Problem (Simultaneous Recurrence): There are two kinds of particles inside a nuclear reactor. In every second, an α particle will split into three β particles, and a β particle will split into an α particle and two β particles. Assume that there is a single α particle in the reactor at time $t = 0$. Let a_r and b_r denote the number of α particles and β particles at the r-th second in the reactor, respectively.

(i) Construct the simultaneous recurrence relations for a_r and b_r.

(ii) Show that

$$\begin{aligned} a_r &= \frac{3}{4}(3^{r-1} + (-1)^r), r \geq 0, \\ b_r &= \frac{3}{4}(3^r - (-1)^r), r \geq 0. \end{aligned}$$

Simultaneous Recurrence

Solution:

(i) Let a_r and b_r denote the number of α particles and β particles at the r-th second in the reactor, respectively. According to the initial condition, $a_0 = 1$ and $b_0 = 0$.

α particle $\rightarrow$ 0(α particle)&3(β particles)

β particle $\rightarrow$ 1(α particle)&2(β particles)

We have the following simultaneous recurrence relations:

$$\begin{aligned} a_r &= 0.a_{r-1} + 1.b_{r-1} \\ &= b_{r-1} \qquad (4) \\ b_r &= 3.a_{r-1} + 2.b_{r-1}, r \geq 1, \qquad (5) \end{aligned}$$

with the initial condition $a_0 = 1$ and $b_0 = 0$.

Simultaneous Recurrence

Solution:

(ii) From Equation (4), using the generating function both sides, we have,

$\sum_{r=1}^{\infty} a_r z^r = \sum_{r=1}^{\infty} b_{r-1} z^r$

or, $(\sum_{r=0}^{\infty} a_r z^r - a_0) = z.\sum_{r=1}^{\infty} b_{r-1} z^{r-1}$

or, $A(z) - 1 = zB(z)$

$$A(z) = zB(z) + 1. \tag{6}$$

Again, from Equation (5), using the generating function both sides, we have, $\sum_{r=1}^{\infty} b_r z^r = 3\sum_{r=1}^{\infty} a_{r-1} z^r + 2\sum_{r=1}^{\infty} b_{r-1} z^r$

or, $(\sum_{r=0}^{\infty} b_r z^r - b_0) = 3z.\sum_{r=1}^{\infty} a_{r-1} z^{r-1} + 2z\sum_{r=1}^{\infty} b_{r-1} z^{r-1}$

or, $B(z) - 0 = 3zA(z) + 2zB(z)$

$$A(z) = \frac{1-2z}{3z} B(z). \tag{7}$$

Simultaneous Recurrence

Solving Equations (6) and (7), we obtain,

$$B(z) = \frac{3z}{1-2z-3z^2}. \tag{8}$$

$$A(z) = \frac{1-2z}{3z} \times \frac{3z}{(1-3z)(1+z)}. \tag{9}$$

Now, from Equation (8),

$B(z) = \frac{3z}{(1-3z)(1+z)} = \frac{\alpha_1}{1-3z} + \frac{\beta_1}{1+z}$, say

$= \frac{3}{4}\frac{1}{1-3z} - \frac{3}{4}\frac{1}{1+z}$

$= \frac{3}{4}\sum_{r=0}^{\infty} 3^r z^r - \frac{3}{4}\sum_{r=0}^{\infty}(-1)^r z^r$.

Hence, we have, $b_r = \frac{3}{4}3^r - \frac{3}{4}(-1)^r$, that is,

$$b_r = \frac{3}{4}(3^r - (-1)^r), r \geq 0.$$

Simultaneous Recurrence

Similarly, we have,
$A(z) = \frac{1-2z}{3z} \times \frac{3z}{(1-3z)(1+z)} = \frac{\alpha_2}{1-3z} + \frac{\beta_2}{1+z}$, say
$= \frac{1}{4}\frac{1}{1-3z} + \frac{3}{4}\frac{1}{1+z}$
$= \frac{1}{4}\sum_{r=0}^{\infty} 3^r z^r + \frac{3}{4}\sum_{r=0}^{\infty}(-1)^r z^r$.
Thus,
$a_r = \frac{1}{4}3^r + \frac{3}{4}(-1)^r$, that is,

$$a_r = \frac{3}{4}(3^{r-1} + (-1)^r), r \geq 0.$$

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Generating Function

Problem: Using the generating function, prove that the n^{th} Fibonacci's number is

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

where the Fibonacci's sequence $(F_0, F_1, F_2, F_3, \ldots, F_r, \ldots)$ is defined as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_r &= F_{r-1} + F_{r-2},\ r \geq 2 \end{aligned}$$

Generating Function

Given that

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 1, \\ F_r &= F_{r-1} + F_{r-2},\ r \geq 2. \end{aligned}$$

Taking the infinite summation of the powers of z^r on both sides starting from $r = 2$, we have,

$$\begin{aligned} \sum_{r=2}^{\infty} F_r z^r &= \sum_{r=2}^{\infty} F_{r-1} z^r + \sum_{r=2}^{\infty} F_{r-2} z^r \\ \Rightarrow (\sum_{r=0}^{\infty} F_r z^r - F_0 - F_1 z) &= z(\sum_{r=1}^{\infty} F_{r-1} z^{r-1} - F_0) + z^2 \sum_{r=2}^{\infty} F_{r-2} z^{r-2} \\ \Rightarrow F(z) - 0 - z &= zF(z) + z^2 F(z), \text{Given:} F_0 = 0, F_1 = 1. \\ \Rightarrow F(z) &= \frac{z}{1 - z - z^2} \qquad (10) \end{aligned}$$

Generating Function

$F(z)$ is the generating function of the numeric function $F = \{F_0, F_1, \ldots, F_r, \ldots\}$.

Let $F(z) = \frac{z}{1-z-z^2} = \frac{z}{(1-\alpha z)(1-\beta z)}$, say, where α and β are the roots of the quadratic equation $1 - z - z^2 = 0$. Then,

$(1-\alpha z)(1-\beta z) = 1 - z - z^2$. So, $1 - (\alpha+\beta)z + \alpha\beta z^2 = 1 - z - z^2$.

Equating the coefficients on both sides, we have,

$$\begin{aligned} \alpha + \beta &= 1 & (11) \\ \alpha\beta &= -1 & (12) \end{aligned}$$

Now,

$$\begin{aligned} \alpha - \beta &= +\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} \\ &= \sqrt{5} & (13) \end{aligned}$$

Generating Function

Solving the equations 11 and 13, we obtain, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
Let $F(z) = \frac{z}{(1-\alpha z)(1-\beta z)} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$, where A and B are the constants to be determined.
Then, $A(1-\beta z) + B(1-\alpha z) = z$. This implies that $(A+B) - (\beta A + \alpha B) = z$.
Equating the coefficients on both sides, we have the following equations

$$\begin{aligned} A + B &= 0 \qquad (14) \\ \beta A + \alpha B &= -1 \qquad (15) \end{aligned}$$

Solving the equations 14 and 15, we obtain, $A = \frac{1}{\alpha-\beta} = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\alpha-\beta} = -\frac{1}{\sqrt{5}}$.

Generating Function

Finally, we have,

$$
\begin{aligned}
F(z) &= \sum_{r=0}^{\infty} F_r z^r \\
&= \frac{1}{\sqrt{5}} \frac{1}{1-\alpha z} - \frac{1}{\sqrt{5}} \frac{1}{1-\beta z} \\
&= \frac{1}{\sqrt{5}} (1-\alpha z)^{-1} - \frac{1}{\sqrt{5}} (1-\beta z)^{-1} \\
&= \frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \alpha^r z^r - \frac{1}{\sqrt{5}} \sum_{r=0}^{\infty} \beta^r z^r \\
\Rightarrow F_r &= \frac{1}{\sqrt{5}} \alpha^r - \frac{1}{\sqrt{5}} \beta^r
\end{aligned}
$$

Hence, $F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n], n \geq 0.$

Generating Function

Problem: Using the generating function, find the number of binary trees with n nodes.