Then, for all z inside C, the expansion of $f(z)$ can be written as

$$f(z) = \frac{a_{-p}}{(z-z_0)^p} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots , \tag{148}$$

with $a_{-p} \neq 0$. Note that $a_n = b_{n+p}$. This is called the Laurent series. The coefficients in the series expansion are given by

$$\begin{aligned} b_n &= \frac{g^n(z)}{n!} = \frac{1}{2\pi i}\oint \frac{g(z)}{(z-z_0)^{n+1}} dz , \\ a_n &= \frac{1}{2\pi i}\oint_C \frac{g(z)}{(z-z_0)^{n+1+p}} dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} , \end{aligned}$$

These are valid for both positive and negative values of n.

The terms in the Laurent series that are of positive powers in $(z - z_0)$ are called the *analytic part*, and the the remainder, with inverse powers in $z - z_0$, are called the *principle part* or the *singular part*.

Before we take some examples, we note that the above integral formulas are not that useful in computing the Laurent series. Instead algebraic tricks will just do fine. Separating the the singular part, we can expand the rest in Taylor series.

Examples: Find the Laurent series of

$$f(z) = \frac{z+1}{z} ,$$

around $z = 0$.

Ans: The answer is

$$f(z) = 1 + \frac{1}{z} .$$

This is valid for $0 < |z| < \infty$.

Examples: Find the Laurent series for

$$f(z) = \frac{z}{z^2+1} ,$$

around $z = i$. Give the region where your answer is valid. Identify the singular (principal) part.

Ans: We can write the function as

$$f(z) = \frac{1}{2}\left(\frac{1}{z-i} + \frac{1}{z+i}\right) .$$

Now, the function $1/(z + i)$ is analytic at $z = i$ and can be Taylor series expanded around $z = i$ as follows

$$\frac{1}{z+i} = \frac{1}{2i}\frac{1}{1+(z-1)/(2i)} = \frac{1}{2i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n .$$

Hence, the Laurent series is given by

$$f(z) = \frac{1}{2}\frac{1}{z-i} + \frac{1}{4i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n \tag{149}$$

The first terms is the principle part. As $z = -i$ is another singular point, the region of validity is $0 < |z - i| < 2$.

4.2.2 Residue:

If $f(z)$ is not analytic at $z = z_0$, then there are two cases in the Laurent series

- in the series expansion (148), there is a positive integer p such that $a_{-p} \neq 0$, but for all integers $k > 0$ all $a_{-p-k} = 0$ – in other words, the series is such that there are only finite terms with negative exponents of $(z - z_0)$.
- there is no lowest values of $(-p)$ - - *i.e.,* there are infinite number of terms with negative exponents of $(z - z_0)$

In the case i), the series is said to have pole of order p and the value of a_{-p} is called the *residue of* $f(z)$ *at* $z = z_0$. In the case ii) where there are infinite numbers of terms with negative exponenst, the function is said to have essential singularity at $z = z_0$.

Example: Find the Laurent series of

$$f(z) = \frac{1}{z(z-2)^3},$$

about the singularities $z = 0$ and $z = 2$. Hence, verify that $z = 0$ is a pole of order 1 and $z = 2$ is a pole of order 3. Find the residue at each pole

Ans: For series about $z = 0$ we write

$$\begin{aligned} f(z) &= -\frac{1}{8z(1-z/2)^3}, \\ &= \frac{1}{8z}\left[1 + (-3)(-z/2) + \frac{(-3)(-4)}{2!}(-z/2)^2 + \frac{(-3)(-4)(-5)}{3!}(-z/2)^3 + \cdots\right], \\ &= -\frac{1}{8z} - \frac{3}{16} - \frac{3z}{16} - \frac{5z^2}{32} + \cdots \end{aligned} \tag{150}$$

As the lowest power of z is -1, the point $z = 0$ is pole of order 1. The residue is the coefficient of z^{-1} which is $-1/8$.

The Laurent series about $z = 2$ is most easily found by letting $z = \xi + 2$ and substituting into the expression for $f(z)$ to obtain

$$\begin{aligned} f(z) = \frac{1}{(2+\xi)\xi^3} &= \frac{1}{2\xi^3(1+\xi/2)}, \\ &= \frac{1}{2\xi^3}\left[1 - (\xi/2) + (\xi/2)^2 - (\xi/2)^3 + (\xi/2)^4 - \cdots\right], \\ &= \frac{1}{2\xi^3} - \frac{1}{4\xi^2} + \frac{1}{8\xi} - \frac{1}{16} + \frac{\xi}{32} - \cdots, \\ &= \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} - \frac{1}{16} + \frac{z-2}{32} \end{aligned} \tag{151}$$

From this series we see thatt $z = 2$ is pole of order 3 and that the residue $f(z)$ at $z = 2$ is 1/8.

4.2.3 Calculating Residue:

If $f(z)$ has a pole of order m at $z = z_0$, then its Laurent series about this point has the form

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{152}$$

On multiplying both sides of the equation by $(z - z_0)^m$, gives

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots \tag{153}$$

Differentiating both sides $m-1$ times, we obtain

$$\frac{d^{m-1}}{dz^{m-1}}\Big[(z-z_0)^m f(z)\Big] = (m-1)!a_{-1} + \sum_{n=1}^{\infty} b_n (z-z_0)^n \,, \tag{154}$$

for some coefficients b_n. In the limit $z \to z_0$, however, the terms in the sum disappear, and after rearranging we get

$$R(z_0) = a_{-1} = \lim_{z\to z_0}\left[\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right] \tag{155}$$

which gives the value of the residue of $f(z)$ at the points $z = z_0$.

An important special case of this formula is that the residue for a function with simple pole at $z = z_0$ is

$$R(z_0) = \lim_{z\to z_0}\big[(z-z_0)f(z)\big] \,.$$

And, if the singularity of $f(z)$ is removable, then

Examples: Given the function

$$f(z) = \frac{e^{iz}}{(z^2+1)^2} \,,$$

find out the poles and their order. Also calculate the residue at $z = i$.

Ans: We can write the function as

$$f(z) = \frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z-i)^2(z+i)^2} \,,$$

So there are poles of order 2 at $z = i$ and $z = -i$. To calculate the residue at $z = i$, we use the above formula

$$\begin{aligned} \frac{d}{dz}\Big[(z-i)^2 f(z)\Big] &= \frac{d}{dz}\frac{e^{iz}}{(z+i)^2} \,, \\ &= \frac{ie^{iz}}{(z+i)^2} - 2\frac{e^{iz}}{(z+i)^3} \,, \end{aligned}$$

Setting $z = i$ we get the residue as

$$R(i) = -\frac{i}{2e}$$

4.2.4 Cauchy Residue Theorem:

If C is a closed contour oriented in a counterclockwise direction, and $f(z)$ is a function that is analytic inside and on C except finite number of isolated singularities $z_1, z_2....z_k$, then the integral of the function over the circle is equal to $2\pi i$ times the sum of all residues

$$\oint_C f(z)dz = 2\pi i \sum_k R_k \tag{156}$$

Proof: Inside C there are finite number of points $z_1, z_2....z_k$, where the function fails to be analytic. In that case one can continuously deform the contour to a union of integrations over contours $C_1, C_2....C_k$ oriented counterclockwise encircling exactly one isolated singularity. Then from from Cauchy theorem

$$\oint f(z)dz = \sum_k \oint_{C_k} f(z)dz \,.$$

Now, take any one of the k integrals. From the previous discussions

$$R(z_k) = a_{-1} = \frac{1}{\pi i}\oint_{C_k} \frac{f(z)}{(z-z_k)^{(-1)+1}}dz = \frac{1}{2\pi i}\oint_C f(z)dz \tag{157}$$

Hence proved

Example: Compute

$$\int_{|z|=2} \frac{5z-2}{z(z-1)} \tag{158}$$

Ans: The function is

$$f(z) = \frac{5z-2}{z(z-1)} \tag{159}$$

has poles at $z = 0, 1$ and the circle $|z| = 2$ closes them both. Now, at $z = 0$ the residue is

$$\lim_{z \to 0} z f(z) = 2 \,,$$

At $z = 1$ the residue is

$$\lim_{z \to 1} (z-1) f(z) = 3 \tag{160}$$

Hence, the result is

$$\int_{|z|=2} \frac{5z-2}{z(z-1)} = 2\pi i(3+2) = 10\pi i \tag{161}$$

Example: Compute

$$\oint_{|z|=1} z^2 \sin(1/z) dz \tag{162}$$

Ans: Using Taylor series expansion we get

$$z^2 \sin(1/z) = z^2 \left(\frac{1}{z} - \frac{1}{3! z^3} + \frac{1}{5! z^5} + \right) = z - \frac{1/6}{z} + \cdots \tag{163}$$

Hence, the residue at $z = 0$ is $-1/6$. Then the result of the integration is $-2\pi i/6 = -\pi i/3$.