Real Analysis

by Siddhartha & Diganta.

1st half

Assignments	20%
Quiz	10%
Mid-Sem	15%
Misc.	5%
50% → 1st ½	

Textbook suggestions:

1. Principles of Real Analysis
   - Aliprantis & Burkinshaw
2. Introduction to Real Analysis
   - Bartle & Sherbert
3. Principles of Mathematical Analysis
   - Rudin
4. A basic course in Real Analysis
   - Kumar & Kumaresan
5. A course in Calculus & Real Analysis
   - Ghorpade & Limaye
6. Mathematical methods for Physicists
   - Arfken & Weber

- Rapid review of set theory
- Countable & uncountable sets
- Real nos. (sequences, extended real nos.)
- Metric spaces (..., compactness)
- Briefly discuss topology, continuity.

Set Theory

+ To define a set as collection of objects leads to contradictions.*
+ In the foundation of set theory the notion of a set is left undefined.

We shall restrict ourselves to some set or space X fixed for a given discussion and considering only sets whose elements are ~~subsets~~ elements of X, or sets whose elements are subsets of X, or sets (families) whose elements are collections of subsets of X, and so forth.

So, a set is considered to be a collection of objects, viewed as a single entity.

* Russell's paradox: Let R be the set of all sets that are not members of themselves. I.e., $R = \{x : x \notin x\}$, then $R \in R \Leftrightarrow R \notin R$.

* Axiomatic Set theory: The existence of the power set of any set is postulated by the axiom of power set.

- Objects of a set A are called elements (or the members or the points) of A.

'$x \in A$' means x belongs to A, i.e., x is an element of set A.

'$x \notin A$' $\longrightarrow$ x doesn't belong to set A.

$\{a, b, c\}$ $\longrightarrow$ a set of elements a, b, c.

Singleton set $\rightarrow$ a set with only one element.

✓✓ Two sets A and B are said to be equal, i.e., $A = B$ iff A & B have precisely the same elements.

✓✓ A is called a subset of B i.e., $A \subseteq B$, if every element of A is also an element of B.

$A = B$ iff $A \subseteq B$ and $B \subseteq A$.

✓✓ If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B.

✓✓ The set without any element is called empty set.

Empty set : ϕ.

If A & B are two sets, then we define

(i) the union $A \cup B$ of A & B to be the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\};$$

(ii) the intersection $A \cap B$ of A & B to be the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\};$$

(iii) the set difference $A \setminus B$ of B from A to be the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

$A \setminus B$: the complement of B relative to A.

A and B are called disjoint sets if $A \cap B = \phi$.

For any sets A, B, C :

① $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;

② $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;

③ $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$;

④ $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$:

Identities ① & ② betn unions & intersections are referred as the distributive laws.

Proof of ① :

$x \in (A \cup B) \cap C \iff x \in A \cup B$ and $x \in C$

$\iff (x \in A$ or $x \in B)$ and $x \in C$

$\iff x \in A \cap C$ or $x \in B \cap C$

$\iff x \in (A \cap C) \cup (B \cap C)$

- Symmetric difference of the two sets A and B is defined as

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

- A family of sets is a nonempty set $\mathcal{F}$ whose members are sets by themselves.

For each element i of a nonempty set I, a subset A_i of a fixed set X is assigned, then $\{A_i\}_{i\in I}$ (or $\{A_i : i \in I\}$ or simply $\{A_i\}$) denotes the family whose members are the sets A_i. Nonempty set I is called the <u>index set</u> of the family and $i \in I$ are called indices.

If $\mathcal{F}$ is a family of sets, then by letting $I = \mathcal{F}$ and $A_i = i \;\forall\; i \in I$,

$\mathcal{F} = \{A_i\}_{i\in I}$.

If $\{A_i\}_{i\in I}$ is a family of sets, then the <u>union</u> of the family is defined to be the set

$$\bigcup_{i\in I} A_i = \{x : \exists\, i \in I \text{ s.t. } x \in A_i\},$$

& intersection of the family by

$$\bigcap_{i\in I} A_i = \{x : x \in A_i \;\forall\; i \in I\}.$$

The <u>distributive laws</u> for general families of sets now can be written as

$$\left(\bigcup_{i\in I} A_i\right) \cap B = \bigcup_{i\in I} (A_i \cap B) \; ;$$

$$\left(\bigcap_{i\in I} A_i\right) \cup B = \bigcap_{i\in I} (A_i \cup B) .$$

A family of sets $\{A_i\}_{i\in I}$ is called <u>pairwise disjoint</u> if for each pair i and j of distinct indices, $A_i \cap A_j = \phi$ for $i \neq j$.

The set of all subsets of a set A is called the <u>power set</u> of A $(\mathcal{P}(A))$.

$\phi \in \mathcal{P}(A)$, $A \in \mathcal{P}(A)$.

Consider a fixed set X. If $P(x)$ is a property (i.e., a well-defined logical sentence) involving $x \in X$, then the set of all $x \in X$ for which $P(x)$ is true is denoted by $\{x \in X : P(x)\}$.

If $A \subseteq X$, then its complement A^c (relative to X) is the set

$$A^c = X \setminus A := \{x \in X : x \notin A\}.$$

$$(A^c)^c = A \ ; \ A \cap A^c = \phi \ ; \ A \cup A^c = X.$$

⑤ $A \setminus B = A \cap B^c$;

⑥ $A \subseteq B$ iff $B^c \subseteq A^c$;

⑦ $(A \cup B)^c = A^c \cap B^c$;

⑧ $(A \cap B)^c = A^c \cup B^c$.

} De Morgan's laws.

Theorem (De Morgan's laws). For a family $\{A_i\}_{i \in I}$ of subsets of a set X, the following identities hold:

$$\Big(\bigcup_{i \in I} A_i\Big)^c = \bigcap_{i \in I} A_i^c \quad \text{and}$$

$$\Big(\bigcap_{i \in I} A_i\Big)^c = \bigcup_{i \in I} A_i^c.$$

Proof idea: $x \in \big(\bigcup_{i \in I} A_i\big)^c \iff x \notin \bigcup_{i \in I} A_i$

$\iff x \notin A_i \ \forall \ i \in I \iff x \in A_i^c \ \forall \ i \in I$

$\iff x \in \bigcap_{i \in I} A_i^c$