The Extended Real nos. $\mathbb{R}^*$

The extended real nos. $\mathbb{R}^*$ are the real nos. with two elements adjoint. $\mathbb{R}^* = \mathbb{R} \cup [+\infty, -\infty]$.

Algebraic op's for the two infinities are:

1. $\infty + \infty = \infty$, $-\infty - \infty = -\infty$.
2. $(\pm\infty)\cdot\infty = \pm\infty$, $(\pm\infty)\cdot(-\infty) = \mp\infty$.
3. $x + \infty = \infty$, $x - \infty = -\infty$ for each $x \in \mathbb{R}$.
4. $x\cdot(\pm\infty) = \pm\infty$ if $x > 0$, $\left(\begin{matrix}\infty - \infty \\ \infty + \infty\end{matrix}\right\}$ undefined$)$

   $x\cdot(\pm\infty) = \mp\infty$ if $x < 0$.
5. $0\cdot\infty = 0$

$\mathbb{R}^*$ is ordered, with ∞ the largest element, and $-\infty$ the smallest element.

6. $-\infty < x < \infty$ for each $x \in \mathbb{R}$.

Every sequence of real nos. has a limit superior and limit inferior in $\mathbb{R}^*$.

<u>Thm</u>. Every increasing sequence of real nos. either converges to a real no. or to $+\infty$.

For a sequence of real nos. $\{x_n\}$, the series $\sum_{n=1}^{\infty} x_n$ is convergent, if the sequence of partial sums $\left\{\sum_{k=1}^{n} x_k\right\}$ converges in $\mathbb{R}$.

A series $\sum_{n\geq 1}^{\infty} x_n$ of real numbers is <u>rearrangement variant</u> if for every 1-to-1 and onto f^n $\sigma : \mathbb{N} \to \mathbb{N}$ (called a permutation of $\mathbb{N}$), the series $\sum_{n\geq 1}^{\infty} x_{\sigma_n}$ converges; (in $\mathbb{R}^*$?) and moreover $\sum_{n\geq 1}^{\infty} x_n = \sum_{n\geq 1}^{\infty} x_{\sigma_n}$.

<u>Thm</u>. If $\{x_n\}$ is a sequence of $\mathbb{R}^+$, then the series $\sum_{n\geq 1}^{\infty} x_n$ is rearrangement variant.

$\{a_{n,m}\}$ is a double sequence with $0 \leq a_{n,m} \leq \infty$ for each pair n,m.

$$\sum_{n\geq 1}^{\infty}\sum_{m\geq 1}^{\infty} a_{n,m} = \lim_{k\to\infty} \sum_{n\geq 1}^{k}\left(\sum_{m\geq 1}^{\infty} a_{n,m}\right).$$

<u>Thm</u>. If $0 \leq a_{n,m} \leq \infty$ $\forall\, m,n \in \mathbb{N}$, then

$$\sum_{n\geq 1}^{\infty}\sum_{m\geq 1}^{\infty} a_{n,m} = \sum_{m\geq 1}^{\infty}\sum_{n\geq 1}^{\infty} a_{n,m}.$$

<u>Thm</u>. Let $0 \leq a_{n,m} \leq \infty$ $\forall\, m,n \in \mathbb{N}$. If $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is bijective then $\sum_{n\geq 1}^{\infty} a_{\sigma_n} = \sum_{n\geq 1}^{\infty}\sum_{m\geq 1}^{\infty} a_{n,m}$.

The Root and Ratio Tests:

Thm. (Root test). Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

(a) if $\alpha < 1$, $\sum a_n$ converges in $\mathbb{R}$;

(b) if $\alpha > 1$, $\sum a_n$ diverges;

(c) if $\alpha = 1$, the test gives no information.

Thm. (Ratio test) The series $\sum a_n$,

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) diverges if $\frac{a_{n+1}}{a_n} \geq 1$ for $n \geq n_0$, $n_0 \in \mathbb{N}$.

(convergence in $\mathbb{R}$)

Exercise:

(a) Consider the series

(i) $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$

(ii) $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \cdots\cdots$