3

Complex numbers and hyperbolic functions

This chapter is concerned with the representation and manipulation of complex numbers. Complex numbers pervade this book, underscoring their wide application in the mathematics of the physical sciences. The application of complex numbers to the description of physical systems is left until later chapters and only the basic tools are presented here.

3.1 The need for complex numbers

Although complex numbers occur in many branches of mathematics, they arise most directly out of solving polynomial equations. We examine a specific quadratic equation as an example.

Consider the quadratic equation

$$z^2 - 4z + 5 = 0. \tag{3.1}$$

Equation (3.1) has two solutions, z_1 and z_2, such that

$$(z - z_1)(z - z_2) = 0. \tag{3.2}$$

Using the familiar formula for the roots of a quadratic equation, (1.4), the solutions z_1 and z_2, written in brief as $z_{1,2}$, are

$$\begin{aligned} z_{1,2} &= \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2} \\ &= 2 \pm \frac{\sqrt{-4}}{2}. \end{aligned} \tag{3.3}$$

Both solutions contain the square root of a negative number. However, it is not true to say that there are no solutions to the quadratic equation. The *fundamental theorem of algebra* states that a quadratic equation will always have two solutions and these are in fact given by (3.3). The second term on the RHS of (3.3) is called an *imaginary* term since it contains the square root of a negative number;

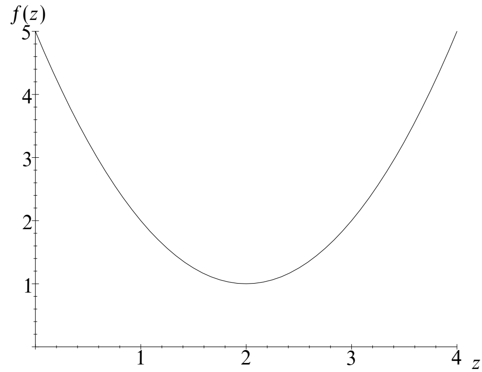

Figure 3.1 The function $f(z) = z^2 - 4z + 5$.

the first term is called a *real* term. The full solution is the sum of a real term and an imaginary term and is called a *complex number*. A plot of the function $f(z) = z^2 - 4z + 5$ is shown in figure 3.1. It will be seen that the plot does not intersect the z-axis, corresponding to the fact that the equation $f(z) = 0$ has no purely real solutions.

The choice of the symbol z for the quadratic variable was not arbitrary; the conventional representation of a complex number is z, where z is the sum of a real part x and i times an imaginary part y, i.e.

$$z = x + iy,$$

where i is used to denote the square root of -1. The real part x and the imaginary part y are usually denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ respectively. We note at this point that some physical scientists, engineers in particular, use j instead of i. However, for consistency, we will use i throughout this book.

In our particular example, $\sqrt{-4} = 2\sqrt{-1} = 2i$, and hence the two solutions of (3.1) are

$$z_{1,2} = 2 \pm \frac{2i}{2} = 2 \pm i.$$

Thus, here $x = 2$ and $y = \pm 1$.

For compactness a complex number is sometimes written in the form

$$z = (x, y),$$

where the components of z may be thought of as coordinates in an xy-plot. Such a plot is called an *Argand diagram* and is a common representation of complex numbers; an example is shown in figure 3.2.

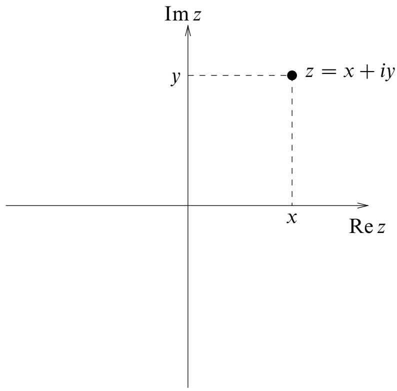

Figure 3.2 The Argand diagram.

Our particular example of a quadratic equation may be generalised readily to polynomials whose highest power (degree) is greater than 2, e.g. cubic equations (degree 3), quartic equations (degree 4) and so on. For a general polynomial $f(z)$, of degree n, the fundamental theorem of algebra states that the equation $f(z) = 0$ will have exactly n solutions. We will examine cases of higher-degree equations in subsection 3.4.3.

The remainder of this chapter deals with: the algebra and manipulation of complex numbers; their polar representation, which has advantages in many circumstances; complex exponentials and logarithms; the use of complex numbers in finding the roots of polynomial equations; and hyperbolic functions.

3.2 Manipulation of complex numbers

This section considers basic complex number manipulation. Some analogy may be drawn with vector manipulation (see chapter 7) but this section stands alone as an introduction.

3.2.1 Addition and subtraction

The addition of two complex numbers, z_1 and z_2, in general gives another complex number. The real components and the imaginary components are added separately and in a like manner to the familiar addition of real numbers:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

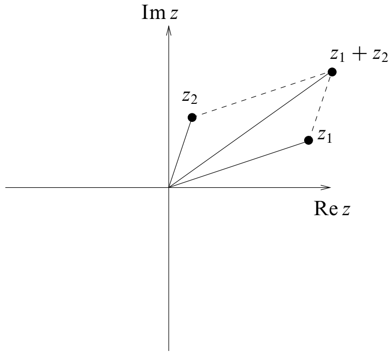

Figure 3.3 The addition of two complex numbers.

or in component notation

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

The Argand representation of the addition of two complex numbers is shown in figure 3.3.

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is itself commutative and associative, i.e.

$$z_1 + z_2 = z_2 + z_1,$$
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

Thus it is immaterial in what order complex numbers are added.

►*Sum the complex numbers* $1 + 2i$, $3 - 4i$, $-2 + i$.

Summing the real terms we obtain

$$1 + 3 - 2 = 2,$$

and summing the imaginary terms we obtain

$$2i - 4i + i = -i.$$

Hence

$$(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i. \blacktriangleleft$$

The subtraction of complex numbers is very similar to their addition. As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

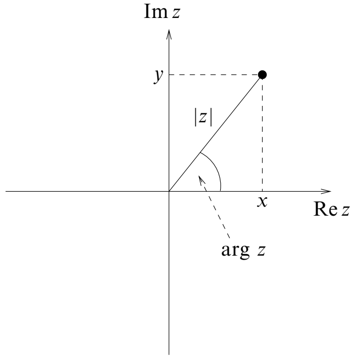

Figure 3.4 The modulus and argument of a complex number.

3.2.2 Modulus and argument

The modulus of the complex number z is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2}. \tag{3.4}$$

Hence the modulus of the complex number is the distance of the corresponding point from the origin in the Argand diagram, as may be seen in figure 3.4.

The argument of the complex number z is denoted by $\arg z$ and is defined as

$$\arg z = \tan^{-1}\left(\frac{y}{x}\right). \tag{3.5}$$

It can be seen that $\arg z$ is the angle that the line joining the origin to z on the Argand diagram makes with the positive x-axis. The anticlockwise direction is taken to be positive by convention. The angle $\arg z$ is shown in figure 3.4. Account must be taken of the signs of x and y individually in determining in which quadrant $\arg z$ lies. Thus, for example, if x and y are both negative then $\arg z$ lies in the range $-\pi < \arg z < -\pi/2$ rather than in the first quadrant ($0 < \arg z < \pi/2$), though both cases give the same value for the ratio of y to x.

►*Find the modulus and the argument of the complex number $z = 2 - 3i$.*

Using (3.4), the modulus is given by

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$$

Using (3.5), the argument is given by

$$\arg z = \tan^{-1}\left(-\tfrac{3}{2}\right).$$

The two angles whose tangents equal -1.5 are -0.9828 rad and 2.1588 rad. Since $x = 2$ and $y = -3$, z clearly lies in the fourth quadrant; therefore $\arg z = -0.9828$ is the appropriate answer. ◄

3.2.3 Multiplication

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers z_1 and z_2 is found by multiplying them out in full and remembering that $i^2 = -1$, i.e.

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned} \tag{3.6}$$

► *Multiply the complex numbers $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.*

By direct multiplication we find

$$\begin{aligned} z_1 z_2 &= (3 + 2i)(-1 - 4i) \\ &= -3 - 2i - 12i - 8i^2 \\ &= 5 - 14i. \blacktriangleleft \end{aligned} \tag{3.7}$$

The multiplication of complex numbers is both commutative and associative, i.e.

$$z_1 z_2 = z_2 z_1, \tag{3.8}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3). \tag{3.9}$$

The product of two complex numbers also has the simple properties

$$|z_1 z_2| = |z_1||z_2|, \tag{3.10}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{3.11}$$

These relations are derived in subsection 3.3.1.

► *Verify that (3.10) holds for the product of $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.*

From (3.7)

$$|z_1 z_2| = |5 - 14i| = \sqrt{5^2 + (-14)^2} = \sqrt{221}.$$

We also find

$$\begin{aligned} |z_1| &= \sqrt{3^2 + 2^2} = \sqrt{13}, \\ |z_2| &= \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}, \end{aligned}$$

and hence

$$|z_1||z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} = |z_1 z_2|. \blacktriangleleft$$

We now examine the effect on a complex number z of multiplying it by ± 1 and $\pm i$. These four multipliers have modulus unity and we can see immediately from (3.10) that multiplying z by another complex number of unit modulus gives a product with the same modulus as z. We can also see from (3.11) that if we

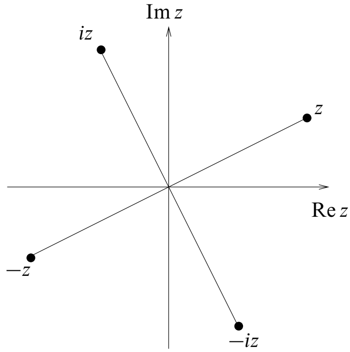

Figure 3.5 Multiplication of a complex number by ± 1 and $\pm i$.

multiply z by a complex number then the argument of the product is the sum of the argument of z and the argument of the multiplier. Hence multiplying z by unity (which has argument zero) leaves z unchanged in both modulus and argument, i.e. z is completely unaltered by the operation. Multiplying by -1 (which has argument π) leads to rotation, through an angle π, of the line joining the origin to z in the Argand diagram. Similarly, multiplication by i or $-i$ leads to corresponding rotations of $\pi/2$ or $-\pi/2$ respectively. This geometrical interpretation of multiplication is shown in figure 3.5.

►*Using the geometrical interpretation of multiplication by i, find the product* $i(1-i)$.

The complex number $1 - i$ has argument $-\pi/4$ and modulus $\sqrt{2}$. Thus, using (3.10) and (3.11), its product with i has argument $+\pi/4$ and unchanged modulus $\sqrt{2}$. The complex number with modulus $\sqrt{2}$ and argument $+\pi/4$ is $1 + i$ and so

$$i(1-i) = 1 + i,$$

as is easily verified by direct multiplication. ◄

The division of two complex numbers is similar to their multiplication but requires the notion of the complex conjugate (see the following subsection) and so discussion is postponed until subsection 3.2.5.

3.2.4 Complex conjugate

If z has the convenient form $x + iy$ then the complex conjugate, denoted by z^*, may be found simply by changing the sign of the imaginary part, i.e. if $z = x + iy$ then $z^* = x - iy$. More generally, we may define the complex conjugate of z as the (complex) number having the same magnitude as z that when multiplied by z leaves a real result, i.e. there is no imaginary component in the product.

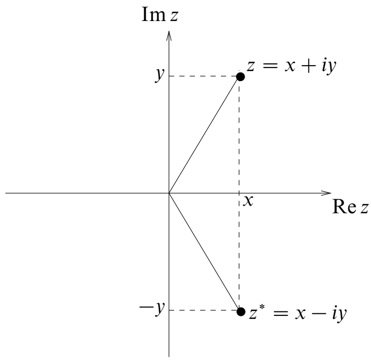

Figure 3.6 The complex conjugate as a mirror image in the real axis.

In the case where z can be written in the form $x + iy$ it is easily verified, by direct multiplication of the components, that the product zz^* gives a real result:

$$zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 = |z|^2.$$

Complex conjugation corresponds to a reflection of z in the real axis of the Argand diagram, as may be seen in figure 3.6.

►*Find the complex conjugate of* $z = a + 2i + 3ib$.

The complex number is written in the standard form

$$z = a + i(2 + 3b);$$

then, replacing i by $-i$, we obtain

$$z^* = a - i(2 + 3b). \blacktriangleleft$$

In some cases, however, it may not be simple to rearrange the expression for z into the standard form $x + iy$. Nevertheless, given two complex numbers, z_1 and z_2, it is straightforward to show that the complex conjugate of their sum (or difference) is equal to the sum (or difference) of their complex conjugates, i.e. $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$. Similarly, it may be shown that the complex conjugate of the product (or quotient) of z_1 and z_2 is equal to the product (or quotient) of their complex conjugates, i.e. $(z_1z_2)^* = z_1^*z_2^*$ and $(z_1/z_2)^* = z_1^*/z_2^*$.

Using these results, it can be deduced that, no matter how complicated the expression, its complex conjugate may *always* be found by replacing every i by $-i$. To apply this rule, however, we must always ensure that all complex parts are first written out in full, so that no i's are hidden.

►*Find the complex conjugate of the complex number $z = w^{(3y+2ix)}$, where $w = x + 5i$.*

Although we do not discuss complex powers until section 3.5, the simple rule given above still enables us to find the complex conjugate of z.

In this case w itself contains real and imaginary components and so must be written out in full, i.e.

$$z = w^{3y+2ix} = (x + 5i)^{3y+2ix}.$$

Now we can replace each i by $-i$ to obtain

$$z^* = (x - 5i)^{(3y-2ix)}.$$

It can be shown that the product zz^* is real, as required. ◄

The following properties of the complex conjugate are easily proved and others may be derived from them. If $z = x + iy$ then

$$(z^*)^* = z, \tag{3.12}$$

$$z + z^* = 2\,\mathrm{Re}\,z = 2x, \tag{3.13}$$

$$z - z^* = 2i\,\mathrm{Im}\,z = 2iy, \tag{3.14}$$

$$\frac{z}{z^*} = \left(\frac{x^2 - y^2}{x^2 + y^2}\right) + i\left(\frac{2xy}{x^2 + y^2}\right). \tag{3.15}$$

The derivation of this last relation relies on the results of the following subsection.

3.2.5 Division

The division of two complex numbers z_1 and z_2 bears some similarity to their multiplication. Writing the quotient in component form we obtain

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}. \tag{3.16}$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator. By definition, this process will leave the denominator as a real quantity. Equation (3.16) gives

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.\end{aligned}$$

Hence we have separated the quotient into real and imaginary components, as required.

In the special case where $z_2 = z_1^*$, so that $x_2 = x_1$ and $y_2 = -y_1$, the general result reduces to (3.15).

►*Express z in the form $x + iy$, when*

$$z = \frac{3 - 2i}{-1 + 4i}.$$

Multiplying numerator and denominator by the complex conjugate of the denominator we obtain

$$\begin{aligned} z &= \frac{(3-2i)(-1-4i)}{(-1+4i)(-1-4i)} = \frac{-11-10i}{17} \\ &= -\frac{11}{17} - \frac{10}{17}i. \blacktriangleleft \end{aligned}$$

In analogy to (3.10) and (3.11), which describe the multiplication of two complex numbers, the following relations apply to division:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad (3.17)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \qquad (3.18)$$

The proof of these relations is left until subsection 3.3.1.

3.3 Polar representation of complex numbers

Although considering a complex number as the sum of a real and an imaginary part is often useful, sometimes the *polar representation* proves easier to manipulate. This makes use of the complex exponential function, which is defined by

$$e^z = \exp z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \qquad (3.19)$$

Strictly speaking it is the function $\exp z$ that is defined by (3.19). The number e is the value of exp(1), i.e. it is just a number. However, it may be shown that e^z and $\exp z$ are equivalent when z is real and rational and mathematicians then *define* their equivalence for irrational and complex z. For the purposes of this book we will not concern ourselves further with this mathematical nicety but, rather, assume that (3.19) is valid for all z. We also note that, using (3.19), by multiplying together the appropriate series we may show that (see chapter 24)

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \qquad (3.20)$$

which is analogous to the familiar result for exponentials of real numbers.

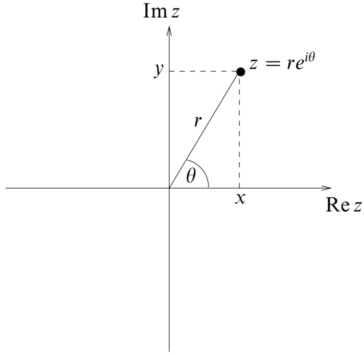

Figure 3.7 The polar representation of a complex number.

From (3.19), it immediately follows that for $z = i\theta$, θ real,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \tag{3.21}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \tag{3.22}$$

and hence that

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.23}$$

where the last equality follows from the series expansions of the sine and cosine functions (see subsection 4.6.3). This last relationship is called *Euler's equation.* It also follows from (3.23) that

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

for all n. From Euler's equation (3.23) and figure 3.7 we deduce that

$$\begin{aligned} re^{i\theta} &= r(\cos\theta + i\sin\theta) \\ &= x + iy. \end{aligned}$$

Thus a complex number may be represented in the polar form

$$z = re^{i\theta}. \tag{3.24}$$

Referring again to figure 3.7, we can identify r with $|z|$ and θ with $\arg z$. The simplicity of the representation of the modulus and argument is one of the main reasons for using the polar representation. The angle θ lies conventionally in the range $-\pi < \theta \leq \pi$, but, since rotation by θ is the same as rotation by $2n\pi + \theta$, where n is any integer,

$$re^{i\theta} \equiv re^{i(\theta+2n\pi)}.$$

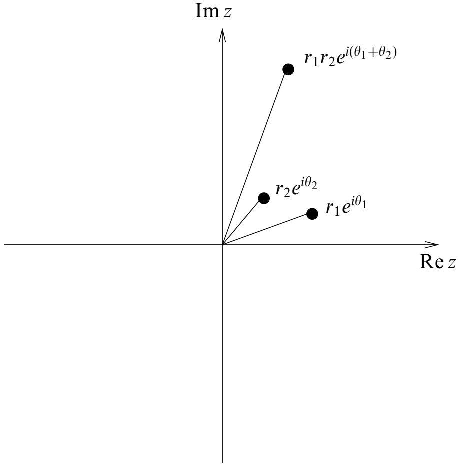

Figure 3.8 The multiplication of two complex numbers. In this case r_1 and r_2 are both greater than unity.

The algebra of the polar representation is different from that of the real and imaginary component representation, though, of course, the results are identical. Some operations prove much easier in the polar representation, others much more complicated. The best representation for a particular problem must be determined by the manipulation required.

3.3.1 Multiplication and division in polar form

Multiplication and division in polar form are particularly simple. The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ is given by

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)}. \end{aligned} \tag{3.25}$$

The relations $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ follow immediately. An example of the multiplication of two complex numbers is shown in figure 3.8.

Division is equally simple in polar form; the quotient of z_1 and z_2 is given by

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}. \tag{3.26}$$

The relations $|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ are again

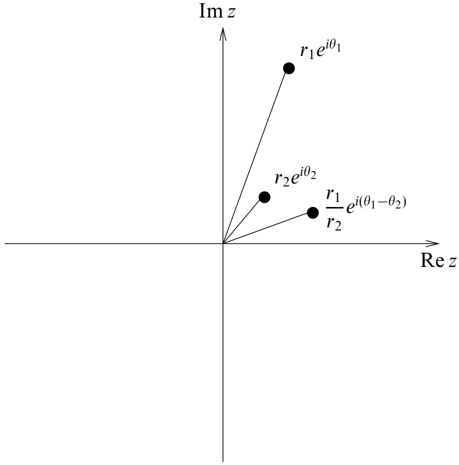

Figure 3.9 The division of two complex numbers. As in the previous figure, r_1 and r_2 are both greater than unity.

immediately apparent. The division of two complex numbers in polar form is shown in figure 3.9.

3.4 de Moivre's theorem

We now derive an extremely important theorem. Since $\left(e^{i\theta}\right)^n = e^{in\theta}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{3.27}$$

where the identity $e^{in\theta} = \cos n\theta + i\sin n\theta$ follows from the series definition of $e^{in\theta}$ (see (3.21)). This result is called *de Moivre's theorem* and is often used in the manipulation of complex numbers. The theorem is valid for all n whether real, imaginary or complex.

There are numerous applications of de Moivre's theorem but this section examines just three: proofs of trigonometric identities; finding the nth roots of unity; and solving polynomial equations with complex roots.

3.4.1 Trigonometric identities

The use of de Moivre's theorem in finding trigonometric identities is best illustrated by example. We consider the expression of a multiple-angle function in terms of a polynomial in the single-angle function, and its converse.

►*Express* $\sin 3\theta$ *and* $\cos 3\theta$ *in terms of powers of* $\cos\theta$ *and* $\sin\theta$.

Using de Moivre's theorem,

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta).\end{aligned} \tag{3.28}$$

We can equate the real and imaginary coefficients separately, i.e.

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \blacktriangleleft\end{aligned}$$

This method can clearly be applied to finding power expansions of $\cos n\theta$ and $\sin n\theta$ for any positive integer n.

The converse process uses the following properties of $z = e^{i\theta}$,

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \tag{3.30}$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta. \tag{3.31}$$

These equalities follow from simple applications of de Moivre's theorem, i.e.

$$\begin{aligned}z^n + \frac{1}{z^n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta\end{aligned}$$

and

$$\begin{aligned}z^n - \frac{1}{z^n} &= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta \\ &= 2i\sin n\theta.\end{aligned}$$

In the particular case where $n = 1$,

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta, \tag{3.32}$$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta. \tag{3.33}$$

►*Find an expression for* $\cos^3\theta$ *in terms of* $\cos 3\theta$ *and* $\cos\theta$.

Using (3.32),

$$\begin{aligned}\cos^3\theta &= \frac{1}{2^3}\left(z+\frac{1}{z}\right)^3\\ &= \frac{1}{8}\left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)\\ &= \frac{1}{8}\left(z^3+\frac{1}{z^3}\right)+\frac{3}{8}\left(z+\frac{1}{z}\right).\end{aligned}$$

Now using (3.30) and (3.32), we find

$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta. \blacktriangleleft$$

This result happens to be a simple rearrangement of (3.29), but cases involving larger values of n are better handled using this direct method than by rearranging polynomial expansions of multiple-angle functions.

3.4.2 Finding the nth roots of unity

The equation $z^2 = 1$ has the familiar solutions $z = \pm 1$. However, now that we have introduced the concept of complex numbers we can solve the general equation $z^n = 1$. Recalling the fundamental theorem of algebra, we know that the equation has n solutions. In order to proceed we rewrite the equation as

$$z^n = e^{2ik\pi},$$

where k is any integer. Now taking the nth root of each side of the equation we find

$$z = e^{2ik\pi/n}.$$

Hence, the solutions of $z^n = 1$ are

$$z_{1,2,\ldots,n} = 1, \quad e^{2i\pi/n}, \quad \ldots, \quad e^{2i(n-1)\pi/n},$$

corresponding to the values $0, 1, 2, \ldots, n-1$ for k. Larger integer values of k do not give new solutions, since the roots already listed are simply cyclically repeated for $k = n, n+1, n+2$, etc.

►*Find the solutions to the equation* $z^3 = 1$.

By applying the above method we find

$$z = e^{2ik\pi/3}.$$

Hence the three solutions are $z_1 = e^{0i} = 1$, $z_2 = e^{2i\pi/3}$, $z_3 = e^{4i\pi/3}$. We note that, as expected, the next solution, for which $k = 3$, gives $z_4 = e^{6i\pi/3} = 1 = z_1$, so that there are only three separate solutions. ◄

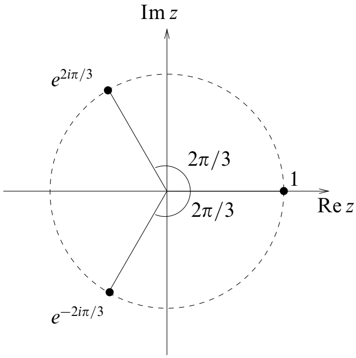

Figure 3.10 The solutions of $z^3 = 1$.

Not surprisingly, given that $|z^3| = |z|^3$ from (3.10), all the roots of unity have unit modulus, i.e. they all lie on a circle in the Argand diagram of unit radius. The three roots are shown in figure 3.10.

The cube roots of unity are often written 1, ω and ω^2. The properties $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ are easily proved.

3.4.3 Solving polynomial equations

A third application of de Moivre's theorem is to the solution of polynomial equations. Complex equations in the form of a polynomial relationship must first be solved for z in a similar fashion to the method for finding the roots of real polynomial equations. Then the complex roots of z may be found.

►*Solve the equation* $z^6 - z^5 + 4z^4 - 6z^3 + 2z^2 - 8z + 8 = 0$.

We first factorise to give

$$(z^3 - 2)(z^2 + 4)(z - 1) = 0.$$

Hence $z^3 = 2$ or $z^2 = -4$ or $z = 1$. The solutions to the quadratic equation are $z = \pm 2i$; to find the complex cube roots, we first write the equation in the form

$$z^3 = 2 = 2e^{2ik\pi},$$

where k is any integer. If we now take the cube root, we get

$$z = 2^{1/3}e^{2ik\pi/3}.$$

To avoid the duplication of solutions, we use the fact that $-\pi < \arg z \leq \pi$ and find

$$z_1 = 2^{1/3},$$
$$z_2 = 2^{1/3}e^{2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),$$
$$z_3 = 2^{1/3}e^{-2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right).$$

The complex numbers z_1, z_2 and z_3, together with $z_4 = 2i$, $z_5 = -2i$ and $z_6 = 1$ are the solutions to the original polynomial equation.

As expected from the fundamental theorem of algebra, we find that the total number of complex roots (six, in this case) is equal to the largest power of z in the polynomial. ◀

A useful result is that the roots of a polynomial with real coefficients occur in conjugate pairs (i.e. if z_1 is a root, then z_1^* is a second distinct root, unless z_1 is real). This may be proved as follows. Let the polynomial equation of which z is a root be

$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Taking the complex conjugate of this equation,

$$a_n^*(z^*)^n + a_{n-1}^*(z^*)^{n-1} + \cdots + a_1^* z^* + a_0^* = 0.$$

But the a_n are real, and so z^* satisfies

$$a_n(z^*)^n + a_{n-1}(z^*)^{n-1} + \cdots + a_1 z^* + a_0 = 0,$$

and is also a root of the original equation.

3.5 Complex logarithms and complex powers

The concept of a complex exponential has already been introduced in section 3.3, where it was assumed that the definition of an exponential as a series was valid for complex numbers as well as for real numbers. Similarly we can define the logarithm of a complex number and we can use complex numbers as exponents.

Let us denote the natural logarithm of a complex number z by $w = \operatorname{Ln} z$, where the notation Ln will be explained shortly. Thus, w must satisfy

$$z = e^w.$$

Using (3.20), we see that

$$z_1 z_2 = e^{w_1}e^{w_2} = e^{w_1+w_2},$$

and taking logarithms of both sides we find

$$\operatorname{Ln}(z_1 z_2) = w_1 + w_2 = \operatorname{Ln} z_1 + \operatorname{Ln} z_2, \tag{3.34}$$

which shows that the familiar rule for the logarithm of the product of two real numbers also holds for complex numbers.

We may use (3.34) to investigate further the properties of $\operatorname{Ln} z$. We have already noted that the argument of a complex number is multivalued, i.e. $\arg z = \theta + 2n\pi$, where n is any integer. Thus, in polar form, the complex number z should strictly be written as

$$z = re^{i(\theta+2n\pi)}.$$

Taking the logarithm of both sides, and using (3.34), we find

$$\operatorname{Ln} z = \ln r + i(\theta + 2n\pi), \tag{3.35}$$

where $\ln r$ is the natural logarithm of the real positive quantity r and so is written normally. Thus from (3.35) we see that $\operatorname{Ln} z$ is itself multivalued. To avoid this multivalued behaviour it is conventional to define another function $\ln z$, the *principal value* of $\operatorname{Ln} z$, which is obtained from $\operatorname{Ln} z$ by restricting the argument of z to lie in the range $-\pi < \theta \leq \pi$.

►*Evaluate* $\operatorname{Ln}(-i)$.

By rewriting $-i$ as a complex exponential, we find

$$\operatorname{Ln}(-i) = \operatorname{Ln}\left[e^{i(-\pi/2+2n\pi)}\right] = i(-\pi/2 + 2n\pi),$$

where n is any integer. Hence $\operatorname{Ln}(-i) = -i\pi/2,\ 3i\pi/2,\ \ldots$. We note that $\ln(-i)$, the principal value of $\operatorname{Ln}(-i)$, is given by $\ln(-i) = -i\pi/2$. ◄

If z and t are both complex numbers then the zth power of t is defined by

$$t^z = e^{z \operatorname{Ln} t}.$$

Since $\operatorname{Ln} t$ is multivalued, so too is this definition.

►*Simplify the expression* $z = i^{-2i}$.

Firstly we take the logarithm of both sides of the equation to give

$$\operatorname{Ln} z = -2i \operatorname{Ln} i.$$

Now inverting the process we find

$$e^{\operatorname{Ln} z} = z = e^{-2i \operatorname{Ln} i}.$$

We can write $i = e^{i(\pi/2+2n\pi)}$, where n is any integer, and hence

$$\begin{aligned}\operatorname{Ln} i &= \operatorname{Ln}\left[e^{i\left(\pi/2+2n\pi\right)}\right]\\ &= i\left(\pi/2 + 2n\pi\right).\end{aligned}$$

We can now simplify z to give

$$\begin{aligned}i^{-2i} &= e^{-2i\times i\left(\pi/2+2n\pi\right)}\\ &= e^{(\pi+4n\pi)},\end{aligned}$$

which, perhaps surprisingly, is a real quantity rather than a complex one. ◄

Complex powers and the logarithms of complex numbers are discussed further in chapter 24.

3.6 Applications to differentiation and integration

We can use the exponential form of a complex number together with de Moivre's theorem (see section 3.4) to simplify the differentiation of trigonometric functions.

► *Find the derivative with respect to x of $e^{3x}\cos 4x$.*

We could differentiate this function straightforwardly using the product rule (see subsection 2.1.2). However, an alternative method in this case is to use a complex exponential. Let us consider the complex number

$$z = e^{3x}(\cos 4x + i\sin 4x) = e^{3x}e^{4ix} = e^{(3+4i)x},$$

where we have used de Moivre's theorem to rewrite the trigonometric functions as a complex exponential. This complex number has $e^{3x}\cos 4x$ as its real part. Now, differentiating z with respect to x we obtain

$$\frac{dz}{dx} = (3+4i)e^{(3+4i)x} = (3+4i)e^{3x}(\cos 4x + i\sin 4x), \tag{3.36}$$

where we have again used de Moivre's theorem. Equating real parts we then find

$$\frac{d}{dx}\left(e^{3x}\cos 4x\right) = e^{3x}(3\cos 4x - 4\sin 4x).$$

By equating the imaginary parts of (3.36), we also obtain, as a bonus,

$$\frac{d}{dx}\left(e^{3x}\sin 4x\right) = e^{3x}(4\cos 4x + 3\sin 4x). \blacktriangleleft$$

In a similar way the complex exponential can be used to evaluate integrals containing trigonometric and exponential functions.

► *Evaluate the integral $I = \int e^{ax}\cos bx\, dx$.*

Let us consider the integrand as the real part of the complex number

$$e^{ax}(\cos bx + i\sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x},$$

where we use de Moivre's theorem to rewrite the trigonometric functions as a complex exponential. Integrating we find

$$\begin{aligned}\int e^{(a+ib)x}\, dx &= \frac{e^{(a+ib)x}}{a+ib} + c \\ &= \frac{(a-ib)e^{(a+ib)x}}{(a-ib)(a+ib)} + c \\ &= \frac{e^{ax}}{a^2+b^2}\left(ae^{ibx} - ibe^{ibx}\right) + c,\end{aligned} \tag{3.37}$$

where the constant of integration c is in general complex. Denoting this constant by $c = c_1 + ic_2$ and equating real parts in (3.37) we obtain

$$I = \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c_1,$$

which agrees with result (2.37) found using integration by parts. Equating imaginary parts in (3.37) we obtain, as a bonus,

$$J = \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c_2. \blacktriangleleft$$

3.7 Hyperbolic functions

The *hyperbolic functions* are the complex analogues of the trigonometric functions. The analogy may not be immediately apparent and their definitions may appear at first to be somewhat arbitrary. However, careful examination of their properties reveals the purpose of the definitions. For instance, their close relationship with the trigonometric functions, both in their identities and in their calculus, means that many of the familiar properties of trigonometric functions can also be applied to the hyperbolic functions. Further, hyperbolic functions occur regularly, and so giving them special names is a notational convenience.

3.7.1 Definitions

The two fundamental hyperbolic functions are $\cosh x$ and $\sinh x$, which, as their names suggest, are the hyperbolic equivalents of $\cos x$ and $\sin x$. They are defined by the following relations:

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \tag{3.38}$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}). \tag{3.39}$$

Note that $\cosh x$ is an even function and $\sinh x$ is an odd function. By analogy with the trigonometric functions, the remaining hyperbolic functions are

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{3.40}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{3.41}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \tag{3.42}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \tag{3.43}$$

All the hyperbolic functions above have been defined in terms of the real variable x. However, this was simply so that they may be plotted (see figures 3.11–3.13); the definitions are equally valid for any complex number z.

3.7.2 Hyperbolic–trigonometric analogies

In the previous subsections we have alluded to the analogy between trigonometric and hyperbolic functions. Here, we discuss the close relationship between the two groups of functions.

Recalling (3.32) and (3.33) we find

$$\cos ix = \tfrac{1}{2}(e^x + e^{-x}),$$

$$\sin ix = \tfrac{1}{2}i(e^x - e^{-x}).$$

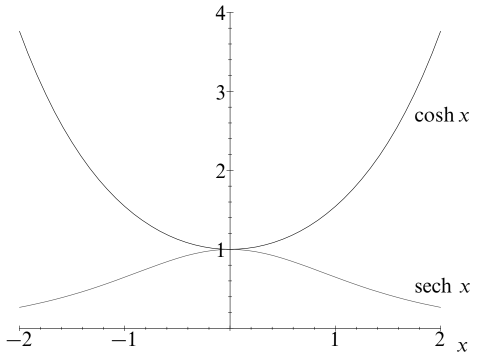

Figure 3.11 Graphs of $\cosh x$ and sech$x$.

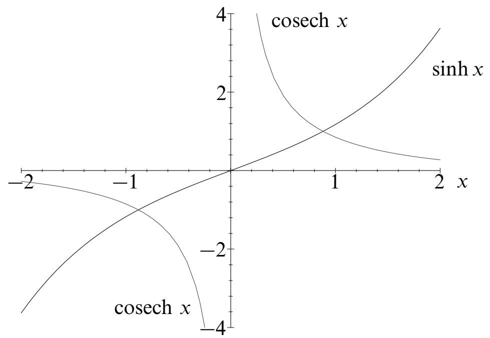

Figure 3.12 Graphs of $\sinh x$ and cosech$x$.

Hence, by the definitions given in the previous subsection,

$$\cosh x = \cos ix, \tag{3.44}$$

$$i \sinh x = \sin ix, \tag{3.45}$$

$$\cos x = \cosh ix, \tag{3.46}$$

$$i \sin x = \sinh ix. \tag{3.47}$$

These useful equations make the relationship between hyperbolic and trigono-

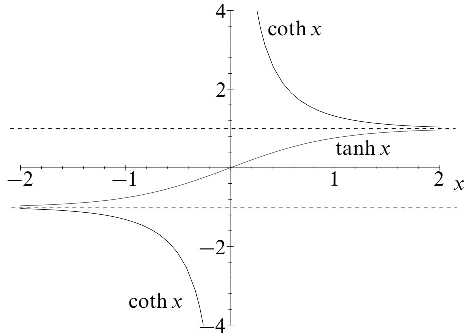

Figure 3.13 Graphs of $\tanh x$ and $\coth x$.

metric functions transparent. The similarity in their calculus is discussed further in subsection 3.7.6.

3.7.3 Identities of hyperbolic functions

The analogies between trigonometric functions and hyperbolic functions having been established, we should not be surprised that all the trigonometric identities also hold for hyperbolic functions, with the following modification. Wherever $\sin^2 x$ occurs it must be replaced by $-\sinh^2 x$, and vice versa. Note that this replacement is necessary even if the $\sin^2 x$ is hidden, e.g. $\tan^2 x = \sin^2 x/\cos^2 x$ and so must be replaced by $(-\sinh^2 x/\cosh^2 x) = -\tanh^2 x$.

►*Find the hyperbolic identity analogous to* $\cos^2 x + \sin^2 x = 1$.

Using the rules stated above $\cos^2 x$ is replaced by $\cosh^2 x$, and $\sin^2 x$ by $-\sinh^2 x$, and so the identity becomes

$$\cosh^2 x - \sinh^2 x = 1.$$

This can be verified by direct substitution, using the definitions of $\cosh x$ and $\sinh x$; see (3.38) and (3.39). ◄

Some other identities that can be proved in a similar way are

$$\text{sech}^2 x = 1 - \tanh^2 x, \tag{3.48}$$

$$\text{cosech}^2 x = \coth^2 x - 1, \tag{3.49}$$

$$\sinh 2x = 2 \sinh x \cosh x, \tag{3.50}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x. \tag{3.51}$$

3.7.4 Solving hyperbolic equations

When we are presented with a hyperbolic equation to solve, we may proceed by analogy with the solution of trigonometric equations. However, it is almost always easier to express the equation directly in terms of exponentials.

►*Solve the hyperbolic equation* $\cosh x - 5\sinh x - 5 = 0$.

Substituting the definitions of the hyperbolic functions we obtain

$$\tfrac{1}{2}(e^x + e^{-x}) - \tfrac{5}{2}(e^x - e^{-x}) - 5 = 0.$$

Rearranging, and then multiplying through by $-e^x$, gives in turn

$$-2e^x + 3e^{-x} - 5 = 0$$

and

$$2e^{2x} + 5e^x - 3 = 0.$$

Now we can factorise and solve:

$$(2e^x - 1)(e^x + 3) = 0.$$

Thus $e^x = 1/2$ or $e^x = -3$. Hence $x = -\ln 2$ or $x = \ln(-3)$. The interpretation of the logarithm of a negative number has been discussed in section 3.5. ◄

3.7.5 Inverses of hyperbolic functions

Just like trigonometric functions, hyperbolic functions have inverses. If $y = \cosh x$ then $x = \cosh^{-1} y$, which serves as a definition of the inverse. By using the fundamental definitions of hyperbolic functions, we can find closed-form expressions for their inverses. This is best illustrated by example.

►*Find a closed-form expression for the inverse hyperbolic function* $y = \sinh^{-1} x$.

First we write x as a function of y, i.e.

$$y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y.$$

Now, since $\cosh y = \frac{1}{2}(e^y + e^{-y})$ and $\sinh y = \frac{1}{2}(e^y - e^{-y})$,

$$\begin{aligned} e^y &= \cosh y + \sinh y \\ &= \sqrt{1 + \sinh^2 y} + \sinh y \\ e^y &= \sqrt{1 + x^2} + x, \end{aligned}$$

and hence

$$y = \ln(\sqrt{1 + x^2} + x). \text{ ◄}$$

In a similar fashion it can be shown that

$$\cosh^{-1} x = \ln(\sqrt{x^2 - 1} + x).$$

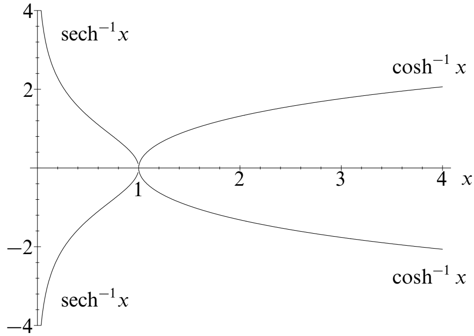

Figure 3.14 Graphs of $\cosh^{-1}x$ and $\operatorname{sech}^{-1}x$.

►*Find a closed-form expression for the inverse hyperbolic function* $y = \tanh^{-1}x$.

First we write x as a function of y, i.e.

$$y = \tanh^{-1}x \qquad \Rightarrow \qquad x = \tanh y.$$

Now, using the definition of $\tanh y$ and rearranging, we find

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \qquad \Rightarrow \qquad (x+1)e^{-y} = (1-x)e^y.$$

Thus, it follows that

$$e^{2y} = \frac{1+x}{1-x} \qquad \Rightarrow \qquad e^y = \sqrt{\frac{1+x}{1-x}},$$
$$y = \ln\sqrt{\frac{1+x}{1-x}},$$
$$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \blacktriangleleft$$

Graphs of the inverse hyperbolic functions are given in figures 3.14–3.16.

3.7.6 Calculus of hyperbolic functions

Just as the identities of hyperbolic functions closely follow those of their trigonometric counterparts, so their calculus is similar. The derivatives of the two basic

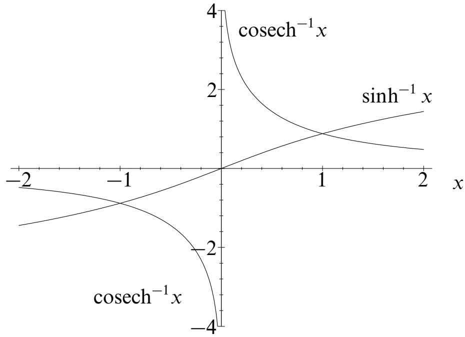

Figure 3.15 Graphs of $\sinh^{-1} x$ and $\mathrm{cosech}^{-1} x$.

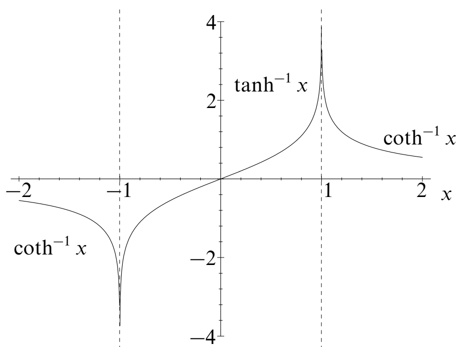

Figure 3.16 Graphs of $\tanh^{-1} x$ and $\coth^{-1} x$.

hyperbolic functions are given by

$$\frac{d}{dx}(\cosh x) = \sinh x, \tag{3.52}$$

$$\frac{d}{dx}(\sinh x) = \cosh x. \tag{3.53}$$

They may be deduced by considering the definitions (3.38), (3.39) as follows.

► *Verify the relation* $(d/dx)\cosh x = \sinh x$.

Using the definition of $\cosh x$,

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

and differentiating directly, we find

$$\begin{aligned}\frac{d}{dx}(\cosh x) &= \tfrac{1}{2}(e^x - e^{-x})\\ &= \sinh x. \blacktriangleleft\end{aligned}$$

Clearly the integrals of the fundamental hyperbolic functions are also defined by these relations. The derivatives of the remaining hyperbolic functions can be derived by product differentiation and are presented below only for completeness.

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x, \tag{3.54}$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x, \tag{3.55}$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x, \tag{3.56}$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x. \tag{3.57}$$

The inverse hyperbolic functions also have derivatives, which are given by the following:

$$\frac{d}{dx}\left(\cosh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 - a^2}}, \tag{3.58}$$

$$\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 + a^2}}, \tag{3.59}$$

$$\frac{d}{dx}\left(\tanh^{-1}\frac{x}{a}\right) = \frac{a}{a^2 - x^2}, \quad \text{for } x^2 < a^2, \tag{3.60}$$

$$\frac{d}{dx}\left(\coth^{-1}\frac{x}{a}\right) = \frac{-a}{x^2 - a^2}, \quad \text{for } x^2 > a^2. \tag{3.61}$$

These may be derived from the logarithmic form of the inverse (see subsection 3.7.5).

► *Evaluate $(d/dx)\sinh^{-1} x$ using the logarithmic form of the inverse.*

From the results of section 3.7.5,

$$\begin{aligned}\frac{d}{dx}\left(\sinh^{-1} x\right) &= \frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right]\\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)\\ &= \frac{1}{\sqrt{x^2+1}}. \blacktriangleleft\end{aligned}$$

3.8 Exercises

3.1 Two complex numbers z and w are given by $z = 3+4i$ and $w = 2-i$. On an Argand diagram, plot

(a) $z+w$, (b) $w-z$, (c) wz, (d) z/w,
(e) z^*w+w^*z, (f) w^2, (g) $\ln z$, (h) $(1+z+w)^{1/2}$.

3.2 By considering the real and imaginary parts of the product $e^{i\theta}e^{i\phi}$ prove the standard formulae for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

3.3 By writing $\pi/12 = (\pi/3)-(\pi/4)$ and considering $e^{i\pi/12}$, evaluate $\cot(\pi/12)$.

3.4 Find the locus in the complex z-plane of points that satisfy the following equations.

(a) $z-c = \rho\left(\dfrac{1+it}{1-it}\right)$, where c is complex, ρ is real and t is a real parameter that varies in the range $-\infty < t < \infty$.
(b) $z = a+bt+ct^2$, in which t is a real parameter and a, b, and c are complex numbers with b/c real.

3.5 Evaluate

(a) $\mathrm{Re}(\exp 2iz)$, (b) $\mathrm{Im}(\cosh^2 z)$, (c) $(-1+\sqrt{3}i)^{1/2}$,
(d) $|\exp(i^{1/2})|$, (e) $\exp(i^3)$, (f) $\mathrm{Im}(2^{i+3})$, (g) i^i, (h) $\ln[(\sqrt{3}+i)^3]$.

3.6 Find the equations in terms of x and y of the sets of points in the Argand diagram that satisfy the following:

(a) $\mathrm{Re}\, z^2 = \mathrm{Im}\, z^2$;
(b) $(\mathrm{Im}\, z^2)/z^2 = -i$;
(c) $\arg[z/(z-1)] = \pi/2$.

3.7 Show that the locus of all points $z = x+iy$ in the complex plane that satisfy

$$|z-ia| = \lambda|z+ia|, \quad \lambda > 0,$$

is a circle of radius $|2\lambda a/(1-\lambda^2)|$ centred on the point $z = ia[(1+\lambda^2)/(1-\lambda^2)]$. Sketch the circles for a few typical values of λ, including $\lambda < 1$, $\lambda > 1$ and $\lambda = 1$.

3.8 The two sets of points $z = a$, $z = b$, $z = c$, and $z = A$, $z = B$, $z = C$ are the corners of two similar triangles in the Argand diagram. Express in terms of $a, b, \ldots, C$