# Sequence of Real nos.

Def$^n$: A sequence $\{x_n\}$ of real nos. is ~~called~~ said to converge to $x \in \mathbb{R}$ if $\forall\ \epsilon > 0$ $\exists\ n_0 \in \mathbb{N}$ (depending on $\epsilon$) s.t.

$$|x - x_n| < \epsilon \quad \forall\ n > n_0.$$

The $x \in \mathbb{R}$ is called the limit of the sequence $\{x_n\}$, and we write $x_n \to x$, or $x = \lim\limits_{n\to\infty} x_n$, or $x = \lim x_n$.

The terms of a sequence $\{x_n\}$ of a set A satisfy a property (P) eventually, if $\exists$ some natural no. $n_0 \in \mathbb{N}$ s.t. $x_n$ satisfies the property (P) $\forall\ n > n_0$. A sequence $\{x_n\}$ of real nos. converges to some $x \in \mathbb{R}$ iff for each $\epsilon > 0$ the terms $x_n$ are eventually $\epsilon$-close to $x$.

If $\lim x_n = x$, then $\lim y_n = x$ for every subsequence $\{y_n\}$ of $\{x_n\}$.

**Thm.** A sequence of real nos. can have at-most one limit.

Proof: Assume that a sequence of real nos. $\{x_n\}$ satisfies $x = \lim x_n$ and $y = \lim x_n$. Let $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ and $|x_n - y| < \varepsilon$ $\forall\, n > n_0$.

Fix $n > n_0$, using $\Delta^{gle}$ ineq.:

$$0 \leq |x-y| \leq |x-x_n| + |y-x_n| < \varepsilon + \varepsilon = 2\varepsilon$$

$\forall\, \varepsilon > 0$. This implies $x = y$.

A sequence of real nos. $\{x_n\}$ is said to be bounded if $\exists$ $M > 0$, $M \in \mathbb{R}$, s.t. $|x_n| \leq M$ $\forall\, n \in \mathbb{N}$. A sequence $\{x_n\}$ of $\mathbb{R}$ is said to be increasing if $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$, and decreasing if $x_{n+1} \leq x_n$ $\forall\, n$. A monotone sequence is either an increasing or a decreasing sequence.

$x_n \uparrow x$ means that $\{x_n\}$ is increasing and $x = \sup\{x_n\}$.

$x_n \downarrow x$ means $\{x_n\}$ is decreasing with $x = \inf\{x_n\}$.

$x_n = c$ $\forall\, n \in \mathbb{N}$ is called a constant sequence.

<u>Thm</u>. Every monotone bounded sequence of real nos. is convergent.

Proof: Assume $\{x_n\}$ is increasing and bounded. $\because$ $\{x_n\}$ is bounded, it follows from the completeness axiom that $x = \sup\{x_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$. We claim $x = \lim x_n$. Indeed, if $\varepsilon > 0$ is given, then $\exists\ n_0 \in \mathbb{N}$ s.t.
$x - \varepsilon < x_{n_0} \leq x$ (Recall: $\sup(A)$ of $A \subset \mathbb{R}$ exists. Then for every $\varepsilon > 0$, $\exists$ some $x \in A$ s.t. $\sup A - \varepsilon < x \leq \sup A$.). $\because$ $\{x_n\}$ is increasing, $|x - x_n| = x - x_n \leq x - x_{n_0} < \varepsilon$ $\forall\ n > n_0$, and thus, $\lim x_n = x$. Similar proof for the decreasing sequence.

An increasing sequence $\{x_n\}$ of real nos. satisfies $x_n \uparrow x$ iff $x = \lim x_n$. The basic convergence properties of real sequences are:

1. Every convergent sequence is bounded.
2. If $x_n = c$ for each $n \in \mathbb{N}$, then $\lim x_n = c$.

3. If the ~~three~~ sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ of $\mathbb{R}$ satisfy $x_n \leq z_n \leq y_n \ \forall\, n \in \mathbb{N}$, and $\lim x_n = \lim y_n = x$, then $\{z_n\}$ converges and $\lim z_n = x$.

For the next properties, we assume $\lim x_n = x$ and $\lim y_n = y$ :

4. For each $\alpha, \beta \in \mathbb{R}$ the sequence $\{\alpha x_n + \beta y_n\}$ converges & $\lim (\alpha x_n + \beta y_n) = \alpha x + \beta y$.
5. The sequence $\{x_n y_n\}$ is convergent and $\text{limit} (x_n y_n) = xy$.
6. If $|y_n| \geq \delta > 0$ holds $\forall\, n \in \mathbb{N}$, and some $\delta > 0$, then $\{x_n / y_n\}$ converges and $\lim x_n/y_n = x/y$.
7. If $x_n \geq y_n$ holds $\forall\, n \geq n_0$, then $x \geq y$.

$x \in \mathbb{R}$ is said to be a <u>limit point</u> (or a <u>cluster point</u>) of a sequence of real nos. $\{x_n\}$ if $\forall\, n \in \mathbb{N}$ and $\varepsilon > 0$, $\exists\, k > n$ (depending on $\varepsilon$ and $n$) s.t. $|x_k - x| < \varepsilon$.

<u>Thm:</u> Let $\{x_n\}$ be a sequence of real nos. Then a real no. $x \in \mathbb{R}$ is a limit point for $\{x_n\}$ iff $\exists$ a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $\lim x_{k_n} = x$.

Defn: Let $\{x_n\}$ be a bounded sequence of $\mathbb{R}$. Then the <u>limit superior</u> of $\{x_n\}$ is defined by
$$\lim\sup x_n = \inf_n \left[\sup_{k \geq n} x_k\right],$$
and the <u>limit inferior</u> of $\{x_n\}$ by
$$\lim\inf x_n = \sup_n \left[\inf_{k \geq n} x_k\right].$$

If we write
$$\sup_{k \geq n} x_k = \bigvee_{k=n}^{\infty} x_k \quad \text{and} \quad \inf_{k \geq n} x_k = \bigwedge_{k=n}^{\infty} x_k,$$

then
$$\lim\sup x_n = \bigwedge_{n=1}^{\infty} \left[\bigvee_{k=1}^{\infty} x_k\right] \quad \text{and}$$
$$\lim\inf x_n = \bigvee_{n=1}^{\infty} \left[\bigwedge_{k=n}^{\infty} x_k\right].$$