Any $x \in \mathbb{R}$ satisfying $x \geq 0$ (i.e., $x \geq 0$ and $x \neq 0$) is called a <u>positive no.</u>

$x < 0$ is called a <u>negative no.</u>

· If $x + \epsilon \geq y$ holds for each $\epsilon \in \mathbb{R}$ and $\epsilon > 0$, then $x \geq y$ holds.

Proof: Suppose the conclusion is not true, i.e., $y - x > 0$. Let $\epsilon = \frac{1}{2}(y-x) > 0$, then hypothesis $x + \epsilon \geq y \Rightarrow$

$$\frac{1}{2}(x+y) = x + \frac{1}{2}(y-x) \geq y$$

$\Rightarrow$ $y - x \leq 0$, which is in contradiction.

<u>Absolute value</u> of a real no. $a \in \mathbb{R}$ is defined as $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. If $a \vee b = \max\{a, b\}$ then $|a| = a \vee (-a)$ $\forall\, a \in \mathbb{R}$. It follows that $|a| = |-a|$ for each $a \in \mathbb{R}$. The absolute value satisfies the properties:

1. $|a| \geq 0$ for each $a \in \mathbb{R}$, and $|a| = 0$ iff $a = 0$.
2. $|ab| = |a| \cdot |b|$ $\forall\, a, b \in \mathbb{R}$; and
3. $|a+b| \leq |a| + |b|$ $\forall\, a, b \in \mathbb{R}$ (Δ-gle ineq.)

Let A be a nonempty subset of $\mathbb{R}$. An upper bound of A is any $a \in \mathbb{R}$ s.t. $x \leq a$ $\forall\ x \in A$; $b \in \mathbb{R}$ is a lower bound for A if $b \leq x$ $\forall\ x \in A$. If A has an upper bound, then A is said to be bounded from above. If A has a lower bound then A is said to be bounded from below. If A is both, bounded from above and below, then A is called a bounded set. A real no. is called a least upper bound (or a supremum) of A if is an upper bound for A, and it is less than or equal to every other upper bound of A. I.e., $x \in \mathbb{R}$ is a least upper bound for A if

i) A is bounded from above by x, and

ii) if A is bounded from above by y, then $x \leq y$.

Similarly $x \in \mathbb{R}$ is the <u>greatest lower bound</u> (or <u>infimum</u>) of a set A if

i) A is bounded from below by x, and

ii) if A is bounded from below by y, then $x \geq y$.

<u>The Completeness axiom.</u>

Axiom 11. Every nonempty set of real nos. that is bounded from above has a least upper bound.

It follows that if $A \subset \mathbb{R}$ is nonempty and bounded from below, then the set $B = \{b \in \mathbb{R} : b \leq x \ \forall\, x \in A\}$ is bounded from above, and so sup B exists, $\sup B = \inf A$.

Note that if set A has a maximum (resp. a minimum) element, then $\max A = \sup A$ (resp. $\min A = \inf A$).

Thm Assume that $A \subset \mathbb{R}$ and $\sup A$ exists. Then for every $\epsilon > 0$, $\exists$ some $x \in A$ s.t. $\sup A - \epsilon < x \leq \sup A$.

Proof: If $\forall\, x \in A$ we have $x \leq \sup A - \epsilon$, then $\sup A - \epsilon$ is an upper bound of A, which is less than the least upper bound. But this is impossible. $\therefore$ $\exists$ some $x \in A$ s.t. $\sup A - \epsilon < x \leq \sup A$.

Corollary: The set of natural nos. $\mathbb{N}$ is unbounded.

Thm (The Archimedean property). If $x, y \in \mathbb{R}^+$, then $\exists$ some natural no. $n \in \mathbb{N}$ s.t. $nx > y$.

Thm. Bet! any two distinct real nos. $\exists$ a rational no.