Lemma. If A is a subset of a metric space, then $A^{\circ} = \left(\overline{A^c}\right)^c$.

Proof: $x \in A^{\circ} \Leftrightarrow \exists\, r > 0$ with $B(x,r) \subseteq A$
$\Leftrightarrow \exists\, r > 0$ with $B(x,r) \cap A^c = \phi$
$\Leftrightarrow x \notin \overline{A^c} \Leftrightarrow x \in \left(\overline{A^c}\right)^c$.

A point x is called an accumulation point of a set A if every open ball $B(x,r)$ contains an element of A distinct from x; i.e., $B(x,r) \cap (A \setminus \{x\}) = \phi$ for each $r > 0$. Notice that x need not be an element of A.

Every accumulation point of a set is automatically a closure point of that set. The set of accumulation points of A is called the derived set of A, and is denoted by A'. $\bar{A} = A \cup A'$.

A set is closed iff it contains its accumulation points.

A sequence $\{x_n\}$ of a metric space (X,d) is said to be convergent to $x \in X$ ($\lim x_n = x$) if $\lim d(x_n, x) = 0$.

Thm: Let A be a subset of a metric space (X,d). Then a point $x \in X$ belongs to $\bar{A}$ iff $\exists$ a sequence $\{x_n\}$ of A s.t. $\lim x_n = x$.

Moreover, if x is an accumulation point of A, then $\exists$ a sequence of A with distinct terms that converges to x.

Proof: Assume that $x \in \bar{A}$. For each n, pick $x_n \in A$ such that $d(x, x_n) < \frac{1}{n}$. Then the sequence $\{x_n\}$ of A satisfies $\lim x_n = x$.

Whereas, if a sequence $\{x_n\}$ of A satisfies $\lim x_n = x$, then for each $r > 0$ $\exists$ some k s.t. $d(x, x_n) < r$ for $n > k$. Thus, $B(x,r) \cap A \neq \phi$ for each $r > 0$, and so $x \in \bar{A}$.

Next, assume that $x \in A'$. Start with choosing some $x_1 \in A$ s.t. $x_1 \neq x$ and $d(x, x_1) < 1$. Now, inductively, if $x_1, x_2, \ldots, x_n \in (A \setminus \{x\})$ have been choosen, pick $x_{n+1} \in A \setminus \{x\}$ s.t. $d(x, x_{n+1}) < \min\left\{\frac{1}{n+1}, d(x, x_n)\right\}$. Then $\{x_n\}$ is a sequence of A satisfying $x_n \neq x_m$ if $n \neq m$ and $\lim x_n = x$.

A subset A of a metric space (X, d) is called dense in X if $\bar{A} = X$. According to the previous theorem, a set A is dense in X iff for every $x \in X$ $\exists$ a sequence $\{x_n\}$ of A s.t. $\lim x_n = x$. Notice that a set A is dense if and only if $V \cap A \neq \phi$ holds for each nonempty ~~set~~ open set V.

A point $x \in X$ is called a boundary point of a set A if every open ball of x contains points from A and A^c; i.e., if $B(x, r) \cap A \neq \phi$ and $B(x, r) \cap A^c \neq \phi$ for all $r > 0$. The set of all boundary points of a set A is denoted by ∂A and is called the boundary of A.

By the def^n, $\partial A = \partial A^c$ holds for every subset A of X.

$$\partial A = \bar{A} \cap \overline{A^c}.$$

Def^n A f^n $f:(X,d) \longrightarrow (Y,\rho)$ between two metric spaces is said to be <u>continuous at a point</u> $a \in X$ if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ (depending on ε) s.t. $\rho(f(x), f(a)) < \varepsilon$ whenever $d(x,a) < \delta$.

The f^n f is said to be <u>continuous on X</u> (or simply <u>continuous</u>) if f is continuous at every point of X.

<u>Thm.</u> For a f^n $f:(X,d) \longrightarrow (Y,\rho)$ bet^n two metric spaces, the following statements are equivalent:

i) f is continuous on X.

ii) $f^{-1}(O)$ is an open subset of X whenever O is an open subset of Y.

iii) If $\lim x_n = x$ holds in X, then $\lim f(x_n) = f(x)$ holds in Y.

iv) $f(\overline{A}) \subseteq \overline{f(A)}$ holds for every subset A of X.

v) $f^{-1}(C)$ is a closed subset of X whenever C is closed subset of Y.