With these parameterizations the required integrals can be written as

$$\int_{C_3} \frac{dz}{z} = \int_0^1 \frac{-R+iR}{(1-t)R+itR}dt + \int_0^1 \frac{-R-iR}{-sR+i(1-s)R}ds\,.$$

By simplifying a little bit, the first integral becomes

$$\begin{aligned}
\int_0^1 \frac{-R+iR}{(1-t)R+itR}dt &= \int_0^1 \frac{2t-1}{1-2t+2t^2}dt + i\int_0^1 \frac{1}{1-2t+2t^2}dt\,, \\
&= \frac{1}{2}\Big[\ln(1-2t+2t^2)\Big]_0^1 + \frac{i}{2}\Big[2\arctan\big(\frac{t-1/2}{1/2}\big)\Big]\,, \\
&= 0 + \frac{i}{2}\big[\frac{\pi}{2} - (-\pi/2)\big] = i\frac{\pi}{2}\,,
\end{aligned}$$

The other integral can be shown to be $i\pi$.

The above examples demonstrate that despite the same limits, the value of integrals depend on paths. In certain circumstances, the result is independent of path as well.

Upper bounds of Moduli in contour integration

Before going to the Cauchy theorem, we state a result without proof.

If M is the upper bound on the value of $|f(z)|$ on a contour C, *i.e.*, $|f(z)| < M$, and L is the length of the path C then the following is true

$$\Big|\int_C f(z)dz\Big| \le \int_C |f(z)||dz| \le M\int_C d\ell = ML\,. \tag{90}$$

Let us see one example of this result.

Example: Let C be the arc of a circle $|z| = 2$, from $z = 2$ to $z = 2i$, that lies on the first quadrant. Show that

$$\Big|\int_C \frac{z+4}{z^3-1}dz\Big| \le \frac{6\pi}{7}\,.$$

Ans: Note that, $|z| = 2$. Hence from triangle inequalities

$$|z+4| \le |z| + 4 = 6\,, \qquad |z^3-1| = \big||z^3|-1\big| = 7\,,$$

Now, the length of the arc is π. Hence we have

$$\Big|\int_C \frac{z+4}{z^3-1}dz\Big| \le \frac{6\pi}{7}$$

4.0.8 Cauchy Integral Theorem:

The theorem states that if $f(z)$ is an analytic function, and the derivative $f'(z)$ is continuous at each point within and on a closed contour C, then

$$\oint_C f(z)dz = 0\,.$$

Proof: The theorem can be proved using the Green's theorem: if p, q are two functions with continuous first derivative within and on a closed contour C in the $x-y$ plane, then

$$\int\int \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)dxdy = \oint_C (pdy - qdx)\,.$$

Hence, if $f(z) = u + iv$ and $dz = dx + idy$, this can be applied to

$$\begin{aligned}
I = \oint_c f(z)dz &= \oint_C (udx - ydy) + i\oint(vdx + udy)\,, \\
&= \int\int\left[\frac{\partial(u)}{dy} + \frac{\partial(v)}{\partial x}\right]dxdy + i\int\int\left[\frac{\partial(-v)}{dy} + \frac{\partial u}{\partial x}\right]dxdy\,.
\end{aligned}$$

Since both u, v satisfy the CR relations, each term in the above equations identically vanish, which proves the theorem.

In the above theorem, the continuity of $f'(z)$ is not necessary condition, the analyticity condition is sufficient. But without that the proof is more involved.

Corollary: If two points A and B on a complex plane are joined by two paths C_1 and C_2 such that they together form a contour, and if the function $f(z)$ is analytic on the contour and in the region enclosed by them (see figure 9), then the value of the integrals are same on the paths C_1 and C_2. This is easy to prove. From the Cauchy therem

$$\begin{aligned} 0 &= \oint f(z)dz\,, \\ &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz\,, \end{aligned}$$

$$\text{hence} \quad \int_{C_1} f(z)dz = \int_{C_2} f(z)dz\,,$$

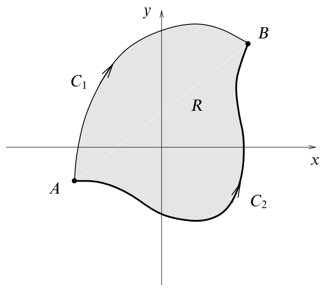

Figure 9: a region enclosed by two paths C_1 and C_2.

Contour Deformation: The Cauchy theorem can be extended to included different types of contour. We need the concept if *simply connected domain*. A simple connected domain D is such that every simple closed curve within this domain only encloses points on D. Another way to define simply connected region is this: any closed path in a simply connected domain can be continuously shrink to a point while remaining within this domain. Domains that are not simply connected are called multiply connected or non-simply connected. See figure 10 where the shaded blue regions indicates simply and non-simply connected regions.

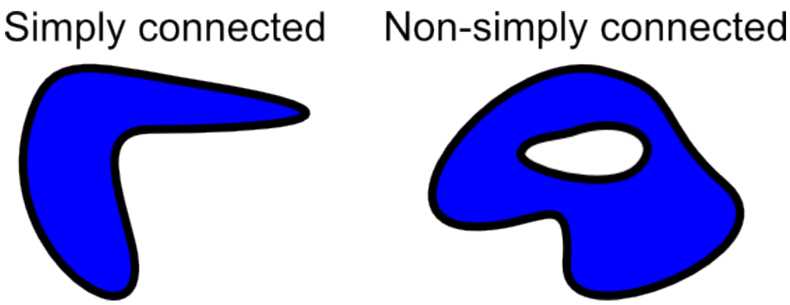

Figure 10: simply and multiply connected domain.

Closed contour with finite intersections: Consider a closed contour with finite number of intersection as shown in figure 11. The curve C is simple curve, and lies in a simply connected domain D, and

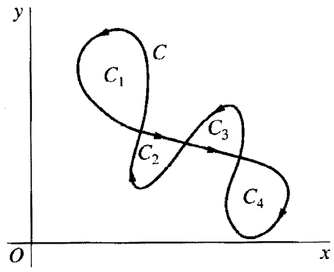

Figure 11: simply and multiply connected domai.

it intersects itself finite number of times forming a finite number of closed contours $C_k(k = 1, 2, 3, 4..)$. If a function $f(z)$ is analytic on C, and within it, then the Cuachy theorem is

$$\oint_C f(z)dz = \sum_k \oint_{C_k} f(z)dz = 0\,.$$

Contour enclosed by a contour: Consider a simple closed contour C, described in the counterclockwise direction. In the interior of C there are simple closed contours $C_1, C_2...$, described in the clockwise direction, such that they are disjoint and their interiors have no points in common. This is shown in figure 12. If a function $f(z)$ is anaytic on all those contours and through the multiply connected region that consists of all points intirior to C an extirior to $C_1, C_2..$ etc, then the Cauchy theorem takes the form

$$\oint_C f(z)dz + \sum_k \oint_{C_k} f(z)dz = 0\,. \tag{91}$$

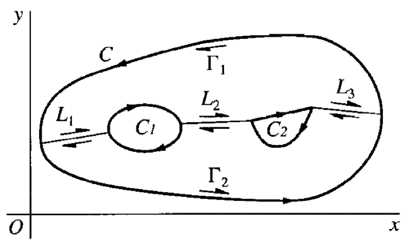

Figure 12:

Another situation is this: There two positively oriented, *i.e.,* counterclockwise contours C and γ where γ is intirior of C. This type of contour is shown in figure 13. If a function $f(z)$ is analytic in the closed region consisting of these contours and all the points between them then the Cauchy theorem reads

$$\oint_C f(z)dz = \oint_\gamma f(z)dz\,. \tag{92}$$

We will prove the result (92). The result can be similarly proven. To prove it, we we consider a contour shown in figure 13. The two closed parallel lines C_1 and C_2 join the contours γ and C, which are cut to accommodate them. The new contour Γ consists of C, C_1, C_2, γ. Since the function $f(z)$ is analytic on and within this contour, applying Cauchy's theoerem we get

$$\oint_\Gamma f(z)dz = 0 \tag{93}$$

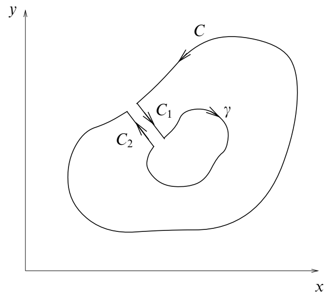

Figure 13:

Now, the integrations along C_1 and C_2 are done in the opposite direction. Therefore, they cancel each other in the limits that they are infinitesimaly closed to each other. Thus we get

$$\oint_C f(z)dz + \oint_\gamma f(z)dz \tag{94}$$

Now changing the direction of γ we get

$$\oint_C f(z)dz = \oint_\gamma f(z)dz\,. \tag{95}$$

4.0.9 Cauchy Integral Formula:

If $f(z)$ is analytic within and on a closed contour C, and z_0 is a point within C then

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}\,. \tag{96}$$

This formula tells that the value of the function anywhere inside a closed contour is uniquely determined by its value on the contour.

To prove the theorem, we use the figure 13 and the result 92, which is given by

$$\oint_C f(z)dz = \oint_\gamma f(z)dz\,. \tag{97}$$

We consider that the contour γ in the figure 13 is a circle. Now, the integrand $f(z)/(z - z_0)$ is analytic between the contours C and γ. Hence from the above result, the integral around γ has same result as that around C. Any point z on γ can be written as $z = z_0 + \rho e^{i\theta}$, and hence $dz = i\rho e^{i\theta}d\theta$. Hence the value of the integral is around γ is given by

$$\begin{aligned} I = \oint_\gamma \frac{f(z)}{z - z_0} &= \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta\,, \\ &= i\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta\,, \end{aligned}$$

In the limit $\rho \to 0$, the result if the integration $I \to i2\pi f(z_0)$. Hence proved.

Corollaries: Suppose that a function $f(z)$ is analytic function everywhere inside and on a simple closed contour C, taken in the counterclockwise sense, and z_0 is any point that is interior to C then

$$f'(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{(z - z_0)^2}dz \tag{98}$$

The n^{th} derivative at z_0 can also be obtained (given that the derivatives exist at that point) as

$$f^n(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial z_0^n}\left(\frac{f(z)}{z-z_0}\right)dz \tag{99}$$

Example 1: Given that C is a positively oriented, *i.e.,* counterclockwise circle $|z| = 2$. Show that

$$\oint_C \frac{zdz}{(9-z^2)(z+i)} = \frac{\pi}{5}$$

Ans: As $|z| = 2$ only $z = i$ is the interior point. Hence we can write

$$\oint_C \frac{zdz}{(9-z^2)(z+i)} = \oint_C \frac{z/(9-z^2)}{(z-(-i))}dz = 2\pi i\frac{-i}{10} = \frac{\pi}{5}$$

Example 2: If C is a positively oriented unit circle $|z| = 1$, and function $f(z) = exp(2z)$ is a analytic function, then show that

$$\oint_C \frac{exp(2z)}{z^4}dz = \frac{8\pi i}{3} \tag{100}$$

Ans: From the Cauchy integral formula

$$\oint_C \frac{exp(2z)}{z^4} = \oint_C \frac{exp(2z)}{(z-z_0)^{3+1}} = \frac{2\pi i}{3!}f^3(0) = \frac{8\pi i}{3}$$

References

[1]