Section 2: Introduction to Basic Algebra

Algebra like arithmetic deals with numbers. Both subjects employ the fundamental operations of addition, subtraction, multiplication, division, raising to a power and taking a root. In both, the same symbols are used to indicate these operations (e.g., + is used for addition, × for multiplication etc.) and the same rules govern their use. In arithmetic we use definite numbers, and we obtain definite numerical results when we perform operations on numbers, but in algebra we are mainly concerned with general expressions and general results in which letters or other symbols are used to represent numbers not named or specified. If a letter is used to represent any number from a given set of numbers, it is called a **variable**. A constant is either a fixed number such as 5 or $\sqrt{2}$, or a letter that represents a fixed (possibly unspecified) number.

Example:
Consider the formula for the area A of a circle of radius r:

$$A = \pi r^2$$

Here, A and r are variables representing the area and the radius of a circle respectively while π is a constant.

2.1 Introduction to Algebraic Expressions

Any collection of numbers or letters standing for numbers (or powers or roots of these), connected by the signs +, -, ×, ÷ is called an expression. Parts of an expression separated by the signs + or – are called **terms**.

An **algebraic expression** is one of the following

(i) a constant
(ii) a variable
(iii) a combination formed by performing one or more algebraic operations (addition, subtraction, multiplication, division, raising to a power or taking a root) on non-zero constants or variables

Examples of algebraic expressions: x, c, $x+c$, $3xy+y^2+2$

A **term** is a product or quotient of one or more variables (or powers or roots of these) and constants.

Examples of terms: x, x^2y, cx, $\frac{x}{y^2}$

In a term of the form $3x$, 3 is called the **numerical coefficient** of x and x is called the **literal coefficient** of 3. When two or more terms (e.g., $-4x^2$ and $6x^2$) have the same literal coefficient, we say that they are **like terms**.

Like terms can always be combined together. This process of combining like terms is called **collecting coefficients**.

Example: $3x^2 - 4xy + 2xy - 10x^2 = 3x^2 - 10x^2 - 4xy + 2xy = -7x^2 - 2xy$

An algebraic expression which consists of only one term is called a **monomial**. A **binomial expression** is an algebraic expression consisting of two terms, and a **trinomial expression** is an algebraic expression consisting of three terms. An algebraic expression with two or more terms is called a **multinomial**.

2.2 Expanding and Factorizing Algebraic Expressions

In algebra, as in arithmetic, the order in which operations are performed is important. Operations are performed from left to right. Brackets are used to indicate that expressions enclosed within them are to be considered as one quantity. Thus, operations within brackets are performed first. All operations of multiplication and division must be performed before those of addition and subtraction. Parenthesis ( ), brackets [ ] and braces { } are used to show the order of performing operation. The general practice is that parentheses are used first and innermost, then brackets, and finally braces.

Note:

1. When an expression within brackets is preceded by the sign +, the brackets may be removed without making any change in the expression. Conversely, any part of an expression may be enclosed within brackets and the sign + prefixed, provided the sign of every term within the brackets remains unaltered.
2. When an expression within brackets is preceded by the sign -, the brackets may be removed if the sign of every term within the brackets is changed. Conversely, any part of an expression may be enclosed within brackets and the sign – prefixed, provided the sign of every term within the brackets is changed.

Multiplying Algebraic Expressions

In order to multiply algebraic expressions, we need to apply the laws of indices of section 1.

Example:

(i) $(3xy^2)(4x^3y^3) = (3 \times 4) \times (x \times x^3) \times (y^2 \times y^3) = 12 \times x^{1+3} \times y^{2+3} = 12x^4y^5$

(ii) $(2x^3y)(-3x^{-5}y^4z) = (2 \times -3) \times (x^3 \times x^{-5}) \times (y \times y^4) \times z = -6x^{-2}y^5z$

(iii) $6x^3(2x^2 - 3xy) = (6x^3 \times 2x^2) + (6x^3 \times -3xy)$ by the distributive law

$= 12x^5 - 18x^4y$

(iv) $(x + 3)(x - 4) = (x + 3)x - (x + 3)4$ by the distributive law

$= (x \times x) + (3 \times x) - (x \times 4) - (3 \times 4)$ by the distributive law

$= x^2 + 3x - 4x - 12$

$= x^2 - x - 12$

Factoring Algebraic Expressions

In this section we will do the reverse of multiplying algebraic expressions. This reverse operation is called **factoring**.

To factor out common terms we apply the distributive property $a(b + c) = ab + ac$ in reverse.

Example:

(i) $-4xy^2 + 12x^2y = 4xy(-y) + 4xy(3x) = 4xy(-y + 3x)$

(ii) $3x^2y + 6x + 5xy + 10 = 3x(xy + 2) + 5(xy + 2) = (3x + 5)(xy + 2)$

Factoring of Trinomials

Consider the trinomial $x^2 + bx + c$ where b and c are integers.
Suppose $x^2 + bx + c = (x + p)(x + q) = x^2 + px + qx + pq = x^2 + (p + q)x + pq$.

From the above we see that if we consider a trinomial $x^2 + bx + c$ where b and c are integers, if we can find integers p and q such that $b = p + q$ and $c = pq$, then the trinomial can be factored as
$x^2 + bx + c = (x + p)(x + q)$.

Now consider the case $ax^2 + bx + c$ (where a, b and c are integers with $a \neq 0$).
Suppose
$ax^2 + bx + c = (px + q)(rx + s) = prx^2 + (ps + qr)x + qs$.

In this case the task is to find integral factors p, r of a and q, s of c such that $ps + qr = b$.

Example:
Factor the following

(i) $x^2 + 7x + 10$

(ii) $x^2 - x - 20$

(iii) $3x^2 - x - 4$

Solution:

(i) $x^2 + 7x + 10$

Here $b = 7$ and $c = 10$. The factors 2 and 5 of 10 are such that their sum $2 + 5 = 7$.

Thus $x^2 + 7x + 10 = (x + 5)(x + 2)$

(Note: The other pairs of factors of 10 are (10, 1), (-10, -1) and (-2, -5), but the sums of these do not add up to 7)

(ii) $x^2 - x - 20$

Here b = -1 and c = -20.

The factors of -20 (and the sum of the factors) are:

20 and -1 (sum 19)

-20 and 1 (sum -19)

4 and -5 (sum -1)

-4 and 5 (sum 1)

10 and -2 (sum 8)

-10 and 2 (sum -8)

Thus the factors of -20 such that the sum of the factors equals -1 are -5 and 4.

Therefore, $x^2 - x - 20 = (x - 5)(x + 4)$

(iii) $3x^2 - x - 4 = (3x - 4)(x + 1)$

Difference of Two Squares

An expression of the form $\boldsymbol{x^2 - a^2}$ is known as **the difference of two squares** and is factored as follows:

$x^2 - a^2 = (x - a)(x + a)$

Other Useful Factorizations

(i) $x^2 + 2xy + y^2 = (x + y)^2$

(ii) $x^2 - 2xy + y^2 = (x - y)^2$

(iii) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Example:

Factor the following:

(i) $9x^2 + 6xy + y^2$

(ii) $16x^2 - 25$

(iii) $9x^2 - 30xy + 25y^2$

(iv) $x^3 - 27$

Solution:

(i) $9x^2 + 6xy + y^2 = (3x + y)^2$ using factorization (i) above

(ii) $16x^2 - 25 = (4x - 5)(4x + 5)$ difference of two squares

(iii) $9x^2 - 30xy + 25y^2 = (3x - 5y)^2$ using factorization (ii) above

(iv) $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$ using factorization (iv) above

Algebraic Fractions

The procedures followed in simplifying arithmetic fractions can be used to simplify algebraic fractions.

Example:

Simplify the following

(i) $\dfrac{x^2+6x+9}{3xyz}.\dfrac{6xy}{x^2-9}$

(ii) $\dfrac{x+3}{x+4} \div \dfrac{xy+3y}{x^2-16}$

(iii) $\dfrac{1}{x^2-1} - \dfrac{1}{(x-1)^2}$

(iv) $\dfrac{4}{x^2+x} + \dfrac{7}{4x+4}$

Solution:

(i) $\dfrac{x^2+6x+9}{3xyz}.\dfrac{6xy}{x^2-9} = \dfrac{(x+3)^2}{3xyz}.\dfrac{6xy}{(x-3)(x+3)} = \dfrac{2(x+3)}{z(x-3)}$

(ii) $\dfrac{x+3}{x+4} \div \dfrac{xy+3y}{x^2-16} = \dfrac{(x+3)}{(x+4)}.\dfrac{(x-4)(x+4)}{y(x+3)} = \dfrac{x-4}{y}$

(iii) $\dfrac{1}{x^2-1} - \dfrac{1}{(x-1)^2} = \dfrac{1}{(x-1)(x+1)} - \dfrac{1}{(x-1)^2} = \dfrac{(x-1)}{(x-1)^2(x+1)} - \dfrac{(x+1)}{(x-1)^2(x+1)}$

$$= \frac{x-1-(x+1)}{(x-1)^2(x+1)} = \frac{-2}{(x-1)^2(x+1)}$$

(iv) $\dfrac{4}{x^2+x} + \dfrac{7}{4x+4} = \dfrac{4}{x(x+1)} + \dfrac{7}{4(x+1)} = \dfrac{16}{4x(x+1)} + \dfrac{7x}{4x(x+1)} = \dfrac{7x+16}{4x(x+1)}$

Evaluating an Expression

An expression is **evaluated** by replacing the variables in the expression with given numbers and performing the indicated operations. By evaluating an expression we obtain a numerical **value** for the expression.

Example:
The value of $9x^2 + 6xy + y^2$ when $x = 1$ and $y = -1$ is $9(1)^2 + 6(1)(-1) + (-1)^2 = 9 - 6 + 1 = 4$.

Equations

An **equation** is a statement that two algebraic expressions are equal. The parts of an equation separated by the sign of equality are called the **sides** of the equation and are distinguished as the right side and the left side.

Example: $2x + 1 = -3$

A numerical value that makes the equation true is said to **satisfy** the equation. A symbol in an equation whose value it is required to find is called the **unknown quantity**. The process of finding the values that satisfy the equation is called **solving the equation**. The values that satisfy an equation are called **roots** of the equation.

Unknown quantities could be either variables or constants. Variables are usually denoted by the letters *x*, *y* and *z* and unknown constants are usually denoted by the letters *c*, *k*.

The process of finding solutions to equations is based on the following axioms.

1. If to equals we add equals, the sums are equal.
2. If from equals we subtract equals, the remainders are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.

Consider the above example:

	$2x + 1 = -3$	
$\therefore$	$2x + 1 - 1 = -3 - 1$	(adding -1 to both sides)
$\therefore$	$2x = -4$	
$\therefore$	$x = -2$	(dividing both sides by 2)

Once a solution is obtained for an equation, the accuracy of the solution should be **verified** by substituting the value in for the unknown.

Consider the above example:
Substituting the value of $x = -2$ into the expression $2x + 1$on the left side of the equation we obtain $2(-2) + 1 = -4 + 1 = -3$ which is the right side of the equation.
This confirms the accuracy of the solution.

Application:
Rs. 3000 is divided between Amal, Sunil and Aruna such that Amal gets Rs. 200 more than Sunil, and Aruna gets twice the amount that Sunil gets. How much does each person get?

Solution:
Suppose the amount Sunil gets is denoted by x. Then we obtain the equation
$(x + 200) + x + 2x = 3000$
$\therefore 4x + 200 = 3000$
$\therefore 4x = 2800$
$\therefore x = 700$

Thus the amount that Amal gets is Rs. 900, the amount that Sunil gets is Rs. 700 and the amount that Aruna gets is Rs. 1400.

2.3 Working with Formulae

One of the most important applications of Algebra is the use of formulae. In every form of applied science and mathematics formulae are applied.

Formulae involve three operations:
(i) Construction (ii) Manipulation (iii) Evaluation

Examples of formulae:

(i) The formula for the perimeter P of a rectangle of length a and breadth b is $P = 2a + 2b$
(ii) The formula for the area A of a rectangle of length a and breadth b is $A = a.b$
(iii) The formula for the surface area A of a sphere of radius r is $A = 4\pi r^2$

From the above examples it can be observed that in a formula**, one quantity is expressed in terms of other quantities and the formula expresses a relationship between the quantities**.

Consider the formula for the volume V of a cylinder with base radius r and altitude h:

$V = \pi r^2 h$.

If it is required to express the altitude *h* of the cylinder in terms of the volume *V* and base radius *r*, we would write

$$h = \frac{V}{\pi r^2}.$$

When one quantity (say *h* in the above formula) is expressed in terms of other quantities, we call the relevant quantity the **subject of the formula**.

Thus *V* is the subject of the formula $V = \pi r^2 h$, and *h* is the subject of the formula $h = \frac{V}{\pi r^2}$.

The process of transforming one formula into another is called '**changing the subject of the formula**'. Algebraic skills are required to transform one formula into another. **Evaluation** is done by substituting given values for the unknowns.

<u>Example</u>:

(i) The time of vibration *t* of a simple pendulum is given by the formula $t = 2\pi\sqrt{\frac{l}{g}}$. Make *l* the subject of the formula.

(ii) If $v^2 = u^2 + 2fs$, make *s* the subject of the formula and find its value when $u = 15$, $v = 20$ and $f = 5$.

(iii) Make *r* the subject of the formula $V = \frac{1}{3}\pi r^2 h$ and evaluate it when $V = 66$ and $h = 7$. (Assume that $\pi = \frac{22}{7}$).

<u>Solution</u>:

(i) $t = 2\pi\sqrt{\frac{l}{g}}$. Therefore $\sqrt{\frac{l}{g}} = \frac{t}{2\pi}$. Thus $\frac{l}{g} = \left(\frac{t}{2\pi}\right)^2$. Hence $l = \frac{gt^2}{4\pi^2}$.

(ii) $v^2 = u^2 + 2fs$. Therefore $2fs = v^2 - u^2$. Thus $s = \frac{v^2 - u^2}{2f}$. When $u = 15$, $v = 20$ and $f = 5$, $s = \frac{400 - 225}{2 \times 5} = \frac{175}{10} = 17.5$.

(iii) $V = \frac{1}{3}\pi r^2 h$. Therefore $r^2 = \frac{3V}{\pi h}$. Thus $r = \sqrt{\frac{3V}{\pi h}}$. When V = 66, h = 7 we obtain $r = \sqrt{\frac{3 \times 66}{\frac{22}{7} \times 7}} = \sqrt{9} = 3$.

2.4 ***Solutions of Equations involving Absolute Values***

We recall from the first section that the **absolute value** of the real number x symbolized by $|x|$ is

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus if we consider the number line, the absolute value of a real number is the distance to the number from the origin.

Properties:

(i) For every real number x, $|x| = |-x|$.

(ii) For any two real numbers a, b, $|a.b| = |a|.|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$.

(iii) If $|a| = |b|$, then $a = \pm b$

Solving Equations involving Absolute Values

Example:

(i) $|3t - 4| - 5 = 6$.

$|3t - 4| = 11$

$3t - 4 = 11$ or $3t - 4 = -11$

$3t = 15$ or $3t = -7$

$t = 5$ or $t = -\frac{7}{3}$

Verification:

When $t = 5$, left side = $|3(5) - 4| - 5 = |15 - 4| - 5 = |11| - 5 = 6 =$ right side

When $t = -\frac{7}{3}$,

left side = $\left|3\left(-\frac{7}{3}\right) - 4\right| - 5 = |-7 - 4| - 5 = |-11| - 5 = 11 - 5 = 6 =$ right side

(ii) $\left|\frac{5x-6}{3x}\right| = 5$

$\frac{5x-6}{3x} = 5$ or $\frac{5x-6}{3x} = -5$

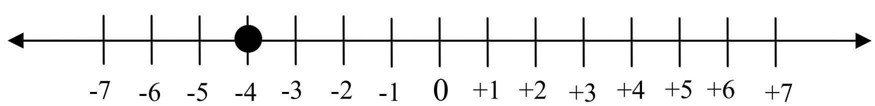
$5x - 6 = 15x$ or $5x - 6 = -15x$
$10x = -6$ or $-20x = -6$
$x = -\frac{3}{5}$ or $x = \frac{3}{10}$.
As in the above example, these solutions can be verified by substituting back into the original equation.

2.5 ***Intervals on the Number Line***

Representation of points on the Number Line

When graphing a point on the number line we simply colour in the corresponding point on the number line.

The example below indicates the point $x = -4$



In mathematics we deal with intervals of numbers. An application of intervals is that they are used to describe the solution sets of inequalities.

Intervals of finite length

Suppose a, b are real numbers such that $a < b$.

(i) $(a, b) = \{x \mid a < x < b\}$. i.e., the interval (a, b) is the set consisting of all real numbers that lie between a and b. This is called an open interval.

(ii) $[a, b) = \{ x \mid a \leq x < b \}$. i.e., the interval $[a, b)$ is the set consisting of all real numbers that lie between a and b, and the number a. This is called an interval closed on the left and open on the right.

(iii) $(a, b] = \{ x \mid a < x \leq b \}$. i.e., the interval $(a, b]$ is the set consisting of all real numbers that lie between a and b, and the number b. This is called an interval open on the left and closed on the right.

(iv) $[a, b] = \{ x \mid a \leq x \leq b \}$. i.e., the interval $[a, b]$ is the set consisting of all real numbers that lie between a and b, and the numbers a and b. This is called a closed interval.

<u>Intervals of infinite length</u>

(i) $(a, \infty) = \{ x \mid x > a \}$. i.e., the interval (a, ∞) is the set of real numbers greater than a.

(ii) $[a, \infty) = \{ x \mid x \geq a \}$. i.e., the interval $[a, \infty)$ is the set of real numbers greater or equal to a.

(iii) $(-\infty, a) = \{ x \mid x < a \}$. i.e., the interval $(-\infty, a)$ is the set of real numbers less than a.

(iv) $(-\infty, a] = \{ x \mid x \leq a \}$. i.e., the interval $(-\infty, a]$ is the set of real numbers less than or equal to a.

2.6 ***Rules of Inequalities, Solutions of Inequalities***

Rules of Inequalities

(i) Trichotomy Property:
For any pair of real numbers a and b, $a < b$ or $a = b$ or $b < a$.

(ii) Transitive Property:
For any real numbers a, b and c; if $a < b$ and $b < c$, then $a < c$.

(iii) Addition Property:
For any real numbers a, b and c; if $a < b$, then $a + c < b + c$.

(iv) Subtraction Property:
For any real numbers a, b and c; if $a < b$, then $a - c < b - c$.

(v) Multiplication Property:
For any real numbers a, b and c;
(a) if $a < b$ and $c > 0$, then $ac < bc$.
(b) if $a < b$ and $c < 0$, then $ac > bc$.

(vi) Division Property:
For any real numbers a, b and c;
(a) if $a < b$ and $c > 0$, then $a \div c < b \div c$.
(b) if $a < b$ and $c < 0$, then $a \div c > b \div c$.

(vii) Power Property:
If $a > b > 0$, then $a^m > b^m$ for $m > 0$.

(viii) Root Property:
If $a > b > 0$, then $\sqrt[m]{a} > \sqrt[m]{b}$ for $m > 0$.

(ix) Reciprocal Property:
If $ab > 0$ and $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

Solutions of Inequalities

In this section we will only consider **linear inequalities in one variable**. Linear inequalities in one variable are inequalities involving one variable in which the exponent of the variable is 1. The properties of inequalities can be used to solve inequalities. **Solving an inequality** means finding all values of the variable for which the inequality is true. The set of all values which makes the inequality true is the **solution of the inequality**.

Examples of linear inequalities in one variable:

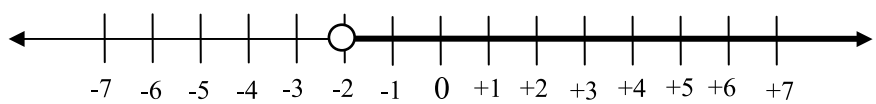
(i) $2x + 1 < -3$
(ii) $2x + 1 > 4x - 3$
(iii) $2x + 1 \geq -4x + 5$

Example:
Solve the following inequalities and illustrate the solution on the number line.

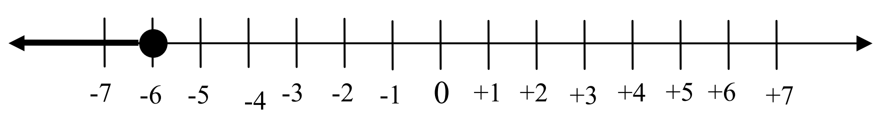
(i) $1 - 2x < 5$
(ii) $4x + 9 \leq 2x - 3$
(iii) $-5 < 3x - 2 \leq 4$

Solution:

(i)
$1 - 2x < 5$
$-1 + 1 - 2x < -1 + 5$ (property (iii) of inequalities)
$-2x < 4$
$x > -2$ (property (vi) of inequalities)



(ii)
$4x + 9 \leq 2x - 3$
$4x + 9 - 2x \leq 2x - 3 - 2x$ (property (iv))
$2x + 9 \leq -3$
$2x + 9 - 9 \leq -3 - 9$ (property (iv))
$2x \leq -12$
$x \leq -6$ (property (vi))



(iii) $-5 < 3x - 2 \le 4$

The above inequality can be written as the following two inequalities

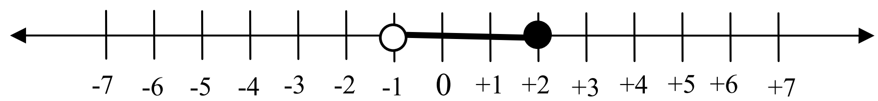
$-5 < 3x - 2$ and $3x - 2 \le 4$

$-5 < 3x - 2$ and $3x - 2 \le 4$

$-5 + 2 < 3x$ and $3x \le 6$

$-3 < 3x$ and $x \le 2$

$-1 < x$ and $x \le 2$



<u>Inequalities that involve Absolute Values</u>

<u>Properties</u>:

(i) For any positive real number a, $|x| < a$ is equivalent to $-a < x < a$.

(ii) For real numbers a and b with $b > 0$, $|x - a| < b$ is equivalent to $-b < x - a < b$

(iii) For any positive real number a, $|x| > a$ is equivalent to $x > a$ or $x < -a$.

(iv) For real numbers a and b with $b > 0$, $|x - a| > b$ is equivalent to $x - a > b$ or $x - a < -b$.

<u>Example</u>:

Solve the following inequalities

(i) $|3x + 5| < 2$

(ii) $|5 - 7x| \ge 9$

<u>Solution</u>:

(i) $|3x + 5| < 2$

$-2 < 3x + 5 < 2$ (property (ii))

$-2 < 3x + 5$ and $3x + 5 < 2$

$-7 < 3x$ and $3x < -3$

$-\frac{7}{3} < x$ and $x < -1$

$-\frac{7}{3} < x < -1$.

Thus the solution set is $\{x \mid -\frac{7}{3} < x < -1\}$ or $(-\frac{7}{3}, -1)$

(ii) $|5 - 7x| \geq 9$

$5 - 7x \geq 9$ or $5 - 7x \leq -9$ (property (iv))

$-7x \geq 4$ or $-7x \leq -14$

$x \leq -\frac{4}{7}$ or $x \geq 2$

Thus the solution is $\{ x \mid x \leq -\frac{4}{7} \text{ or } x \geq 2\}$