14

First-order ordinary differential equations

Differential equations are the group of equations that contain derivatives. Chapters 14–21 discuss a variety of differential equations, starting in this chapter and the next with those ordinary differential equations (ODEs) that have closed-form solutions. As its name suggests, an ODE contains only ordinary derivatives (no partial derivatives) and describes the relationship between these derivatives of the *dependent variable*, usually called y, with respect to the *independent variable*, usually called x. The solution to such an ODE is therefore a function of x and is written $y(x)$. For an ODE to have a closed-form solution, it must be possible to express $y(x)$ in terms of the standard elementary functions such as $\exp x$, $\ln x$, $\sin x$ etc. The solutions of some differential equations cannot, however, be written in closed form, but only as an infinite series; these are discussed in chapter 16.

Ordinary differential equations may be separated conveniently into different categories according to their general characteristics. The primary grouping adopted here is by the *order* of the equation. The order of an ODE is simply the order of the highest derivative it contains. Thus equations containing dy/dx, but no higher derivatives, are called first order, those containing d^2y/dx^2 are called second order and so on. In this chapter we consider first-order equations, and in the next, second- and higher-order equations.

Ordinary differential equations may be classified further according to *degree*. The degree of an ODE is the power to which the highest-order derivative is raised, after the equation has been rationalised to contain only integer powers of derivatives. Hence the ODE

$$\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2y = 0,$$

is of third order and second degree, since after rationalisation it contains the term $(d^3y/dx^3)^2$.

The *general solution* to an ODE is the most general function $y(x)$ that satisfies the equation; it will contain *constants of integration* which may be determined by the application of some suitable *boundary conditions*. For example, we may be told that for a certain first-order differential equation, the solution $y(x)$ is equal to zero when the parameter x is equal to unity; this allows us to determine the value of the constant of integration. The *general solutions* to nth-order ODEs, which are considered in detail in the next chapter, will contain n (essential) arbitrary constants of integration and therefore we will need n boundary conditions if these constants are to be determined (see section 14.1). When the boundary conditions have been applied, and the constants found, we are left with a *particular solution* to the ODE, which obeys the given boundary conditions. Some ODEs of degree greater than unity also possess *singular solutions*, which are solutions that contain no arbitrary constants and cannot be found from the general solution; singular solutions are discussed in more detail in section 14.3. When any solution to an ODE has been found, it is always possible to check its validity by substitution into the original equation and verification that any given boundary conditions are met.

In this chapter, firstly we discuss various types of first-degree ODE and then go on to examine those higher-degree equations that can be solved in closed form. At the outset, however, we discuss the general form of the solutions of ODEs; this discussion is relevant to both first- and higher-order ODEs.

14.1 General form of solution

It is helpful when considering the general form of the solution of an ODE to consider the inverse process, namely that of obtaining an ODE from a given group of functions, each one of which is a solution of the ODE. Suppose the members of the group can be written as

$$y = f(x, a_1, a_2, \ldots, a_n), \tag{14.1}$$

each member being specified by a different set of values of the parameters a_i. For example, consider the group of functions

$$y = a_1 \sin x + a_2 \cos x; \tag{14.2}$$

here $n = 2$.

Since an ODE is required for which *any* of the group is a solution, it clearly must not contain any of the a_i. As there are n of the a_i in expression (14.1), we must obtain $n+1$ equations involving them in order that, by elimination, we can obtain one final equation without them.

Initially we have only (14.1), but if this is differentiated n times, a total of $n+1$ equations is obtained from which (in principle) all the a_i can be eliminated, to give one ODE satisfied by all the group. As a result of the n differentiations, $d^n y/dx^n$ will be present in one of the $n+1$ equations and hence in the final equation, which will therefore be of nth order.

In the case of (14.2), we have

$$\frac{dy}{dx} = a_1 \cos x - a_2 \sin x,$$
$$\frac{d^2y}{dx^2} = -a_1 \sin x - a_2 \cos x.$$

Here the elimination of a_1 and a_2 is trivial (because of the similarity of the forms of y and d^2y/dx^2), resulting in

$$\frac{d^2y}{dx^2} + y = 0,$$

a second-order equation.

Thus, to summarise, a group of functions (14.1) with n parameters satisfies an nth-order ODE in general (although in some degenerate cases an ODE of less than nth order is obtained). The intuitive converse of this is that the general solution of an nth-order ODE contains n arbitrary parameters (constants); for our purposes, this will be assumed to be valid although a totally general proof is difficult.

As mentioned earlier, external factors affect a system described by an ODE, by fixing the values of the dependent variables for particular values of the independent ones. These externally imposed (or *boundary*) conditions on the solution are thus the means of determining the parameters and so of specifying precisely which function is the required solution. It is apparent that the number of boundary conditions should match the number of parameters and hence the order of the equation, if a unique solution is to be obtained. Fewer independent boundary conditions than this will lead to a number of undetermined parameters in the solution, whilst an excess will usually mean that no acceptable solution is possible.

For an nth-order equation the required n boundary conditions can take many forms, for example the value of y at n different values of x, or the value of any $n-1$ of the n derivatives dy/dx, d^2y/dx^2, ..., d^ny/dx^n together with that of y, all for the same value of x, or many intermediate combinations.

14.2 First-degree first-order equations

First-degree first-order ODEs contain only dy/dx equated to some function of x and y, and can be written in either of two equivalent standard forms,

$$\frac{dy}{dx} = F(x, y), \qquad A(x, y)\, dx + B(x, y)\, dy = 0,$$

where $F(x, y) = -A(x, y)/B(x, y)$, and $F(x, y)$, $A(x, y)$ and $B(x, y)$ are in general functions of both x and y. Which of the two above forms is the more useful for finding a solution depends on the type of equation being considered. There are several different types of first-degree first-order ODEs that are of interest in the physical sciences. These equations and their respective solutions are discussed below.

14.2.1 Separable-variable equations

A separable-variable equation is one which may be written in the conventional form

$$\frac{dy}{dx} = f(x)g(y), \tag{14.3}$$

where $f(x)$ and $g(y)$ are functions of x and y respectively, including cases in which $f(x)$ or $g(y)$ is simply a constant. Rearranging this equation so that the terms depending on x and on y appear on opposite sides (i.e. are separated), and integrating, we obtain

$$\int \frac{dy}{g(y)} = \int f(x)\, dx.$$

Finding the solution $y(x)$ that satisfies (14.3) then depends only on the ease with which the integrals in the above equation can be evaluated. It is also worth noting that ODEs that at first sight do not appear to be of the form (14.3) can sometimes be made separable by an appropriate factorisation.

►*Solve*

$$\frac{dy}{dx} = x + xy.$$

Since the RHS of this equation can be factorised to give $x(1 + y)$, the equation becomes separable and we obtain

$$\int \frac{dy}{1+y} = \int x\, dx.$$

Now integrating both sides separately, we find

$$\ln(1 + y) = \frac{x^2}{2} + c,$$

and so

$$1 + y = \exp\left(\frac{x^2}{2} + c\right) = A\exp\left(\frac{x^2}{2}\right),$$

where c and hence A is an arbitrary constant. ◄

Solution method. *Factorise the equation so that it becomes separable. Rearrange it so that the terms depending on x and those depending on y appear on opposite sides and then integrate directly. Remember the constant of integration, which can be evaluated if further information is given.*

14.2.2 Exact equations

An *exact* first-degree first-order ODE is one of the form

$$A(x,y)\,dx + B(x,y)\,dy = 0 \quad \text{and for which} \quad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{14.4}$$

In this case $A(x,y)\,dx + B(x,y)\,dy$ is an exact differential, $dU(x,y)$ say (see section 5.3). In other words

$$A\,dx + B\,dy = dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy,$$

from which we obtain

$$A(x,y) = \frac{\partial U}{\partial x}, \tag{14.5}$$

$$B(x,y) = \frac{\partial U}{\partial y}. \tag{14.6}$$

Since $\partial^2 U/\partial x \partial y = \partial^2 U/\partial y \partial x$ we therefore require

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{14.7}$$

If (14.7) holds then (14.4) can be written $dU(x,y) = 0$, which has the solution $U(x,y) = c$, where c is a constant and from (14.5) $U(x,y)$ is given by

$$U(x,y) = \int A(x,y)\,dx + F(y). \tag{14.8}$$

The function $F(y)$ can be found from (14.6) by differentiating (14.8) with respect to y and equating to $B(x,y)$.

►*Solve*

$$x\frac{dy}{dx} + 3x + y = 0.$$

Rearranging into the form (14.4) we have

$$(3x + y)\,dx + x\,dy = 0,$$

i.e. $A(x,y) = 3x + y$ and $B(x,y) = x$. Since $\partial A/\partial y = 1 = \partial B/\partial x$, the equation is exact, and by (14.8) the solution is given by

$$U(x,y) = \int (3x + y)\,dx + F(y) = c_1 \qquad \Rightarrow \qquad \frac{3x^2}{2} + yx + F(y) = c_1.$$

Differentiating $U(x,y)$ with respect to y and equating it to $B(x,y) = x$ we obtain $dF/dy = 0$, which integrates immediately to give $F(y) = c_2$. Therefore, letting $c = c_1 - c_2$, the solution to the original ODE is

$$\frac{3x^2}{2} + xy = c. \blacktriangleleft$$

Solution method. *Check that the equation is an exact differential using (14.7) then solve using (14.8). Find the function $F(y)$ by differentiating (14.8) with respect to y and using (14.6).*

14.2.3 Inexact equations: integrating factors

Equations that may be written in the form

$$A(x, y)\, dx + B(x, y)\, dy = 0 \qquad \text{but for which} \quad \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \tag{14.9}$$

are known as inexact equations. However, the differential $A\,dx + B\,dy$ can always be made exact by multiplying by an *integrating factor* $\mu(x, y)$, which obeys

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}. \tag{14.10}$$

For an integrating factor that is a function of both x and y, i.e. $\mu = \mu(x, y)$, there exists no general method for finding it; in such cases it may sometimes be found by inspection. If, however, an integrating factor exists that is a function of either x or y alone then (14.10) can be solved to find it. For example, if we assume that the integrating factor is a function of x alone, i.e. $\mu = \mu(x)$, then (14.10) reads

$$\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{d\mu}{dx}.$$

Rearranging this expression we find

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx = f(x)\, dx,$$

where we require $f(x)$ also to be a function of x only; indeed this provides a general method of determining whether the integrating factor μ is a function of x alone. This integrating factor is then given by

$$\mu(x) = \exp\left\{\int f(x)\, dx\right\} \qquad \text{where} \qquad f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right). \tag{14.11}$$

Similarly, if $\mu = \mu(y)$ then

$$\mu(y) = \exp\left\{\int g(y)\, dy\right\} \qquad \text{where} \qquad g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right). \tag{14.12}$$

►*Solve*

$$\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}.$$

Rearranging into the form (14.9), we have

$$(4x + 3y^2)\,dx + 2xy\,dy = 0, \tag{14.13}$$

i.e. $A(x,y) = 4x + 3y^2$ and $B(x,y) = 2xy$. Now

$$\frac{\partial A}{\partial y} = 6y, \qquad \frac{\partial B}{\partial x} = 2y,$$

so the ODE is not exact in its present form. However, we see that

$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x},$$

a function of x alone. Therefore an integrating factor exists that is also a function of x alone and, ignoring the arbitrary constant of integration, is given by

$$\mu(x) = \exp\left\{2\int \frac{dx}{x}\right\} = \exp(2\ln x) = x^2.$$

Multiplying (14.13) through by $\mu(x) = x^2$ we obtain

$$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 4x^3\,dx + (3x^2y^2\,dx + 2x^3y\,dy) = 0.$$

By inspection this integrates immediately to give the solution $x^4 + y^2x^3 = c$, where c is a constant. ◄

Solution method. *Examine whether $f(x)$ and $g(y)$ are functions of only x or y respectively. If so, then the required integrating factor is a function of either x or y only, and is given by (14.11) or (14.12) respectively. If the integrating factor is a function of both x and y, then sometimes it may be found by inspection or by trial and error. In any case, the integrating factor μ must satisfy (14.10). Once the equation has been made exact, solve by the method of subsection 14.2.2.*

14.2.4 Linear equations

Linear first-order ODEs are a special case of inexact ODEs (discussed in the previous subsection) and can be written in the conventional form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{14.14}$$

Such equations can be made exact by multiplying through by an appropriate integrating factor in a similar manner to that discussed above. In this case, however, the integrating factor is always a function of x alone and may be expressed in a particularly simple form. An integrating factor $\mu(x)$ must be such that

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x), \tag{14.15}$$

which may then be integrated directly to give

$$\mu(x)y = \int \mu(x)Q(x)\,dx. \tag{14.16}$$

The required integrating factor $\mu(x)$ is determined by the first equality in (14.15), i.e.

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu P y,$$

which immediately gives the simple relation

$$\frac{d\mu}{dx} = \mu(x)P(x) \qquad \Rightarrow \qquad \mu(x) = \exp\left\{\int P(x)\,dx\right\}. \tag{14.17}$$

►*Solve*

$$\frac{dy}{dx} + 2xy = 4x.$$

The integrating factor is given immediately by

$$\mu(x) = \exp\left\{\int 2x\,dx\right\} = \exp x^2.$$

Multiplying through the ODE by $\mu(x) = \exp x^2$ and integrating, we have

$$y\exp x^2 = 4\int x\exp x^2\,dx = 2\exp x^2 + c.$$

The solution to the ODE is therefore given by $y = 2 + c\exp(-x^2)$. ◄

Solution method. *Rearrange the equation into the form (14.14) and multiply by the integrating factor $\mu(x)$ given by (14.17). The left- and right-hand sides can then be integrated directly, giving y from (14.16).*

14.2.5 Homogeneous equations

Homogeneous equation are ODEs that may be written in the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right), \tag{14.18}$$

where $A(x,y)$ and $B(x,y)$ are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if, for any λ, it obeys

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

For example, if $A = x^2y - xy^2$ and $B = x^3 + y^3$ then we see that A and B are both homogeneous functions of degree 3. In general, for functions of the form of A and B, we see that for both to be homogeneous, and of the same degree, we require the sum of the powers in x and y in each term of A and B to be the same (in this example equal to 3). The RHS of a homogeneous ODE can be written as a function of y/x. The equation may then be solved by making the substitution $y = vx$, so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v).$$

This is now a separable equation and can be integrated directly to give

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}. \tag{14.19}$$

►*Solve*

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$

Substituting $y = vx$ we obtain

$$v + x\frac{dv}{dx} = v + \tan v.$$

Cancelling v on both sides, rearranging and integrating gives

$$\int \cot v \, dv = \int \frac{dx}{x} = \ln x + c_1.$$

But

$$\int \cot v \, dv = \int \frac{\cos v}{\sin v} \, dv = \ln(\sin v) + c_2,$$

so the solution to the ODE is $y = x \sin^{-1} Ax$, where A is a constant. ◄

Solution method. *Check to see whether the equation is homogeneous. If so, make the substitution $y = vx$, separate variables as in (14.19) and then integrate directly. Finally replace v by y/x to obtain the solution.*

14.2.6 Isobaric equations

An isobaric ODE is a generalisation of the homogeneous ODE discussed in the previous section, and is of the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)}, \tag{14.20}$$

where the equation is dimensionally consistent if y and dy are each given a weight m relative to x and dx, i.e. if the substitution $y = vx^m$ makes it separable.

►*Solve*

$$\frac{dy}{dx} = \frac{-1}{2yx}\left(y^2 + \frac{2}{x}\right).$$

Rearranging we have

$$\left(y^2 + \frac{2}{x}\right) dx + 2yx\, dy = 0.$$

Giving y and dy the weight m and x and dx the weight 1, the sums of the powers in each term on the LHS are $2m+1$, 0 and $2m+1$ respectively. These are equal if $2m+1=0$, i.e. if $m = -\frac{1}{2}$. Substituting $y = vx^m = vx^{-1/2}$, with the result that $dy = x^{-1/2}\, dv - \frac{1}{2}vx^{-3/2}\, dx$, we obtain

$$v\, dv + \frac{dx}{x} = 0,$$

which is separable and may be integrated directly to give $\frac{1}{2}v^2 + \ln x = c$. Replacing v by $y\sqrt{x}$ we obtain the solution $\frac{1}{2}y^2 x + \ln x = c$. ◄

Solution method. *Write the equation in the form $A\, dx + B\, dy = 0$. Giving y and dy each a weight m and x and dx each a weight 1, write down the sum of powers in each term. Then, if a value of m that makes all these sums equal can be found, substitute $y = vx^m$ into the original equation to make it separable. Integrate the separated equation directly, and then replace v by yx^{-m} to obtain the solution.*

14.2.7 Bernoulli's equation

Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad \text{where } n \neq 0 \text{ or } 1. \tag{14.21}$$

This equation is very similar in form to the linear equation (14.14), but is in fact non-linear due to the extra y^n factor on the RHS. However, the equation can be made linear by substituting $v = y^{1-n}$ and correspondingly

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}.$$

Substituting this into (14.21) and dividing through by y^n, we find

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which is a linear equation and may be solved by the method described in subsection 14.2.4.

►*Solve*
$$\frac{dy}{dx} + \frac{y}{x} = 2x^3y^4.$$

If we let $v = y^{1-4} = y^{-3}$ then
$$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dv}{dx}.$$
Substituting this into the ODE and rearranging, we obtain
$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3,$$
which is linear and may be solved by multiplying through by the integrating factor (see subsection 14.2.4)
$$\exp\left\{-3\int\frac{dx}{x}\right\} = \exp(-3\ln x) = \frac{1}{x^3}.$$
This yields the solution
$$\frac{v}{x^3} = -6x + c.$$
Remembering that $v = y^{-3}$, we obtain $y^{-3} = -6x^4 + cx^3$. ◄

Solution method. *Rearrange the equation into the form (14.21) and make the substitution $v = y^{1-n}$. This leads to a linear equation in v, which can be solved by the method of subsection 14.2.4. Then replace v by y^{1-n} to obtain the solution.*

14.2.8 Miscellaneous equations

There are two further types of first-degree first-order equation that occur fairly regularly but do not fall into any of the above categories. They may be reduced to one of the above equations, however, by a suitable change of variable.

Firstly, we consider
$$\frac{dy}{dx} = F(ax + by + c), \tag{14.22}$$
where a, b and c are constants, i.e. x and y *only* appear on the RHS in the particular combination $ax + by + c$ and not in any other combination or by themselves. This equation can be solved by making the substitution $v = ax + by + c$, in which case
$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bF(v), \tag{14.23}$$
which is separable and may be integrated directly.

►*Solve*

$$\frac{dy}{dx} = (x + y + 1)^2.$$

Making the substitution $v = x + y + 1$, we obtain, as in (14.23),

$$\frac{dv}{dx} = v^2 + 1,$$

which is separable and integrates directly to give

$$\int \frac{dv}{1 + v^2} = \int dx \quad \Rightarrow \quad \tan^{-1} v = x + c_1.$$

So the solution to the original ODE is $\tan^{-1}(x + y + 1) = x + c_1$, where c_1 is a constant of integration. ◄

Solution method. *In an equation such as (14.22), substitute $v = ax+by+c$ to obtain a separable equation that can be integrated directly. Then replace v by $ax + by + c$ to obtain the solution.*

Secondly, we discuss

$$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g}, \tag{14.24}$$

where a, b, c, e, f and g are all constants. This equation may be solved by letting $x = X + \alpha$ and $y = Y + \beta$, where α and β are constants found from

$$a\alpha + b\beta + c = 0 \tag{14.25}$$

$$e\alpha + f\beta + g = 0. \tag{14.26}$$

Then (14.24) can be written as

$$\frac{dY}{dX} = \frac{aX + bY}{eX + fY},$$

which is homogeneous and can be solved by the method of subsection 14.2.5. Note, however, that if $a/e = b/f$ then (14.25) and (14.26) are not independent and so cannot be solved uniquely for α and β. However, in this case, (14.24) reduces to an equation of the form (14.22), which was discussed above.

►*Solve*

$$\frac{dy}{dx} = \frac{2x - 5y + 3}{2x + 4y - 6}.$$

Let $x = X + \alpha$ and $y = Y + \beta$, where α and β obey the relations

$$2\alpha - 5\beta + 3 = 0$$

$$2\alpha + 4\beta - 6 = 0,$$

which solve to give $\alpha = \beta = 1$. Making these substitutions we find

$$\frac{dY}{dX} = \frac{2X - 5Y}{2X + 4Y},$$

which is a homogeneous ODE and can be solved by substituting $Y = vX$ (see subsection 14.2.5) to obtain

$$\frac{dv}{dX} = \frac{2 - 7v - 4v^2}{X(2 + 4v)}.$$

This equation is separable, and using partial fractions we find

$$\int \frac{2 + 4v}{2 - 7v - 4v^2}\, dv = -\frac{4}{3}\int \frac{dv}{4v - 1} - \frac{2}{3}\int \frac{dv}{v + 2} = \int \frac{dX}{X},$$

which integrates to give

$$\ln X + \tfrac{1}{3}\ln(4v - 1) + \tfrac{2}{3}\ln(v + 2) = c_1,$$

or

$$X^3(4v - 1)(v + 2)^2 = \exp 3c_1.$$

Remembering that $Y = vX$, $x = X + 1$ and $y = Y + 1$, the solution to the original ODE is given by $(4y - x - 3)(y + 2x - 3)^2 = c_2$, where $c_2 = \exp 3c_1$. ◄

Solution method. *If in (14.24) $a/e \neq b/f$ then make the substitution $x = X + \alpha$, $y = Y + \beta$, where α and β are given by (14.25) and (14.26); the resulting equation is homogeneous and can be solved as in subsection 14.2.5. Substitute $v = Y/X$, $X = x - \alpha$ and $Y = y - \beta$ to obtain the solution. If $a/e = b/f$ then (14.24) is of the same form as (14.22) and may be solved accordingly.*

14.3 Higher-degree first-order equations

First-order equations of degree higher than the first do not occur often in the description of physical systems, since squared and higher powers of first-order derivatives usually arise from resistive or driving mechanisms, when an acceleration or other higher-order derivative is also present. They do sometimes appear in connection with geometrical problems, however.

Higher-degree first-order equations can be written as $F(x, y, dy/dx) = 0$. The most general standard form is

$$p^n + a_{n-1}(x, y)p^{n-1} + \cdots + a_1(x, y)p + a_0(x, y) = 0, \qquad (14.27)$$

where for ease of notation we write $p = dy/dx$. If the equation can be solved for one of x, y or p then either an explicit or a parametric solution can sometimes be obtained. We discuss the main types of such equations below, including Clairaut's equation, which is a special case of an equation explicitly soluble for y.

14.3.1 Equations soluble for p

Sometimes the LHS of (14.27) can be factorised into the form

$$(p - F_1)(p - F_2)\cdots(p - F_n) = 0, \qquad (14.28)$$

where $F_i = F_i(x, y)$. We are then left with solving the n first-degree equations $p = F_i(x, y)$. Writing the solutions to these first-degree equations as $G_i(x, y) = 0$, the general solution to (14.28) is given by the product

$$G_1(x, y)G_2(x, y) \cdots G_n(x, y) = 0. \tag{14.29}$$

►*Solve*

$$(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp + 2xy^2 = 0. \tag{14.30}$$

This equation may be factorised to give

$$[(x + 1)p - y][(x^2 + 1)p - 2xy] = 0.$$

Taking each bracket in turn we have

$$(x + 1)\frac{dy}{dx} - y = 0,$$
$$(x^2 + 1)\frac{dy}{dx} - 2xy = 0,$$

which have the solutions $y - c(x + 1) = 0$ and $y - c(x^2 + 1) = 0$ respectively (see section 14.2 on first-degree first-order equations). Note that the arbitrary constants in these two solutions can be taken to be the same, since only one is required for a first-order equation. The general solution to (14.30) is then given by

$$[y - c(x + 1)]\left[y - c(x^2 + 1)\right] = 0. \blacktriangleleft$$

Solution method. *If the equation can be factorised into the form (14.28) then solve the first-order ODE $p - F_i = 0$ for each factor and write the solution in the form $G_i(x, y) = 0$. The solution to the original equation is then given by the product (14.29).*

14.3.2 Equations soluble for x

Equations that can be solved for x, i.e. such that they may be written in the form

$$x = F(y, p), \tag{14.31}$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to y, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy}.$$

This results in an equation of the form $G(y, p) = 0$, which can be used together with (14.31) to eliminate p and give the general solution. Note that often a singular solution to the equation will be found at the same time (see the introduction to this chapter).

►*Solve*

$$6y^2p^2 + 3xp - y = 0. \tag{14.32}$$

This equation can be solved for x explicitly to give $3x = (y/p) - 6y^2p$. Differentiating both sides with respect to y, we find

$$3\frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12yp,$$

which factorises to give

$$\left(1 + 6yp^2\right)\left(2p + y\frac{dp}{dy}\right) = 0. \tag{14.33}$$

Setting the factor containing dp/dy equal to zero gives a first-degree first-order equation in p, which may be solved to give $py^2 = c$. Substituting for p in (14.32) then yields the general solution of (14.32):

$$y^3 = 3cx + 6c^2. \tag{14.34}$$

If we now consider the first factor in (14.33), we find $6p^2y = -1$ as a possible solution. Substituting for p in (14.32) we find the singular solution

$$8y^3 + 3x^2 = 0.$$

Note that the singular solution contains no arbitrary constants and cannot be found from the general solution (14.34) by any choice of the constant c. ◄

Solution method. *Write the equation in the form (14.31) and differentiate both sides with respect to y. Rearrange the resulting equation into the form $G(y, p) = 0$, which can be used together with the original ODE to eliminate p and so give the general solution. If $G(y, p)$ can be factorised then the factor containing dp/dy should be used to eliminate p and give the general solution. Using the other factors in this fashion will instead lead to singular solutions.*

14.3.3 Equations soluble for y

Equations that can be solved for y, i.e. are such that they may be written in the form

$$y = F(x, p), \tag{14.35}$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to x, so that

$$\frac{dy}{dx} = p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{dp}{dx}.$$

This results in an equation of the form $G(x, p) = 0$, which can be used together with (14.35) to eliminate p and give the general solution. An additional (singular) solution to the equation is also often found.

►*Solve*

$$xp^2 + 2xp - y = 0. \tag{14.36}$$

This equation can be solved for y explicitly to give $y = xp^2 + 2xp$. Differentiating both sides with respect to x, we find

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p,$$

which after factorising gives

$$(p+1)\left(p + 2x\frac{dp}{dx}\right) = 0. \tag{14.37}$$

To obtain the general solution of (14.36), we consider the factor containing dp/dx. This first-degree first-order equation in p has the solution $xp^2 = c$ (see subsection 14.3.1), which we then use to eliminate p from (14.36). Thus we find that the general solution to (14.36) is

$$(y - c)^2 = 4cx. \tag{14.38}$$

If instead, we set the other factor in (14.37) equal to zero, we obtain the very simple solution $p = -1$. Substituting this into (14.36) then gives

$$x + y = 0,$$

which is a singular solution to (14.36). ◄

Solution method. *Write the equation in the form (14.35) and differentiate both sides with respect to x. Rearrange the resulting equation into the form $G(x, p) = 0$, which can be used together with the original ODE to eliminate p and so give the general solution. If $G(x, p)$ can be factorised then the factor containing dp/dx should be used to eliminate p and give the general solution. Using the other factors in this fashion will instead lead to singular solutions.*

14.3.4 Clairaut's equation

Finally, we consider Clairaut's equation, which has the form

$$y = px + F(p) \tag{14.39}$$

and is therefore a special case of equations soluble for y, as in (14.35). It may be solved by a similar method to that given in subsection 14.3.3, but for Clairaut's equation the form of the general solution is particularly simple. Differentiating (14.39) with respect to x, we find

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF}{dp}\frac{dp}{dx} \quad \Rightarrow \quad \frac{dp}{dx}\left(\frac{dF}{dp} + x\right) = 0. \tag{14.40}$$

Considering first the factor containing dp/dx, we find

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad y = c_1 x + c_2. \tag{14.41}$$

Since $p = dy/dx = c_1$, if we substitute (14.41) into (14.39) we find $c_1x + c_2 = c_1x + F(c_1)$. Therefore the constant c_2 is given by $F(c_1)$, and the general solution to (14.39) is

$$y = c_1x + F(c_1), \tag{14.42}$$

i.e. the general solution to Clairaut's equation can be obtained by replacing p in the ODE by the arbitrary constant c_1. Now, considering the second factor in (14.40), we also have

$$\frac{dF}{dp} + x = 0, \tag{14.43}$$

which has the form $G(x, p) = 0$. This relation may be used to eliminate p from (14.39) to give a singular solution.

►*Solve*

$$y = px + p^2. \tag{14.44}$$

From (14.42) the general solution is $y = cx + c^2$. But from (14.43) we also have $2p + x = 0 \Rightarrow p = -x/2$. Substituting this into (14.44) we find the singular solution $x^2 + 4y = 0$. ◄

Solution method. *Write the equation in the form (14.39), then the general solution is given by replacing p by some constant c, as shown in (14.42). Using the relation $dF/dp+x = 0$ to eliminate p from the original equation yields the singular solution.*

14.4 Exercises

14.1 A radioactive isotope decays in such a way that the number of atoms present at a given time, $N(t)$, obeys the equation

$$\frac{dN}{dt} = -\lambda N.$$

If there are initially N_0 atoms present, find $N(t)$ at later times.

14.2 Solve the following equations by separation of the variables:

(a) $y' - xy^3 = 0$;
(b) $y' \tan^{-1} x - y(1 + x^2)^{-1} = 0$;
(c) $x^2y' + xy^2 = 4y^2$.

14.3 Show that the following equations either are exact or can be made exact, and solve them:

(a) $y(2x^2y^2 + 1)y' + x(y^4 + 1) = 0$;
(b) $2xy' + 3x + y = 0$;
(c) $(\cos^2 x + y \sin 2x)y' + y^2 = 0$.

14.4 Find the values of α and β that make

$$dF(x, y) = \left(\frac{1}{x^2 + 2} + \frac{\alpha}{y}\right) dx + (xy^\beta + 1)\, dy$$

an exact differential. For these values solve $F(x, y) = 0$.

14.5 By finding suitable integrating factors, solve the following equations:

(a) $(1 - x^2)y' + 2xy = (1 - x^2)^{3/2}$;
(b) $y' - y\cot x + \operatorname{cosec} x = 0$;
(c) $(x + y^3)y' = y$ (treat y as the independent variable).

14.6 By finding an appropriate integrating factor, solve

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}.$$

14.7 Find, in the form of an integral, the solution of the equation

$$\alpha\frac{dy}{dt} + y = f(t)$$

for a general function $f(t)$. Find the specific solutions for

(a) $f(t) = H(t)$,
(b) $f(t) = \delta(t)$,
(c) $f(t) = \beta^{-1}e^{-t/\beta}H(t)$ with $\beta < \alpha$.

For case (c), what happens if $\beta \to 0$?

14.8 A series electric circuit contains a resistance R, a capacitance C and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation

$$R\frac{dq}{dt} + \frac{q}{C} = V(t).$$

Assuming that initially there is no charge on the capacitor, and given that $V(t) = V_0 \sin\omega t$, find the charge on the capacitor as a function of time.

14.9 Using tangential–polar coordinates (see exercise 2.20), consider a particle of mass m moving under the influence of a force f directed towards the origin O. By resolving forces along the instantaneous tangent and normal and making use of the result of exercise 2.20 for the instantaneous radius of curvature, prove that

$$f = -mv\frac{dv}{dr} \qquad \text{and} \qquad mv^2 = fp\frac{dr}{dp}.$$

Show further that $h = mpv$ is a constant of the motion and that the law of force can be deduced from

$$f = \frac{h^2}{mp^3}\frac{dp}{dr}.$$

14.10 Use the result of exercise 14.9 to find the law of force, acting towards the origin, under which a particle must move so as to describe the following trajectories:

(a) A circle of radius a that passes through the origin;
(b) An equiangular spiral, which is defined by the property that the angle α between the tangent and the radius vector is constant along the curve.

14.11 Solve

$$(y - x)\frac{dy}{dx} + 2x + 3y = 0.$$

14.12 A mass m is accelerated by a time-varying force $\alpha\exp(-\beta t)v^3$, where v is its velocity. It also experiences a resistive force ηv, where η is a constant, owing to its motion through the air. The equation of motion of the mass is therefore

$$m\frac{dv}{dt} = \alpha\exp(-\beta t)v^3 - \eta v.$$

Find an expression for the velocity v of the mass as a function of time, given that it has an initial velocity v_0.

14.13 Using the results about Laplace transforms given in chapter 13 for df/dt and $tf(t)$, show, for a function $y(t)$ that satisfies

$$t\frac{dy}{dt} + (t-1)y = 0 \qquad (*)$$

with $y(0)$ finite, that $\bar{y}(s) = C(1+s)^{-2}$ for some constant C.

Given that

$$y(t) = t + \sum_{n=2}^{\infty} a_n t^n,$$

determine C and show that $a_n = (-1)^{n-1}/(n-1)!$. Compare this result with that obtained by integrating $(*)$ directly.

14.14 Solve

$$\frac{dy}{dx} = \frac{1}{x+2y+1}.$$

14.15 Solve

$$\frac{dy}{dx} = -\frac{x+y}{3x+3y-4}.$$

14.16 If $u = 1 + \tan y$, calculate $d(\ln u)/dy$; hence find the general solution of

$$\frac{dy}{dx} = \tan x \cos y\ (\cos y + \sin y).$$

14.17 Solve

$$x(1-2x^2y)\frac{dy}{dx} + y = 3x^2y^2,$$

given that $y(1) = 1/2$.

14.18 A reflecting mirror is made in the shape of the surface of revolution generated by revolving the curve $y(x)$ about the x-axis. In order that light rays emitted from a point source at the origin are reflected back parallel to the x-axis, the curve $y(x)$ must obey

$$\frac{y}{x} = \frac{2p}{1-p^2},$$

where $p = dy/dx$. By solving this equation for x, find the curve $y(x)$.

14.19 Find the curve with the property that at each point on it the sum of the intercepts on the x- and y-axes of the tangent to the curve (taking account of sign) is equal to 1.

14.20 Find a parametric solution of

$$x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = 0$$

as follows.

(a) Write an equation for y in terms of $p = dy/dx$ and show that

$$p = p^2 + (2px+1)\frac{dp}{dx}.$$

(b) Using p as the independent variable, arrange this as a linear first-order equation for x.

(c) Find an appropriate integrating factor to obtain

$$x = \frac{\ln p - p + c}{(1-p)^2},$$

which, together with the expression for y obtained in (a), gives a parameterisation of the solution.

(d) Reverse the roles of x and y in steps (a) to (c), putting $dx/dy = p^{-1}$, and show that essentially the same parameterisation is obtained.

14.21 Using the substitutions $u = x^2$ and $v = y^2$, reduce the equation

$$xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2 - 1)\frac{dy}{dx} + xy = 0$$

to Clairaut's form. Hence show that the equation represents a family of conics and the four sides of a square.

14.22 The action of the control mechanism on a particular system for an input $f(t)$ is described, for $t \geq 0$, by the coupled first-order equations:

$$\dot{y} + 4z = f(t),$$
$$\dot{z} - 2z = \dot{y} + \tfrac{1}{2}y.$$

Use Laplace transforms to find the response $y(t)$ of the system to a unit step input, $f(t) = H(t)$, given that $y(0) = 1$ and $z(0) = 0$.

Questions 23 to 31 are intended to give the reader practice in choosing an appropriate method. The level of difficulty varies within the set; if necessary, the hints may be consulted for an indication of the most appropriate approach.

14.23 Find the general solutions of the following:

(a) $\dfrac{dy}{dx} + \dfrac{xy}{a^2 + x^2} = x$; (b) $\dfrac{dy}{dx} = \dfrac{4y^2}{x^2} - y^2$.

14.24 Solve the following first-order equations for the boundary conditions given:

(a) $y' - (y/x) = 1$, $y(1) = -1$;
(b) $y' - y\tan x = 1$, $y(\pi/4) = 3$;
(c) $y' - y^2/x^2 = 1/4$, $y(1) = 1$;
(d) $y' - y^2/x^2 = 1/4$, $y(1) = 1/2$.

14.25 An electronic system has two inputs, to each of which a constant unit signal is applied, but starting at different times. The equations governing the system thus take the form

$$\dot{x} + 2y = H(t),$$
$$\dot{y} - 2x = H(t-3).$$

Initially (at $t = 0$), $x = 1$ and $y = 0$; find $x(t)$ at later times.

14.26 Solve the differential equation

$$\sin x\frac{dy}{dx} + 2y\cos x = 1,$$

subject to the boundary condition $y(\pi/2) = 1$.

14.27 Find the complete solution of

$$\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} + \frac{A}{x} = 0,$$

where A is a positive constant.

14.28 Find the solution of

$$(5x + y - 7)\frac{dy}{dx} = 3(x + y + 1).$$

14.29 Find the solution $y = y(x)$ of

$$x\frac{dy}{dx} + y - \frac{y^2}{x^{3/2}} = 0,$$

subject to $y(1) = 1$.

14.30 Find the solution of

$$(2\sin y - x)\frac{dy}{dx} = \tan y,$$

if (a) $y(0) = 0$, and (b) $y(0) = \pi/2$.

14.31 Find the family of solutions of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0$$

that satisfy $y(0) = 0$.

14.5 Hints and answers

14.1 $N(t) = N_0 \exp(-\lambda t)$.

14.3 (a) exact, $x^2y^4 + x^2 + y^2 = c$; (b) IF $= x^{-1/2}$, $x^{1/2}(x + y) = c$; (c) IF $= \sec^2 x$, $y^2 \tan x + y = c$.

14.5 (a) IF $= (1 - x^2)^{-2}$, $y = (1 - x^2)(k + \sin^{-1} x)$; (b) IF $= \operatorname{cosec} x$, leading to $y = k\sin x + \cos x$; (c) exact equation is $y^{-1}(dx/dy) - xy^{-2} = y$, leading to $x = y(k + y^2/2)$.

14.7 $y(t) = e^{-t/\alpha} \int^t \alpha^{-1} e^{t'/\alpha} f(t')dt'$; (a) $y(t) = 1 - e^{-t/\alpha}$; (b) $y(t) = \alpha^{-1}e^{-t/\alpha}$; (c) $y(t) = (e^{-t/\alpha} - e^{-t/\beta})/(\alpha - \beta)$. It becomes case (b).

14.9 Note that, if the angle between the tangent and the radius vector is α, then $\cos\alpha = dr/ds$ and $\sin\alpha = p/r$.

14.11 Homogeneous equation, put $y = vx$ to obtain $(1 - v)(v^2 + 2v + 2)^{-1}\, dv = x^{-1}\, dx$; write $1 - v$ as $2 - (1 + v)$, and $v^2 + 2v + 2$ as $1 + (1 + v)^2$; $A[x^2 + (x + y)^2] = \exp\{4\tan^{-1}[(x + y)/x]\}$.

14.13 $(1 + s)(d\bar{y}/ds) + 2\bar{y} = 0$. $C = 1$; use separation of variables to show directly that $y(t) = te^{-t}$.

14.15 The equation is of the form of (14.22), set $v = x + y$; $x + 3y + 2\ln(x + y - 2) = A$.

14.17 The equation is isobaric with weight $y = -2$; setting $y = vx^{-2}$ gives $v^{-1}(1 - v)^{-1}(1 - 2v)\, dv = x^{-1}\, dx$; $4xy(1 - x^2y) = 1$.

14.19 The curve must satisfy $y = (1-p^{-1})^{-1}(1-x+px)$, which has solution $x = (p-1)^{-2}$, leading to $y = (1 \pm \sqrt{x})^2$ or $x = (1 \pm \sqrt{y})^2$; the singular solution $p' = 0$ gives straight lines joining $(\theta, 0)$ and $(0, 1 - \theta)$ for any θ.

14.21 $v = qu + q/(q - 1)$, where $q = dv/du$. General solution $y^2 = cx^2 + c/(c - 1)$, hyperbolae for $c > 0$ and ellipses for $c < 0$. Singular solution $y = \pm(x \pm 1)$.

14.23 (a) Integrating factor is $(a^2+x^2)^{1/2}$, $y = (a^2+x^2)/3 + A(a^2+x^2)^{-1/2}$; (b) separable, $y = x(x^2 + Ax + 4)^{-1}$.

14.25 Use Laplace transforms; $\bar{x}s(s^2 + 4) = s + s^2 - 2e^{-3s}$; $x(t) = \frac{1}{2}\sin 2t + \cos 2t - \frac{1}{2}H(t - 3) + \frac{1}{2}\cos(2t - 6)H(t - 3)$.

14.27 This is Clairaut's equation with $F(p) = A/p$. General solution $y = cx + A/c$; singular solution, $y = 2\sqrt{Ax}$.

14.29 Either Bernoulli's equation with $n = 2$ or an isobaric equation with $m = 3/2$; $y(x) = 5x^{3/2}/(2 + 3x^{5/2})$.

14.31 Show that $p = (Ce^x - 1)^{-1}$, where $p = dy/dx$; $y = \ln[C - e^{-x})/(C - 1)]$ or $\ln[D - (D - 1)e^{-x}]$ or $\ln(e^{-K} + 1 - e^{-x}) + K$.

15

Higher-order ordinary differential equations

Following on from the discussion of first-order ordinary differential equations (ODEs) given in the previous chapter, we now examine equations of second and higher order. Since a brief outline of the general properties of ODEs and their solutions was given at the beginning of the previous chapter, we will not repeat it here. Instead, we will begin with a discussion of various types of higher-order equation. This chapter is divided into three main parts. We first discuss linear equations with constant coefficients and then investigate linear equations with variable coefficients. Finally, we discuss a few methods that may be of use in solving general linear or non-linear ODEs. Let us start by considering some general points relating to *all* linear ODEs.

Linear equations are of paramount importance in the description of physical processes. Moreover, it is an empirical fact that, when put into mathematical form, many natural processes appear as higher-order linear ODEs, most often as second-order equations. Although we could restrict our attention to these second-order equations, the generalisation to nth-order equations requires little extra work, and so we will consider this more general case.

A linear ODE of general order n has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x). \qquad (15.1)$$

If $f(x) = 0$ then the equation is called *homogeneous*; otherwise it is *inhomogeneous*. The first-order linear equation studied in subsection 14.2.4 is a special case of (15.1). As discussed at the beginning of the previous chapter, the general solution to (15.1) will contain n arbitrary constants, which may be determined if n boundary conditions are also provided.

In order to solve any equation of the form (15.1), we must first find the general solution of the *complementary equation*, i.e. the equation formed by setting

$f(x) = 0$:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \tag{15.2}$$

To determine the general solution of (15.2), we must find n linearly independent functions that satisfy it. Once we have found these solutions, the general solution is given by a linear superposition of these n functions. In other words, if the n solutions of (15.2) are $y_1(x)$, $y_2(x), \dots, y_n(x)$, then the general solution is given by the linear superposition

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x), \tag{15.3}$$

where the c_m are arbitrary constants that may be determined if n boundary conditions are provided. The linear combination $y_c(x)$ is called the *complementary function* of (15.1).

The question naturally arises how we establish that any n individual solutions to (15.2) are indeed linearly independent. For n functions to be linearly independent over an interval, there must not exist *any* set of constants c_1, $c_2, \dots, c_n$ such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0 \tag{15.4}$$

over the interval in question, except for the trivial case $c_1 = c_2 = \cdots = c_n = 0$.

A statement equivalent to (15.4), which is perhaps more useful for the practical determination of linear independence, can be found by repeatedly differentiating (15.4), $n-1$ times in all, to obtain n simultaneous equations for c_1, $c_2, \dots, c_n$:

$$\begin{aligned}
c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) &= 0 \\
c_1 y_1{}'(x) + c_2 y_2{}'(x) + \cdots + c_n y_n{}'(x) &= 0 \\
&\ \vdots \\
c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)}(x) &= 0,
\end{aligned} \tag{15.5}$$

where the primes denote differentiation with respect to x. Referring to the discussion of simultaneous linear equations given in chapter 8, if the determinant of the coefficients of c_1, $c_2, \dots, c_n$ is non-zero then the only solution to equations (15.5) is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. In other words, the n functions $y_1(x)$, $y_2(x), \dots, y_n(x)$ are linearly independent over an interval if

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1{}' & y_2{}' & & \vdots \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0 \tag{15.6}$$

over that interval; $W(y_1, y_2, \dots, y_n)$ is called the *Wronskian* of the set of functions. It should be noted, however, that the vanishing of the Wronskian does not guarantee that the functions are linearly dependent.

If the original equation (15.1) has $f(x) = 0$ (i.e. it is homogeneous) then of course the complementary function $y_c(x)$ in (15.3) is already the general solution. If, however, the equation has $f(x) \neq 0$ (i.e. it is inhomogeneous) then $y_c(x)$ is only one part of the solution. The general solution of (15.1) is then given by

$$y(x) = y_c(x) + y_p(x), \tag{15.7}$$

where $y_p(x)$ is the *particular integral*, which can be *any* function that satisfies (15.1) directly, provided it is linearly independent of $y_c(x)$. It should be emphasised for practical purposes that *any* such function, no matter how simple (or complicated), is equally valid in forming the general solution (15.7).

It is important to realise that the above method for finding the general solution to an ODE by superposing particular solutions assumes crucially that the ODE is linear. For non-linear equations, discussed in section 15.3, this method cannot be used, and indeed it is often impossible to find closed-form solutions to such equations.

15.1 Linear equations with constant coefficients

If the a_m in (15.1) are constants rather than functions of x then we have

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x). \tag{15.8}$$

Equations of this sort are very common throughout the physical sciences and engineering, and the method for their solution falls into two parts as discussed in the previous section, i.e. finding the complementary function $y_c(x)$ and finding the particular integral $y_p(x)$. If $f(x) = 0$ in (15.8) then we do not have to find a particular integral, and the complementary function is by itself the general solution.

15.1.1 Finding the complementary function $y_c(x)$

The complementary function must satisfy

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0 \tag{15.9}$$

and contain n arbitrary constants (see equation (15.3)). The standard method for finding $y_c(x)$ is to try a solution of the form $y = Ae^{\lambda x}$, substituting this into (15.9). After dividing the resulting equation through by $Ae^{\lambda x}$, we are left with a polynomial equation in λ of order n; this is the *auxiliary equation* and reads

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{15.10}$$

In general the auxiliary equation has n roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. In certain cases, some of these roots may be repeated and some may be complex. The three main cases are as follows.

(i) *All roots real and distinct.* In this case the n solutions to (15.9) are $\exp \lambda_m x$ for $m = 1$ to n. It is easily shown by calculating the Wronskian (15.6) of these functions that if all the λ_m are distinct then these solutions are linearly independent. We can therefore linearly superpose them, as in (15.3), to form the complementary function

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}. \tag{15.11}$$

(ii) *Some roots complex.* For the special (but usual) case that all the coefficients a_m in (15.9) are real, if one of the roots of the auxiliary equation (15.10) is complex, say $\alpha + i\beta$, then its complex conjugate $\alpha - i\beta$ is also a root. In this case we can write

$$\begin{aligned} c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} &= e^{\alpha x}(d_1 \cos \beta x + d_2 \sin \beta x) \\ &= A e^{\alpha x} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} (\beta x + \phi), \end{aligned} \tag{15.12}$$

where A and ϕ are arbitrary constants.

(iii) *Some roots repeated.* If, for example, λ_1 occurs k times ($k > 1$) as a root of the auxiliary equation, then we have not found n linearly independent solutions of (15.9); formally the Wronskian (15.6) of these solutions, having two or more identical columns, is equal to zero. We must therefore find $k-1$ further solutions that are linearly independent of those already found and also of each other. By direct substitution into (15.9) we find that

$$xe^{\lambda_1 x}, \quad x^2 e^{\lambda_1 x}, \quad \ldots, \quad x^{k-1} e^{\lambda_1 x}$$

are also solutions, and by calculating the Wronskian it is easily shown that they, together with the solutions already found, form a linearly independent set of n functions. Therefore the complementary function is given by

$$y_c(x) = (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} + c_{k+1} e^{\lambda_{k+1} x} + c_{k+2} e^{\lambda_{k+2} x} + \cdots + c_n e^{\lambda_n x}. \tag{15.13}$$

If more than one root is repeated the above argument is easily extended. For example, suppose as before that λ_1 is a k-fold root of the auxiliary equation and, further, that λ_2 is an l-fold root (of course, $k > 1$ and $l > 1$). Then, from the above argument, the complementary function reads

$$\begin{aligned} y_c(x) = {} & (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} \\ & + (c_{k+1} + c_{k+2} x + \cdots + c_{k+l} x^{l-1}) e^{\lambda_2 x} \\ & + c_{k+l+1} e^{\lambda_{k+l+1} x} + c_{k+l+2} e^{\lambda_{k+l+2} x} + \cdots + c_n e^{\lambda_n x}. \end{aligned} \tag{15.14}$$

►*Find the complementary function of the equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x. \tag{15.15}$$

Setting the RHS to zero, substituting $y = Ae^{\lambda x}$ and dividing through by $Ae^{\lambda x}$ we obtain the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0.$$

The root $\lambda = 1$ occurs twice and so, although e^x is a solution to (15.15), we must find a further solution to the equation that is linearly independent of e^x. From the above discussion, we deduce that xe^x is such a solution, so that the full complementary function is given by the linear superposition

$$y_c(x) = (c_1 + c_2 x)e^x. \blacktriangleleft$$

Solution method. *Set the RHS of the ODE to zero (if it is not already so), and substitute $y = Ae^{\lambda x}$. After dividing through the resulting equation by $Ae^{\lambda x}$, obtain an nth-order polynomial equation in λ (the auxiliary equation, see (15.10)). Solve the auxiliary equation to find the n roots, $\lambda_1, \lambda_2, \ldots, \lambda_n$, say. If all these roots are real and distinct then $y_c(x)$ is given by (15.11). If, however, some of the roots are complex or repeated then $y_c(x)$ is given by (15.12) or (15.13), or the extension (15.14) of the latter, respectively.*

15.1.2 Finding the particular integral $y_p(x)$

There is no generally applicable method for finding the particular integral $y_p(x)$ but, for linear ODEs with constant coefficients and a simple RHS, $y_p(x)$ can often be found by inspection or by assuming a parameterised form similar to $f(x)$. The latter method is sometimes called the *method of undetermined coefficients*. If $f(x)$ contains only polynomial, exponential, or sine and cosine terms then, by assuming a trial function for $y_p(x)$ of similar form but one which contains a number of undetermined parameters and substituting this trial function into (15.9), the parameters can be found and $y_p(x)$ deduced. Standard trial functions are as follows.

(i) If $f(x) = ae^{rx}$ then try

$$y_p(x) = be^{rx}.$$

(ii) If $f(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be zero) then try

$$y_p(x) = b_1 \sin rx + b_2 \cos rx.$$

(iii) If $f(x) = a_0 + a_1 x + \cdots + a_N x^N$ (some a_m may be zero) then try

$$y_p(x) = b_0 + b_1 x + \cdots + b_N x^N.$$

(iv) If $f(x)$ is the sum or product of any of the above then try $y_p(x)$ as the sum or product of the corresponding individual trial functions.

It should be noted that this method fails if any term in the assumed trial function is also contained within the complementary function $y_c(x)$. In such a case the trial function should be multiplied by the smallest integer power of x such that it will then contain no term that already appears in the complementary function. The undetermined coefficients in the trial function can now be found by substitution into (15.8).

Three further methods that are useful in finding the particular integral $y_p(x)$ are those based on Green's functions, the variation of parameters, and a change in the dependent variable using knowledge of the complementary function. However, since these methods are also applicable to equations with variable coefficients, a discussion of them is postponed until section 15.2.

►*Find a particular integral of the equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

From the above discussion our first guess at a trial particular integral would be $y_p(x) = be^x$. However, since the complementary function of this equation is $y_c(x) = (c_1 + c_2x)e^x$ (as in the previous subsection), we see that e^x is already contained in it, as indeed is xe^x. Multiplying our first guess by the lowest integer power of x such that the result does not appear in $y_c(x)$, we therefore try $y_p(x) = bx^2e^x$. Substituting this into the ODE, we find that $b = 1/2$, so the particular integral is given by $y_p(x) = x^2e^x/2$. ◄

Solution method. *If the RHS of an ODE contains only functions mentioned at the start of this subsection then the appropriate trial function should be substituted into it, thereby fixing the undetermined parameters. If, however, the RHS of the equation is not of this form then one of the more general methods outlined in subsections 15.2.3–15.2.5 should be used; perhaps the most straightforward of these is the variation-of-parameters method.*

15.1.3 Constructing the general solution $y_c(x) + y_p(x)$

As stated earlier, the full solution to the ODE (15.8) is found by adding together the complementary function and any particular integral. In order to illustrate further the material discussed in the last two subsections, let us find the general solution to a new example, starting from the beginning.

►*Solve*

$$\frac{d^2y}{dx^2} + 4y = x^2 \sin 2x. \tag{15.16}$$

First we set the RHS to zero and assume the trial solution $y = Ae^{\lambda x}$. Substituting this into (15.16) leads to the auxiliary equation

$$\lambda^2 + 4 = 0 \qquad \Rightarrow \qquad \lambda = \pm 2i. \tag{15.17}$$

Therefore the complementary function is given by

$$y_c(x) = c_1 e^{2ix} + c_2 e^{-2ix} = d_1 \cos 2x + d_2 \sin 2x. \tag{15.18}$$

We must now turn our attention to the particular integral $y_p(x)$. Consulting the list of standard trial functions in the previous subsection, we find that a first guess at a suitable trial function for this case should be

$$(ax^2 + bx + c)\sin 2x + (dx^2 + ex + f)\cos 2x. \tag{15.19}$$

However, we see that this trial function contains terms in $\sin 2x$ and $\cos 2x$, both of which already appear in the complementary function (15.18). We must therefore multiply (15.19) by the smallest integer power of x which ensures that none of the resulting terms appears in $y_c(x)$. Since multiplying by x will suffice, we finally assume the trial function

$$(ax^3 + bx^2 + cx)\sin 2x + (dx^3 + ex^2 + fx)\cos 2x. \tag{15.20}$$

Substituting this into (15.16) to fix the constants appearing in (15.20), we find the particular integral to be

$$y_p(x) = -\frac{x^3}{12}\cos 2x + \frac{x^2}{16}\sin 2x + \frac{x}{32}\cos 2x. \tag{15.21}$$

The general solution to (15.16) then reads

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= d_1 \cos 2x + d_2 \sin 2x - \frac{x^3}{12}\cos 2x + \frac{x^2}{16}\sin 2x + \frac{x}{32}\cos 2x. \end{aligned}$$ ◄

15.1.4 Linear recurrence relations

Before continuing our discussion of higher-order ODEs, we take this opportunity to introduce the discrete analogues of differential equations, which are called *recurrence relations* (or sometimes *difference equations*). Whereas a differential equation gives a prescription, in terms of current values, for the new value of a dependent variable at a point only infinitesimally far away, a recurrence relation describes how the next in a sequence of values u_n, defined only at (non-negative) integer values of the 'independent variable' n, is to be calculated.

In its most general form a recurrence relation expresses the way in which u_{n+1} is to be calculated from all the preceding values $u_0, u_1, \ldots, u_n$. Just as the most general differential equations are intractable, so are the most general recurrence relations, and we will limit ourselves to analogues of the types of differential equations studied earlier in this chapter, namely those that are linear, have