$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \cdot dx$$

Hint: Start with $\oint \frac{e^{iz}}{x} dz$

February 6, 2023

Indefinite integral

$$\int \frac{\sin x}{x} \cdot dx = i \cdot \frac{\Gamma(0, ix) - \Gamma(0, -ix)}{2}$$

can NOT solve Analytically !!

Start with $\oint \frac{e^{i}z}{z} \cdot dz$

Start with $\oint \frac{e^i z}{z} \cdot dz$

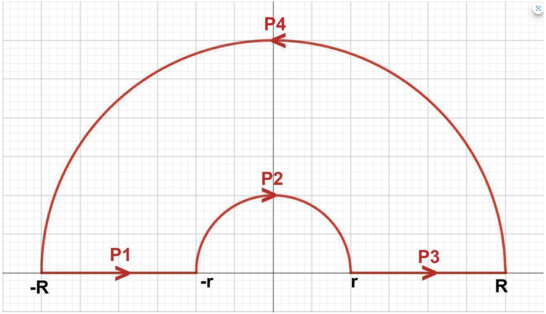

Figure: Our contour

Start with $\oint \frac{e^i z}{z} \cdot dz$

- Why this contour ?

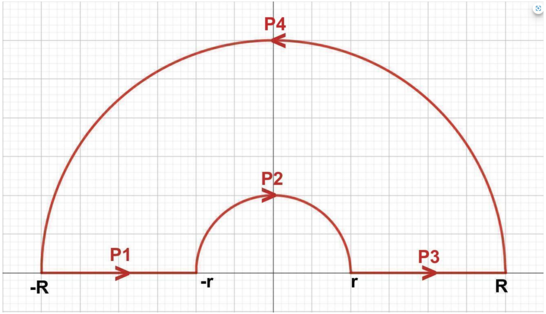

Figure: Our contour

Start with $\oint \frac{e^i z}{z} \cdot dz$

- Why this contour ?
- Is the function Analytical ??

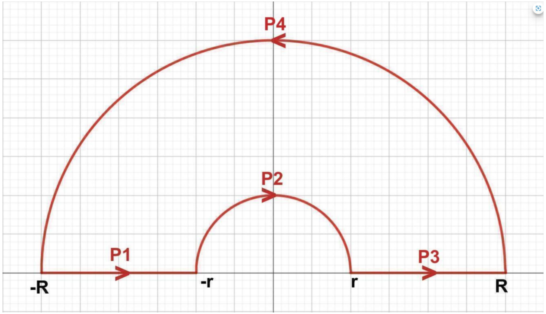

Figure: Our contour

Start with $\oint \frac{e^i z}{z} \cdot dz$

- Why this contour ?
- Is the function Analytical ?? $\oint \frac{e^i z}{z} \cdot dz = 0$

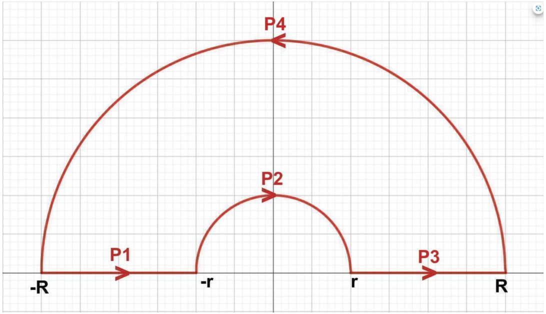

Figure: Our contour

WHY $\frac{e^{iz}}{z}$??

$$\oint \frac{e^{iz}}{z} dz = I_1 + I_2 + I_3 + I_4$$

$$\oint \frac{e^{iz}}{z} dz = I_1 + I_2 + I_3 + I_4$$
$$= \int_{P_1} \frac{e^{iz}}{z} \cdot dz + \int_{P_3} \frac{e^{iz}}{z} \cdot dz + I_2 + I_4$$

$$\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= I_1 + I_2 + I_3 + I_4 \\
&= \int_{P_1} \frac{e^{iz}}{z} \cdot dz + \int_{P_3} \frac{e^{iz}}{z} \cdot dz + I_2 + I_4 \\
&= \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx + \int_{r}^{R} \frac{e^{ix}}{x} \cdot dx + I_2 + I_4
\end{aligned}$$

$$\oint \frac{e^{iz}}{z} dz = I_1 + I_2 + I_3 + I_4$$
$$= \int_{P_1} \frac{e^{iz}}{z} \cdot dz + \int_{P_3} \frac{e^{iz}}{z} \cdot dz + I_2 + I_4$$
$$= \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx + \int_{r}^{R} \frac{e^{ix}}{x} \cdot dx + I_2 + I_4$$

Let $t = -x,$ and swap limits

$$\oint \frac{e^{iz}}{z} dz = \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx - \int_{-R}^{-r} \frac{e^{-ix}}{x} \cdot dx + I_2 + I_4$$

$$\oint \frac{e^{iz}}{z} dz = \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx - \int_{-R}^{-r} \frac{e^{-ix}}{x} \cdot dx + I_2 + I_4$$
$$= \int_{-R}^{-r} \frac{e^{ix} - e^{-ix}}{x} \cdot dx + I_2 + I_4$$

$$\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx - \int_{-R}^{-r} \frac{e^{-ix}}{x} \cdot dx + I_2 + I_4 \\
&= \int_{-R}^{-r} \frac{e^{ix} - e^{-ix}}{x} \cdot dx + I_2 + I_4 \\
&= 2i \cdot \int_{-R}^{-r} \frac{e^{-ix} - e^{ix}}{2x} \cdot dx + I_2 + I_4
\end{aligned}$$

$$
\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx - \int_{-R}^{-r} \frac{e^{-ix}}{x} \cdot dx + I_2 + I_4 \\
&= \int_{-R}^{-r} \frac{e^{ix} - e^{-ix}}{x} \cdot dx + I_2 + I_4 \\
&= 2i \cdot \int_{-R}^{-r} \frac{e^{-ix} - e^{ix}}{2x} \cdot dx + I_2 + I_4 \\
&= 2i \cdot \int_{-R}^{-r} \frac{\sin(x)}{x} \cdot dx + I_2 + I_4
\end{aligned}
$$

$$
\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{-R}^{-r} \frac{e^{ix}}{x} \cdot dx - \int_{-R}^{-r} \frac{e^{-ix}}{x} \cdot dx + I_2 + I_4 \\
&= \int_{-R}^{-r} \frac{e^{ix} - e^{-ix}}{x} \cdot dx + I_2 + I_4 \\
&= 2i \cdot \int_{-R}^{-r} \frac{e^{-ix} - e^{ix}}{2x} \cdot dx + I_2 + I_4 \\
&= 2i \cdot \int_{-R}^{-r} \frac{\sin(x)}{x} \cdot dx + I_2 + I_4 \\
&= I_{-R}^{-r} + I_2 + I_4
\end{aligned}
$$

$$P_4(\theta) = R \cdot e^{i\theta} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$P_4(\theta) = R \cdot e^{i\theta} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$\oint \frac{e^{iz}}{z} dz = \int_{P_4} \frac{e^{iz}}{z} \cdot dz + I_2 + I_{-R}^{-r}$$

$$P_4(\theta) = R \cdot e^{i\theta} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{P_4} \frac{e^{iz}}{z} \cdot dz + I_2 + I_{-R}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot d(P_4(\theta)) + I_2 + 2i \cdot I_{-R}^{-r}
\end{aligned}$$

$$P_4(\theta) = R \cdot e^{i\theta} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{P_4} \frac{e^{iz}}{z} \cdot dz + I_2 + I_{-R}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot d(P_4(\theta)) + I_2 + 2i \cdot I_{-R}^{-r} \\
&= i \cdot \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot P_4(\theta) \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r}
\end{aligned}$$

$$P_4(\theta) = R \cdot e^{i\theta} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{P_4} \frac{e^{iz}}{z} \cdot dz + I_2 + I_{-R}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot d(P_4(\theta)) + I_2 + 2i \cdot I_{-R}^{-r} \\
&= i \cdot \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot P_4(\theta) \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r} \\
&= i \cdot \int_0^{\pi} e^{iR[\cos(\theta) + i \cdot \sin(\theta)]} \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r}
\end{aligned}$$

$$P_4(\theta) = R \cdot e^{i\theta} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$\begin{aligned}
\oint \frac{e^{iz}}{z} dz &= \int_{P_4} \frac{e^{iz}}{z} \cdot dz + I_2 + I_{-R}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot d(P_4(\theta)) + I_2 + 2i \cdot I_{-R}^{-r} \\
&= i \cdot \int_0^{\pi} \frac{e^{i \cdot P_4(\theta)}}{P_4(\theta)} \cdot P_4(\theta) \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r} \\
&= i \cdot \int_0^{\pi} e^{iR[\cos(\theta) + i \cdot \sin(\theta)]} \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r}
\end{aligned}$$

$$\oint \frac{e^{iz}}{z} dz = i \cdot \int_0^{\pi} e^{Ri\cos(\theta)} \cdot e^{-R \cdot \sin(\theta)} \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r}$$

$$\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz = \left[\lim_{R\to\infty} i\cdot\int_0^{\pi} e^{Ri\cos(\theta)}\cdot e^{-R\cdot\sin(\theta)}\cdot d\theta + I_2 + 2i\cdot I_{-R}^{-r}\right]$$

$$\begin{aligned}\lim_{R\to\infty}\oint\frac{e^{iz}}{z}dz &= \left[\lim_{R\to\infty} i\cdot\int_0^{\pi} e^{Ri\cos(\theta)}\cdot e^{-R\cdot\sin(\theta)}\cdot d\theta + I_2 + 2i\cdot I_{-R}^{-r}\right] \\ &= \left[\lim_{R\to\infty} i\cdot\int_0^{\pi} e^{Ri\cos(\theta)}\cdot e^{-R\cdot\sin(\theta)}\cdot d\theta\right] + I_2 + 2i\cdot I_{-\infty}^{-r}\end{aligned}$$

$$
\begin{aligned}
\lim_{R\to\infty} \oint \frac{e^{iz}}{z} dz &= \left[\lim_{R\to\infty} i \cdot \int_0^{\pi} e^{Ri\cos(\theta)} \cdot e^{-R\cdot\sin(\theta)} \cdot d\theta + I_2 + 2i \cdot I_{-R}^{-r} \right] \\
&= \left[\lim_{R\to\infty} i \cdot \int_0^{\pi} e^{Ri\cos(\theta)} \cdot e^{-R\cdot\sin(\theta)} \cdot d\theta \right] + I_2 + 2i \cdot I_{-\infty}^{-r} \\
&= 0 + I_2 + 2i \cdot I_{-\infty}^{-r}
\end{aligned}
$$

$$\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz = I_2 + 2i \cdot I_{-\infty}^{-r}$$

$$\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz = I_2 + 2i\cdot I_{-\infty}^{-r}$$
$$= \int_{P_2}\frac{e^{iz}}{z}\cdot dz + 2i\cdot I_{-\infty}^{-r}$$

$$
\begin{aligned}
\lim_{R\to\infty} \oint \frac{e^{iz}}{z} dz &= I_2 + 2i \cdot I_{-\infty}^{-r} \\
&= \int_{P_2} \frac{e^{iz}}{z} \cdot dz + 2i \cdot I_{-\infty}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_2(\theta)}}{P_2(\theta)} \cdot d(P_2(\theta)) + 2i \cdot I_{-\infty}^{-r}
\end{aligned}
$$

$$P_2(\theta) = r \cdot e^{i(\pi-\theta)} \qquad \theta \text{ from 0 to } \pi$$

$$\begin{aligned}
\lim_{R\to\infty} \oint \frac{e^{iz}}{z} dz &= I_2 + 2i \cdot I_{-\infty}^{-r} \\
&= \int_{P_2} \frac{e^{iz}}{z} \cdot dz + 2i \cdot I_{-\infty}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_2(\theta)}}{P_2(\theta)} \cdot d(P_2(\theta)) + 2i \cdot I_{-\infty}^{-r} \\
&= -i \cdot \int_0^{\pi} \frac{e^{i \cdot P_2(\theta)}}{P_2(\theta)} \cdot P_2(\theta) \cdot d\theta + 2i \cdot I_{-\infty}^{-r}
\end{aligned}$$

$$P_2(\theta) = r \cdot e^{i(\pi-\theta)} \qquad \theta \text{ from } 0 \text{ to } \pi$$

$$\begin{aligned}
\lim_{R\to\infty} \oint \frac{e^{iz}}{z} dz &= I_2 + 2i \cdot I_{-\infty}^{-r} \\
&= \int_{P_2} \frac{e^{iz}}{z} \cdot dz + 2i \cdot I_{-\infty}^{-r} \\
&= \int_0^{\pi} \frac{e^{i \cdot P_2(\theta)}}{P_2(\theta)} \cdot d(P_2(\theta)) + 2i \cdot I_{-\infty}^{-r} \\
&= -i \cdot \int_0^{\pi} \frac{e^{i \cdot P_2(\theta)}}{P_2(\theta)} \cdot P_2(\theta) \cdot d\theta + 2i \cdot I_{-\infty}^{-r} \\
&= -i \cdot \int_0^{\pi} e^{ir[\cos(\theta)+i \cdot \sin(\theta)]} \cdot d\theta + 2i \cdot I_{-\infty}^{-r}
\end{aligned}$$

$$\lim_{r\to 0}\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz = \lim_{r\to 0}\left[-i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta + 2i\cdot I_{-\infty}^{-r}\right]$$

$$\lim_{r\to 0}\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz = \lim_{r\to 0}\left[-i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta + 2i\cdot I_{-\infty}^{-r}\right]$$

$$= \left[\lim_{r\to 0} -i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta\right] + 2i\cdot I_{-\infty}^{0}$$

$$
\begin{aligned}
\lim_{r\to 0}\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz &= \lim_{r\to 0}\left[-\, i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta + 2i\cdot I_{-\infty}^{-r}\right] \\
&= \left[\lim_{r\to 0} -i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta\right] + 2i\cdot I_{-\infty}^{0} \\
&= -i\cdot\int_0^{\pi} 1\cdot d\theta + 2i\cdot I_{-\infty}^{0} = 0
\end{aligned}
$$

$$\lim_{r\to 0}\lim_{R\to\infty}\oint \frac{e^{iz}}{z}dz = \lim_{r\to 0}\left[-i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta + 2i\cdot I_{-\infty}^{-r}\right]$$

$$= \left[\lim_{r\to 0} -i\cdot\int_0^{\pi} e^{ir[\cos(\theta)+i\cdot\sin(\theta)]}\cdot d\theta\right] + 2i\cdot I_{-\infty}^{0}$$

$$= -i\cdot\int_0^{\pi} 1\cdot d\theta + 2i\cdot I_{-\infty}^{0} = 0$$

$$\Rightarrow 2\cdot I_{-\infty}^{0} = \pi$$

$$2 \cdot \int_{-\infty}^{0} \frac{\sin(x)}{x} \cdot dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \cdot dx = \pi$$