Show that $(C[0,1], d_\infty)$ is a metric space.

$$d_\infty(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|, \text{ where } f(x), g(x) \in C[0,1].$$

(1)

if $\max_{x\in[0,1]} |f(x) - g(x)| = 0$

then $|f(x) - g(x)| = 0 \ \forall x \in [0,1]$

i.e, $f(x) = g(x) \ \forall x \in [0,1]$

$\Rightarrow d_\infty(f,g) = 0$ iff $f = g \ \forall x \in [0,1]$.

Otherwise $d_\infty(f,g) > 0$ if $f \neq g$, $f, g \in C[0,1]$.

(II)

Now

$$d_\infty(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$$

$$= \max_{x\in[0,1]} |g(x) - f(x)|$$

$$= d_\infty(g,f) \quad \forall f, g \in C[0,1].$$

(III)

Now,

$$|f(x) - g(x)| = |f(x) - h(x) + h(x) - g(x)| \quad \forall x \in [0,1]$$

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \quad \forall x \in [0,1].$$

$$\max_{x\in[0,1]} |f(x) - g(x)| \leq \max_{x\in[0,1]} |f(x) - h(x)| + \max_{x\in[0,1]} |h(x) - g(x)|$$

$$d_\infty(f,g) \leq d_\infty(f,h) + d_\infty(h,g).$$

Hence $(C[0,1], d_\infty)$ is a metric space.

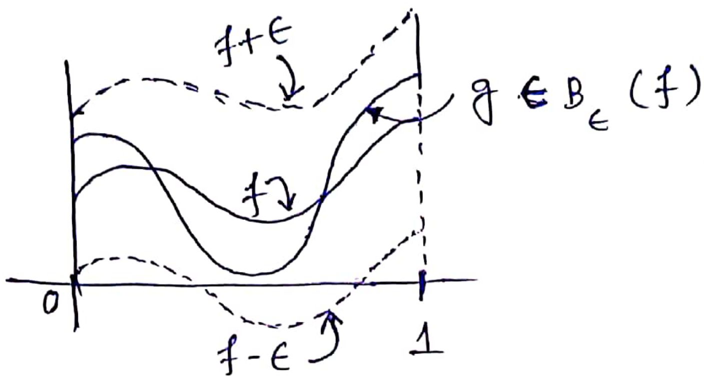
Consider, $f \in C[0,1]$ and $\epsilon > 0$

In $C[0,1]$, an open ball with centre at f ~~to~~gether with radius 'ϵ' defined as $B_\epsilon(f)$. (With respect to the metric d_∞).

$$= \{ g \in C[0,1] \; ; \; d_\infty(f,g) < \epsilon \}$$

$$= \{ g \in C[0,1] \; ; \; \max_{x \in [0,1]} |f(x) - g(x)| < \epsilon \}$$

$B_\epsilon(f)$



d_p is another norm defined on $C[0,1]$.

Let $f, g \in C[0,1]$

$$d_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p \, dx \right)^{1/p}.$$

Similarly

$$d_q(f,g) = \left(\int_0^1 |f(x) - g(x)|^q \, dx \right)^{1/q}.$$

But the sup-norm, i.e d_∞ over $\mathbb{R}^n$ can be defined as,

$$d_\infty(X,Y) = d_\infty\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big)$$

$$= \max_{i=1(1)n} |x_i - y_i| \quad ; \text{ where } X, Y \in \mathbb{R}^n.$$

and the $p(\geq 1)$-norm over $\mathbb{R}^n$ defined by

$$d_p(X,Y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

Now

(X, d) be a metric space, and (X, d_1) is another metric space. Now d, d_1 are two metrices defined over X.

Now, d_0 and d_1 ~~can~~ are called equivalent, if $\exists$ two constants $M \geq m > 0$ such that,

$$m\,d(x,y) \leq d_1(x,y) \leq M\,d(x,y) \quad \forall\, x, y \in X.$$

holds.