Cosets

Definition (Left Coset)

Let H be a subgroup of a group $\langle G, \cdot \rangle$. The left cosets of G relative of H are defined by

$$g \cdot H = \{g \cdot h | h \in H\}, \forall g \in G.$$

If $\cdot = +$, then

$$g \cdot H = g + H = \{g + h | h \in H\}.$$

Definition (Right Coset)

Let H be a subgroup of a group $\langle G, \cdot \rangle$. The right cosets of G relative of H are defined by

$$H \cdot g = \{h \cdot g | h \in H\}, \forall g \in G.$$

Cosets

Example

Let $\underline{3} = \{1, 2, 3\}$ be a finite set. Considering all $3! = 6$ permutations on $\underline{3}$, define a set $S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$. Then, S_3 forms a group under permutation composition (multiplication). Also, S_3 is called a symmetric group of degree 3. Find the left and right cosets of S_3 relative to a subgroup $H = \{e, (1\,2)\} \subseteq S_3$, where e is the identity permutation defined on $\underline{3}$.

Group

Problem: If H be a subgroup of a group $\langle G, \circ \rangle$ and $h \in H$, then $h \circ H = H \circ h = H$.

Problem: For each g in a group $[G, .]$, the set $N_g = \{h | h.g.h^{-1} = g\}$ is called the *normalizer* of g. Show that N_g is a subgroup of G for every g.

Subgroup

Theorem

The left (right) cosets of a group G relative to a subgroup H form a partition of G. Moreover, all of the left or right cosets of G relative to H have equal number of elements.