4 Analytic function

If a function $f(z)$ is differentiable and single valued in a region of the complex plane, the function is said to be analytic in that region. If a function $f(z)$ is analytic everywhere in the finite complex plane, it is called an entire function. If the derivative of a function does not exist at a point $z = z_0$, then it called a *singular point* of that function – we will return to more example of singular points later.

The necessary and sufficient conditions for a function $f(z) = u + iv$ to be analytic are stated as follows

- The partial derivative of the function $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$ must satisfy the CR conditions discussed in the previous calss.
- The four partial derivatives and must also be continuous.

If two functions are analytic in a domain, then their sum and products are also analytic in that domain. The quotient is also analytic provided that the function the denominator does not vanish.

4.0.1 Example: $f(z) = z^2$ is an analytic function:

To show this note that $f(z) = z^2 = (x^2 - y^2) + 2xyi$. So we can identify $u = x^2 - y^2$ and $v = 2xy$. One caneasily check that the CR conditions are satisfied and the four partial derivatives are continuous

4.0.2 Example: $f(z) = 1/z^*$ is not a analytic function:

For this function we have $u = x$ and $v = -y$. Hence the CR conditions are

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

Since the CR is not satisfied, it is not analytic. But note that the function is a continuous function

4.0.3 Harmonic function:

The existence of derivative of an analytic function has the following interesting consequence. The real and the imaginary part of the analytic function satisfy the following set of equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\,.$$

These are 2D Laplace's equations and in this case u, v are harmonic conjugate of each other. A harmonic function is a one that has continuous partial derivative. In fact, a function $f(z) = u + iv$ is analytic in a domain if and only if the real functions are harmonic conjugate of each other.

4.0.4 Example:

To give an example of how to apply this, consider the function

$$u = y^3 - 3x^2 y\,,$$

This function has continuous partial derivative in the entire $x - y$ plane. To find the harmonic conjugate v note that u and v are related by the CR conditions given in the previous class

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,. \tag{71}$$

From the first equation we get

$$\frac{\partial v}{\partial y} = -6xy\,,$$

which when integrated by keeping the x fixed gives

$$v = -3xy^2 + \phi(x)\,,$$

where $\phi(x)$ must be constant with respect to y. The second CR condition implies that

$$3y^2 - 3x^2 = 3y^2 - \phi'(x)\,,$$

so that $\phi'(x) = 3x^2$. One then obtains $\phi(x) = x^3 + c$, where c is an real constant. Hence we get the expression $v = -3xy^2 + x^3 + c$ which is the harmonic conjugate of u. Therefore the complex function is

$$f(z) = y^3 - 3x^2y + i(-3xy^2 + x^3 + c)\,.$$

4.0.5 Derivatives of Analytic function:

While discussing CR relations, we saw that one can calculate the derivative of a complex function by taking partial derivatives of the real and the imaginary parts. Now we will see that the rules of derivatives are same as for the real functions. So, if a function $f(z)$ is analytic then

$$\frac{d}{dz}f(z) = f'(z) = \frac{\partial f}{\partial x}\,, \tag{72}$$

$$\frac{d}{dz}\big[f(z)g(z)\big] = \frac{\partial}{\partial}\big[f(z)g(z)\big] = f'(z)g(z) + f(z)g'(z)\,, \tag{73}$$

$$\frac{dz}{dz} = 1\,, \quad \frac{dz^n}{dz} = nz^{n-1}\,. \tag{74}$$

We can also calculate the derivative of a logarithm in the following way:

$$\ln z = \ln r + i\theta + i2n\pi\,.$$

So we can identify $u = \ln r$ and $v = \theta + 2n\pi$. We can see that the real functions u, v satisfy the CR conditions

$$\frac{\partial u}{\partial x} = \frac{x}{r^2} = \frac{\partial u}{\partial x}\,, \qquad \frac{\partial u}{\partial y} = \frac{y}{r^2} = -\frac{\partial v}{\partial x}\,,$$

So the CR equations are satisfied everywhere except at $r = 0$. Then the derivative is

$$\frac{d}{dz}\ln z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{x - iy}{r^2} = \frac{1}{x + iy} = \frac{1}{z}\,. \tag{75}$$

As logarithmic functions are multivalued, there are some more to this story that we will discuss later.

4.0.6 Point at infinity:

In complex variable, infinity is regarded as a single point and is treated after making a change of variable from z to $\omega = 1/z$. For example, the behavior of $f(z) = z$ is discussed as that of $1/\omega$ as $\omega \to 0$.

4.0.7 Integrals:

Integrals of complex variables are defined on curves in the complex plane , rather than interval of real line. There are different types of curves. A set of points $z = (x, y)$ in the complex plane is said to be an *arc* if

$$x = x(t)\,, \qquad y = y(t)\,, a \le t \le b$$

where $x(t), y(t)$ are continuous functions of real parameter t that is bounded between a, b. A arc is called *simple arc* or a Jordon arc, if it does not intersect itself. A simple arc is called *simple closed curve* or a Jordon curve, if $z(a) = z(b)$, *i.e.,* $x(a) = x(b)$ and $y(a) = y(b)$. A simple closed curve is called positively oriented if it is traced counterclockwise when the parameter t ranges from a to b. For exmample the circle $z = e^{i\theta}$ is a simple closed curve.

If a arc C is defined by $z(t) = x(t) + iy(t)$ where $a \le t \le b$, and both $x'(t)$ and $y'(t)$ are continuous in that interval, the arc is called differentiable. A *smooth arc* is defined by a function $z(t)$ is the one where $z'(t)$ is continuous for $a \le t \le b$ and $z'(t) \neq 0$ for $a < t < b$. A contour is formed when a number of smooth arcs are joined from end to end. If $z(t)$ is a smooth arc and the initial and the final value is same then it is called the *simple closed contour*.

One can express the integral of a complex variable as a sum of real integrals.

Examples: Evaluate the complex integral of $f(z) = 1/z$ along the circle $|z| = R$, starting and finishing at $z = R$.

Ans: The path is shown in figure 9a. With reference to this figure, the curve is parametrized as

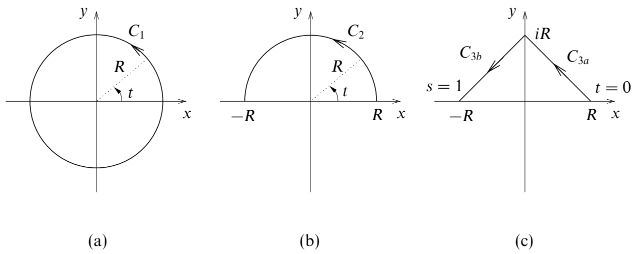

Figure 8: different types of contour.

$$z(t) = R\cos t + i \sin t\,, \qquad 0 \le t \le 2\pi\,.$$

The function is given by

$$f(z) = \frac{x - iy}{x^2 + y^2}\,,$$

so that the real and the imaginary part of the function is

$$u = \frac{R\cos t}{R^2}\,, \qquad v = -\frac{R\sin t}{R^2}\,.$$

Now, $dz = -R\sin tdt + R\cos tdt$. Hence we can write the integration separately for the real and the imaginary parts as

$$\begin{aligned}\int_C \frac{1}{z}dz &= \int_0^{2\pi} \frac{\cos t}{R}(-R\sin t)dt - \int_0^{2\pi} \frac{-\cos t}{R} R\cos tdt \\ &+ i\int_0^{2\pi} \frac{\cos t}{R} R\cos tdt + i\int_0^{2\pi} \frac{-\sin t}{R}(-R\sin t)dt \\ &= 0 + 0 + i\pi + i\pi = 2\pi i \end{aligned} \tag{76}$$

In the above example the integral began and ended at the same point. We will now see and example where it is not the case. **Example:**Evaluate the complex integral of $f(z) = 1/z$ along the paths b and c shown in figure 9 *i)* first along the contour C_2 consisting of the semi-circle $|z| = R$ in the half-plane $h \ge 0$, and *ii)* the contour C_3 made up of the two straight lines C_{3a} and C_{3b}. Ans: *i)* In this case the procedure is the same as before, expect that now $0 \le t \le \pi$. Hence the answer is given bu

$$\int_{C_2} \frac{dz}{z} = \pi i\,,$$

ii) there are two straight lines now

$$\begin{aligned} C_{3a} : z &= (1-t)R + itR\,, \quad 0 \le t \le 1\,, \\ C_{3b} : z &= -sR + i(1-s)R\,, \quad 0 \le s \le 1\,, \end{aligned}$$

With these parameterizations the required integrals can be written as

$$\int_{C_3} \frac{dz}{z} = \int_0^1 \frac{-R+iR}{(1-t)R+itR} dt + \int_0^1 \frac{-R-iR}{-sR+i(1-s)R} ds \,.$$

By simplifying a little bit, the first integral becomes

$$\begin{aligned}
\int_0^1 \frac{-R+iR}{(1-t)R+itR} dt &= \int_0^1 \frac{2t-1}{1-2t+2t^2} dt + i\int_0^1 \frac{1}{1-2t+2t^2} dt \,, \\
&= \frac{1}{2}\left[\ln(1-2t+2t^2)\right]_0^1 + \frac{i}{2}\left[2\arctan\left(\frac{t-1/2}{1/2}\right)\right] , \\
&= 0 + \frac{i}{2}\left[\frac{\pi}{2} - (-\pi/2)\right] = i\frac{\pi}{2} \,,
\end{aligned}$$

The other integral can be shown to be $i\pi$.