Discrete Structures (Monsoon 2022)

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Discrete Numeric Functions and Generating Functions

Numeric Function

Definition

A numeric function a is written as $a_0, a_1, a_2, \cdots, a_r, \cdots$ to denote the values of the function at $0, 1, 2, \cdots, r, \cdots$

Example: $a_r = 7r^3 + 1, r \geq 0$.
Then, $a = (1, 8, 57, 190, 449, 876, 1513, 2402, 3585, 5104, 7001, \cdots)$

Numeric Function

Right Shift

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function and i be a positive integer.
- $S^i.a$ denotes a numeric function such that its value at r is 0 for $r = 0, 1, 2, \cdots, i-1$; and is a_{r-i} for $r \geq i$.
- If $b = S^i.a$, then

$$b_r = \begin{cases} 0, & 0 \leq r \leq i-1 \\ a_{r-i}, & r \geq i \end{cases}$$

- $S =$ shift; $S^i \leftarrow$ right shift

Numeric Function

Left Shift

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function and i be a positive integer.
- $S^{-i}.a$ denotes a numeric function such that its value at r is a_{r+i} for $r \geq 0$.
- If $c = S^{-i}.a$, then

$$c_r = a_{r+i}, r \geq 0.$$

Numeric Function

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Forward Difference

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function.
- The forward difference of a is defined as $\triangle a$.
- If $b = \triangle a$, then
$$b_r = a_{r+1} - a_r, r \geq 0.$$
Thus, we have:
$$\begin{aligned} b_0 &= a_1 - a_0 \\ b_1 &= a_2 - a_1 \\ b_2 &= a_3 - a_2 \\ &\vdots \end{aligned}$$

Numeric Function

Backward Difference

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function.
- The backward difference of a is defined as ∇a.
- If $c = \nabla a$, then

$$c_r = \begin{cases} a_0, & r = 0 \\ a_r - a_{r-1}, & r \geq 1 \end{cases}$$

  Thus, we have:

$$\begin{aligned} c_0 &= a_0 \\ c_1 &= a_1 - a_0 \\ c_2 &= a_2 - a_1 \\ \vdots & \quad \vdots \end{aligned}$$

Numeric Function

Problem: Let a be a numeric function such that

$$a_r = \begin{cases} 2, & 0 \leq r \leq 3 \\ 2^{-r} + 5, & r \geq 4 \end{cases}$$

(a) Determine $S^2 a$ and $S^{-2} a$.
(b) Determine $\triangle a$ and ∇a.

Convolution

Definition

Let a and b be two numeric functions. The *convolution* of a and b, defined by $a * b$, is a numeric function c such that $c = a * b$, where

$$\begin{aligned} c_r &= a_0 b_r + a_1 b_{r-1} + \cdots + a_{r-1} b_1 + a_r b_0 \\ &= \sum_{i=0}^{r} a_i b_{r-i}. \end{aligned}$$

Numeric Function

Problem: Consider the problem of determining c_r, the number of sequences of length r that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters.

Solution: Let $a_r =$ the number of sequences of length r that are made up from English letters $\{x, y, z\}$;
$b_r =$ the number of sequences of length r that are made up from Greek letters $\{\alpha, \beta\}$.
Then, we have,

$$\begin{aligned} a_r &= 3^r, r \geq 0 \\ b_r &= 2^r, r \geq 0 \end{aligned}$$

Then, for $c = a * b$, we have,

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} 3^i 2^{r-i}, r \geq 0.$$

Generating Function

Tests for Convergence

Whether an infinite series is convergent or not, the following tests are available (see `http://home.iitk.ac.in/~psraj/mth101/lecture_notes/Lecture11-13.pdf`):

- Comparison Test
- Cauchy Test
- Ratio Test
- Root Test
- Leibniz Test

Generating Function

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Definition

For a numeric function $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$, define an infinite series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

which is called generating function (G.F.) of the numeric function a and denoted by

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

provided that the series $A(z)$ is convergent, where z is a variable.

Generating Function

Properties

Let a, b, c be the numeric functions.

- If $a_r = z^r$, $r \geq 0$, then $A(z) = \frac{1}{1-z^2}$.
- If $b = \alpha a$, where α is a constant, then $B(z) = \alpha A(z)$.
- If $c = a + b$, then $C(z) = A(z) + B(z)$.
- If a is a numeric function and $A(z)$ is its generating function and $b_r = \alpha^r a_r$ for a numeric function b and α is a constant, then $B(z) = A(\alpha z)$.
- If $b = S^i.a$, then $B(z) = z^i.A(z)$
- If $c = S^{-i}.a$, then

$$C(z) = z^{-i}[A(z) - a_0 - a_1 z - a_2 z^2 - \cdots - a_{i-1} z^{i-1}]$$