* $f: A \to B$ is a function from a set A to set B. $A \xrightarrow{f} B$ $x \in A \to f(x) \in B$.

$f: A \to B$ is a specific "rule" that assigns to each $x \in A$ a unique element $y \in B$. $\underset{\text{output}}{y} = f(\underset{\text{input}}{x})$ is called the value of function f at x (or image of x under f).

For $f: A \to B$, set A is domain of f and the set $\{y \in B : \exists\, x \in A \text{ w/ } y = f(x)\}$ is the range of f. Understood that A & B are nonempty sets.

Two f's $f: A \to B$ and $g: A \to B$ are said to be equal, i.e., $f = g$, if $f(x) = g(x)$ holds true $\forall\, x \in A$.

$f: A \to B$ is onto (or surjective) if $\text{range}(f) = B$; i.e., $\forall\, y \in B\ \exists$ (at least one) $x \in A$ s.t. $y = f(x)$.

$f: A \to B$ is one-to-one (or injective) if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Let $f: X \to Y$ be a f^n. If $A \subseteq X$, then image $f(A)$ of A under f is the subset of Y defined by

$$f(A) = \{ y \in Y : \exists x \in A \text{ s.t. } y = f(x) \}.$$

If $B \subseteq Y$, then the inverse image (or pre-image) $f^{-1}(B)$ of B under f is ~~the subset of Y defined~~ the subset of X defined by

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}.$$

Regarding images and pre-images of sets, ($\{A_i\}_{i \in I}$ is a family of subsets of X and $\{B_i\}_{i \in I}$ a family of subsets of Y) we have:

9. $f\left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i)$;

10. $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$;

11. $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$;

12. $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$;

13. $f^{-1}(B^c) = (f^{-1}(B))^c$.

Given two f^ns $f: X \to Y$ and $g: Y \to Z$, their <u>composition</u> $g \circ f$ is the f^n
$g \circ f: X \longrightarrow Y$ defined by
$(g \circ f)(x) = g(f(x))$ for each $x \in X$.

If $f: X \to Y$ is <u>one-to-one</u> and <u>onto</u> f^n, then $\forall\, y \in Y$ $\exists$ a unique $x \in X$ s.t. $y = f(x)$; $x = f^{-1}(y)$; then f is called bijective f^n and it's inverse is also a f^n $f^{-1}: Y \to X$.
$(f \circ f^{-1})(y) = y$ $\forall\, y \in Y$ and
$(f^{-1} \circ f)(x) = x$ $\forall\, x \in X$.

$f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$, where $I_X : X \to X$ and $I_Y : Y \to Y$ denote the identity f^ns, i.e., $I_X(x) = x$ and $I_Y(y) = y$ $\forall\, x \in X$ and $\forall\, y \in Y$.

Consider $N = \{1, 2, 3, \ldots\}$ be the set of natural numbers. Any f^n $x : N \to X$ is called a <u>sequence</u> of X. $x_n := x(n)$.

n^{th} term of the sequence.

We denote sequence x as $\{x_n\}_{n \in N}$ and consider it as both f^n and subset of X.

A <u>subsequence</u> of a sequence $\{x_n\}$ is a sequence $\{y_n\}$ for which $\exists$ a strictly increasing sequence $\{k_n\}$ of natural numbers (i.e., $1 \leq k_1 < k_2 < k_3 < \ldots$) s.t. $y_n = x_{k_n}$ holds for each n.

Consider $\{A_i\}_{i\in I}$ is a family of sets, then the <u>Cartesian product</u> $\prod_{i\in I} A_i$ (or $\prod A_i$) is defined to be the set consisting of all f^{n}s $f: I \longrightarrow \bigcup_{i\in I} A_i$ s.t. $x_i = f(i) \in A_i$ for each $i \in I$. Such a function is called a <u>choice f^{n}</u> & is denoted by $(x_i)_{i\in I}$ or (x_i).

If a family of sets consists of two sets, A and B, then the Cartesian product of the sets A & B is designated by $A \times B$:

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\},$$

(a,b) are <u>ordered pairs</u>.

$$(a_1, b_1) = (a_2, b_2) \text{ iff } a_1 = a_2,\ b_1 = b_2.$$

$$A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) : a_i \in A_i \ \forall\, i \in \{1, \dots, n\}\}.$$

If $A_1 = A_2 = \dots = A_n = A$, then $A_1 \times \dots \times A_n =: A^n$.

If the family $\{A_i\}_{i\in I}$ of sets satisfies $A_i = A \ \forall\, i \in I$, then $\prod_{i\in I} A_i =: A^I$, i.e.,

$$A^I = \{f \mid f: I \longrightarrow A\}.$$

Q. When is the Cartesian product of a family of sets $\{A_i\}_{i\in I}$ nonempty?

→ If the Cartesian product is non-empty, then each A_i must be nonempty.

Q. If each A_i is nonempty, is then the Cartesian product $\prod_{i\in I} A_i$ nonempty?

→ .... "Axiom of choice" needs to be considered, usual axioms of set theory not enough.

Axiom of choice. If $\{A_i\}_{i\in I}$ is a nonempty family of sets s.t. A_i is nonempty for each $i \in I$, then $\prod A_i$ is nonempty.

If $\{A_i\}_{i\in I}$ is a nonempty family of pairwise disjoint sets s.t. $A_i \neq \phi$ for each $i\in I$, then $\exists$ a set $E \subseteq \bigcup_{i\in I} A_i$ s.t. $E \cap A_i$ consists of precisely one element for each $i \in I$.