Q. When is the Cartesian product of a family of sets $\{A_i\}_{i \in I}$ nonempty?

→ If the Cartesian product is non-empty, then each A_i must be nonempty.

Q. If each A_i is nonempty, is then the Cartesian product $\prod_{i \in I} A_i$ nonempty?

→ .... "Axiom of choice" needs to be considered, usual axioms of set theory not enough.

$\boxed{\text{Axiom of choice}}$. If $\{A_i\}_{i \in I}$ is a nonempty family of sets s.t. A_i is nonempty for each $i \in I$, then $\prod A_i$ is nonempty.

If $\{A_i\}_{i \in I}$ is a nonempty family of pairwise disjoint sets s.t. $A_i \neq \phi$ for each $i \in I$, then $\exists$ a set $E \subseteq \bigcup_{i \in I} A_i$ s.t. $E \cap A_i$ consists of precisely one element for each $i \in I$.

Relation & order

A (binary) relation on a set X means a subset R of $X \times X$, i.e., $R \subseteq X \times X$. If $(x,y) \in R$, then xRy (x is in the relation R with y).

A relation R on set X is called an <u>equivalence relation</u> if it satisfies the following three properties:

a) xRx $\forall x \in X$ (reflexivity)

b) If xRy, then yRx (symmetry)

c) If xRy and yRz (transitivity)

Consider R is an equivalence rel^n on a set X. The equivalence class determined by $x \in X$ is defined as

$$[x] = \{y \in X : xRy\}.$$

Any two equivalence classes are either disjoint or they coincide.

$\because x \in [x]\ \forall\ x \in X$, R partitions X. That is, $\exists$ a family $\{A_i\}_{i\in I}$ of pairwise disjoint sets (here, the family of equivalence classes) s.t. $X = \bigcup_{i\in I} A_i$. Conversely, if a family $\{A_i\}_{i\in I}$ of pairwise disjoint sets ($A_i \cap A_j = \phi$ for all $i \neq j$) partitions X (i.e., $X = \bigcup_{i\in I} A_i$), then by letting

$$R = \{(x,y) \in X \times X : \exists\ i \in I \text{ s.t. } x \in A_i \ \&\ y \in A_i\}$$

an equivalence relation is defined on X whose equivalence classes are precisely the sets A_i.

Order relation

A relation "$\leq$" on a set X is said to be a <u>partial order</u> for X (or that X is partially ordered by $\leq$) if it satisfies the following three properties:

1) $x \leq x$ holds for every $x \in X$ (reflexivity)
2) If $x \leq y$ & $y \leq x$, then $x = y$ (antisymmetry).
3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity)

$y \geq x$ is alternative notation for $x \leq y$.
A set equipped w/ an order relation is called a partially ordered set.

If X is a partially ordered set. $Y \subseteq X$ is said to be a <u>chain</u> if for every pair $x, y \in Y$, either $x \leq y$ or else $y \leq x$; a chain is also called a <u>totally ordered set</u>. If $Y \subseteq X$ s.t. $x \leq u$ holds for all $x \in Y$ and some $u \in X$, then u is called an <u>upper bound</u> of Y. ~~An~~ $m \in X$ is called a <u>maximal</u> element of X whenever $m \leq x$ implies $x = m$. (Alert: A partially ordered set may contain more than ~~one~~ one maximal element.)

Zorn's Lemma. If every chain in a partially ordered set X has an upper bound in X, then X has a maximal.