International Institute of Information Technology Hyderabad

Discrete Structures (MA5.101)

Assignment on "Discrete Numeric Functions and Generating Functions"
Deadline: November 30, 2022 (Wednesday), 23:55 PM
Total Marks: 100
Instructions: Submit ONLY handwritten scanned pdf file in the moodle under Assignments directory.

1. Let

$$a_r = \begin{cases} 1, r = 0 \\ 3, r = 1 \\ 5, r = 2 \\ 0, r \geq 3 \end{cases}$$
$$c_r = 7^r \text{ for all } r$$

Given that $c = a * b$, that is c is the convolution of numeric functions a and b. Derive b.

[10]

2. Using the generating function, show that solution of the following recurrence relation

$$a_k - 7a_{k-1} + 10a_{k-2} = 3^k$$

with initial conditions $a_0 = 0$ and $a_1 = 1$, is

$$a_k = \frac{8}{3}2^k - \frac{9}{2}3^k + \frac{11}{6}5^r$$

[10]

3. Consider an air traffic-control system in which the desired altitude of an aircraft, a_r, is computed by a computer every second and is compared with the actual altitude of the aircraft, b_{r-1}, determined by a tracking radar 1 second earlier. Depending on whether a_r is larger or smaller than b_{r-1}, the altitude of the aircraft will be changed accordingly. Specifically, the change in altitude at the r-th second, $b_r - b_{r-1}$, is proportional to the difference $a_r - b_{r-1}$. That is,

$$b_r - b_{r-1} = K(a_r - b_{r-1})$$

where K is a proportional constant.
(a) Determine b_r, given that $a_r = 100(\frac{3}{2})^2$, $K = 2$, and $b_0 = 0$.
(b) Determine b_r, given that

$$a_r = \begin{cases} 100(\frac{3}{2})^r, 0 \leq r \leq 9 \\ 100(\frac{3}{2})^{10}, r \geq 10 \end{cases}$$

$K = 2$, and $b_0 = 0$.

[10 + 10 = 20]

4. Using the generating function, find the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$.

[10]

5. Let a_r denote the total dollar assets of a company in the r^{th} year. Clearly, $a_r - a_{r-1}$ is the increase in assets during the r^{th} year. If the increase in assets during each year is always five times the increase during the previous year, what are the total assets in the r^{th} year? It is given that $a_0 = 3$ and $a_1 = 7$.

[10]

6. [**Cauchy's Root Test**] Recall the Cauchy's Root test for convergence of an infinite series. Suppose we have an infinite series $\sum u_n$. Define $L = \lim_{n\to\infty} |u_n|^{\frac{1}{n}}$. Then,

- If $L < 1$, the series is absolutely convergent (and hence convergent).
- If $L > 1$, the series is divergent.
- If $L = 1$, the series may be divergent, conditionally convergent, or absolutely convergent.

Determine if the following series is convergent or divergent:

$$\sum_{n=0}^{\infty} \left(\frac{5n - 4n^3}{9n^3 + 2} \right)^n$$

[10]

7. [**Comparison Test**] Show that the following series is divergent:

$$\frac{1}{a.1^2 + b} + \frac{2}{a.2^2 + b} + \frac{3}{a.3^2 + b} + \cdots + \frac{n}{a.n^2 + b} + \cdots$$

[10]

8. [**D'Alembert's Ratio Test**] Consider the following hypergeometric series:

$$1 + \frac{\alpha}{1}.\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{1.2}.\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3}.\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$$

for positive values of α, β, γ and x.

Show that:

(a) The series converges for $x < 1$ and diverges for $x > 1$.
(b) For $x = 1$, the series is converget if $\gamma - \alpha - \beta > 0$ and divergent if $\gamma - \alpha - \beta \leq 0$.

[10 + 10 = 20]

All the best!!!