A *group* is an algebraic structure $(G, \circ)$ which satisfies the following conditions:

(A1): Closure: $\forall\, a, b \in G : a \circ b \in G$

(A2) Associativity: $\forall\, a, b, c \in G :$
$$(a \circ b) \circ c = a \circ (b \circ c).$$

(A3) Identity: $\exists\, e \in G$ such that
$$\forall\, a \in G : e \circ a = a = a \circ e.$$

(A4) Inverse: $\forall\, a \in G : \exists\, a^{-1} \in G$
such that $a \circ a^{-1} = e = a^{-1} \circ a$.

An *abelian group* is a group $(G, \circ)$ where $\forall\, a, b \in G : a \circ b = b \circ a$, (commutativity) holds.

- Identity element $e \in G$ is unique.
- Inverse element $a^{-1} \in G$ for each $a \in G$ is unique.

$e = e \circ f = f$ for $e, f \in G$ s.t. e, f are identity elements of G.

A ring $(R, +, \circ)$ is a nonempty set with two binary operations (usually written as addition $+$ and multiplication $\circ$) such that for all

$a, b, c \in R$,

(A1) $a + b \in R$ (closure under $+$)

(A2) $(a+b)+c = a+(b+c)$

(A3) $a+b = b+a$

(A4) $\exists\ 0 \in R$ s.t. $a+0 = a$.

(A5) For each $a \in R$, $\exists$ an additive inverse $-a \in R$ s.t. $a+(-a) = 0$.

(A6) $a \circ b \in R$ (closure under $\circ$)

(A7) $(a \circ b) \circ c = a \circ (b \circ c)$

(A8) $a \circ (b+c) = a \circ b + b \circ c$.

A <u>ring with identity</u> is a ring $(R, +, \circ)$ that contains a multiplicative identity element $1 \in R$ s.t. $\forall\ a \in R$: $1 \circ a = a$.

✱✱ The Real Numbers $\mathbb{R} = (-\infty, \infty)$

The set of real nos. $\equiv$ the real line.

What axioms characterize the real nos.?

- ↳ the field axioms
- ↳ the order axioms
- ↳ the completeness axiom

The set of real nos. is referred to as the one and only "complete ordered field".

The real nos. are elements of a nonempty set $\mathbb{R}$ equipped with two operations, $+$ and $\cdot$ from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, called addition & multiplication, that satisfy the following axioms :—

Field axioms

Let $x, y, z \in \mathbb{R}$.

Axiom 1 : $x + y = y + x$ & $xy = yx$
(the commutative laws)

Axiom 2: $x + (y+z) = (x+y) + z$ & $x(yz) = (xy)z$
(the associative laws)

Axiom 3. $x(y+z) = xy + xz$ (the distributive law)

Axiom 4. $\exists$ an element $0 \in \mathbb{R}$ s.t.
$x + 0 = x \ \forall x \in \mathbb{R}$.

Axiom 5. For each $x \in \mathbb{R}$ $\exists$ an element
$-x \in \mathbb{R}$ s.t. $x + (-x) = 0$.

Axiom 6. $\exists$ an element $1 \in \mathbb{R}$ with $1 \neq 0$
satisfying $1 \cdot x = x$ for all $x \in \mathbb{R}$.

Axiom 7. For each $x \neq 0$ $\exists$ an element
$x^{-1} \in \mathbb{R}$ s.t. $x x^{-1} = 1$.

Note that it can be shown that 0, x^{-1}, and $-x = (-1) \cdot x$ ~~are~~ ~~be~~ uniquely determined.

<u>Order axioms</u>

Axiom 8. For any $x, y \in \mathbb{R}$, either
$x \geq y$ or $y \geq x$ holds.

Axiom 9. If $x \geq y$, then $x + z \geq y + z$
holds for each $z \in \mathbb{R}$.

Axiom 10. If $x \geq y$ and $z \geq 0$, then
$xz \geq yz$.