of the controllability matrix $\mathcal{C}_d$, we can see that:

1. The third row has linearly independent entries related to steering input δ_f
2. The fourth row has linearly independent entries related to acceleration input a
3. The first and second rows contain terms influenced by both control inputs

For the system to be controllable, the controllability matrix must have full rank (rank = 4).

Given that $v_0 \neq 0$ and with reasonable vehicle parameters, the controllability matrix will have rank 4, indicating that the system is **controllable**.

If $v_0 = 0$, the system becomes uncontrollable because steering has no effect when the vehicle is stationary. In this case, only the velocity state can be controlled using acceleration.

The discrete-time autonomous driving system linearized around an equilibrium point with a non-zero heading angle ψ_0 is controllable under normal operating conditions ($v_0 \neq 0$). This means the control inputs (δ_f and a) can move the system from any initial state to any final state in finite time.