Class 9, 10/1/23

* Limits: $f(z)$ is a function of complex variable z. The limit of $f(z)$ as z approaches z_0 is a number w_0

$$\lim_{z \to z_0} f(z) = w_0$$

means the following. For each positive number ϵ there is a positive number δ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

So if you make z arbitrarily close to z_0 then the function $f(z)$ will be close to w_0

* If limits exist then it is unique – we will not prove this – but it means the following. Suppose

$$\lim_{z \to z_0} f(z) = w_0^1 \ \& \ \lim_{z \to z_0} f(z) = w_0^2$$

then one can prove that $w_0^1 = w_0^2$

Example: Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$

Ans: To prove it we have to show that $\epsilon > 0$ & $\delta > 0$ exist such that whenever

$0 < |z-(1+i)| < \delta$ we will have $|(2+i)z-(1+3i)| < \epsilon$. So we have to find suitable ϵ & δ.

Start with the inequality $|(2+i)z-(1+3i)| < \epsilon$

$$\Rightarrow |2+i|\left|z - \frac{1+3i}{2+i}\right| < \epsilon$$

$$\Rightarrow \sqrt{5}\,|z-(1+i)| < \epsilon$$

$$\Rightarrow |z-(1+i)| < \epsilon/\sqrt{5}$$

Hence for any $\epsilon > 0$ we can have $\delta = \epsilon/\sqrt{5}$ and then this is valid.

* Limit independent of the path of approach: In the above examples $\lim_{z\to z_0}$ means that the z is allowed to approach in arbitrary manner, not along some particular direction. In the next example we show this

Let $f(z) = z/z^*$ and we want to know if $z\to 0$ limit exist.

Let us approach $z\to 0$ along the real axis; ie $z = x+0i$ then

$$\lim_{z\to 0} f(z) = \frac{x+0i}{x-0i} = 1$$

If we approach along the Im(z) axis then $z = 0 + iy$

$$\lim_{z \to 0} f(z) = \frac{0 + iy}{0 - iy} = -1.$$

So there is no unique limit. Hence the limit does not exist.

* Some Theorems:

1. Suppose $f(z) = u(x,y) + iv(x,y)$; $z_0 = x_0 + iy_0$, & $w_0 = u_0 + iv_0$

Then $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y) \to (x_0, y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y) \to (x_0, y_0)} v(x,y) = v_0$$

2. Following can also be proven: Suppose $\lim_{z \to z_0} f(z) = w_0$ & $\lim_{z \to z_0} F(z) = W_0$

then

* $\lim_{z \to z_0} [f(z) + F(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} F(z) = w_0 + W_0$

* $\lim_{z \to z_0} [f(z) F(z)] = \lim_{z \to z_0} f(z) \lim_{z \to z_0} F(z) = w_0 W_0$

* if $w_0 \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} F(z)} = \frac{w_0}{W_0}$

* For a Polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$$\lim_{z \to z_0} P(z) = P(z_0)$$

* Limits involving Infinity: If z_0 and w_0 are points in the z and w planes, then

$$* \lim_{z \to z_0} f(z) = \infty \text{ if and only if } \lim_{z \to z_0} f\left(\tfrac{1}{z}\right) = 0$$

and

$$* \lim_{z \to \infty} f(z) = w_0 \text{ if and only if } \lim_{z \to 0} f\left(\tfrac{1}{z}\right) = w_0$$

$$* \lim_{z \to \infty} f(z) = \infty \text{ if and only if } \lim_{z \to 0} \frac{1}{f(1/z)} = 0.$$

Example: Calculate the limit $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$

Ans $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{\lim_{z \to i} [(3+i)z^4 - z^2 + 2z]}{\lim_{z \to i} [z + i]}$

Now $\lim_{z \to i} z^4 = i^4 = 1$, $\lim_{z \to i} z^2 = -1$ using these we get

$$= \frac{4+3i}{1+i} = \frac{7}{2} - \frac{1}{2}i$$

Example: Compute the limit $\lim_{z \to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i}$

Ans: Note that if we apply the limit for the quotient then we get for the denominator $\lim_{z \to 1+\sqrt{3}i} (z-1-\sqrt{3}i) = 0$. But we can show that the limit exist

$$\lim_{z \to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = \lim_{z \to 1+\sqrt{3}i} \frac{(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)}{z-1-\sqrt{3}i}$$

$$= \lim_{z \to 1+\sqrt{3}i} (z-1+\sqrt{3}i) = 2\sqrt{3}i$$

* Continuity: A function f is continuous at point z_0 if $\lim_{z \to z_0} f(z)$ exist, $f(z_0)$ exists, & $\lim_{z \to z_0} f(z) = f(z_0)$

Example: Let $f(z) = \begin{cases} z^2 & \text{for } z < 0 \\ z-1 & \text{for } x \geq 0 \end{cases}$, then the $\lim_{z \to 0} f(z)$ does not exist

Since at $z=0$ the $f(z)$ is continuos.

Example: Suppose $f(z) = \frac{z^2-1}{z-1}$. Is the function continuos at $z=1$?

Ans: $\lim_{z \to 1} \frac{z^2-1}{z-1} = \lim_{z \to 1} (z+1) = 2$.

But at $z=1$, $f(1)$ is not defined. Hence not continuos.

* A function that is not continuos at a point is called a discontinuos function.

* Steps of checking continuity at a point: (i) Calculate the limit $\lim_{z \to z_0} f(z)$ (ii) calculate $f(z_0)$. Compare the two values. If they are equal then the function is continuos.

* Derivatives: The derivative of $f(z)$ at $z = z_0$ is defined by

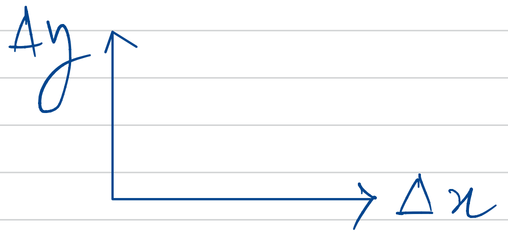
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ provided that this limit exists. When the derivative at z_0 exist then the function is called differentiable.

* An important about derivative about complex function is that even if there exist continuos partial derivatives of all orders of the real & imaginary parts, the derivative of the function itself may not exist.

Example: Let $f(z) = |z|^2$. In this case

$$\underset{\Delta z \to 0}{Lt} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \underset{\Delta z \to 0}{Lt} \frac{(z+\Delta z)(z^* + \Delta z^*) - zz^*}{\Delta z}$$



$$= \underset{z \to 0}{Lt} \frac{\cancel{zz^*} + z\Delta z^* + z^*\Delta z + \Delta z \Delta z^* - \cancel{zz^*}}{\Delta z}$$

$$\underset{\Delta z \to 0}{Lt} \left[z^* + \Delta z^* + z \frac{\Delta z^*}{\Delta z} \right]$$

Now to find the limit we approach $\Delta z \to 0$ in two ways - along the real and imaginary axis. Now $\Delta z = \Delta x + i\Delta y$. If the approach is along the horizontal axis then

$$\Delta z^* = \Delta x + i0 = \Delta z$$

So $$\underset{\Delta z \to 0}{Lt} \left[z^* + \Delta z^* + z \frac{\Delta z^*}{\Delta z} \right] = z^* + \Delta z^* + z$$

For approach along the vertical axis it can be shown that

$$\lim_{\Delta z \to 0}\left[z^* + \Delta z^* + z\,\frac{\Delta z^*}{\Delta z}\right] = z^* + \Delta z^* - z$$

Since limit must be unique

$$z^* + \Delta z^* + z = z^* + \Delta z^* - z$$

$$\Rightarrow \quad z^* + z = z^* - z$$

$$\Rightarrow z = 0.$$

Hence derivative only exist at $z=0$, and not in other points.

But note that $f(z) = |z|^2 = x^2+y^2+i0$. The $u = x^2+y^2$, $v=0$ has derivative at all points.

* Cauchy-Riemann Equations: This is a set of equations that determines if derivative of a function exists. To obtain this set of equations we take the limit

$$\lim_{\delta z \to 0}\frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta z \to 0}\frac{\delta f(z)}{\delta z}$$

in two different approaches - along $\delta x = 0$ & along $\delta y = 0$

Now $\delta z = \delta x + i\delta y$. Since we can write $f(z) = u + iv$

$$\delta f = \delta u + i\delta v$$

Hence $$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

Now first we take the limit along $\delta y = 0$ with $\delta x \to 0$

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\substack{\delta x\to 0\\ \delta y = 0}}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Similarly along the path $\delta x = 0$ with $\delta y = 0$ we get

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\substack{\delta x = 0\\ \delta y\to 0}}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta x}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since these two limits must be same, equating we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real & the imaginary parts seperately we get

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

These are the Cauchy-Riemann conditions. If these conditions are satisfied and the partial derivative of $u(x,y)$ & $v(x,y)$ exist then the derivative $\boxed{df/dz = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$

* CR Equations in Polar Form: $z = re^{i\theta}$ & $f(z) = u(r,\theta) + i\,v(r,\theta)$. Then the CR equations are

$$\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Example: Determine if $f(z) = z^2$ is differentiable everywhere.

Ans: $f(z) = z^2 = x^2 - y^2 + i\,2xy \Rightarrow u = x^2 - y^2,\ v = 2xy$

$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$ Hence CR satisfied.

The derivative is $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2iy$

* The CR conditions itself are not the sufficient conditions for $f'(z)$ to exist. It also needs some conditions about the continuity of u & v

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be defined through some region in the complex plane, and suppose the first order partial derivative of $u(x,y)$ & $v(x,y)$ with respect to x & y exist everywhere in that region. If the partial derivative satisfy the CR equations at $z_0 = x_0 + iy_0$ the the derivative $f'(z)$ exist at $z_0 = x_0 + iy_0$.