Metric spaces

A <u>metric</u> (or a <u>distance</u>) d on a nonempty set X is a fn $d: X \times X \longrightarrow \mathbb{R}$ satisfying three properties:

a) $d(x,y) \geq 0 \quad \forall\, x,y \in X$ and
$d(x,y) = 0 \Leftrightarrow x = y$;

b) $d(x,y) = d(y,x) \quad \forall\, x,y \in X$;

c) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in X$.

The pair (X,d) is called a <u>metric space</u>.

In a metric space (X,d) the inequality

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

holds for all points $x,y,z \in X$.

Examples:

a) The set of real nos. $\mathbb{R}$ equipped with the distance $d(x,y) = |x-y| \quad \forall\, x,y \in \mathbb{R}$.

b) Euclidean space $\mathbb{R}^n$ equipped with the distance

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$.
This distance is called the Euclidean distance.

If Y is a subset of a metric space (X,d), then Y equipped with the distance d also becomes a metric space.

Fix a metric space (X,d). If $x \in X$, then the <u>open ball</u> at x with radius $r>0$ is the set $B(x,r) = \{y \in X : d(x,y) < r\}$. ~~The open subsets of X~~ A subset A of X is called <u>open</u> if for every $x \in A$, $\exists$ some $r > 0$ s.t. $B(x,r) \subseteq A$.

Every open ball $B(x,r)$ is an open set. If $y \in B(x,r)$, then the open ball $B(y,r_1)$, where $r_1 = r - d(x,y) > 0$ satisfies $B(y,r_1) \subseteq B(x,r)$.
Because, $z \in B(y,r_1)$ implies
$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + r_1 = r$, and so
$z \in B(x,r)$.

<u>Thm.</u> For a metric space (X,d) the following statement holds:
i) X and ϕ are open sets.
ii) Arbitrary union of open sets are open sets.
iii) Finite intersections of open sets are open sets.

A point x is called an <u>interior point</u> of a subset A if $\exists$ an open ball: $B(x,r)$ such that $B(x,r) \subseteq A$. Set of all interior points of A is denoted by A^0 and is called the <u>interior</u> of A; clearly, $A^0 \subseteq A$. A^0 is the largest open subset of X included in A. Also, note that A is <u>open</u> iff $A = A^0$.

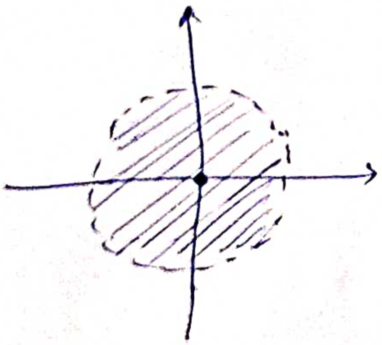
Metric space $(\mathbb{R}^2, d_p)$

$$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

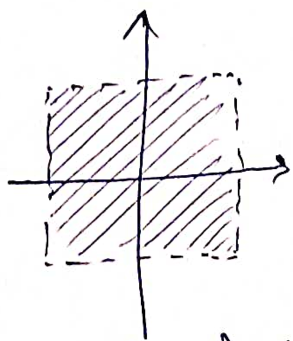
$$d_1((x_1,y_1),(x_2,y_2)) = |x_2-x_1|+|y_2-y_1|$$

$$d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_2-x_1|, |y_2-y_1|\}.$$

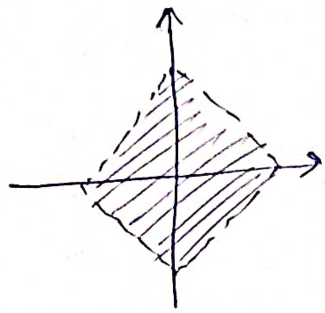
$$d_\infty(\, , \,) \le d_2(\, , \,) \le d_1(\, , \,) \le 2d_\infty(\, , \,).$$



$d(\, , \,) < L$



$d_\infty(\, , \,) < L$



$d_1(\, , \,) < L.$

A subset A of a metric space (X, d) is called <u>closed</u> if its complement A^c $(= X \setminus A)$ is an open set.

<u>Thm</u>. For a metric space (X, d).

i) X and ϕ are closed sets.

ii) Arbitrary intersections of closed sets are closed sets.

iii) Finite unions of closed sets are closed sets.

Note that a set A is open iff A^c is closed; and a set A is closed iff A^c is open.

~~A set which is not necessarily closed~~

A set which is not closed is not necessarily open, and vice versa.

A point $x \in X$ is called a <u>closure point</u> of a subset A of X if every open ball at x contains (at least) one element of A; i.e., $B(x, r) \cap A \neq \phi \;\; \forall r > 0$. The set of all closure points of A is denoted as $\bar{A}$, and is called the <u>closure</u> of A; clearly, $A \subseteq \bar{A}$.

<u>Thm</u>. For every subset A of a metric space, $\bar{A}$ is the smallest closed set that includes A.

An immediate consequence of the previous theorem:

A set A is closed iff $A = \overline{A}$.

Every set of the form $A = \{x \in X : d(x,a) \leq r\}$, called the closed ball at a with radius r, is a closed set. Assume $d(a,x) > r$, put $r_1 = d(x,a) - r > 0$. If $d(y,x) < r_1$, then

$$d(a,y) \geq d(a,x) - d(y,x) > d(a,x) - r_1 = r,$$

which shows that A^c is open, hence, A is closed.

Notice that in a discrete metric space, $\overline{B(a,r)}$ may be a proper subset of $\{x \in X : d(x,a) \leq r\}$. However, in the Euclidean space $\mathbb{R}^n$, the closure of every open ball of radius r is the closed ball of radius r. (Why? Think about it.)

Lemma. If A is subset of a metric space, then $A^{\circ} = (\overline{A^c})^c$.

Lemma. If A is a subset of a metric space, then $A^\circ = \left(\overline{A^c}\right)^c$.

Proof:
$$x \in A^\circ \iff \exists\, r>0 \text{ with } B(x,r) \subseteq A$$
$$\iff \exists\, r>0 \text{ with } B(x,r) \cap A^c = \phi$$
$$\iff x \notin \overline{A^c} \iff x \in \left(\overline{A^c}\right)^c.$$

A point x is called an accumulation point of a set A if every open ball $B(x,r)$ contains an element of A distinct from x; i.e., $B(x,r) \cap (A \setminus \{x\}) = \phi$ for each $r>0$. Notice that x need not be an element of A.

Every accumulation point of a set is automatically a closure point of that set. The set of accumulation points of A is called the derived set of A, and is denoted by A'. $\overline{A} = A \cup A'$.

A set is closed iff it contains its accumulation points.

A sequence $\{x_n\}$ of a metric space (X,d) is said to be convergent to $x \in X$ ($\lim x_n = x$) if $\lim d(x_n, x) = 0$.