PROBLEM:- Using generating function, find the number of binary trees with 'n' nodes.

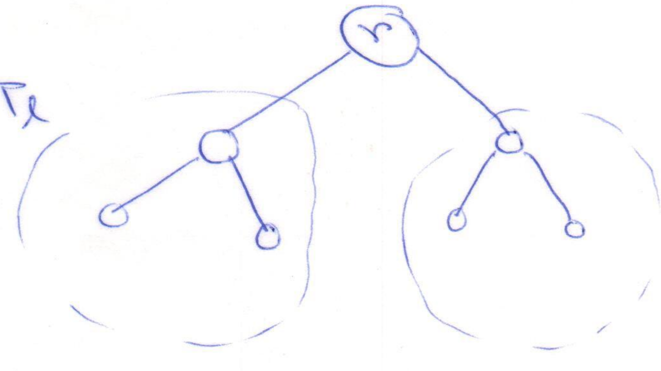
Solution:- A binary tree T_n with 'n' nodes is empty, if $n=0$.

If $n>0$, it is a triple (T_l, r, T_{n-l-1}) where $\underline{r}$ is a designated 'distinguished' node, called the root of the binary tree, T_n.

T_l is a binary tree of 'l' nodes, for some $l = 0, 1, \ldots, n-1$, called the left subtree of T_n.

T_{n-l-1} is a binary tree of '$n-l-1$' nodes, called the right subtree of the tree T_n.

T_l



T_{n-l-1}

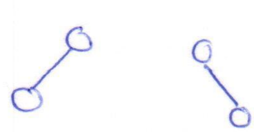
(n nodes)

Let x_n denote the number of distinct binary trees T_n of 'n' nodes.
From the definition of x_n, it follows that

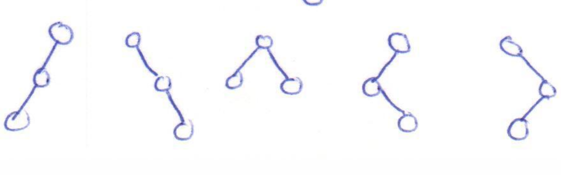
$x_0 = 1$ (empty tree),

$x_1 = \underline{1}$,

$x_2 = 2$,



$x_3 = 5$,



From the definition of T_n, we deduce the recurrence relation:

$$x_n = \sum_{l=0}^{n-1} x_l \, x_{n-l-1} \quad \cdots\cdots (1)$$

Consider the generating function:

$$X(s) = \sum_{r=0}^{\infty} x_r s^r.$$

$$\therefore \; X^2(s) = \left(\sum_{r=0}^{\infty} x_r s^r \right)^2$$

$$= \left(\sum_{r=0}^{\infty} x_r s^r \right) \cdot \left(\sum_{l=0}^{\infty} x_l s^l \right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} x_l \, x_{k-l} \right) s^k$$

$$\cdots\cdots\cdots (2)$$

We have the convolution that if

$$z_k = \sum_{i=0}^{k} x_i \, y_{k-i}, \quad k = 0, 1, 2, \cdots,$$

then

$$Z(s) = X(s) \cdot Y(s).$$

Substitute $Y(s) = X(s)$. Then we have,

$$X(s) \cdot X(s) = \sum_{r=0}^{\infty} z_r s^r$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} x_i \cdot x_{k-i} \right) s^k$$

$$[\because \; y_{k-i} = x_{k-i}]$$

$$\cdots\cdots (3)$$

We have $x_k = \sum_{l=0}^{k-1} x_l \, x_{k-l-1}$

$$\Rightarrow \quad x_{k+1} = \sum_{l=0}^{k} x_l \, x_{k-l} \quad \cdots\cdots (4)$$

Then using (4), from (2) & (3), we obtain,

$$X^2(s) = \sum_{k=0}^{\infty} x_{k+1} s^k$$

$$= \left(\sum_{k=-1}^{\infty} x_{k+1} s^{k+1} \right) \cdot \frac{1}{s} - x_0 \cdot \frac{1}{s}$$

$$= \frac{1}{s} \cdot \sum_{k=0}^{\infty} x_k s^k - \frac{x_0}{s} \quad [\because x_0 = 1]$$

$$= \frac{1}{s} \cdot X(s) - \frac{1}{s}$$

$$\Rightarrow \quad s X^2(s) - X(s) + 1 = 0 \quad \cdots\cdots (5)$$

This is a quadratic equation in $X(s)$.

$$\therefore \quad X(s) = \frac{+1 \pm \sqrt{1-4s}}{2s} \quad \cdots\cdots (6)$$

The binomial expansion of $\sqrt{1-4s}$ gives us:

$$(1-4s)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4s)^k$$

$$= 1 + \frac{1}{2} \cdot (-4s) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} (-4s)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} (-4s)^3 + \cdots\cdots$$

$$= 1 - \frac{1}{2} \cdot 4s + \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)}{2!} (4s)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} (4s)^3 + \cdots$$

$$= 1 - \frac{1}{2} \cdot (4s) - \frac{\frac{1}{2} \cdot \frac{1}{2}}{2!} (4s)^2 - \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2}}{3!} (4s)^3 - \cdots$$

The coefficient of s^k, $k \geq 1$, in this series can be written as

$$-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\cdots\left(\frac{1}{2}-\overline{k-1}\right)}{k!}(4)^k$$

$$\left[\because \frac{1}{2}-\overline{k-1} = \frac{3}{2}-k = \frac{3-2k}{2} = -\frac{2k-3}{2}\right]$$

$$= -\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2k-3}{2}\right)}{k!}4^k$$

$$= -\frac{\left(\frac{1}{2}\right)^k \cdot 1\cdot 3\cdot 5\cdot\cdots(2k-3)}{k!}\left(4^k\right)$$

$$= -\frac{1\cdot 3\cdot 5\cdot\cdots(2k-3)}{2^k\cdot k!}\times 2^{2k}$$

$$= -\frac{1\cdot 3\cdot 5\cdot\cdots(2k-3)}{k\,(k-1)!}\times 2^k$$

$$= -\frac{2}{k}\left(\frac{1\cdot 3\cdot 5\cdot\cdots(2k-3)\cdot 2^{k-1}}{(k-1)!}\right)$$

$$= -\frac{2}{k}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots(2k-3)\cdot 2^{k-1}\cdot (k-1)!}{(k-1)!\times(k-1)!}$$

$$= -\frac{2}{k}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots(2k-3)\cdot\left[\overbrace{2\cdot 2\cdots 2}^{(k-1)\text{ times}}\cdot\overbrace{(k-1)(k-2)\cdots 3\cdot 2\cdot 1}^{(k-1)\text{ times}}\right]}{(k-1)!\;(k-1)!}$$

$$= -\frac{2}{k}\cdot\frac{1\cdot 3\cdot 5\cdots(2k-3)\cdot 2(k-1)\cdot 2(k-2)\cdots(2\cdot 3)\cdot(2\cdot 2)\cdot(2\cdot 1)}{(k-1)!\;(k-1)!}$$

$$= -\frac{2}{k}\cdot\frac{(2k-2)(2k-3)(2k-4)\cdots\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}{(k-1)!\;(k-1)!}$$

$$= -\left(\frac{2}{k}\right) \cdot \frac{(2k-2)!}{(k-1)!\,(k-1)!}$$

$$= -\frac{2}{k} \cdot \binom{2k-2}{k-1}$$

so that $(1-4s)^{1/2} = 1 - \sum_{k=1}^{\infty} \left(\frac{2}{k}\right)\binom{2k-2}{k-1} s^k$ .... (7)

Since x_k represents the number of the binary trees T_k of 'k' nodes, so we choose the sign of $X(s)$ so that x_n becomes +ve.

Hence, taking

$$X(s) = \frac{1-\sqrt{1-4s}}{2s}, \text{ we have}$$

$$X(s) = \frac{1}{2s} \sum_{k=1}^{\infty} \left(\frac{2}{k}\right)\binom{2k-2}{k-1} s^k$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \cdot \binom{2k-2}{k-1} s^{k-1} \quad (\text{Put } k-1=n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \binom{2n}{n} s^n$$

$$= \sum_{n=0}^{\infty} x_n s^n$$

$$\Rightarrow \boxed{x_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0} \text{ ------ (8)}$$

Using the Stirling's formula,

$n! \simeq \sqrt{2\pi} \cdot e^{-n} \cdot n^{n+1/2}$, from (8), we obtain

$$x_n = \frac{1}{n+1} \cdot \frac{(2n)!}{n!\, n!}$$

$$\simeq \frac{1}{n+1} \cdot \frac{\sqrt{2\pi} \cdot e^{-2n} \cdot (2n)^{2n+1/2}}{\sqrt{2\pi} \cdot e^{-n} \cdot n^{n+1/2} \cdot \sqrt{2\pi} \cdot e^{-n} \cdot n^{n+1/2}}$$

$$= \frac{1}{n+1} \cdot \frac{2^{2n+1/2} \cdot n^{2n+1/2}}{\sqrt{2\pi} \cdot n^{2n+1}}$$

$$= \frac{1}{n+1} \cdot \frac{2^{2n}}{\sqrt{\pi} \cdot n^{1/2}}$$

$$= \frac{1}{n+1} \cdot \frac{4^n}{\sqrt{\pi n}}$$

i.e, $$\boxed{x_n \approx \frac{4^n}{n\sqrt{\pi n}} \quad \text{as } n \to \infty} \quad \ldots\ldots (9)$$

Again, we can re-write above x_n as

$$x_n = \frac{1}{n+1} \cdot \frac{4^n}{\sqrt{\pi n}}$$

$$= \frac{4^n}{\sqrt{\pi} \cdot \sqrt{n}} \cdot \frac{1}{n} \cdot \frac{1}{(1+1/n)}$$

$$= \frac{4^n}{\sqrt{\pi} \cdot n^{3/2}} \left(1 + 1/n\right)^{-1}$$

$$= \frac{4^n}{n^{3/2} \cdot \sqrt{\pi}} \left(1 - 1/n + 1/n^2 - \cdots\cdots\right)$$

$$= \left(\frac{4^n}{n^{3/2} \cdot \sqrt{\pi}}\right)\left(1 + O(1/n)\right) \ldots\ldots (10). \quad \square$$