Lecture 6, 3/2/23

* **Vector Operators in Cylindrical Polar & Spherical Polar Coordinates:**

See Riley-Hobson-Bence - Uploaded in Moodle.

* **A complete Example - Electromagnetic Wave;**

Note added seperately -

Lecture 7, 5/1/23

Surface & Volume Integrals; An integral over a surface is given as

$$\int \phi \, d\vec{s} \quad \text{or} \quad \int \vec{V} \cdot d\vec{s} \quad \text{or} \quad \int \vec{V} \times d\vec{s}$$

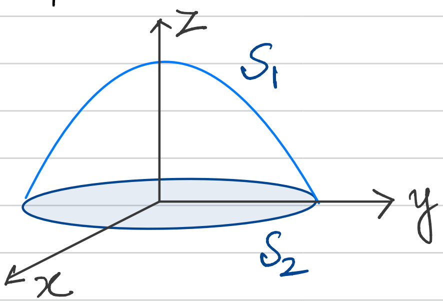
Now the differential $d\vec{s} = \hat{n}\, dA$, where $dA = |d\vec{s}|$ is the magnitude and $\hat{n}$ is the unit vector that is normal to the surface. For a closed surface, the direction of the vector is outward normal. For open surface, the direction of $\hat{n}$ determined by the direction of the perimeter - by the right-hand thumb rule

Volume integrals are of the type $\int \vec{v}\, dv$ or $\int \phi\, dv$ where dv is a scalar quantity.

* Gauss's Divergence Theorem: It relates the closed surface integral of a vector to the volume integral of the divergence of the vector. The theorem states that

$$\oiint_S \vec{F} \cdot \vec{ds} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dv$$

Example: Given a vector field $\vec{F} = \hat{k}(z+R)$ and a solid hemisphericall ball, $x^2+y^2+z^2 \leq R$ check the Gauss's theorem.



Soln: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dv = \iiint_V dv = \frac{2}{3}\pi R^3$

Now, the left hand side of the Gauss's theorem is given by

$$\oiint \vec{F} \cdot \vec{ds} = \iint \vec{F} \cdot \vec{ds_1} + \iint \vec{F} \cdot \vec{ds_2}$$

$$= \iint \vec{F} \cdot \hat{n}_1\, ds_1 + \iint \vec{F} \cdot \hat{n}_2\, ds_2$$

On S_1 the unit vector $\hat{n}_1 = \frac{1}{R}(\hat{i}x + \hat{j}y + \hat{k}z)$

$$\vec{F}.\hat{n}_1 = \frac{z(z+R)}{R} = R\cos\theta(1+\cos\theta)$$

Spherical Polar coordinate
$z = R\cos\theta$

Hence $\int \vec{F}.\hat{n}_1 ds_1 = \int \vec{F}.\hat{n}_1 (R d\theta)(R\sin\theta\, d\phi)$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, (R^2 \sin\theta)\, R\cos\theta\,(1+\cos\theta)$$

$$= 2\pi R^3 \left[-\frac{1}{3}\cos^3\theta - \frac{1}{2}\cos\theta\right]_0^{\pi/2} = \frac{5}{3}\pi R^3$$

dr
$r\,d\theta$
$r\sin\theta\, d\phi$

For the second surface $\hat{n}_2 = -\hat{k}$

$$\int \vec{F}.\hat{n}_2\, dS_2 = \int \vec{F}.\hat{n}_2\, dr.\, r\, d\theta$$

$$= \int_0^R r\,dr \int_0 d\phi\,(-R)$$

$$= \frac{R^2}{2}.\, 2\pi\,(-R) = -\pi R^2$$

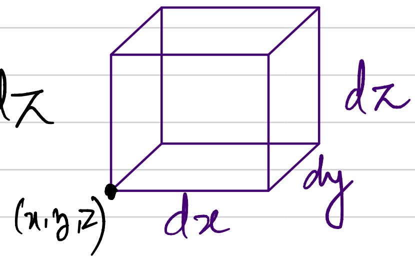
In the S_2 plane the area element is given by $dr.\, r\sin\frac{\pi}{2}\, d\phi$
$= r\,dr\,d\phi$

Adding the two results $= \frac{2}{3}\pi R^3$.

Proof of Divergence theorem: Theorem Statement $\int_V \vec{\nabla}\cdot\vec{F}\, dv = \oint \vec{F}\cdot ds$

Any finite volume V that is closed by the surface S can be divided into infinitesimally small cubes of sides dx, dy & dz.

Consider infinitesimally small volume of sides dx, dy & dz



Lets discuss the R.H.S of the theorem

$$\int \vec{F}\cdot ds = \sum \vec{F}\cdot\hat{n}_i\, \Delta S_i$$

the sum is over all the point (x, y, z) around which infinitesimal volume is considered. First consider vector through the dydz surface. Contribution from the two sides given by

$$\left[F_x(x+dx, y, z) - F(x, y, z)\right] dy\, dz$$

$$= \frac{\partial F_x}{\partial x}\, dx\, dy\, dz$$

The sign is negative because the surface normal are opposite on either sides.

Similar results hold for the other two surfaces - can be summed. Now we have to sum these up for all the volume element consider - The contributions from the adjacent walls cancel each other leaving only the contribution from the overall surface.

So
$$\oint \vec{F} \cdot \hat{n}\, ds = \sum \vec{F} \cdot \hat{n}_i \Delta S_i$$
$$= \sum \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx\, dy\, dz$$
$$= \int_V \vec{\nabla} \cdot \vec{F}\, dv \qquad \text{proven!}$$

Example: Given $\vec{F} = r^2 \hat{r}$ test the divergence theorem for a sphere of radius R

Soln: Using the $\vec{\nabla}$ operator for spherical polars
$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 F_r \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^4 \right) = 4r$$
$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \,. \text{ So } \int_V \vec{\nabla} \cdot \vec{F}\, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} 4r^3 \sin\theta\, dr\, d\theta\, d\phi = 4\pi R^4$$

For RHS; On the surface $\vec{F} = R^2 \hat{r}$ & $d\vec{s} = \hat{r} R^2 \sin\theta\, d\theta\, d\phi \Rightarrow \oint \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \int_0^{\pi} R^4 \sin\theta\, d\theta\, d\phi = 4\pi R^4$

Green's theorem in Plane: It relates the line integral over a closed curve c to a double integral over the surface enclosed by the curve. This is a 2D version of Stoke's theorem that we will study next.

Statement: Let $\vec{F} = F_x(x,y)\hat{i} + F_y(x,y)\hat{j}$ be any differentiable function in the two dimension in x-y plane in an area S that is bounded by a curve c.

Then the theorem states that

→ the path c is anti-clockwise

$$\iint_S \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) ds = \oint_C \left(F_x\, dx + F_y\, dy \right)$$

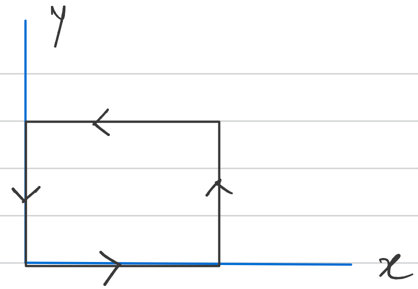
Proof: Very easy to prove in a rectangle (Exercise)

Example: Prove Green's theorem for $\vec{r} = -\hat{i}y + \hat{j}x$ for a square of side L in the first quadrant with vertex at the origin

Soln: The integrand of the surface integral is $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2$

Then the result of the integral $\oiint 2\, dA = 2L^2$

For the R.H.S of the equation $\oint (F_x dx + F_y dy)$ the contour is shown in the figure.



$$\oint (F_x dx + F_y dy) = \int_0^L 0.dx + \int (-L)\, dx + \int_L^0 0.dx + \int_0^L L dy$$

$$= 2L^2$$

Example 2: Same as the previous problem, but now the curve C is a circle of radius R centered at the origin.

Soln: The LHS $= \iint_S 2.dA = 2\pi R^2$

To do the RHS, $x = R\cos\theta$, $y = R\sin\theta$, then $dx = -R\sin\theta$, $dy = R\cos\theta$

and the RHS is $\oint_C (-y\,dx + x\,dy) = \int_0^{2\pi} (R^2\sin^2\theta + R^2\cos^2\theta)\, d\theta = 2\pi R^2$

Stoke's Theorem: This is 3D extension of Green's theorem. For a 3D surface s closed by a simple curve c the theorem states that

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{r}$$

Example: Prove Stoke's theorem for $\vec{F} = -y\hat{i} + x\hat{j}$ for a hemisphere of radius R with $z > 0$ bounded by a circle of radius R lying in the x-y plane with the center at the origin.

Soln: $\oint \vec{F} \cdot d\vec{r} = \oint_C (F_x dx + F_y dy) = \oint_C (-y dx + x dy) = \int_0^{2\pi} (R^2 \sin^2\theta + R^2 \cos^2\theta) d\theta = 2\pi R^2$

The LHS of the Stoke's theorem is $\vec{\nabla} \times \vec{F} = 2\hat{k}$

Area differential $d\vec{s} = \hat{r} R^2 \sin\theta \, d\theta \, d\varphi$. Hence the LHS of the theorem is

$$\int_0^{2\pi} \int_0^{\pi/2} \hat{k} \cdot \hat{r} \; 2R^2 \sin\theta \, d\theta \, d\varphi \qquad \left\{ \hat{k} \cdot \hat{r} = \cos\theta . \right.$$

$$= 2R^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = 2\pi R^2 \sin^2\theta \Big|_0^{\pi/2} = 2\pi R^2$$