Section 3: Solving Equations

Simple Equations

An **equation** is a statement that two algebraic expressions are equal to each other.
The two sides of the equality sign of an equation are known as the **left side** (l.s) and the **right side** (r.s.) of the equation.

Examples of Equations:

(i) $2x + 3 + 4x + 7 = 6x + 10$
(ii) $5x + 3 = 13$

If the two sides of an equation are always equal to each other irrespective of what values we give to the symbols (unknowns), the equation is called an **identity**. Thus example (i) above is an identity. If the two sides of an equation are equal only for a particular value or values of the symbols (unknowns), the equation is called an **equation of condition** or simply an **equation.** Thus example (ii) above is an equation since it is true only when x = 2. We say that the value 2 **satisfies** the equation.

A letter in an equation, whose value it is required to find is called an **unknown quantity** of the equation. The process of finding the value of an unknown is called **solving the equation**. A value so found is called a **root** or **solution** of the equation.

An equation which when reduced to a simple form involves no power of the unknown quantity higher than the first is called a **simple equation**. Thus example (ii) above is a simple equation.

Simple equations can be solved using the following axioms:

1. If to equals we add equals the sums are equal.

   i.e., If $a = b$ and $c = d$, then $a + c = b + d$

2. If from equals we subtract equals the remainders are equal.

   i.e., If $a = b$ and $c = d$, then $a - c = b - d$

3. If equals are multiplied by equals the products are equal.

   i.e., If $a = b$ and $c = d$, then $a \times c = b \times d$

4. If equals are divided by equals the quotients are equal.

   i.e., If $a = b$ and $c = d$, then $a \div c = b \div d$

   Note: Here $c, d \neq 0$

Example:

Consider	$5x + 3 = 13$
Subtracting 3 from both sides (axiom 2) we obtain	$5x + 3 - 3 = 13 – 3$
Thus	$5x = 10$
Dividing both sides by 5 (axiom 4) we obtain	$x = 2$

3.1 Solving Quadratic Equations

An equation which contains the square of the unknown quantity but no higher power is called a **quadratic equation** or an equation of the second degree.

An equation of the form $ax^2 + bx + c = 0$ where a, b and c are real numbers and $a \neq 0$ is called a **general quadratic equation** in the variable x.

A quadratic equation written in the form $ax^2 + bx + c = 0$ (where the right side is equal to 0) is said to be in **standard form**.

Solving Quadratic Equations by Factoring

The solution of a quadratic equation by factoring depends on the following result:

If the product of two numbers is zero, then one (or both) of the numbers is zero; i.e., if $a \times b = 0$ then $a = 0$ or $b = 0$ (or both $a = 0$ and $b = 0$)

Example:
If $(x – 3)(x + 2) = 0$ then $x – 3 = 0$ or $x + 2 = 0$; i.e., $x = 3$ or $x = -2$

To solve quadratic equations by factoring

1. Write the quadratic equation in the standard form $ax^2 + bx + c = 0$
2. Factor $ax^2 + bx + c$
3. Set each factor equal to 0 and solve the resulting linear equations
4. Check the results by substituting back into the original equation

Example:

(i) Solve $x^2 - 3x - 10 = 0$

(ii) Solve $9x^2 + 12x = -4$

(iii) Find the lengths of the sides of a right triangle in which the hypotenuse is respectively 1cm and 8cm longer than the other two sides.

Solution:

(i) $x^2 - 3x - 10 = 0$

$(x – 5)(x + 2) = 0$	by factoring
$x - 5 = 0$ or $x + 2 = 0$	by setting each factor equal to zero
$x = 5$ or $x = -2$	by solving for x

Thus the solutions of the quadratic equation $x^2 - 3x - 10 = 0$ are x = 5 and x = -2.
To verify these solutions we substitute x = 5 and x = -2 back into the original equation.

x = 5: Left side = $(5)^2 - 3(5) - 10 = 25 - 15 - 10 = 0$ = right side
x = -2 Left side = $(-2)^2 - 3(-2) - 10 = 4 + 6 - 10 = 0$ = right side

(ii) $9x^2 + 12x = -4$
$9x^2 + 12x + 4 = 0$ by adding 4 on both sides to obtain the general form
$(3x + 2)(3x + 2) = 0$ by factoring
$3x + 2 = 0$ or $3x + 2 = 0$ by setting each factor equal to zero
$x = -\frac{2}{3}$ or $x = -\frac{2}{3}$ by solving for x

Here the solution to the equation is a unique value $x = -\frac{2}{3}$. This is always the case when the factors are identical and we say that the root $-\frac{2}{3}$ is of multiplicity two.

As in question (i) we can verify the solution $x = -\frac{2}{3}$ by substituting it back into the original equation.

(iii) Let the length of the hypotenuse be denoted by x cm. Then the other two sides are of length (x – 1) cm and (x – 8) cm respectively. Therefore, by Pythagoras' theorem,
$(x - 1)^2 + (x - 8)^2 = x^2$
Expanding the equation we obtain
$x^2 - 2x + 1 + x^2 - 16x + 64 = x^2$
Therefore,
$x^2 - 18x + 65 = 0$
$(x - 13)(x - 5) = 0$
$x - 13 = 0$ or $x - 5 = 0$
$x = 13$ or $x = 5$
If x = 5 then x – 8 = -3. Since the length of a side cannot be a negative value, x = 5 cannot be an answer.
Therefore x = 13, x – 1 = 12 and x – 8 = 5; i.e., the lengths of the three sides are respectively 13cm, 12cm and 5cm.

Solving Quadratic Equations by the Method of Completing the Square

The equation $x^2 = 16$ is an instance of the simplest form of quadratic equations.
Taking the square root on both sides we obtain $\pm x = \pm 4$.
From this we obtain x = 4 or x = -4.

The solution to the equation $(x - 3)^2 = 16$ can be found in a similar manner by taking the square root on both sides.

$x - 3 = \pm 4$
$x - 3 = 4$ or $x - 3 = -4$
$x = 7$ or $x = -1$.

Quadratic equations that are not in the above form can be put into this form by a technique called **completing the square**; i.e., we make the left side of the equation a perfect square.

Example:

$x^2 + 14x + 24 = 0$
$x^2 + 14x + 24 - 24 = 0 - 24$
$x^2 + 14x = -24$
$x^2 + 14x + \left(\frac{14}{2}\right)^2 = -24 + \left(\frac{14}{2}\right)^2$
$x^2 + 14x + 49 = -24 + 49$
$(x + 7)^2 = 25$
$x + 7 = \pm 5$
$x + 7 = 5$ or $x + 7 = -5$
$x + 7 - 7 = 5 - 7$ or $x + 7 - 7 = -5 - 7$
$x = -2$ or $x = -12$

The solutions can be verified by substituting $x = -2$, $x = -12$ back into the original equation.

We will illustrate the steps involved in solving quadratic equations by completing the square by another example.

Example:

$$3x^2 - 12x - 36 = 0$$

Step 1:
Divide both sides of the equation by the coefficient of x^2 (if it is not already equal to 1)

$$\frac{3x^2}{3} - \frac{12x}{3} - \frac{36}{3} = 0$$

$$x^2 - 4x - 12 = 0$$

Step 2:
Move the constant term to the right side of the equal sign and all terms containing x's into the left side.

$$x^2 - 4x = 12$$

Step 3:
Complete the square on the left side by adding the square of one half the coefficient of the first degree term to both sides.

$$x^2 - 4x + \left(\frac{-4}{2}\right)^2 = 12 + \left(\frac{-4}{2}\right)^2$$
$$x^2 - 4x + 4 = 12 + 4$$
$$x^2 - 4x + 4 = 16$$
$$(x - 2)^2 = 16$$

Step 4:
Take the square root of both sides and solve for x

$$x - 2 = \pm 4$$
$$x - 2 = 4 \text{ or } x - 2 = -4$$
$$x = 6 \text{ or } x = -2$$

Therefore, the roots of the equation $3x^2 - 12x - 36 = 0$ are $x = 6$ and $x = -2$

<u>The Quadratic Formula</u>

We will construct a formula called the **quadratic formula** which can be used to obtain the roots of any quadratic equation, by solving the general quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$, using the method of completing the square.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{0}{a}$$ Step 1, divide each term by the coefficient a of x^2

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ Step 2, subtract the constant term from both sides

$$x^2 + \frac{b}{a}x + \left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = -\frac{c}{a} + \left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2$$ Step 3, complete the square

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$ Step 4, take the square root of both sides and solve for x

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Therefore, the solutions of $ax^2 + bx + c = 0$ are

$x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

> **The quadratic formula** for solving $ax^2 + bx + c = 0$ where $a \neq 0$, is
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>Example</u>:

(i) Solve $7x^2 - 6x - 1 = 0$

(ii) If the distance s (in metres) travelled by a cyclist in t seconds is given by $s = t^2 - 5t$, how long will he take to travel a distance of 36 m?

<u>Solution</u>:

(i) For the equation $7x^2 - 6x - 1 = 0$, $a = 7$, $b = -6$ and $c = -1$.
Substituting these values into the quadratic formula

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we obtain

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(7)(-1)}}{2(7)}$$

$$x = \frac{6 \pm \sqrt{36 + 28}}{14}$$

$$x = \frac{6 \pm \sqrt{64}}{14} = \frac{6 \pm 8}{14}$$

$x = \dfrac{14}{14}$ or $x = \dfrac{-2}{14} = -\dfrac{1}{7}$

Therefore, $x = 1$ and $x = -\dfrac{1}{7}$ are the solutions of the quadratic equation $7x^2 - 6x - 1 = 0$

(ii) Since $s = 36$ m, the quadratic equation to be solved is $36 = t^2 - 5t$
The general form of this equation is $t^2 - 5t - 36 = 0$
Here $a = 1$, $b = -5$ and $c = -36$
Substituting these values into the quadratic formula $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we obtain

$$t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-36)}}{2(1)} = \frac{5 \pm \sqrt{25+144}}{2(1)} = \frac{5 \pm \sqrt{169}}{2}$$

Thus $t = \frac{5+13}{2} = 9$ or $t = \frac{5-13}{2} = -4$.

Since t can take only positive values, the time the cyclist takes to travel a distance of 36 m is 9 s.

3.2 Solving Simultaneous Equations

When two or more equations are satisfied by the same values of the unknown quantities they are called **simultaneous equations**.

Example:

Consider the equation

$5x - 4y = 1$ -------------------- (i)

that contains two unknown quantities which are denoted by x and y respectively.

Adding $4y$ to both sides we obtain

$5x - 4y + 4y = 1 + 4y$

i.e., $5x = 1 + 4y$

Therefore by dividing both sides by 5 we obtain

$x = \frac{1+4y}{5}$ ------------------ (ii)

We see that by substituting different values for y we obtain different values for x such that the pair of values satisfies the given equation.

For example, when $y = 1$, we obtain $x = \frac{1+4(1)}{5} = 1$ from (ii).

Substituting $y = 1$, $x = 1$ in the original equation (i) we obtain

left side = 5(1) – 4(1) = 5 – 4 = 1 = right side

Thus $x = 1$, $y = 1$ is a solution of the equation.

When $y = 6$, $x = \frac{1+4(6)}{5} = 5$ from (ii).

Substituting $y = 6$, $x = 5$ in the original equation we obtain

left side = 5(5) – 4(6) = 25 – 24 = 1 = right side

Therefore $x = 5$, $y = 6$ is also a solution of the equation.

We can obtain several pairs of values for x and y that satisfy the given equation by continuing in this manner.

Now suppose we have a second equation of the same kind such as

$x + 2y = 3$ ------------------------- (iii)

From this we obtain $x = 3 - 2y$ --------------------(iv)

If we wish to find the solution to both equations

$5x - 4y = 1$ -------------------- (i)

and $x + 2y = 3$ -------------------- (iii)

then the values of x we obtain from $x = \frac{1+4y}{5}$ and $x = 3 - 2y$ should be identical.

Thus $\frac{1+4y}{5} = 3 - 2y$.

Multiplying both sides by 5 we obtain $1 + 4y = 15 - 10y$.

Thus $4y + 10y = 15 - 1$

Therefore $14y = 14$ and hence $y = 1$.

Substituting this value into (iv) we obtain $x = 3 - 2(1) = 1$.

Earlier we verified that $x = 1$, $y = 1$ is a solution of the equation $5x - 4y = 1$ ------------ (i)

Substituting these values into the equation $x + 2y = 3$ --------------------(iii)

we obtain

left side = 1 + 2(1) = 3 = right side.

Thus the solution to the pair of simultaneous equations

$5x - 4y = 1$ ------------ (i)

$x + 2y = 3$ ------------- (iii)

is $x = 1$, $y = 1$.

We note here that since the two equations are simultaneously true, any equation formed by combining them using the axioms for solving equations will be satisfied by the values of x and y which satisfy the original equation.

In this section we will only be considering how to solve pairs of linear simultaneous equations in 2 unknowns using the following two methods

(i) Elimination Method

(ii) Substitution Method

Our first object will be to obtain an equation which involves only one of the unknown quantities.

3.2.1 Elimination Method

In the elimination method we eliminate one of the unknown quantities by multiplying one or both the equations by appropriate constants and adding them together or subtracting one from the other.

Example:

(i) $2x - 5y = 3$ ------------------- (a)

$4x - 2y = -2$ ------------------- (b)

(ii) $2x - 5y = 3$ ------------------- (a)
$3x – 2y = -1$ ------------------- (b)

(iii) Half the sum of two numbers equals 20 and three times their difference equals 18. Find the two numbers.

<u>Solution</u>:

(i) $2x - 5y = 3$ ------------------- (a)
$4x – 2y = -2$ ------------------- (b)

Multiplying equation (a) by -2 we obtain
$-4x + 10y = -6$ ------------------- (c)

Adding the left sides of equations (b) and (c) together and the right sides together (axiom 1) we obtain

$8y = - 8$
Therefore $y = -1$
Substituting $y = -1$ into equation (a) and solving for x we obtain
$2x – 5(-1) = 3$
$2x + 5 = 3$
$2x = -2$
$x = -1$

Note: We would obtain the same value $x = -1$, if we substitute $y = -1$ into equation (b) instead of into equation (a).

We verify the solution $x = -1$, $y = -1$ by substituting these values back into the equations (a) and (b).
Equation (a): left side = 2(-1) – 5(-1) = -2 + 5 = 3 = right side
Equation (b): left side = 4(-1) – 2(-1) = -4 + 2 = -2 = right side

(ii) $2x - 5y = 3$ ------------------- (a)
$3x – 2y = -1$ ------------------- (b)

Multiplying (a) by 3 and (b) by 2 we obtain
$6x – 15y = 9$ ------------------- (c)
$6x – 4y = -2$ ------------------- (d)

Subtracting (c) from (d) we obtain
$- 4y – (-15y) = -2 - 9$
Therefore
$11y = -11$ and hence $y = -1$
Substituting this value into equation (a) we obtain
$2x – 5(-1) = 3$
$2x + 5 = 3$

$2x + 5 – 5 = 3 – 5$
$2x = -2$
$x = -1$

We now verify the solution $x = -1$, $y = -1$ by substituting them back into the equations (a) and (b).
Equation (a): left side = $2(-1) – 5(-1) = -2 + 5 = 3$ = right side
Equation (b): left side = $3(-1) – 2(-1) = -3 + 2 = -1$ = right side

(iii) Let the two numbers be x and y respectively. Then the two simultaneous equations are
$\frac{1}{2}(x+y) = 20$ ----------------- (a)
$3(x – y) = 18$ ------------------ (b)
Multiplying equation (a) by 2 and equation (b) by $\frac{1}{3}$ we obtain
$x + y = 40$ ---------------- (c)
$x – y = 6$ ------------------ (d)
Adding equations (c) and (d) we obtain
$2x = 46$
Thus $x = 23$
Substituting $x = 23$ into equation (d) we obtain
$23 – y = 6$
Therefore $y = 17$.
The accuracy of the solution $x = 23$, $y = 17$ can be verified as before by substituting these values back into the equations (a) and (b).

3.2.2 Substitution Method

In this method we use one equation to find the value of one of the unknown in terms of the other and then substitute this value into the other equation, thus eliminating one unknown.

Example:

(i) Three pencils and four pens cost Rs. 72. A pencil and three pens cost Rs. 44. Determine the cost of a pencil and a pen.

(ii) The sum of two numbers is 34 while their difference is 10. What are the two numbers?

Solution:
(i) Suppose the cost of a pencil is Rs. x and the cost of a pen is Rs. y. Then we get the following simultaneous equations
$3x + 4y = 72$ ----------------- (a)
$x + 3y = 44$ ----------------- (b)

From (b) we obtain $x = 44 - 3y$.
Substituting this into (a) we obtain
$3(44 - 3y) + 4y = 72$
This is an equation in only the variable y.
$132 - 9y + 4y = 72$
$132 - 5y = 72$
Thus $5y = 132 - 72 = 60$
Hence $y = 12$.
Substituting this into equation (a) we obtain
$3x + 4(12) = 72$
$3x + 48 = 72$
$3x = 24$
$x = 8$
Therefore the cost of a pencil is Rs. 8 and the cost of a pen is Rs. 12.
Note: The accuracy of the solution $x = 8$, $y = 12$ can be verified by substituting these values back into the equations (a) and (b),

(ii) Let the two numbers be x and y respectively. Then we obtain the simultaneous equations
$x + y = 34$ ----------------- (a)
$x - y = 10$ ----------------- (b)
From equation (b) we obtain $x = y + 10$.
Substituting this value for x in equation (a) we obtain
$(y + 10) + y = 34$
$2y + 10 = 34$
$2y = 24$
$y = 12$
Substituting this in equation (b) we obtain
$x - 12 = 10$
Thus $x = 22$
The accuracy of the solution $x = 22$, $y = 12$ can be easily verified by substituting the values back into the equations (a) and (b).

Note: Problem (ii) could have been solved more easily by the method of elimination by adding equations (a) and (b), thus eliminating the unknown y. Therefore, given a situation, the student should discern which method would be more appropriate to use.