Lecture 3, 27/12/22

For $t_1 = 0$ and $t_2 = 2\pi$ the particle makes one full rotation - shown by points A & B on the plot. Total arc length is given by $s = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$.

Integration of Vectors: Two points to remember

1. The integral and the integrand have the same nature
2. For a indefinite integral, the constant of integration has the same nature as the integral.

For indefinite integral

$$\int \vec{a}(u)\, du = \vec{A}(u) + \vec{b} \qquad \left\{ \begin{array}{l} \text{Assuming that} \\ \vec{a}(u) = d\vec{A}(u)/u \end{array} \right.$$

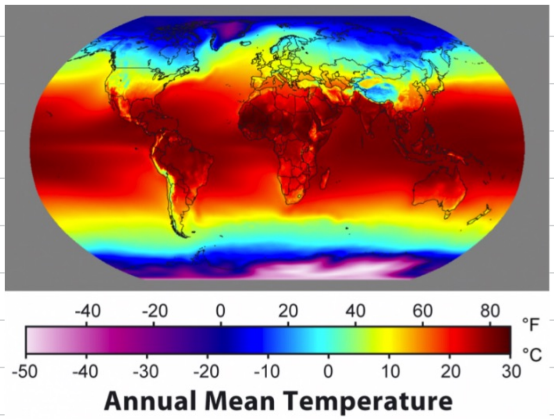
For definite integral

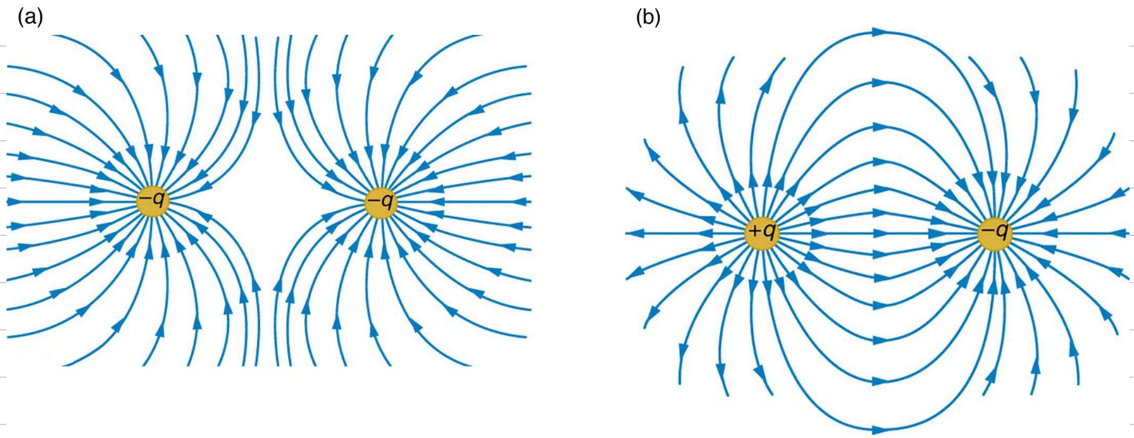
$$\int_{u_1}^{u_2} \vec{a}(u) du = \vec{A}(u_2) - \vec{A}(u_1)$$

Scalar and Vector Field: A scalar field, that is defined in a given region of space, is a continuous function that gives a number at each point in that region.
We denote a scalar field by $\phi(x, y, z)$ which is a function of the

coordinates x, y, z. Example of scalar field is temperature $T(x, y, z)$ in a given region, ~~say~~ inside a room, surface of earth etc.

A vector field is a continuous function that is defined in a given region of space, and associates a vector quantity at each point in that region. We denote a vector field by $\vec{v}(x, y, z)$. Example of vector field is velocity vector field in a flowing river or ocean, electric field created by a static charge, magnetic field of a magnet or flowing electric current, etc.





Line Integral in Scalar Field: There is a scalar field $\phi(x,y,z)$ and we want to integrate $\phi(x,y,z)$ along a line or a curve that is parametrized by a scalar variable u. The line/curve is defined by the position vector

$$\vec{r}(u) = \hat{i}\, r_x(u) + \hat{j}\, r_y(u) + \hat{k}\, r_z(u)$$

The integral is given by

$$\int \phi(x,y,z)\,|d\vec{r}(u)| = \int \phi(x,y,z) \left| \frac{d\vec{r}(u)}{du} \right| du = \int \phi(x,y,z)\, \frac{ds}{du}\, du = \int \phi(x,y,z)\, ds$$

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Here $|d\vec{r}(u)/du| = ds/du$ is the length of the tangent vector to the curve.

Example: You want to integrate the temperature function on the surface of the earth along the coast line of India.

Example: Find the line integral of $\phi(x,y,z) = (x^2+y^2+z)$ over a curve defined by $\vec{r}(u) = \hat{i}\cos u + \hat{j}\sin u + u$, for $0 < u < 2\pi$

Sol^n: The parameter of the curve is u. First we express the integrand in terms of u: $\phi(x,y,z) = x^2+y^2+z = (\cos u)^2 + (\sin u)^2 + u = 1+u.$

$$\frac{ds}{du} = \left|\frac{d\vec{r}}{du}\right| = \sqrt{(-\sin u)^2 + (\cos u)^2 + 1} = \sqrt{2}.$$

Hence substituting in the formula

$$\int (x^2+y^2+z)\,ds = \int_0^{2\pi} (1+u)\sqrt{2}\,du = \left[\sqrt{2}u + \frac{\sqrt{2}u^2}{2}\right]_0^{2\pi} = 2\pi\sqrt{2}(1+\pi)$$

Example: Consider a semicircle of radius a on X-Y plane. Find the length.

Soln: The semi-circle can be parametrized by $x = a\cos\theta$, $y = a\sin\theta$ where $0<\theta<\pi$.

The length is $\displaystyle\int 1\,|d\vec{r}(\theta)| = \int 1\,ds = \int_0^{\pi} 1\,\frac{ds}{d\theta}\,d\theta$

Now $\vec{r} = \hat{i}\,a\cos\theta + \hat{j}\,a\sin\theta \Rightarrow \dfrac{ds}{d\theta} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$

Hence length $= \displaystyle\int_0^{\pi} a\,d\theta = a\pi.$

Line Integral in Vector Field: Consider a force field $\vec{F}(x,y,z)$. The line integral of $\vec{F}(x,y,z)$ along a line $\vec{r}(u)$, where u is a scalar that parametrizes the line is given by

$$\int \vec{F}(x,y,z) \cdot d\vec{r}(u) = \int \vec{F} \cdot \frac{d\vec{r}}{du}\, du$$

Example where such line integral may arise: You want to calculate the work done to move a point charge q in a given electric potential along a given path.

Example: Let $\vec{F} = \hat{i}\, xe^{y} + \hat{j} z^2 + \hat{k}\, xy$. The line along which the integration to be performed is $\vec{r}(u) = \hat{i} u + \hat{j} u^2 + \hat{k} u^3$ for $0 < u < 1$.

Soln: The force field along the curve is $\vec{F}(\vec{r}(u)) = \hat{i}\, ue^{u^2} + \hat{j} u^6 + \hat{k} u^3$

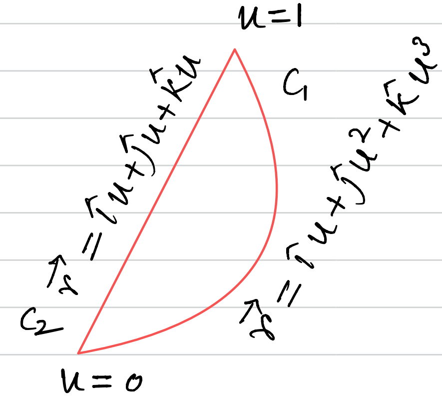
So the integral is

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{du}\, du = \int_0^1 \left(\hat{i}\, ue^{u^2} + \hat{j} u^6 + \hat{k} u^3\right) \cdot \left(\hat{i} + \hat{j}\, 2u + 3u^2\right) du$$

$$= \int_0^1 \left(ue^{u^2} + 2u^7 + 3u^5\right) du = \frac{1}{4}(1+3e)$$

Example: We do the previous integral again but for curve

$$\vec{r}(u) = \hat{i}u + \hat{j}u + \hat{k}u \quad \text{for } 0<u<1$$



The answer is given by

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \left(\hat{i}\,ue^u + \hat{j}\,u^2 + \hat{k}\,u^2\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) du$$

$$= \int_0^1 \left(ue^u + 2u^2\right) = 5/3.$$

* Note that the integration result is different for different curves. This is because in case of vector fields the integration depends on the path along which the integration is done.

* Are there vector fields in which the line integral is independent of the path and only depends on the end points? - To answer this question we need the concepts of (i) partial derivatives (ii) gradiant of a scalar field.

Partial Derivatives: A function $f = f(x, y)$ is function of x & y.

Partial Derivative w.r.to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

A similar definition for $\partial f / \partial y$.

Example: $f(x, y) = 2x^3y^2 + y^2$ Then we have

$$\frac{\partial f}{\partial x} = 6x^2y^2 \quad ; \quad \frac{\partial f}{\partial y} = 4x^3y + 2y$$

2nd derivatives: $\frac{\partial^2 f}{\partial x^2} = 12xy^2 \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 4x^3 + 2$

Mixed derivatives: $\frac{\partial^2 f}{\partial y \partial x} = 12x^2y \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = 12x^2y = \frac{\partial^2 f}{\partial y \partial x}$

This is generalized to functions of more than two variables.

Chain Rule: Let a function $f = f(x, y)$. The total differential of the function is given by

$$df = f(x+dx, y+dy) - f(x, y)$$

using partial derivative we can write

$$df = [f(x+dx, y+dy) - f(x, y+dy)] + [f(x, y+dy) - f(x,y)]$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \begin{cases} \partial f/\partial x \text{ \& } \partial f/\partial y \text{ evaluated at some point between} \\ x \text{ \& } x+dx, \; y \text{ \& } y+dy \end{cases}$$

If $f = f(x(u), y(u))$ then the derivative w.r.t. to u can be written as

$$\frac{df}{du} = \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du}$$

If x & y depend on two parameters r, θ, ie for $f = f(x(r,\theta), y(r,\theta))$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \quad \& \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

Example: Given that $\frac{dx}{dt} = u(t, x(t))$. Determine $\frac{d^2x}{dt^2}$ in terms of u and its partial derivatives.

Soln: $$\frac{d^2x}{dt^2} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$$