The n^{th} derivative at z_0 can also be obtained (given that the derivatives exist at that point) as

$$f^n(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial z_0^n}\left(\frac{f(z)}{z-z_0}\right)dz \tag{105}$$

Example 1: Given that C is a positively oriented, *i.e.,* counterclockwise circle $|z| = 2$. Show that

$$\oint_C \frac{zdz}{(9-z^2)(z+i)} = \frac{\pi}{5}$$

Ans: As $|z| = 2$ only $z = i$ is the interior point. Hence we can write

$$\oint_C \frac{zdz}{(9-z^2)(z+i)} \quad = \quad \oint_C \frac{z/(9-z^2)}{(z-(-i))}dz = 2\pi i\frac{-i}{10} = \frac{\pi}{5}$$

Example 2: If C is a positively oriented unit circle $|z| = 1$, and function $f(z) = exp(2z)$ is a analytic function, then show that

$$\oint_C \frac{exp(2z)}{z^4}dz = \frac{8\pi i}{3} \tag{106}$$

Ans: From the Cauchy integral formula

$$\oint_C \frac{exp(2z)}{z^4} = \oint_C \frac{exp(2z)}{(z-z_0)^{3+1}} = \frac{2\pi i}{3!}f^3(0) = \frac{8\pi i}{3}$$

4.1 Taylor Series

If $f(z)$ is any function that is analytic inside and on a circle C of radius R centred at the a point $z = z_0$, and z is a point inside C, then

$$f(z) = \sum_{n=0}^{n=\infty} a_n(z-z_0)^n \ , \tag{107}$$

where the coefficients a_n are given by $f^{(n)}(z_0)/n!$. This is called the Taylor series that is valid inside the region of analyticity and it can be shown that for a point z_0 this series is unique.

Proof: To prove the theorem we note that from Cauchy formula that if $f(z)$ is analytic inside and on C then we may write

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z}d\xi \, , \tag{108}$$

where ξ lies on C. We can expand $(\xi - z)^{-1}$ as a series in $(z - z_0)/(\xi - z_0)$

$$\begin{aligned}
\frac{1}{\xi - z} &= \frac{1}{(\xi - z_0) - (z - z_0)} \, , \\
&= \frac{1}{(\xi - z_0)\Big[1 - (z - z_0)/(\xi - z_0)\Big]} \, , \\
&= \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n \, ,
\end{aligned} \tag{109}$$

Substituting this in the Cauchy formula, we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n d\xi\,, \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi\,, \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \frac{2\pi i}{n!}f^n(z_0)\,, \\
&= \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^n(z_0)}{n!}\,, \qquad (110)
\end{aligned}$$

Hence proved. For $z_0 = 0$ this is called Maclurian series.

4.1.1 Convergence Theorm

Given a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (111)$$

There is a number $R \geq 0$ such that

1. If $R > 0$ then the series converges absolutely to an analytic function for $|z - z_0| < R$
2. The series diverges for $|z-z_0| > R$. Here R is called the radius of convergence and the disk $|z-z_0| < R$ is called the disk of convergence.

If $R = \infty$ the function is called entire function. For $R = 0$ the series only converges at the point $z = z_0$. In this case the series does not represent an anlytic function on any disk around z_0. The theorem does not say anything about what happened $|z - z_0| = R$. Often, but not always, one can find R.

Example: Show that for $|z| < \infty$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}\,,$$

Ans: Here the function is $f(z) = e^z$. This function is analytic everywhere in the complex plane. The n-th derivative $f^n(0) = 1$. Hence for $z_0 = 0$ we can write

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}\,. \qquad (112)$$

Note that for $z = x + i0$ this gives the usual Taylor series expansion

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,. \qquad (113)$$

Exercise: Show that for $|z| < \infty$

$$\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\,, \qquad \cos z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} \qquad (114)$$

Use the series expansion of e^z or calculate the n-th derivatives.

Example: Show that for $|z| < 1$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty}$$

Ans: The n-th derivative of the function is

$$f^n(z) = \frac{n!}{(1-z)^{n+1}} , \quad n = 0, 1, 2, 3... \tag{115}$$

Henece $f^n(0) = n!$. Substituting in the Taylor/Maclurian formula

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n ,$$

Example: Expand the function

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5} ,$$

around $z = 0$. Ans: The function f has ingularity at $z = 0$. So convergent Taylor series not expected. We write the funtion as

$$f(z) = \frac{1}{z^3} \frac{2(1+z^2) - 1}{1+z^2} = \frac{1}{z^3} \left[2 - \frac{1}{1+z^2} \right]$$

Now, from the previous result for $1/(1-z)$ we get

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = 1 - z^2 + z^4 - z^6 + ... \tag{116}$$

Hence the final result is

$$f(z) = \frac{1}{z^3}(2 - 1 + z^2 - z^4 +) \tag{117}$$

The series is valid for $0 < |z| < 1$.

Example: Radius of Convergence: Given the function

$$f(z) = \frac{e^z}{1-z} ,$$

Find the Taylor series expansion about $z = 0$ and give the radius of convergence.

Ans: We already know the series expansion of the numerator and the denominator individually. We use them to get

$$f(z) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \right) \left(1 + z + z^2 + z^3 + ... \right) .$$

The biggest disk that one can have around $z = 0$ where f is analytic is $|z| < 1$. Therefore, the radius of convergence is $R = 1$.

Example: What is the radius of convergence for

$$f(z) = \frac{1}{1-z} ,$$

around $z = 5$?

Ans: Note that $z = 1$ has singularity. So the radius of convergence is $R = 4$.

4.2 Singularities of complex function

A singular point $z = z_0$ of a function $f(z)$ is point in the complex plane where the function fails to be analytic. There are following types of singularities

- **Isolated Singularity:** If a function $f(z)$ has singular point $z = z_0$, but the function is analytic at all points in some neighbourhood containing z_0 then it is called a isolated singularity. The most important type of singularity is called *pole*. If a function $f(z)$ has the form

$$f(z) = \frac{g(z)}{(z-z_0)^n} ,$$

where n is a positive integer, $g(z)$ is analytic at all points in some neighborhood containing $z = z_0$ and $g(z) \neq 0$, then $f(z)$ is said to have a pole of order n at $z = z_0$. Pole of order 1 is also called simple pole.

- **Essential Singularity:** If for a function

$$f(z) = \frac{g(z)}{(z - z_0)^n} \, ,$$

there is no finite value of n such that

$$\lim_{z \to z_0} \Big[(z - z_0)^n f(z) \Big] \tag{118}$$

is finite, then the function $f(z)$ has essential singularity at $z = z_0$.

- **Removable Singularity:** If at $z = z_0$ the function $f(z)$ has a $0/0$ form but $\lim_{z \to z_0} f(z)$ exist and is independent of the approach to z_0 then it is a removable singularity.

Examples: Find the singularities of the following tow functions

$$i) \;\; f(z) = \frac{1}{1 - z} - \frac{1}{1 + z} \, , \qquad ii) f(z) = \tanh(z) \, .$$

Ans: The first function has two poles at $z = 1$ and $z = -1$ and each of them are of order 1. For the second function, one can write

$$f(z) = \tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \, .$$

The singularity appears when $e^z = -e^{-z}$ or equivalently

$$e^z = e^{i(2n+1)\pi} e^{-z} \, ,$$

Equating the arguments we get $z = (n + 1/2)\pi i$ for all integers n. To find the order of the pole we calculate the limit

$$\lim_{z \to (n+1/2)\pi i} \big[z - (n + \frac{1}{2})\pi i \big] \frac{\sinh z}{\cosh z} = 1 \, ,$$

Hence it is a simple pole.

Example: Show that $f(z) = \sin z / z$ has a removable singularity at $z = 0$.

Ans: Note that at $z = 0$ the function has a $0/0$ form. But the limit $\lim_{z \to 0} \sin z / z = 1$. Hence it is removable. One can calculate the limit by expanding the function in z around 0.

Example: The function $f(z) = e^{1/z}$ has essential singularity at $z = 0$ since

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + ... \tag{119}$$

has infinitely many terms that are singular at $z = 0$.

4.2.1 Laurent Series:

So far we have discussed functions that are analytic within a disk of convergence. However, if a function has singularity at $z = z_0$ then Taylor series expansion can not be done.

References

[1]