Discrete Structures (Monsoon 2022)

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Infinite Series and Convergence Tests

Comparison Tests of Convergence

Theorem

Let $\sum u_n$ and $\sum v_n$ be two infinite series of positive terms. If $\sum v_n$ is convergent, then $\sum u_n$ is also convergent, provided that
(i) $u_n \leq v_n$, for all $n \geq m$, m being some fixed finite number
(ii) $\lim_{n\to\infty} \frac{u_n}{v_n} = k$, a finite positive non-zero number.

Comparison Tests of Convergence

Problem: Let $\sum u_n$ be infinite series of positive terms, $u_n = \frac{1}{n^2+1}$. Test whether the given series $\sum u_n$ is convergent or divergent.

Comparing this series with the convergent series for which $v_n = \frac{1}{n^2}$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{u_n}{v_n} &= \lim_{n\to\infty} \frac{n^2}{n^2+1} \\ &= 1, \text{ a finite quantity} \end{aligned}$$

Therefore, by “Comparison Test”, the series is **convergent.**

Cauchy's Root Test

Theorem

- *An infinite series $\sum u_n$ of positive terms is* ***convergent****, if*

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} < 1.$$

- *An infinite series $\sum u_n$ of positive terms is* ***divergent****, if*

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} > 1.$$

Cauchy's Root Test

Problem: Find whether the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

is convergent or divergent.

We have, $u_n=\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}$. Then,

$$u_n^{\frac{1}{n}}=\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-1}=\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}\cdot\left(\frac{n+1}{n}\right)^{-1}$$

Therefore, $\lim_{n\to\infty} u_n^{\frac{1}{n}}=1.(e-1)^{-1}$, since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$. Thus, $\lim_{n\to\infty} u_n^{\frac{1}{n}}<1$, since $e=2.71828...>2$. By Cauchy's root test, the series is **convergent**.

D'Alembert's Ratio Test

Theorem

Let $\sum u_n$ be an infinite series of positive terms. Let

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l.$$

- $\sum u_n$ *is* ***convergent****, if* $l < 1$.
- $\sum u_n$ *is* ***divergent****, if* $l > 1$.

D'Alembert's Ratio Test

Problem: Examine the convergency or divergency of the following series:

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^5}{10} + \cdots + \frac{x^n}{n^2+1} + \cdots$$

We have, $u_n = \frac{x^n}{n^2+1}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$. Then,

$$\begin{aligned}
\lim_{n\to\infty} \frac{u_n}{u_{n+1}} &= \lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} \cdot \frac{1}{x} \\
&= \lim_{n\to\infty} \frac{(n^2+2n+2)/n^2}{(n^2+1)/n^2} \cdot \frac{1}{x} \\
&= \lim_{n\to\infty} \frac{1+2/n+2/n^2}{1+1/n^2} \cdot \frac{1}{x} = \frac{1}{x}
\end{aligned}$$

Hence, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$. Thus, if $x < 1$, the series is **convergent**, and if $x > 1$, the series is **divergent**.