Predicate Logic Examples

1 Exercises

1.1 Question 1

Given ∀x A(x) ∧ ∀x B(x), deduce ∀y A(y) ∧ B(y)

1		$\forall x A(x) \land \forall x B(x)$	
2		$\forall x A(x)$	
3		$\forall x B(x)$	
4	z	$A(z)$	∀E, 2
5		$B(z)$	∀E, 3
6		$A(z) \land B(z)$	∧I, 4, 5
7		$\forall y(A(y) \land B(y))$	∀I, 4–6

1.2 Question 2

Given $\forall x \forall y P(x, y)$, deduce ∀y ∀x P(x,y)

1			$\forall x \forall y P(x, y)$	
2	b		T	
3		a	$\forall\ y P(a, y)$	∀E, 1
4			$P(a, b)$	∀E, 3
5			$\forall\ x P(x, b)$	∀I, 3–4
6			$\forall\ y\ \forall\ x P(x, y)$	∀I, 2–5

1.3 Question 3

From ∃x Bird(x) , ∀x Bird(x) → Sings(x), deduce ∃x Sings(x)

1		$\exists\ xBird(x)$	
2		$\forall\ xBird(x) \to Sings(x)$	
3	a	$Bird(a)$	
4		$Bird(a) \to Sings(a)$	∀E, 2
5		$Sings(a)$	⇒E, 3, 4
6		$\exists x Sings(x)$	∃I, 5
7		$\exists x Sings(x)$	∃E, 1, 3–6

1.4 Question 4

From ∃x ∃y P(x,y), infer ∃y ∃x P(x,y)

1			$\exists\ x \exists\ yP(x,y)$	
2	a		$\exists\ yP(a,y)$	
3		b	$P(a,b)$	
4			$\exists\ xP(x,b)$	∃I, 3
5			$\exists\ y \exists\ xP(x,y)$	∃I, 4
6			$\exists\ y \exists\ xP(x,y)$	∃E, 2, 3–5
7			$\exists\ y \exists\ xP(x,y)$	∃E, 1, 2–6

1.5 Question 5

From $\exists\ x \forall\ y\ P(x,y)$ infer $\forall\ y \exists\ x\ P(x,y)$

1			$\exists\ x \forall\ y\ P(x,y)$	
2	b		$\top$	
3		a	$\forall\ y\ P(a,y)$	
4			$P(a,b)$	∀E, 3
5			$\exists\ x\ P(x,b)$	∃I, 4
6			$\exists\ x\ P(x,b)$	∃E, 1, 3–5
7			$\forall\ y \exists\ x\ P(x,y)$	∀I, 6

Alternative

1	$\exists\, x\, \forall\, y\; P(x,y)$	
2	$\quad a \mid \forall\, y\; P(a,y)$	
3	$\quad\quad b \mid P(a,b)$	$\forall$E, 2
4	$\quad\quad\quad \mid \exists\, x\; P(x,b)$	$\exists$I, 3
5	$\quad\quad \forall\, y\, \exists\, x\; P(x,y)$	$\forall$I, 4
6	$\forall\, y\, \exists\, x\; P(x,y)$	$\exists$E, 1, 2–5

The opposite deduction is not correct.

1	$\forall\, y \exists\, x\; L(x,y)$	
2	$\quad b \mid \exists\, x\; L(x,b)$	$\forall$E, 1
3	$\quad\quad a \mid L(a,b)$	
4	$\quad\quad\quad \mid L(a,b)$	R, 3
5	$\quad\quad \forall\, y\;\; L(y,b)$	$\forall$I, 4

Step 5 is an incorrect application of $\forall I$ (a is occurring in 3, which is undischarged)

1.6 Quantifier and Negation

Given : $\neg(\exists\, x\; P(x)) \longleftrightarrow\; \forall\, x\, \neg\; P(x)$ Forward Direction Proof

1	$\neg(\exists\, x\; P(x))$	
2	$\quad a \mid$	
3	$\quad\quad\quad \mid P(a)$	
4	$\quad\quad\quad \mid \exists\, x\; P(x)$	$\exists$I, 3
5	$\quad\quad\quad \mid \bot$	$\neg$E, 1, 4
6	$\quad\quad \neg P(a)$	$\neg$I, 3–5
7	$\forall x \neg P(x)$	$\forall$I, 6

Converse Direction Proof

$$
\begin{array}{r|l l}
1 & \forall x \neg P(x) & \\
\hline
2 & \quad \exists\ x\ P(x) & \\
3 & \quad\quad a \quad P(a) & \\
4 & \quad\quad\quad \neg P(a) & \forall\text{E}, 1 \\
5 & \quad\quad\quad \perp & \neg\text{E}, 3, 4 \\
6 & \quad\quad \perp & \exists\text{E}, 2, 3\text{–}5 \\
7 & \neg(\exists\ x\ P(x)) & \neg\text{I}, 2\text{–}6
\end{array}
$$