Assignment 5

Instructions

1. Add any specific instructions for assignment
2. Naming of submissions should strictly follow the format **rollnumber_A4**. Penalty will be imposed for not following the same.
3. Plagiarism will attract penalty.

Problem 1

The Fibonacci sequence $f_1, f_2, f_3 \dots$ is defined as follows.

$$f_1 = f_2 = 1$$
$$f_r = f_{r-1} + f_{r-2} \ \forall \ r \geq 3$$

Prove that the following propositions hold for all positive integers n.

a) $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$

b) $f_{n+1}^2 + f_n^2 = f_{2n+1}$

c) For any positive integers m and n, show that $f_{n+m} = f_m f_{n+1} + f_{m-1} f_n$

Solution:
Part (a)

We see that:

$$f_2 f_0 - f_1^2 = 1 \times 0 - 1 = -1 = (-1)^1$$

so the proposition holds for $n = 1$. We also see that:

$$f_3 f_1 - f_2^2 = 2 \times 1 - 1 = (-1)^2$$

so the proposition holds for $n = 2$. Suppose the proposition is true for $n = k$, that is:

$$f_{k+1} f_{k-1} - f_k^2 = (-1)^k$$

It remains to be shown that it follows from this that the proposition is true for $n = k + 1$, that is:

$$f_{k+2} f_k - f_{k+1}^2 = (-1)^{k+1}$$

So:

$$\begin{aligned}
f_{k+2}f_k - f_{k+1}^2 &= \left(f_k + f_{k+1}\right) f_k - f_{k+1}^2 \\
&= f_k^2 + f_k f_{k+1} - f_{k+1}^2 \\
&= f_k^2 + f_k f_{k+1} - f_{k+1}\left(f_k + f_{k-1}\right) \\
&= f_k^2 + f_k f_{k+1} - f_k f_{k+1} - f_{k+1} f_{k-1} \\
&= f_k^2 - f_{k+1} f_{k-1} \\
&= (-1)\left(f_{k+1} f_{k-1} - f_k^2\right) \\
&= (-1)(-1)^k \\
&= (-1)^{k+1}
\end{aligned}$$

Part (b)

We see that the formulation works for $n = 1$ and $n = 2$. Since for $n = 1$ we have

$$f_2^2 + f_1^2 = 1 + 1 = 2 = f_3$$

And for $n = 2$ we have,

$$f_3^2 + f_2^2 = 4 + 1 = 5 = f_5$$

Now we assume that it works for $P(n)$ and $P(n-1)$, we need to prove that it works for $P(n+1)$. That is, we assume that the following are true

$$\begin{aligned}
f_{2n-1} &= f_{n-1}^2 + f_n^2 \\
f_{2n+1} &= f_n^2 + f_{n+1}^2
\end{aligned}$$

Now, notice that

$$\begin{aligned}
f_{2n+3} &= f_{2n+2} + f_{2n+1} \\
&= f_{2n+1} + f_{2n} + f_{2n+1} \\
&= 2f_{2n+1} + f_{2n} \\
&= 2f_{2n+1} + f_{2n+1} - f_{2n-1} \\
&= 3f_{2n+1} - f_{2n-1} \\
&= 3(f_n^2 + f_{n+1}^2) - (f_{n-1}^2 + f_n^2) \qquad \text{(By Induction Hypothesis)} \\
&= 2f_n^2 + 3f_{n+1}^2 - f_{n-1}^2 \\
&= (2f_n^2 + 2f_{n+1}^2) + f_{n+1}^2 - f_{n-1}^2 \\
&= (f_{n+1} + f_n)^2 + (f_{n+1} - f_n)^2 + f_{n+1}^2 - f_{n-1}^2 \\
&= f_{n+2}^2 + f_{n-1}^2 + f_{n+1}^2 - f_{n-1}^2 \\
&= f_{n+1}^2 + f_{n+2}^2
\end{aligned}$$

Part (c)

We can see that $P(1)$ and $P(2)$ clearly hold. $P(1)$ is the case:

$$\begin{aligned} f_{m+1} &= f_{m-1} + f_m \\ &= f_{m-1} \times 1 + f_m \times 1 \\ &= f_{m-1} f_1 + f_m f_2 \\ &= f_{m-1} f_n + f_m f_{n+1} \end{aligned}$$

and so $P(1)$ is seen to hold. $P(2)$ is the case:

$$\begin{aligned} f_{m+2} &= f_{m+1} + f_m \\ &= f_{m-1} + f_m + f_m \\ &= f_{m-1} \times 1 + f_m \times 2 \\ &= f_{m-1} f_2 + f_m f_3 \\ &= f_{m-1} f_n + f_m f_{n+1} \end{aligned}$$

and so $P(2)$ is seen to hold.

Now, we need to show that $P(n+1)$ is true, when $P(n)$ and $P(n-1)$ is true. That is, Assuming the following statements are true.

$$\begin{aligned} f_{m+n} &= f_m f_{n+1} + f_{m-1} f_n \\ f_{m+n-1} &= f_m f_n + f_{m-1} f_{n-1} \end{aligned}$$

We need to show that $f_{m+n+1} = f_m f_{n+2} + f_{m-1} f_{n+1}$ Now

$$\begin{aligned} f_{m+n+1} \quad &= f_{m+n} + f_{m+n-1} \\ &= f_{m-1} f_n + f_m f_{n+1} + f_{m-1} f_{n-1} + f_m f_n \\ &= f_{m-1} \left(f_n + f_{n-1}\right) + f_m \left(f_{n+1} + f_n\right) \\ &= f_{m-1} f_{n+1} + f_m f_{n+2} \end{aligned}$$

6 marks for each part [18 points]

Problem 2

For any set T whose elements are positive integers, define $\phi(T)$ to be the square of the product of the elements of T. For example, if $T = \{1, 2, 5, 6\}$, then $\phi(T) = (1 \cdot 2 \cdot 5 \cdot 6)^2 = 60^2 = 3600$. For any positive integer n, consider all nonempty subsets S of $\{1, 2, \ldots, n\}$ that do not contain two consecutive integers. Prove that Σ, the sum of all the $\phi(S)$'s of these subsets is given as

$$\Sigma = (n+1)! - 1$$

.

Solution : For this problem, we will need strong induction, assuming that $P(n)$ and $P(n-1)$ are true, in order to prove that $P(n+1)$ is true.

First of all, it is clear that the base case works, for $n = 1$ we notice that there is just one subset possible, which is the set itself, therefore $\Sigma = 1 = 2! - 1$. Also for $n = 2$ The possible subsets are $\{\{1\}, \{2\}\}$. Therefore, $\Sigma = 5 = 3! - 1$. Now we proof for general n.

Consider the new set $S' = \{1, 2, \ldots, n+1\}$. Now any subset X of S' can be one of two types.

a) The subset does not contain $n + 1$.

b) The subset contains $n + 1$.

Let us call the sum of the ϕs on the subsets of type (a) as Σ_1, and the sum of the ϕs on the subsets of type (b) as Σ_2.

We notice that for type (a), since the set does not contain $n + 1$, it's as if we do not even have $n + 1$ in our set, therefore we are just solving $P(n)$. Therefore we have

$$\Sigma_1 = (n + 1)! - 1$$

For type (b), since we assume that we have $n + 1$ in our subset, therefore we cannot choose n (Since the subsets cannot have two consecutive integers). Therefore the problem reduces down to solving $P(n - 1)$ but all subsets will have an extra element of $n + 1$ in them as well. We also need to consider the case, where the subset contains only $n + 1$, since that will not be counted when solving $P(n - 1)$ since we consider only the non empty subsets of $\{1, 2, \ldots, n - 1\}$. Therefore we have the following expression for Σ_2.

$$\begin{aligned}
\Sigma_2 &= (n + 1)^2\{(n - 1 + 1)! - 1\} + (n + 1)^2 \\
&= (n + 1)^2 \cdot n! \\
&= (n + 1) \cdot (n + 1)!
\end{aligned}$$

Therefore Σ is given as

$$\begin{aligned}
\Sigma &= \Sigma_1 + \Sigma_2 \\
&= (n + 1)! - 1 + (n + 1)(n + 1)! \\
&= (n + 1 + 1)(n + 1)! - 1 \\
&= (n + 2)! - 1
\end{aligned}$$

as desired [10 points]

Problem 3

Consider a function refl that reflects a binary tree (each node's left subtree becomes its right subtree, and vice versa). Prove by induction on the set of all binary trees that the reflection is its own inverse, *i.e.*, for all binary trees T,

$$\operatorname{refl}(\operatorname{refl}(T)) = T.$$

Solution:

As a base case, consider a tree with a single node. As the reflection doesn't change it, it is its own inverse.

We will proceed by induction on the height $h(T)$ of a tree. Consider an arbitrary tree $T = (b, L, R)$ with root node b and left and right subtrees L and R respectively. Clearly, $h(T) = 1 + \max(h(L), h(R)) \implies h(L), h(R) < h(T)$.

We have that $\operatorname{refl}(T) = (b, \operatorname{refl}(L), \operatorname{refl}(R))$, and so $\operatorname{refl}(\operatorname{refl}(T)) = (b, \operatorname{refl}(\operatorname{refl}(L)), \operatorname{refl}(\operatorname{refl}(R)))$. However, since $h(L)$ and $h(R)$ are strictly less than $h(T)$, the induction hypothesis tells us that $\operatorname{refl}(\operatorname{refl}(L)) = L$ and $\operatorname{refl}(\operatorname{refl}(R)) = R$. Thus we have $\operatorname{refl}(\operatorname{refl}(T)) = (b, L, R) = T$, QED.

[10 points]

Problem 4

Prove the following

1. The sum of all degrees of an undirected graph is even.
2. In a graph $G = (V, D)$, let $d(u, v)$ be the length of shortest path connecting the two vertices u and v. Prove that the function d satisfies the following:
   $d(u, v) + d(v, w) \geq d(u, w)$ where $u, v, w \in V$

Solution:

1. Each edge ends at two vertices. If we begin with just the vertices and no edges, every vertex has degree zero, so the sum of those degrees is zero, an even number. Now add edges one at a time, each of which connects one vertex to another, or connects a vertex to itself (if you allow that). Either the degree of two vertices is increased by one (for a total of two) or one vertex's degree is increased by two. In either case, the sum of the degrees is increased by two, so the sum remains even.
2. If you simply connect the paths from u to v to the path connecting v to w you will have a valid path of length d(u, v) + d(v, w). Since we are looking for the path of minimal length, if there is a shorter path it will be shorter than this one, so the triangle inequality will be satisfied.

5 each

[10 points]

Problem 5

For a simple undirected graph $G = (V, D)$, Let n be the number of vertices and x_i be the degree of the vertex v_i. For the given series of $(x_1, x_2, \ldots, x_n)$, sketch a graph with the given degrees. If not possible, explain why.

1. (4, 3, 2, 2)
2. (2, 1, 1, 0)
3. (4, 4, 4, 4, 2)
4. (4, 2, 2, 1, 1)
5. (3, 2, 1)

Solution:
We have mentioned **simple** and by defination loops are not allowed.

1. Not possible: sum of degrees is not even
2. Possible (Check answer if it is satisfying)
3. Not possible: the four 4s must be adjacent to every other vertex, but the 1 won't allow this.
4. Possible (Check answer if it is satisfying)
5. Not Possible: The 3 vertex graph cannot have degree more than 2.

[10 points]

Problem 6

On an 8*8 chessboard, A new piece is proposed we call "Jack". Jack can move either (3 squares vertically and 2 squares horizontally) or (2 squares vertically and 3 squares horizontally)

1. Define a relation that relates all squares to the squares Jack can go to in a single move
2. Show that Jack can move to any square of the same color
3. Show that Jack can move from any square of the board to any other square

You can consider the chessboard as a set of coordinate pairs:

$$V = \{(x, y)|x, y \in \mathbb{N} \text{ and } 0 \leq x, y, \leq 7\}$$

Adjacent squares have different colors (basically a conventional chessboard)

Solution:

A

$$R = \{((x, y), (x \pm 3, y \pm 2))|\forall(x, y) \in V\} \cup \{((x, y), (x \pm 2, y \pm 3))|\forall(x, y) \in V\}$$

B

Transitive closure is reachability

Claim: For valid (x, y), $(x + 1, y + 1)$, Jack can reach from (x, y) to $(x - 1, y + 1)$

Proof,

moves: $(x, y) \to (x + 2, y + 3)$ $(x + 2, y + 3) \to (x - 1, y + 1)$

Claim: For valid (x, y), $(x + 1, y + 1)$, Jack can reach from (x, y) to $(x + 1, y - 1)$

Proof,

moves: $(x, y) \to (x + 3, y + 2)$ $(x + 3, y + 2) \to (x + 1, y - 1)$

Claim: For valid (x, y), $(x + 1, y + 1)$, Jack can reach from (x, y) to $(x - 1, y - 1)$

Proof,

moves: $(x, y) \to (x - 3, y + 2)$ $(x - 3, y + 2) \to (x - 1, y - 1)$

Claim: For valid (x, y), $(x + 1, y + 1)$, Jack can reach from (x, y) to $(x + 1, y + 1)$

Proof,

moves: $(x, y) \to (x + 3, y - 2)$ $(x + 3, y - 2) \to (x + 1, y + 1)$

A square on diagonal line that goes through (x, y) is typically of the form $(x \pm a, y \pm a)$

This means that Jack can move to any diagonal square from a given square.

In a chessboard, you can go from any black diagonal to any other black diagonal using a black diagonal (similarly for white diagonals)

In other words, the transitive closure of this covers all squares of the same color

C

Given results from B, if Jack can change colors, it can move to any square of the board to any other square

In a single move

$$(x, y) \to (x + 2, y + 3)$$

This will lead to a change of color, because

(x, y) and $(x+2, y)$ are the same color
but $(x+2, y)$ and $(x+2, y+3)$ are different colors
thus (x, y) and $(x+2, y+3)$ are different colors