Discrete Structures (MA5.101)

Assignment 2 Key

Question 1

Solution

Let $a(x), b(x), c(x)$ represent the generating functions corresponding to a_r, b_r, c_r respectively. Then we have

$$
\begin{aligned}
c(x) &= a(x)b(x) \\
\implies \frac{1}{1-7x} &= (1+3x+5x^2)b(x) \\
\implies b(x) &= \frac{1}{(1-7x)(1-\alpha x)(1-\beta x)} \\
\implies b(x) &= \frac{A}{1-7x} + \frac{B}{1-\alpha x} + \frac{C}{1-\beta x} \\
\implies b(x) &= \sum_r (A7^r + B\alpha^r + C\beta^r)x^r \\
\implies b_r &= A7^r + B\alpha^r + C\beta^r
\end{aligned}
$$

where

$$
\begin{aligned}
A + B + C &= 1 \\
(\alpha+\beta)A + (7+\beta)B + (7+\alpha)C &= 0 \\
\alpha\beta A + 7\beta B + 7\alpha C &= 0.
\end{aligned}
$$

This is solved by

$$
\begin{aligned}
A &= \frac{49}{(\alpha-7)(\beta-7)} \\
B &= \frac{\alpha^2}{(\alpha-7)(\alpha-\beta)} \\
C &= \frac{\beta^2}{(\beta-7)(\alpha-\beta)}
\end{aligned}
$$

Rubric

4: Finding expression for $b(x), b_r$
4: Writing linear equations for A, B, C
2: Solving for A, B, C

Question 2

Solution

Let $a(x)$ be the generating function corresponding to a_k. We have

$$a_k - 7a_{k-1} + 10a_{k-2} = 3^k,$$

which means that

$$\begin{aligned}
(a(x) - a_0 - a_1x) - 7(xa(x) - a_0x) + 10x^2a(x) &= \frac{1}{1-3x} - 1 - 3x \\
\implies (1 - 7x + 10x^2)a(x) + (-a_0 + 1) + (-a_1 + a_0 + 3)x &= \frac{1}{1-3x} \\
\implies (1-2x)(1-5x)a(x) &= \frac{1}{1-3x} - (1+2x) \\
\implies a(x) &= \frac{x + 6x^2}{(1-3x)(1-2x)(1-5x)} \\
\implies a(x) &= \frac{A}{1-3x} + \frac{B}{1-2x} + \frac{C}{1-5x} \\
\implies a_r = A3^r + B2^r + C5^r
\end{aligned}$$

where

$$\begin{aligned}
A + B + C &= 0 \\
7A + 8B + 5C &= -1 \\
10A + 15B + 6C &= 6
\end{aligned}$$

We can see that $A = -\frac{9}{2}, B = \frac{8}{3}, C = \frac{11}{6}$ satisfies these equations, QED.

Rubric

4: Finding expression for $a(x), a_r$
4: Writing linear equations for A, B, C
2: Solving for A, B, C

Question 3

Solution

We have that $b_r - b_{r-1} = K(a_r - b_{r-1})$, or

$$b_r + (K-1)b_{r-1} = Ka_r.$$

Therefore,

$$(b(x) - b_0) + (K-1)(xb(x)) = K(a(x) - a_0),$$

or

$$b(x) = \frac{2(a(x) - a_0)}{(1+x)},$$

substituting the values K and b_0.

Part (a)

Here, we have

$$a(x) = 100\left(\frac{3}{2}\right)^2\left(\frac{1}{1-x} - 1\right),$$

which gives us

$$\begin{aligned}
b(x) &= 200\left(\frac{3}{2}\right)^2\left(\frac{x}{(1-x)(1+x)}\right) \\
&= \frac{1}{2}\cdot 200\left(\frac{3}{2}\right)^2\left(\frac{1}{1-x} - \frac{1}{1+x}\right) \\
\implies b_r = 100\left(\frac{3}{2}\right)^2(1-(-1)^r)
\end{aligned}$$

Part (b)

Here, we have

$$a(x) = 100\left[\frac{1-\left(\frac{3}{2}x\right)^{10}}{1-\frac{3}{2}x} + \left(\frac{3}{2}\right)^{10}\frac{x^{10}}{1-x}\right]$$

which gives us

$$b(x) = 200\left[\left(1 - \left(\frac{3}{2}x\right)^{10}\right)\frac{1}{(1-\frac{3}{2}x)(1+x)} + \left(\frac{3}{2}x\right)^{10}\frac{1}{(1-x)(1+x)}\right]$$

For the first partial fraction, we have

$$\begin{aligned}
A + B &= 1 \\
A - \frac{3}{2}B &= 0,
\end{aligned}$$

which gives us $A = \frac{3}{5}, B = \frac{2}{5}$.
For the second one, we have $A = B = \frac{1}{2}$.

Therefore,

$$\begin{aligned}
b(x) &= 200\left[\left(\frac{\frac{3}{5}}{1-\frac{3}{2}x} + \frac{\frac{2}{5}}{1+x}\right) + \left(\frac{3}{2}x\right)^{10}\left(\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x} - \frac{\frac{3}{5}}{1-\frac{3}{2}x} - \frac{\frac{2}{5}}{1+x}\right)\right] \\
&= 200\left[\left(\frac{\frac{3}{5}}{1-\frac{3}{2}x} + \frac{\frac{2}{5}}{1+x}\right) + \left(\frac{3}{2}x\right)^{10}\left(\frac{\frac{1}{2}}{1-x} - \frac{\frac{3}{5}}{1-\frac{3}{2}x} - \frac{\frac{1}{10}}{1+x}\right)\right] \\
\implies b_r &= 200\left[\left(\frac{3}{5}\left(\frac{3}{2}\right)^r + \frac{2}{5}(-1)^r\right) + \left\{\left(\frac{3}{2}\right)^{10}\left(\frac{1}{2} - \frac{3}{5}\left(\frac{3}{2}\right)^{r-10} - \frac{1}{10}(-1)^{r-10}\right)\right\}_{r\geq 10}\right]
\end{aligned}$$

Rubric

Part (a)

4: Writing $b(x)$ in terms of \$a(x)
4: Writing $a(x)$
2: Solving for $b(x), b_r$

Part (b)

4: Finding expression for $b(x), b_r$
3: First PF decomposition
3: Second PF decomposition

[for PF decompositions, 1 for writing equations and 2 for solving]

Question 4

Solution

Let

$$a_r = 1^3 + \cdots + r^3,$$

and let $a(x)$ be the corresponding generating function.

Now, since $a_r = a_{r-1} + r^3$, we can say

$$\begin{aligned}
(a(x) - a_0) &= xa(x) + \sum r^3 x^r \\
a(x) &= \frac{1}{1-x}\sum r^3 x^r
\end{aligned}$$

Now, knowing that $\frac{1}{1-x} = \sum x^r$, we can differentiate and multiply by x to get

$$\frac{x}{(1-x)^2} = \sum r x^r,$$

and repeat this to get

$$\frac{x(1+x)}{(1-x)^3} = \sum r^2 x^r$$

and once more, ending up with

$$\frac{x(1+4x+x^2)}{(1-x)^4} = \sum r^3 x^r.$$

Therefore we have

$$a(x) = \frac{x(1+4x+x^2)}{(1-x)^5},$$

which we decompose to

$$a(x) = \frac{-1}{(1-x)^2} + \frac{7}{(1-x)^3} + \frac{-12}{(1-x)^4} + \frac{6}{(1-x)^5}.$$

Using the fact that

$$\frac{1}{(1-x)^k} = \sum \binom{r+k-1}{k-1} x^r,$$

we get

$$a(x) = \sum \left(-\binom{r+1}{1} + 7\binom{r+2}{2} - 12\binom{r+3}{3} + 6\binom{r+4}{4} \right) x^r,$$

so

$$a_r = \frac{r^2(r+1)^2}{4}.$$

Rubric

2: Writing summation expression for $a(x)$
3: Writing closed form for $a(x)$
3: PF decomposition of $a(x)$
2: Obtaining a_r

Question 5

Solution

It is given that $a_r - a_{r-1} = 5(a_{r-1} - a_{r-2})$. We rearrange this to

$$a_r - 6a_{r-1} + 5a_{r-2} = 0.$$

Letting $a(x)$ be the corresponding generating function, we have

$$\begin{aligned}
(a(x) - a_0 - a_1 x) - 6(xa(x) - a_1 x) + 5(x^2 a(x)) &= 0 \\
(1 - 6x + 5x^2)a(x) &= 3 + 49x \\
a(x) &= \frac{3 + 49x}{(1 - 5x)(1 - x)} \\
&= \frac{A}{1 - 5x} + \frac{B}{1 - x},
\end{aligned}$$

where

$$\begin{aligned}
A + B &= 3 \\
A + 5B &= -49,
\end{aligned}$$

which gives us $A = 16, B = -13$. Thus

$$a_r = 16 \cdot 5^r - 13.$$

Rubric

3: Write recursive expression for a_r
4: Write expression for $a(x)$
3: PF decomposition

Question 6

Solution

We have the series $\sum u_n$, where

$$u_n = \left(\frac{-4n^3 + 5n}{9n^3 + 2} \right)^n.$$

Since $L = \lim_{n\to\infty} |u_n|^{\frac{1}{n}}$, we get

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \left(\frac{-4n^3 + 5n}{9n^3 + 2} \right)^n \right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left| \frac{4n^3 - 5n}{9n^3 + 2} \right| \\
&= \frac{4}{9}.
\end{aligned}$$

Since $L < 1$, we can see that $\sum u_n$ is absolutely convergent.

Rubric

4: Writing expression for L
4: Solving for L
2: Applying Cauchy's Root Test

Question 7

Solution

We can rewrite the given series as

$$\frac{1}{a}\sum_n \frac{n}{n^2+\frac{b}{a}} = \frac{1}{a}\sum_n u_n.$$

Now, let $k = \left\lceil \frac{b}{a} \right\rceil$. Then we know that

$$u_n > \frac{n}{n^2+k} = \frac{1}{n+\frac{k}{n}}.$$

Now, for all $n > k$, we have

$$\frac{1}{n+\frac{k}{n}} > \frac{1}{n+1}.$$

We know that $\sum_n \frac{1}{n+1}$ diverges. Therefore, $\sum_{n>k} u_n$ also diverges [Cauchy's Comparison Test].
Since removing a finite number of terms from an infinite sum, and multiplying it by a finite factor, does not affect its convergence, $\frac{1}{a}\sum_n u_n$ does not converge either, QED.

Rubric

5: Finding series smaller than given
3: Stating that smaller series diverges
2: Applying Cauchy's Comparison Test

Question 8

Solution

Part (a)

Each term in the given sum is

$$u_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \cdot \frac{\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} x^n.$$

We can therefore calculate $L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$ as

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{\frac{\alpha(\alpha+1)\cdots(\alpha+n)}{(n+1)!} \cdot \frac{\beta(\beta+1)\cdots(\beta+n)}{\gamma(\gamma+1)\cdots(\gamma+n)} x^{n+1}}{\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \cdot \frac{\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} x^n} \right| \\ &= \lim_{n\to\infty} \frac{n+\alpha}{n+1} \cdot \frac{n+\beta}{n+\gamma} x \\ &= x. \end{aligned}$$

Therefore, by D'Alembert's Ratio test, if $x < 1$, then the series absolutely converges, and if $x > 1$, then it diverges.

Part (b)

We have

$$\begin{aligned} \rho_n &= n\left(\frac{u_n}{u_{n+1}} - 1\right) \\ &= n\left(\frac{n+1}{n+\alpha} \cdot \frac{n+\gamma}{n+\beta} - 1\right) \\ &= n \cdot \frac{(1+\gamma-\alpha-\beta)n + \gamma - \alpha\beta}{(n+\alpha)(n+\beta)} \end{aligned}$$

We can see that $L = \lim_{n\to\infty} \rho_n = 1 + \gamma - \alpha - \beta$.

Therefore, if $\gamma - \alpha - \beta > 0$, then $L > 1$. Therefore there must exist a $c > 1$ for which $\rho_n \geq c$ for all $n > N$ (since ρ_n can be made to approach arbitrary close to L, which is greater than 1).

Conversely, if $\gamma - \alpha - \beta < 0$, then $L < 1$, which means that $\rho_n < 1$ for all $n > N$ (since, again, ρ_n can be made to approach arbitrarily close to L, which is less than 1.)

Rubric

Part (a)

4: Writing expression for L
4: Showing $L = x$
2: Applying D'Alembert's Ratio Test

Part (b)

4: Writing expression for ρ_n
2: Showing $L = 1 + \gamma - \alpha - \beta$
2: Proving case $L > 1$
2: Proving case $L < 1$