(ii) If the unit vectors vary as the values of the coordinates change (i.e. are not constant in direction throughout the whole space) then the derivatives of these vectors appear as contributions to $\nabla^2\mathbf{a}$.

Cartesian coordinates are an example of the first case in which each component satisfies $(\nabla^2\mathbf{a})_i = \nabla^2 a_i$. In this case (10.41) can be applied to each component separately:

$$[\nabla\times(\nabla\times\mathbf{a})]_i = [\nabla(\nabla\cdot\mathbf{a})]_i - \nabla^2 a_i. \qquad (10.42)$$

However, cylindrical and spherical polar coordinates come in the second class. For them (10.41) is still true, but the further step to (10.42) cannot be made.

More complicated vector operator relations may be proved using the relations given above.

►*Show that*

$$\nabla\cdot(\nabla\phi\times\nabla\psi) = 0,$$

where ϕ and ψ are scalar fields.

From the previous section we have

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}).$$

If we let $\mathbf{a} = \nabla\phi$ and $\mathbf{b} = \nabla\psi$ then we obtain

$$\nabla\cdot(\nabla\phi\times\nabla\psi) = \nabla\psi\cdot(\nabla\times\nabla\phi) - \nabla\phi\cdot(\nabla\times\nabla\psi) = 0, \qquad (10.43)$$

since $\nabla\times\nabla\phi = 0 = \nabla\times\nabla\psi$, from (10.37). ◄

10.9 Cylindrical and spherical polar coordinates

The operators we have discussed in this chapter, i.e. grad, div, curl and ∇^2, have all been defined in terms of Cartesian coordinates, but for many physical situations other coordinate systems are more natural. For example, many systems, such as an isolated charge in space, have spherical symmetry and spherical polar coordinates would be the obvious choice. For axisymmetric systems, such as fluid flow in a pipe, cylindrical polar coordinates are the natural choice. The physical laws governing the behaviour of the systems are often expressed in terms of the vector operators we have been discussing, and so it is necessary to be able to express these operators in these other, non-Cartesian, coordinates. We first consider the two most common non-Cartesian coordinate systems, i.e. cylindrical and spherical polars, and go on to discuss general curvilinear coordinates in the next section.

10.9.1 Cylindrical polar coordinates

As shown in figure 10.7, the position of a point in space P having Cartesian coordinates x, y, z may be expressed in terms of cylindrical polar coordinates

ρ, ϕ, z, where

$$x = \rho \cos \phi, \qquad y = \rho \sin \phi, \qquad z = z, \tag{10.44}$$

and $\rho \geq 0$, $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$. The position vector of P may therefore be written

$$\mathbf{r} = \rho \cos \phi \, \mathbf{i} + \rho \sin \phi \, \mathbf{j} + z \, \mathbf{k}. \tag{10.45}$$

If we take the partial derivatives of $\mathbf{r}$ with respect to ρ, ϕ and z respectively then we obtain the three vectors

$$\mathbf{e}_\rho = \frac{\partial \mathbf{r}}{\partial \rho} = \cos \phi \, \mathbf{i} + \sin \phi \, \mathbf{j}, \tag{10.46}$$

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = -\rho \sin \phi \, \mathbf{i} + \rho \cos \phi \, \mathbf{j}, \tag{10.47}$$

$$\mathbf{e}_z = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}. \tag{10.48}$$

These vectors lie in the directions of increasing ρ, ϕ and z respectively but are not all of unit length. Although $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ form a useful set of basis vectors in their own right (we will see in section 10.10 that such a basis is sometimes the *most* useful), it is usual to work with the corresponding *unit* vectors, which are obtained by dividing each vector by its modulus to give

$$\hat{\mathbf{e}}_\rho = \mathbf{e}_\rho = \cos \phi \, \mathbf{i} + \sin \phi \, \mathbf{j}, \tag{10.49}$$

$$\hat{\mathbf{e}}_\phi = \frac{1}{\rho} \mathbf{e}_\phi = -\sin \phi \, \mathbf{i} + \cos \phi \, \mathbf{j}, \tag{10.50}$$

$$\hat{\mathbf{e}}_z = \mathbf{e}_z = \mathbf{k}. \tag{10.51}$$

These three unit vectors, like the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, form an orthonormal triad at each point in space, i.e. the basis vectors are mutually orthogonal and of unit length (see figure 10.7). Unlike the fixed vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, however, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ change direction as P moves.

The expression for a general infinitesimal vector displacement $d\mathbf{r}$ in the position of P is given, from (10.19), by

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \rho} \, d\rho + \frac{\partial \mathbf{r}}{\partial \phi} \, d\phi + \frac{\partial \mathbf{r}}{\partial z} \, dz \\ &= d\rho \, \mathbf{e}_\rho + d\phi \, \mathbf{e}_\phi + dz \, \mathbf{e}_z \\ &= d\rho \, \hat{\mathbf{e}}_\rho + \rho \, d\phi \, \hat{\mathbf{e}}_\phi + dz \, \hat{\mathbf{e}}_z. \end{aligned} \tag{10.52}$$

This expression illustrates an important difference between Cartesian and cylindrical polar coordinates (or non-Cartesian coordinates in general). In Cartesian coordinates, the distance moved in going from x to $x + dx$, with y and z held constant, is simply $ds = dx$. However, in cylindrical polars, if ϕ changes by $d\phi$, with ρ and z held constant, then the distance moved is *not* $d\phi$, but $ds = \rho \, d\phi$.

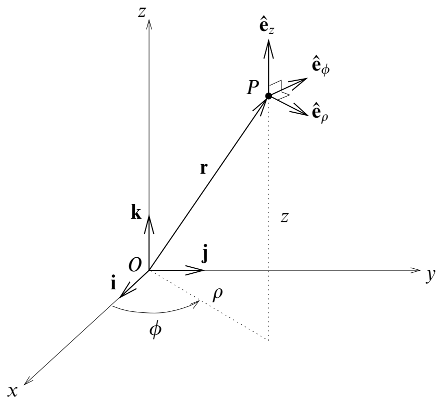

Figure 10.7 Cylindrical polar coordinates ρ, ϕ, z.

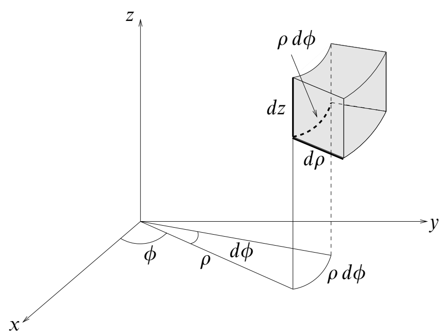

Figure 10.8 The element of volume in cylindrical polar coordinates is given by $\rho\, d\rho\, d\phi\, dz$.

Factors, such as the ρ in $\rho\, d\phi$, that multiply the coordinate differentials to give distances are known as *scale factors*. From (10.52), the scale factors for the ρ-, ϕ- and z- coordinates are therefore 1, ρ and 1 respectively.

The magnitude ds of the displacement $d\mathbf{r}$ is given in cylindrical polar coordinates by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2,$$

where in the second equality we have used the fact that the basis vectors are orthonormal. We can also find the volume element in a cylindrical polar system (see figure 10.8) by calculating the volume of the infinitesimal parallelepiped

$$\nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z$$

$$\nabla\cdot\mathbf{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z}$$

$$\nabla\times\mathbf{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Table 10.2 Vector operators in cylindrical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

defined by the vectors $d\rho\,\hat{\mathbf{e}}_\rho$, $\rho\,d\phi\,\hat{\mathbf{e}}_\phi$ and $dz\,\hat{\mathbf{e}}_z$:

$$dV = |d\rho\,\hat{\mathbf{e}}_\rho\cdot(\rho\,d\phi\,\hat{\mathbf{e}}_\phi\ \times\ dz\,\hat{\mathbf{e}}_z)| = \rho\,d\rho\,d\phi\,dz,$$

which again uses the fact that the basis vectors are orthonormal. For a simple coordinate system such as cylindrical polars the expressions for $(ds)^2$ and dV are obvious from the geometry.

We will now express the vector operators discussed in this chapter in terms of cylindrical polar coordinates. Let us consider a vector field $\mathbf{a}(\rho,\phi,z)$ and a scalar field $\Phi(\rho,\phi,z)$, where we use Φ for the scalar field to avoid confusion with the azimuthal angle ϕ. We must first write the vector field in terms of the basis vectors of the cylindrical polar coordinate system, i.e.

$$\mathbf{a} = a_\rho\,\hat{\mathbf{e}}_\rho + a_\phi\,\hat{\mathbf{e}}_\phi + a_z\,\hat{\mathbf{e}}_z,$$

where a_ρ, a_ϕ and a_z are the components of $\mathbf{a}$ in the ρ-, ϕ- and z- directions respectively. The expressions for grad, div, curl and ∇^2 can then be calculated and are given in table 10.2. Since the derivations of these expressions are rather complicated we leave them until our discussion of general curvilinear coordinates in the next section; the reader could well postpone examination of these formal proofs until some experience of using the expressions has been gained.

►*Express the vector field* $\mathbf{a} = yz\,\mathbf{i} - y\,\mathbf{j} + xz^2\,\mathbf{k}$ *in cylindrical polar coordinates, and hence calculate its divergence. Show that the same result is obtained by evaluating the divergence in Cartesian coordinates.*

The basis vectors of the cylindrical polar coordinate system are given in (10.49)–(10.51). Solving these equations simultaneously for $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we obtain

$$\begin{aligned}\mathbf{i} &= \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi\\ \mathbf{j} &= \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi\\ \mathbf{k} &= \hat{\mathbf{e}}_z.\end{aligned}$$

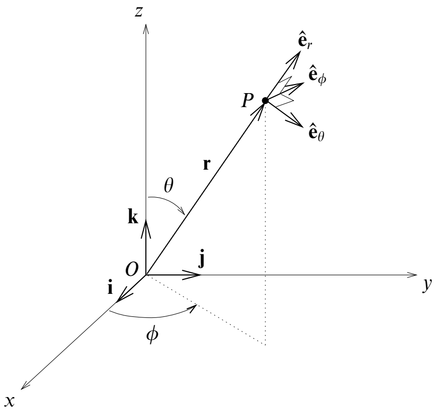

Figure 10.9 Spherical polar coordinates r, θ, ϕ.

Substituting these relations and (10.44) into the expression for $\mathbf{a}$ we find

$$\begin{aligned}\mathbf{a} &= z\rho\sin\phi\,(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi) - \rho\sin\phi\,(\sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi) + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z \\ &= (z\rho\sin\phi\cos\phi - \rho\sin^2\phi)\,\hat{\mathbf{e}}_\rho - (z\rho\sin^2\phi + \rho\sin\phi\cos\phi)\,\hat{\mathbf{e}}_\phi + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z.\end{aligned}$$

Substituting into the expression for $\nabla\cdot\mathbf{a}$ given in table 10.2,

$$\begin{aligned}\nabla\cdot\mathbf{a} &= 2z\sin\phi\cos\phi - 2\sin^2\phi - 2z\sin\phi\cos\phi - \cos^2\phi + \sin^2\phi + 2z\rho\cos\phi \\ &= 2z\rho\cos\phi - 1.\end{aligned}$$

Alternatively, and much more quickly in this case, we can calculate the divergence directly in Cartesian coordinates. We obtain

$$\nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2zx - 1,$$

which on substituting $x = \rho\cos\phi$ yields the same result as the calculation in cylindrical polars. ◀

Finally, we note that similar results can be obtained for (two-dimensional) polar coordinates in a plane by omitting the z-dependence. For example, $(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2$, while the element of volume is replaced by the element of area $dA = \rho\,d\rho\,d\phi$.

10.9.2 Spherical polar coordinates

As shown in figure 10.9, the position of a point in space P, with Cartesian coordinates x, y, z, may be expressed in terms of spherical polar coordinates r, θ, ϕ, where

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{10.53}$$

and $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The position vector of P may therefore be written as

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}.$$

If, in a similar manner to that used in the previous section for cylindrical polars, we find the partial derivatives of $\mathbf{r}$ with respect to r, θ and ϕ respectively and divide each of the resulting vectors by its modulus then we obtain the unit basis vectors

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},\\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.
\end{aligned}$$

These unit vectors are in the directions of increasing r, θ and ϕ respectively and are the orthonormal basis set for spherical polar coordinates, as shown in figure 10.9.

A general infinitesimal vector displacement in spherical polars is, from (10.19),

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi; \tag{10.54}$$

thus the scale factors for the r-, θ- and ϕ- coordinates are 1, r and $r\sin\theta$ respectively. The magnitude ds of the displacement $d\mathbf{r}$ is given by

$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2,$$

since the basis vectors form an orthonormal set. The element of volume in spherical polar coordinates (see figure 10.10) is the volume of the infinitesimal parallelepiped defined by the vectors $dr\,\hat{\mathbf{e}}_r$, $r\,d\theta\,\hat{\mathbf{e}}_\theta$ and $r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi$ and is given by

$$dV = |dr\,\hat{\mathbf{e}}_r\cdot(r\,d\theta\,\hat{\mathbf{e}}_\theta \times r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi)| = r^2\sin\theta\,dr\,d\theta\,d\phi,$$

where again we use the fact that the basis vectors are orthonormal. The expressions for $(ds)^2$ and dV in spherical polars can be obtained from the geometry of this coordinate system.

We will now express the standard vector operators in spherical polar coordinates, using the same techniques as for cylindrical polar coordinates. We consider a scalar field $\Phi(r,\theta,\phi)$ and a vector field $\mathbf{a}(r,\theta,\phi)$. The latter may be written in terms of the basis vectors of the spherical polar coordinate system as

$$\mathbf{a} = a_r\,\hat{\mathbf{e}}_r + a_\theta\,\hat{\mathbf{e}}_\theta + a_\phi\,\hat{\mathbf{e}}_\phi,$$

where a_r, a_θ and a_ϕ are the components of $\mathbf{a}$ in the r-, θ- and ϕ- directions respectively. The expressions for grad, div, curl and ∇^2 are given in table 10.3. The derivations of these results are given in the next section.

As a final note, we mention that, in the expression for $\nabla^2\Phi$ given in table 10.3,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi$$

$$\nabla\cdot\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta\ a_\phi \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Table 10.3 Vector operators in spherical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

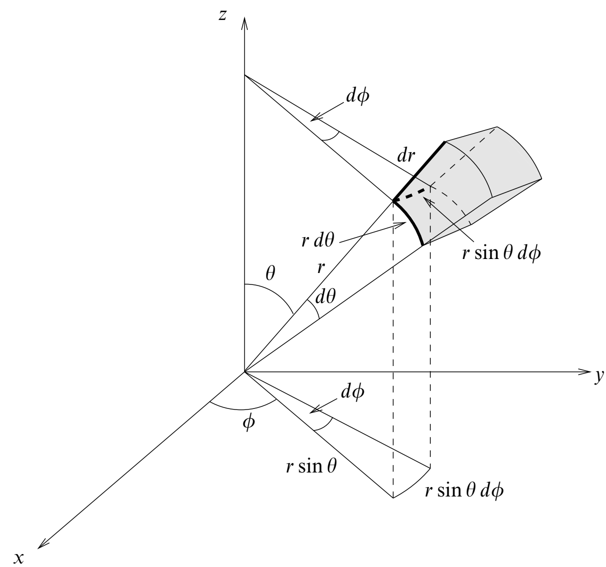

Figure 10.10 The element of volume in spherical polar coordinates is given by $r^2\sin\theta\,dr\,d\theta\,d\phi$.

we can rewrite the first term on the RHS as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi),$$

which can often be useful in shortening calculations.