Also, since $\bigvee_{k=n+1}^{\infty} x_k \leq \bigvee_{k=n}^{\infty} x_k$ and

$\bigwedge_{k=n}^{\infty} x_k \leq \bigwedge_{k=n+1}^{\infty} x_k$ for each n, it

follows that

$$\bigvee_{k=n}^{\infty} x_k \downarrow \limsup x_n \text{ and}$$

$$\bigwedge_{k=n}^{\infty} x_k \uparrow \liminf x_n .$$

<u>Thm</u> : If $\{x_n\}$ is a bounded sequence, then $\liminf x_n$ and $\limsup x_n$ are the smallest and largest limit points of $\{x_n\}$. In particular,

$$\liminf x_n \leq \limsup x_n .$$

Proof: Let $\{x_n\}$ be a bounded sequence of $\mathbb{R}$. Put $s = \limsup x_n$. We shall first attempt to show: s is the largest limit point of $\{x_n\}$. Other case can be shown in a similar manner.

To show that s is a limit point: Let $m \in \mathbb{N}$, $\varepsilon > 0$. $\because \bigvee_{k=n}^{\infty} x_k \downarrow_n s$, $\exists$ $n > m$ s.t. $s \leq \bigvee_{k=n}^{\infty} x_k < s + \varepsilon$. This implies $\exists$ some $k \geqslant n > m$ s.t. $s - \varepsilon < x_k < s + \varepsilon$. $\therefore$ s is a limit point of $\{x_n\}$.

Now we show that s is the largest limit point. Let x be a limit point of $\{x_n\}$, let $\varepsilon > 0$.

~~To finish the~~ Then for each $n \in \mathbb{N}$, $\exists\, m > n$ s.t. $x - \varepsilon < x_m < x + \varepsilon$. It follows that

$$x - \varepsilon < \bigvee_{k=n}^{\infty} x_k \text{ for each } n, \text{ and so,}$$

$$x - \varepsilon \leq \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k = \ell \text{ for each } \varepsilon > 0.$$

Thus, $x \leq \ell$, and that completes the proof.

__Corollary__ (Bolzano-Weierstrass). Every bounded sequence of $\mathbb{R}$ has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence. By the previous theorem, $\{x_n\}$ has a limit point which is the limit of a convergent subsequence of $\{x_n\}$.

If $\lim x_n = x$, then x is the only limit point of $\{x_n\}$, and hence, $\limsup x_n = \liminf x_n = x$ holds. Furthermore,

__Thm__. A bounded sequence $\{x_n\}$ of real numbers converges iff $\liminf x_n = \limsup x_n = x$. In this case, $\lim x_n = x$.

A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be a <u>Cauchy sequence</u> if for each $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ (depending on ε) s.t. $|x_n - x_m| < \varepsilon$ $\forall$ $n, m > n_0$.

A Cauchy sequence must necessarily be bounded. Also, it should be clear that every convergent sequence is a Cauchy sequence. The converse is also true; "The real numbers form a complete metric space".

<u>Thm</u>. A sequence of real nos. converges iff it is a Cauchy sequence.

Proof. Need to show that if $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ converges in $\mathbb{R}$. Every bounded sequence of $\mathbb{R}$ has a convergent sequence. $\exists$ a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ s.t. $\lim x_{k_n} = x$. Let $\varepsilon > 0$, choose n_0 s.t. $|x_{k_n} - x| < \varepsilon$ and $|x_n - x_m| < \varepsilon$ $\forall$ $n, m > n_0$. If $n > n_0$, then $k_n \geq n \geq n_0$, & so $|x_n - x| \leq |x_n - x_{k_n}| + |x_{k_n} - x| < 2\varepsilon$.

Hence $\lim x_n = x$.

Let $\{f_n\}$ be a sequence of real-valued f^{n}s. defined on nonempty set X. Suppose that $\exists$ a real-valued f^n g s.t. $|f_n(x)| \leq g(x)$ for all $x \in X$ and all n. Then for each fixed $x \in X$, the sequence of real nos. $\{f_n(x)\}$ is bounded. Thus, $\limsup f_n(x)$ and $\liminf f_n(x)$ both exist in $\mathbb{R}$. $\limsup f_n$ and $\liminf f_n$ can be defined for each $x \in X$ as

$$(\limsup f_n)(x) = \limsup f_n(x)$$

and

$$(\liminf f_n)(x) = \liminf f_n(x).$$