Lecture 5, 31/12/22

Example: Let $\vec{F} = \hat{i}\, 3x^2 y \sin z + \hat{j}\, x^3 \sin z + \hat{k}\, x^3 y \cos z$. Is this conservative?

Soln: $\dfrac{\partial F_2}{\partial x_1} = \dfrac{\partial F_y}{\partial x} = 3x^2 \sin z$; $\dfrac{\partial F_1}{\partial x_2} = \dfrac{\partial F_x}{\partial y} = 3x^2 \sin z$

Similarly $\dfrac{\partial F_3}{\partial x_1} = 3x^2 y \cos z = \dfrac{\partial F_1}{\partial x_3}$

& $\dfrac{\partial F_3}{\partial x_2} = x^3 \cos z = \dfrac{\partial F_2}{\partial x_3}$

Hence it is conservative.

* **$\vec{\nabla}$ operator:** The gradiant $\vec{\nabla} = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$ looks like a vector but is not a vector in the usual sense. It is a vector operator.

* **Divergence of a vector:** The divergence of a vector $\vec{V} = \hat{i}\, v_x + \hat{j}\, v_y + \hat{k}\, v_z$ is given by

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\, v_x + \hat{j}\, v_y + \hat{k}\, v_z\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Example: For a force field $\vec{F} = \vec{r}/|\vec{r}|^3$ (for all $\vec{r} \neq 0$) find $\vec{\nabla}.\vec{F}$.

Soln: We can write $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$= \hat{i} \frac{x}{(x^2+y^2+z^2)^{3/2}} + \hat{j} \frac{y}{(x^2+y^2+z^2)^{3/2}} + \hat{k} \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial F_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{1}{|\vec{r}|^3} - \frac{3x^2}{|\vec{r}|^5}$$

We can similarly calculate $\partial F_2/\partial y$ & $\partial F_3/\partial z$ and the results are easy to guess. Hence $\vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{3}{|\vec{r}|^3} - \frac{3(x^2+y^2+z^2)}{|\vec{r}|^5} = 0$

The $\vec{\nabla}.\vec{F} = 0$ for all $\vec{r}$ apart from $\vec{r} = 0$, where it diverges. Meaning of this result is little tricky - we may come back to this later.

Examples: Consider two vectors $\vec{v} = \hat{k}$, $\vec{v} = z\hat{k}$, $v = \frac{1}{z}\hat{k}$ Calculate the divergences.

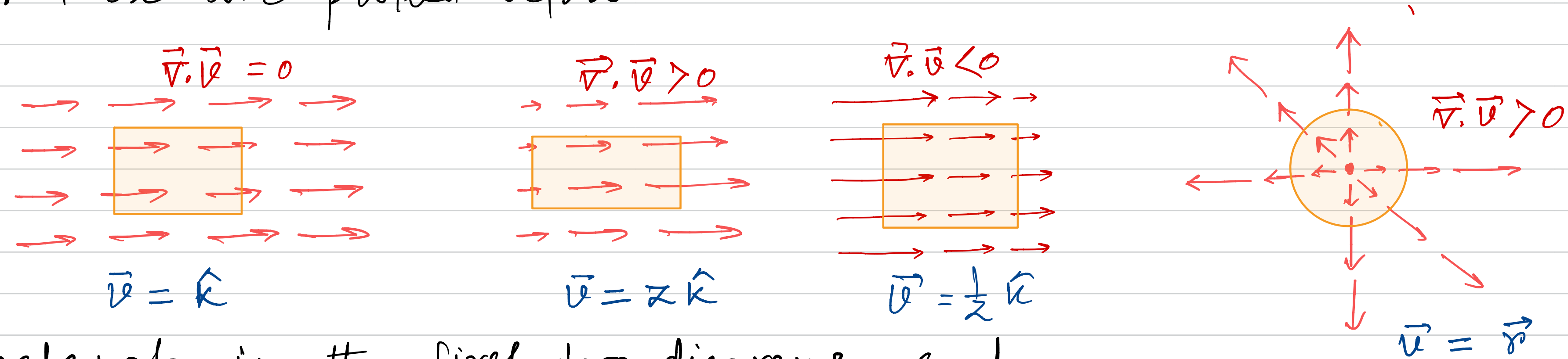
Soln: First : $\vec{\nabla}.\vec{v} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0$

2nd : $\vec{\nabla}.v = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 1$; For third : $\vec{\nabla}.\vec{v} < 0$

Example: Let $\vec{v} = \hat{i}x + \hat{j}y + \hat{k}z$. Then calculate the divergence

$$\vec{\nabla}\cdot v = 3$$

Physical Interpretation: In the previous examples we have considered four vectors. These are plotted below



The rectangle in the first two diagrams, and the circle in the third diagram corresponds to infinitesimal volumes. The vectors, which are denoted by the arrows indicate flow of some quantity - let say water. The divergence correspond to the rate (w.r.to coordinates) of flow of water through the volume.

* Concrete Interpretation: See Arfken.

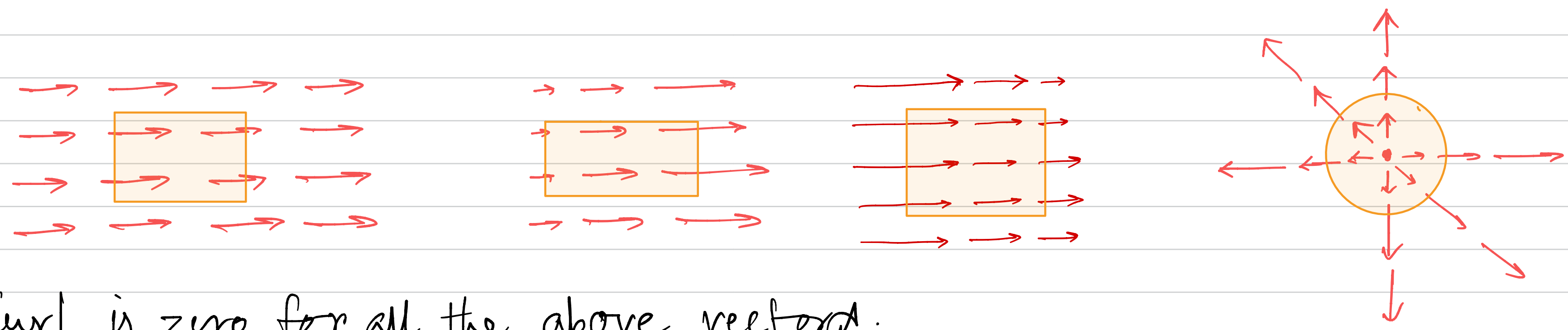
* **Solenoidal Vector:** Any vector, say $\vec{B}$, for which $\vec{\nabla}\cdot\vec{B} = 0$.

Curl of a Vector: Defined by

$$\vec{\nabla}\times\vec{V} = \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

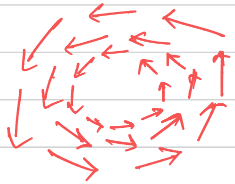
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Interpretation: The curl of a vector is a measure of how much the vector curls around a point. For example, the following vectors have zero curl



Curl is zero for all the above vectors.

Following Vector fields have non-zero curl



* Vector $\vec{v}$ for which $\vec{\nabla} \times \vec{v} = 0$ is called irrotational

Example: Consider flow of water in a river. If you drop a paper-boat and it starts to circle or rolate in the water then the velocity vector of the water has curl.

* **Example:** $\vec{F} = \hat{i}y - \hat{j}x$. Calculate the divergence and curl.

Soln: $\vec{\nabla} \cdot \vec{F} = 0$ and $\vec{\nabla} \times \vec{F} = -2\hat{k}$. So the vector swirl. The direction of curl is perpendicular to the plane of the swirl.

* Properties of gradiant, divergence and curl:

Linear differential operator properties: $\vec{\nabla}(a\phi + \psi) = a\vec{\nabla}\phi + \nabla\psi$; $a \to$ constant

$$\vec{\nabla} \cdot (a\vec{F} + \vec{G}) = a\,\vec{\nabla} \cdot \vec{F} + \vec{\nabla} \cdot \vec{G} \quad ; \quad \vec{\nabla} \times (a\vec{F} + G) = a\vec{\nabla} \times \vec{F} + \vec{\nabla} \times \vec{G}$$

Leibeniz Properties:

$$\vec{\nabla}(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\phi\vec{F}) = (\vec{\nabla}\phi)\cdot\vec{F} + \phi(\vec{\nabla}\cdot\vec{F})$$

$$\vec{\nabla}\times(\phi\vec{F}) = (\vec{\nabla}\phi)\times\vec{F} + \phi(\vec{\nabla}\times\vec{F})$$

$$\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = (\vec{\nabla}\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$$

* More identities involving two operations of $\vec{\nabla}$ can be found in books

There are also two other properties for $\vec{\nabla}(\vec{F}\cdot\vec{G})$ & $\vec{\nabla}\times(\vec{F}\times\vec{G})$.

Theorem: A vector field that is conservative is irrotational. In other words if ϕ is a scalar field & $\vec{F}$ is a vector field then

$$\vec{F} = \vec{\nabla}\phi \quad \text{implies that} \quad \vec{\nabla}\times\vec{F} = 0 \quad \text{and vice versa}$$

Proof (incomplete): For $\vec{F} = \vec{\nabla}\phi$ the components are

$$F_x = \partial\phi/\partial x, \quad F_y = \partial\phi/\partial y \quad \& \quad F_z = \partial\phi/\partial z.$$

Now

$$\vec{\nabla}\times\vec{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \hat{i}\left(\frac{\partial\phi}{\partial y\partial z} - \frac{\partial\phi}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial\phi}{\partial z\partial x} - \frac{\partial\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial\varphi}{\partial x\partial y} - \frac{\partial\varphi}{\partial y\partial x}\right) = 0$$

Theorem: Any divergence free field can be written as a curl of another vector field ie if $\vec{\nabla}.\vec{F}=0$ then $\vec{F}=\vec{\nabla}\times\vec{A}$ and vice versa.

Proof: Same as the previous theorem (incomplete) Exercise

* The previous two theorem imply

1. Curl of gradiant is zero: $\vec{\nabla}\times(\vec{\nabla}\phi)=0$
2. Divergence of a curl is zero: $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$

* The Laplacian: The operator $\vec{\nabla}$ is first order derivative operator w.r.to the coordinates. By operating $\vec{\nabla}$ twice. second-order derivative can be obtained. For example

$$\vec{\nabla}.\vec{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \equiv \nabla^2\phi$$

$$\nabla^2 = \vec{\nabla}.\vec{\nabla} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$ is called the Laplacian.

* It can be shown that there are just two types of fundamental second

derivative - the Laplacian and gradiant-of-divergence (ie $\vec{\nabla}(\nabla\cdot\vec{V})$)

Here note that $\vec{\nabla}(\vec{\nabla}\cdot\vec{V}) \neq \nabla^2\vec{V}$

* Identities: Two important identities involving Laplacian are given below

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2 V$$

Fore more identities see Arfken or Reily-Hobson-Bence.