Section 8: Indices and Logarithms

We first recall the definition and properties of indices found in Section 1 of Part I.

8.1 Indices

8.1.1 Integral and Rational Indices

Let a denote a real number and n denote a natural number. We denote the product $\underbrace{a \times a \times ...... \times a}_{n \text{ times}}$ by a^n.

a^n is called the **n^{th} power of a**. a is called the **base** and n the **index** (exponent) of a^n. The second and third powers of a number are known as its square and cube respectively.

We extend this to negative indices as follows:
$a^{-n} = \dfrac{1}{a^n}$, where n is a natural number and $a \neq 0$
We define $a^0 = 1$ for all $a \neq 0$.

8.1.2 The Laws of Indices

Let a and b denote real numbers and m and n denote integers. Then

(i) $a^n . a^m = a^{n+m}$

(ii) $(a^n)^m = a^{nm}$

(iii) $(ab)^n = a^n . b^n$

(iv) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ if $b \neq 0$

(v) $\dfrac{a^n}{a^m} = \begin{cases} a^{n-m} & \text{if } n \geq m \text{ and } a \neq 0 \\ \dfrac{1}{a^{m-n}} & \text{if } n < m \text{ and } a \neq 0 \end{cases}$

(vi) $a^0 = 1$ if $a \neq 0$

(vii) $a^{-n} = \dfrac{1}{a^n}$ or $\dfrac{1}{a^{-n}} = a^n$ if $a \neq 0$

Roots and Surds

Let n be a natural number and a a real number. The n^{th} root of a denoted by $\sqrt[n]{a}$ is defined as follows:

Case 1: If n is an odd number, then the n^{th} root of a is defined to be the real number b such that $b^n = a$. In this case we write $b = \sqrt[n]{a}$.

Case 2: If n is an even number, then the n^{th} root of a where a is a non-negative number is defined to be the non-negative real number b such that $b^n = a$ and we write $b = \sqrt[n]{a}$.

In $\sqrt[n]{a}$, $\sqrt{\ }$ is called the **radical sign**, a the **radicand** and n the **index** of the radical.
When $n = 2$ we denote $\sqrt[n]{a}$ by $\sqrt{a}$. $\sqrt{a}$ is called the **square root** of a.
$\sqrt[3]{a}$ is called the **cube root** of a.

If m, n are both integers with n a positive number, and if $\sqrt[n]{a}$ is defined then $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

Properties:
Let a, b be real numbers and n, m be natural numbers. Then

(i) $\sqrt[n]{a^n} = a$ if n is odd, and $\sqrt[n]{a^n} = |a|$, if n is even.

(ii) $\sqrt[n]{ab} = \sqrt[n]{a} . \sqrt[n]{b}$ if n is odd or if n is even, provided that a, b are non-negative.

(iii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ if n is odd and $b \neq 0$, or if n is even, provided that a is non-negative and b is positive.

(iv) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$, if m, n are both odd; otherwise, a must be non-negative.

A **surd** is a root of rational number which cannot be exactly determined.

Example: $\sqrt{3}$, $\sqrt[3]{5}$ are surds, but $\sqrt{4}$, $\sqrt[3]{27}$ are not surds.

Example:

(a) Simplify the following expressions

(i) $\dfrac{1}{16^{-1.5}}$

(ii) $\left(\dfrac{2^{-3}\sqrt{x}}{2^2 x^{-0.5}}\right)^{-2}$

(iii) $\dfrac{(3^{2n} \times 3^{-1}) + 3^{2n+1}}{(6^2 + 2^6)^{\frac{1}{2}}}$

(iv) $\dfrac{(20 \times 10^{-3}) + .08}{2 \times 0.0002}$

(b) Solve the equation $4^x + 2^x - 6 = 0$

Simplifying Surds

A surd is said to be simplified if it is expressed in terms of the smallest possible surd.

Example: $\sqrt{45}$ when simplified equals $3\sqrt{5}$ since $\sqrt{45} = \sqrt{9 \times 5}$

Arithmetic Operations on Surds

Any two or more surds which are n^{th} roots (square roots or cube roots etc) may be multiplied or divided by combining them under the same root sign and then simplifying as above.

Only 'like surds' may be added or subtracted. For example $2\sqrt{3}$ and $-\sqrt{3}$ are like surds but $2\sqrt{3}$ and $3\sqrt{2}$ are not like surds. To add and subtract surds they should first be written in simplified form.

Example:

(i) $(2\sqrt{5} + 3\sqrt{7})(3\sqrt{20} - 4\sqrt{7}) = (2\sqrt{5} \times 3\sqrt{20}) - (2\sqrt{5} \times 4\sqrt{7}) + (3\sqrt{7} \times 3\sqrt{20}) - (3\sqrt{7} \times 4\sqrt{7})$

$$= 6\sqrt{100} - 8\sqrt{35} + 9\sqrt{140} - 12\sqrt{49} = 60 - 8\sqrt{35} + 18\sqrt{35} - 84 = -24 + 10\sqrt{35}$$

Rationalizing the Denominator

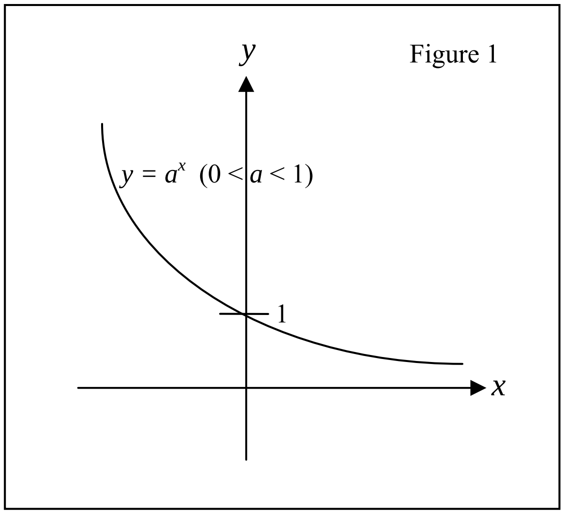
If a quotient with a surd in the denominator is converted into a number with a rational number as the denominator, we call the process rationalizing the denominator.

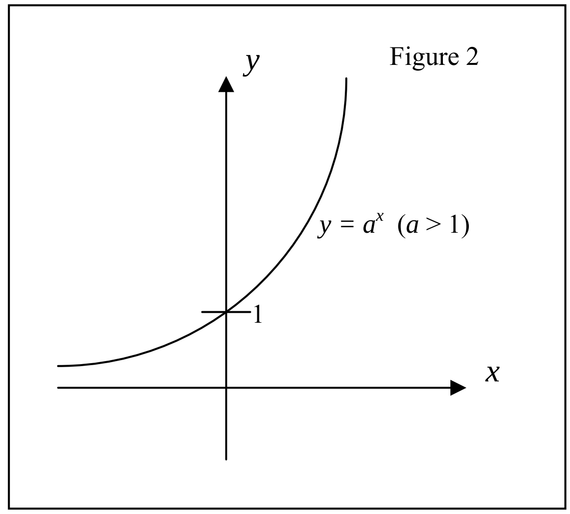
Example: Rationalize the denominator of $\dfrac{\sqrt{5}}{-\sqrt{3} + \sqrt{5}}$

Solution:

$$\frac{\sqrt{5}}{-\sqrt{3}+\sqrt{5}} = \frac{\sqrt{5}}{-\sqrt{3}+\sqrt{5}} \cdot \frac{-\sqrt{3}-\sqrt{5}}{-\sqrt{3}-\sqrt{5}} = \frac{\sqrt{5}(-\sqrt{3}-\sqrt{5})}{(-\sqrt{3}+\sqrt{5})(-\sqrt{3}-\sqrt{5})} = \frac{-\sqrt{15}-\sqrt{25}}{-\sqrt{9}-\sqrt{25}} = \frac{\sqrt{15}-5}{-8} = \frac{5-\sqrt{15}}{8}$$

8.1.3 The graph of a^x for $a > 0$





Observe that for any $a > 0$, since $a^0 = 1$, the graph always passes through the point (0, 1). Also, if $a > 1$, then a^x increases as x increases and if $0 < a < 1$, then a^x decreases as x increases.

8.2 Logarithms

8.2.1 Definition

If a number N can be expressed in the form a^x, the index x is called the **logarithm** of N to the base a. The logarithm of N to the base a is usually written $\log_a N$.
Thus

$$\mathbf{x = \log_a N} \text{ if and only if } \mathbf{N = a^x}$$

Example:

(i) Since $64 = 4^3$, $3 = \log_4 64$
(ii) Since $128 = 2^7$, $7 = \log_2 128$
(iii) Since $10000 = 10^4$, $4 = \log_{10} 10000$

Any number may be taken as the base of a system of logarithms. Logarithms to the base 10 are known as **Common Logarithms.**

Since every logarithm is an index, the rules that govern the use of logarithms can be deduced from the Laws of Indices.

8.2.2 Laws of Logarithms

(i) $\log_a 1 = 0$ for $a \neq 0$.

(ii) $\log_a xy = \log_a x + \log_a y$

(iii) $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

(iv) $\log_a x^r = r \log_a x$

(v) $\log_a x^{\frac{1}{r}} = \dfrac{1}{r}\log_a x$

Proof:

(i) Since $a^0 = 1$ for $a \neq 0$, $\log_a 1 = 0$

(ii) Suppose $M = \log_a x$ and $N = \log_a y$
Then $x = a^M$ and $y = a^N$
$xy = a^M . a^N = a^{M+N}$.
Therefore, $\log_a xy = M + N = \log_a x + \log_a y$

(iii) Suppose $M = \log_a x$ and $N = \log_a y$
Then $x = a^M$ and $y = a^N$
$\dfrac{x}{y} = \dfrac{a^M}{a^N} = a^{M-N}$.
Therefore, $\log_a \dfrac{x}{y} = M - N = \log_a x - \log_a y$

(iv) Suppose $x = a^M$. Then $M = \log_a x$.
$x^r = (a^M)^r = a^{Mr}$
Therefore, $\log_a x^r = Mr = r \log_a x$.

Common Logarithms

Since $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, …
the numbers 1, 2, 3, 4, …. are the common logarithms of 10, 100, 1000, 10000 … respectively.
Also since $10^{-1} = 0.1$, $10^{-2} = 0.01$, $10^{-3} = 0.001$, …
the common logarithms of 0.1, 0.01, 0.001, … are respectively -1, -2, -3 …

It is clear that the common logarithms of all numbers which are exact powers of 10 are either positive or negative integers. In the case of numbers which are not exact powers of 10, the logarithms will always lie between two consecutive integers and will therefore be partly integral and partly fractional. The integral part of a logarithm is called the **characteristic** and the fractional part when expressed as a decimal is called the **mantissa.**

8.2.3 Change of Base

Suppose we are given $\log_a x$ and we wish to find $\log_b x$.
Let $\log_b x = N$. Then $x = b^N$.
Therefore $\log_a x = \log_a b^N$.
i.e., $\log_a x = N \log_a b$.
Therefore, $N = \dfrac{\log_a x}{\log_a b}$

i.e., $\mathbf{\log_b x = \dfrac{\log_a x}{\log_a b}}$.

8.2.4 Graph of $\log_a x$ for $a > 1$

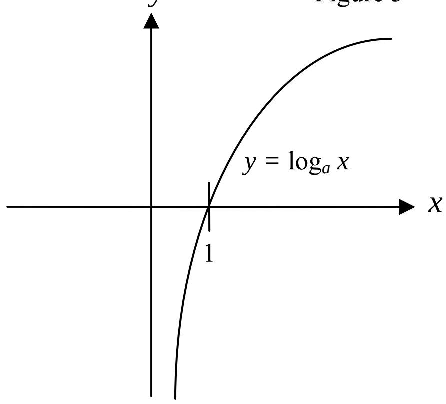


Figure 3

Examples:

(a) Simplify the following

(i) $\log_{10}(0.0001)$
(ii) $\log_2 \sqrt{256}$
(iii) $2\log_5 75 - 2\log_5 3$
(iv) $\log_3 (\sqrt{x\ y}\ .\ \sqrt[3]{x^{-3} y^6})$

(b) If $\log_{10} 5 = 0.6990$, determine $\log_{10} \left(\dfrac{50}{(0.25)(1.25)^3}\right)$

(c) Find the value of x in the following

(i) $6\log_a 3 + 4\log_a x - \log_a 9 = 2\log_a 25$

(ii) $\log_a x = \dfrac{1}{2}[\log_a 9 + \log_a 12 - \log_a 3]$

Solutions:

(a)

(i) $\log_{10}(0.0001) = \log_{10} 10^{-4} = -4$

(ii) $\log_2 \sqrt{256} = \log_2 16 = \log_2 2^4 = 4 \log_2 2 = 4$

(iii) $2\log_5 75 - 2\log_5 3 = 2(\log_5 75 - \log_5 3) = 2\log_5 \frac{75}{3} = 2\log_5 25 = 2\log_5 5^2 = 4$

(iv) $\log_3(\sqrt{x\ y}\ .\ \sqrt[3]{x^{-3}y^6}) = \log_3(\sqrt{x}\sqrt{y})(x^{-1}y^2) = \log_3 \frac{y^{\frac{5}{2}}}{\sqrt{x}} = \frac{5}{2}\log_3 y - \frac{1}{2}\log_3 x$

(b) $\log_{10}(\frac{50}{(0.25)(1.25)^3}) = \log_{10}(5 \times 10) - [\log_{10}\frac{25}{100} + 3\log_{10}\frac{125}{100}]$

$= \log_{10}5 + \log_{10}10 - [\log_{10}5^2 - \log_{10}10^2 + 3\log_{10}5^3 - 3\log_{10}100]$

$= \log_{10}5 + 1 - 2\log_{10}5 + 2 - 9\log_{10}5 + 6$

$= 9 - 10\log_{10}5 = 9 - (10 \times 0.6990) = 9 - 6.99 = 2.01$

(c)

(i) $6\log_a 3 + 4\log_a x - \log_a 9 = 2\log_a 25$

$\log_a 3^6 + \log_a x^4 - \log 3^2 = \log_a 5^4$

$\log_a \frac{3^6 x^4}{3^2} = \log_a 5^4$

$\log_a (3x)^4 = \log_a 5^4$

$(3x)^4 = 5^4$

$3x = 5$

$x = \frac{5}{3}$

(ii) $\log_a x = \frac{1}{2}[\log_a 9 + \log_a 12 - \log_a 3]$

$\log_a x = \frac{1}{2}[\log_a \frac{9 \times 12}{3}] = \frac{1}{2}\log_a 36 = \log_a \sqrt{36} = \log_a 6$