Discrete Structures (Monsoon 2022)

Ashok Kumar Das

Associate Professor
IEEE Senior Member
Center for Security, Theory and Algorithmic Research
International Institute of Information Technology, Hyderabad (IIIT Hyderabad)

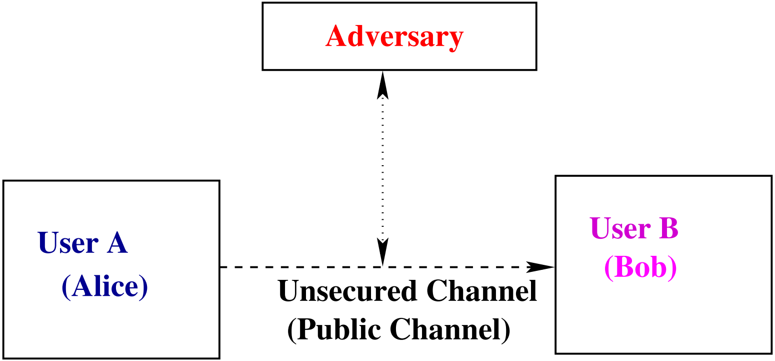
E-mail: *ashok.das@iiit.ac.in*
URL: `http://www.iiit.ac.in/people/faculty/ashokkdas`
`https://sites.google.com/view/iitkgpakdas/`

Intro to Cryptography

What is Cryptography?

- Cryptography is the study of **mathematical techniques** related to aspects of information security such as confidentiality, data integrity, entity authentication, message authentication (data origin authentication) and non-repudiation.

Consider the following simple two-party communication model:



Introduction to Cryptography

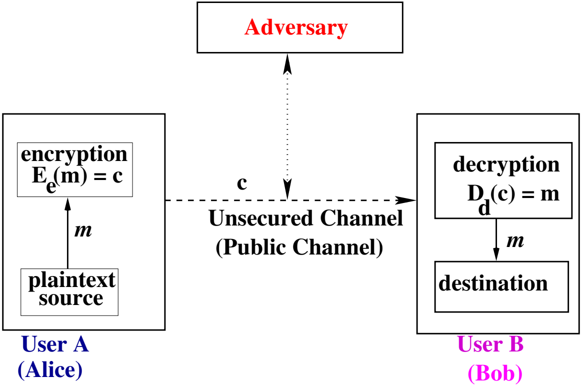
- An “**adversary**” is an entity in a two-party communication which is neither the sender nor the receiver, and which tries to defeat the information security service being provided between the sender and the receiver.
- A “**channel**” is a means of conveying information from one entity to another entity.
- An “**unsecured (public) channel**” is one from which parties other than the sender and the receiver can reorder, delete, insert, or read the data being transmitted.
- A “**secured channel**” is one from which an adversary does not have the ability to reorder, delete, insert, or read the data being transmitted.

Types of adversary

- A “**passive adversary**” is an adversary who is only capable of reading information from an unsecured channel.
- An “**active adversary**” is an adversary who is capable to transmit, alter, or delete information on an unsecured channel.

Introduction to Cryptography

Consider two-party communication model with encryption:



$E_e(\cdot)/D_d(\cdot)$: encryption/decryption transformation using the encryption key e and decryption key d; $D_d = E_e^{-1}$; m: plaintext message and c: ciphertext message

Introduction to Cryptography

- **Security of the scheme**
  - Depends entirely on the secrecy of the key
  - Does not depend on the secrecy of the algorithm (Needs to be public for criticism!)
- Hence, we make the **assumptions** as follows:
  - Algorithms for encryption/decryption are known to the public
  - Keys used are kept secret

Introduction to Cryptography

Definition

An encryption scheme (cipher or cryptosystem) is said to be ***breakable*** if a third party, without prior knowledge of the key pair (e, d), can systematically recover plaintext from corresponding ciphertext within some appropriate time frame.

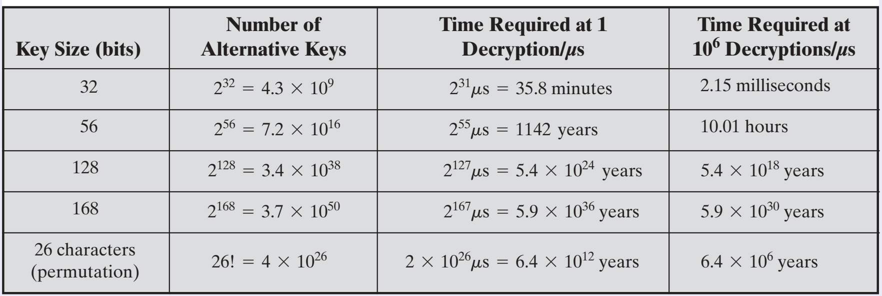
Goal: We want this problem for an adversary (attacker) to be NP-hard (computationally infeasible).

Definition (Brute-force attack)

An encryption scheme can be broken by trying all possible keys to see which one the communicating parties are using (assuming that the class of encryption functions is public knowledge).
This is called an exhaustive search of the key space.

Introduction to Cryptography

What is meant by “Security lies in the keys” (using brute-force attack)



Key Size (bits)	Number of Alternative Keys	Time Required at 1 Decryption/μs	Time Required at 10^6 Decryptions/μs
32	$2^{32} = 4.3 \times 10^9$	$2^{31}\mu s = 35.8$ minutes	2.15 milliseconds
56	$2^{56} = 7.2 \times 10^{16}$	$2^{55}\mu s = 1142$ years	10.01 hours
128	$2^{128} = 3.4 \times 10^{38}$	$2^{127}\mu s = 5.4 \times 10^{24}$ years	5.4×10^{18} years
168	$2^{168} = 3.7 \times 10^{50}$	$2^{167}\mu s = 5.9 \times 10^{36}$ years	5.9×10^{30} years
26 characters (permutation)	$26! = 4 \times 10^{26}$	$2 \times 10^{26}\mu s = 6.4 \times 10^{12}$ years	6.4×10^6 years

Symmetric-Key Encryption

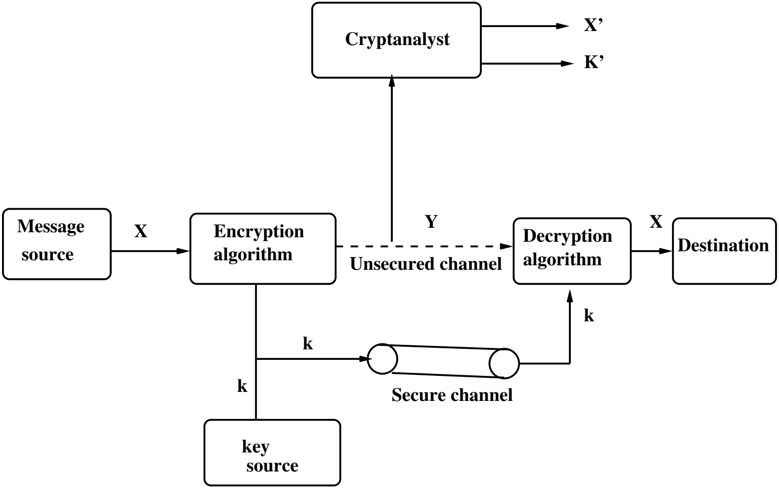
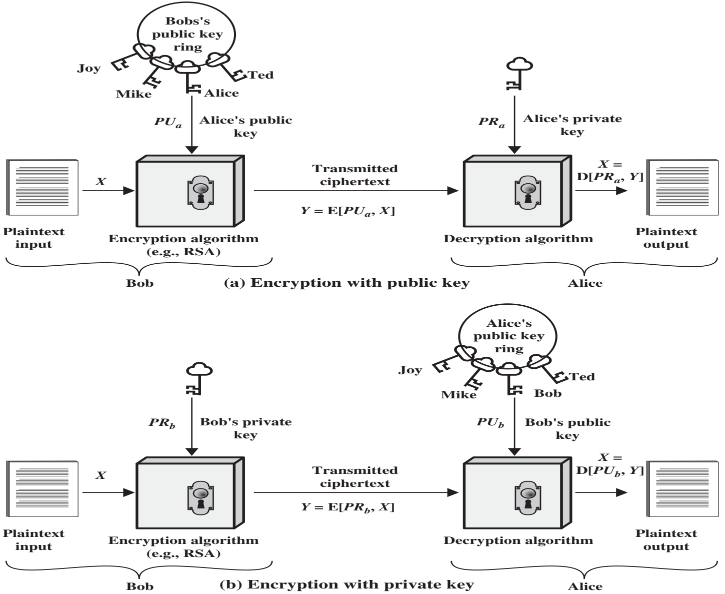


Figure: Model of conventional encryption

Public-Key Cryptography



Elliptic Curve Cryptography (ECC)

Elliptic Curve Cryptography (ECC)

- ECC makes use of the elliptic curves (not ellipses) in which the variables and coefficients are all restricted to elements of a finite field.
- Two family of elliptic curves are used in ECC:
  - prime curves defined over Z_p, that is, $GF(p)$, p being a prime.
  - binary curves constructed over $GF(2^n)$.

Elliptic curves over modulo a prime $GF(p)$

Definition

Let $p > 3$ be a prime. The elliptic curve $y^2 = x^3 + ax + b$ over Z_p is the set $E_p(a, b)$ of solutions $(x, y) \in E_p(a, b)$ to the congruence

$$y^2 \quad = \quad x^3 + ax + b \pmod{p},$$

where $a, b \in Z_p$ are constants such that $4a^3 + 27b^2 \neq 0 \pmod{p}$, together with a special point $\mathcal{O}$ called the point at infinity (or zero point).

- An elliptic curve $E_p(a, b)$ over Z_p (p prime, $p > 3$) will have roughly p points on it.
- More precisely, a well-known theorem due to Hasse asserts that the number of points on $E_p(a, b)$, which is denoted by $\#E$, satisfies the following inequality:

$$p + 1 - 2\sqrt{p} \leq \#E \leq p + 1 + 2\sqrt{p}.$$

Elliptic Curve Cryptography (ECC)

References

- N. Koblitz. Elliptic Curve Cryptosystems. Mathematics of Computation, Vol. 48, pp. 203-209, 1987.
- V. Miller. Uses of elliptic curves in cryptography. Advances in Cryptology - CRYPTO'85, Lecture Notes in Computer Science (LNCS), Springer, Vol. 218, pp. 417-426, 1986.
- Douglas R. Stinson. Cryptography: Theory and Practice, Chapman & Hall/CRC, 2^{nd} Edition, 2005.

Elliptic Curve Cryptography (ECC)

Elliptic curves over modulo a prime $GF(p)$

Finding an inverse

- The inverse of a point $P = (x_P, y_P) \in E_p(a, b)$ is $-P = (x_P, -y_P)$, where $-y$ is the additive inverse of y.
- For example, if $p = 13$, the inverse of $(4, 2)$ is $(4, -2) \pmod{13}$ $= (4, 11)$.

Elliptic Curve Cryptography (ECC)

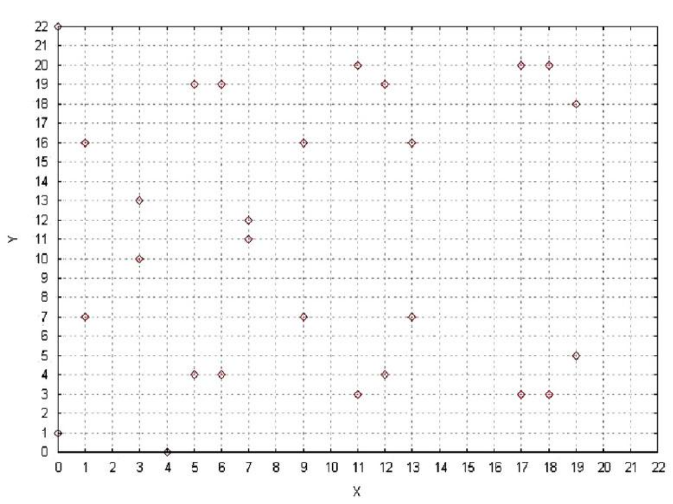
Finding all points on an elliptic curve

Algorithm: EllipticCurvePoints (p, a, b)

```
$x \leftarrow 0$
while $x < p$ do
    $w \leftarrow (x^3 + ax + b) \pmod{p}$
    if $w$ is a perfect square in $Z_p$) then
        Output $(x, \sqrt{w}), (x, -\sqrt{w})$
    end if
    $x \leftarrow x + 1$
end while
```

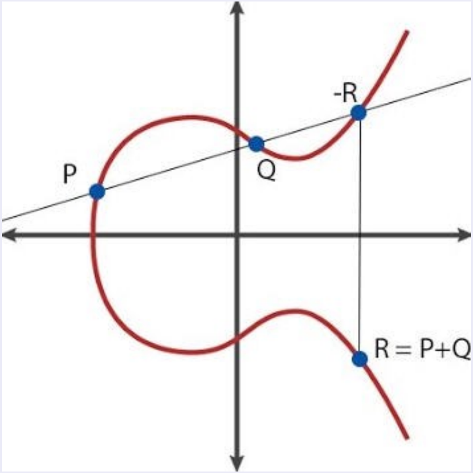
Elliptic Curve Cryptography (ECC)

Example of elliptic curve in case of $y^2 = x^3 + x + 1 \pmod{23}$.



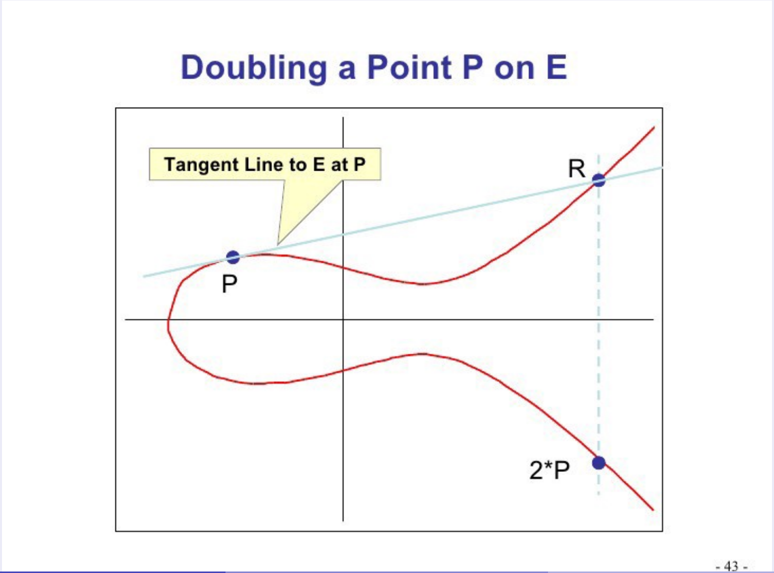
Elliptic Curve Cryptography (ECC)

Point addition on elliptic curve over finite field $GF(p)$



Elliptic Curve Cryptography (ECC)

Doubling on elliptic curve over finite field $GF(p)$



Doubling a Point P on E

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Elliptic Curve Cryptography (ECC)

Point addition on elliptic curve over finite field $GF(p)$

Let G be the base point on $E_p(a, b)$ whose order be n, that is, $nG = G + G + \ldots + G(n\, times) = \mathcal{O}$.
If $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ be two points on elliptic curve $y^2 = x^3 + ax + b \pmod p$, $R = (x_R, y_R) = P + Q$ is computed as follows:

$$\begin{aligned} x_R &= (\lambda^2 - x_P - x_Q) \pmod p, \\ y_R &= (\lambda(x_P - x_R) - y_P) \pmod p, \\ \text{where } \lambda &= \begin{cases} \frac{y_Q - y_P}{x_Q - x_P} \pmod p, \text{if } P \neq -Q \ \ \textbf{[Point Addition]} \\ \frac{3{x_P}^2 + a}{2y_P} \pmod p, \text{if } P = Q. \ \ \textbf{[Point Doubling]} \end{cases} \end{aligned}$$

Elliptic Curve Cryptography (ECC)

Scalar multiplication on elliptic curve over finite field $GF(p)$

If $P = (x_P, y_P)$ be a point on elliptic curve $y^2 = x^3 + ax + b \pmod p$, then then $5P$ is computed as $5P = P + P + P + P + P$.
Think about optimization method?

Reference: N Tiwari, S Padhye. Provable Secure Multi-Proxy Signature Scheme without Bilinear Maps. International Journal of Network Security, Vol. 17, No. 1, pp. 288-293, 2015.

Elliptic Curve Cryptography (ECC)

Problem: Consider two points $P = (11, 3)$ and $Q = (9, 7)$ in the elliptic curve $E_{23}(1, 1)$. Compute $P + Q$ and $2P$.

In order to compute $R = P + Q = (x_R, y_R)$, we first compute λ as

$$\begin{aligned} \lambda &= \frac{7-3}{9-11} \pmod{23} \\ &= -2 \pmod{23} \\ &= 21. \end{aligned} \tag{1}$$

Thus, x_R and y_R are derived as

$$x_R = (21^2 - 11 - 9) (\text{mod } 23) = 7,$$
$$y_R = (21(11-7) - 3) (\text{mod } 23) = 12.$$

As a result, $P + Q = (7, 12)$.

Elliptic Curve Cryptography (ECC)

Problem: Consider two points $P = (11, 3)$ and $Q = (9, 7)$ in the elliptic curve $E_{23}(1, 1)$. Compute $P + Q$ and $2P$.

In order to compute $R = 2P = (x_R, y_R)$, we must first derive λ as follows:

$$\lambda = \frac{3(11^2) + 1}{2 \times 3} \pmod{23} = 7.$$

Hence, $R = P + P = (x_R, y_R)$ is computed as

$$x_R = (7^2 - 11 - 11) \pmod{23} = 4,$$
$$y_R = (7(11 - 4) - 3) \pmod{23} = 0,$$

and, thus $2P = (4, 0)$.

Elliptic Curve Cryptography (ECC)

Elliptic Curve Computational Problems

Elliptic Curve Discrete Logarithm Problem (ECDLP)

- Let $E_p(a, b)$ be an elliptic curve modulo a prime p.
- Given two points $P \in E_p(a, b)$ and $Q = kP \in E_p(a, b)$, for some positive integer k, where $Q = kP$ represent the point P on elliptic curve $E_p(a, b)$ be added to itself k times.
- Then the elliptic curve discrete logarithm problem (ECDLP) is to determine k given P and Q.
- It is computationally easy to calculate Q given k and P, but it is computationally infeasible to determine k given Q and P, when the prime p is large.

Elliptic Curve Cryptography (ECC)

Elliptic Curve Discrete Logarithm Problem (ECDLP)

In other words, ECDLP can be also formally defined as follows. For any PPT algorithm, say A (in the security parameter l), $Pr[A(P, Q) = k] < \epsilon(l)$, where $\epsilon(l)$ is a negligible function depending on l.

References:

- Vanga Odelu, **Ashok Kumar Das**, and Adrijit Goswami. "A secure effective key management scheme for dynamic access control in a large leaf class hierarchy," in ***Information Sciences (Elsevier)***, Vol. 269, No. C, pp. 270-285, 2014. (2020 SCI Impact Factor: 6.795) [This article has been downloaded or viewed 484 times since publication during the period October 2013 to September 2014]
- **Ashok Kumar Das**, Nayan Ranjan Paul, and Laxminath Tripathy. "Cryptanalysis and improvement of an access control in user hierarchy based on elliptic curve cryptosystem," in ***Information Sciences (Elsevier)***, Vol. 209, No. C, pp. 80 - 92, 2012. (2020 SCI Impact Factor: 6.795)

Elliptic Curve Cryptography (ECC)

Definition (Elliptic curve computational Diffie-Hellman problem (ECCDHP))

Let $P \in E_p(a, b)$ be a point in $E_p(a, b)$. The ECCDHP states that given the points $k_1.P \in E_p(a, b)$ and $k_2.P \in E_p(a, b)$ where $k_1, k_2 \in Z_p^*$, it is computationally infeasible to compute $k_1 k_2.P$, where $Z_p^* = \{1, 2, \cdots, p - 1\}$.

Elliptic Curve Cryptography (ECC)

Definition (Elliptic curve decisional Diffie-Hellman problem (ECDDHP))

Let $P \in E_p(a, b)$ be a point in $E_p(a, b)$. The ECDDHP states that given a quadruple $(P, k_1.P, k_2.P, k_3.P)$, decide whether $k_3 = k_1 k_2$ or a uniform value, where $k_1, k_2, k_3 \in Z_p^*$.

The ECDLP, ECCDHP and ECDDHP are computationally infeasible when p is large. To make ECDLP, ECCDHP and ECDDHP intractable, p should be chosen at least 160-bit prime.

Further Readings (Cryptography and Network Security)

- William Stallings, “Cryptography and Network Security: Principles and Practices”, Pearson Education, 2010.
- Behrouz A. Forouzan, “Cryptography and Network Security”, Special Indian Edition.
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- Neal Koblitz, “A course in number theory and cryptography”, Springer.

Thank you