Equality Axioms and Relations

Discrete Structures

December 2022

1 Exercises

1.1 Question 1

Given the equality axioms

$$\forall x \, (x = x) \tag{EQ1}$$

$$\forall x \, (\forall y \, ((x = y) \to (\alpha[z/x] \to \alpha[z/y]))) \tag{EQ2[α]}$$

where α is a formula and z is a variable, use the axioms as inference rules to deduce the following.

1. **Symmetry of Equality**

$$\forall x \, (\forall y \, ((x = y) \to (y = x))) \tag{EQsymm}$$

1	$\top$	
2	u $\quad \forall x \, (\forall y \, ((x = y) \to ((x = u) \to (y = u))))$	EQ2$[z = u]$
3	$\quad \forall y \, ((u = y) \to ((u = u) \to (y = u)))$	$\forall$E(x/u), 2
4	$\quad v$ $\quad (u = v) \to ((u = u) \to (v = u))$	$\forall$E(y/v), 3
5	$\quad\quad\quad (u = v)$	
6	$\quad\quad\quad (u = u) \to (v = u)$	$\to$E, 4, 5
7	$\quad\quad\quad \forall x \, (x = x)$	EQ1
8	$\quad\quad\quad (u = u)$	$\forall$E, 7
9	$\quad\quad\quad (v = u)$	$\to$E, 6, 8
10	$\quad\quad (u = v) \to (v = u)$	$\to$I, 5–9
11	$\quad \forall y \, ((u = y) \to (y = u))$	$\forall$I(v/y), 4–10
12	$\forall x \, (\forall y \, ((x = y) \to (y = x)))$	$\forall$I(u/x), 2–11

2. **Transitivity of Equality**

$$\forall w\ (\forall x\ (\forall y\ ((x = y) \to ((y = w) \to (x = w)))))\qquad \text{(EQtrans)}$$

1	$\top$	
2	w $\quad \forall x\ (\forall y\ ((x = y) \to ((x = w) \to (y = w))))$	EQ2$[z = w]$
3	$\quad v$ $\quad \top$	
4	$\quad\quad u$ $\quad \forall y\ ((u = y) \to ((u = w) \to (y = w)))$	$\forall$E(x/u), 2
5	$\quad\quad\quad (u = v) \to ((u = w) \to (v = w))$	$\forall$E(y/v), 4
6	$\quad\quad\quad\quad (v = u)$	
7	$\quad\quad\quad\quad (u = v)$	EQsymm, 6
8	$\quad\quad\quad\quad (u = w) \to (v = w)$	$\to$E, 5, 7
9	$\quad\quad\quad (v = u) \to ((u = w) \to (v = w))$	$\to$E, 6–8
10	$\quad\quad \forall y\ ((v = y) \to ((y = w) \to (v = w)))$	$\forall$I(u/y), 4–9
11	$\quad \forall x\ (\forall y\ ((x = y) \to ((y = w) \to (x = w))))$	$\forall$I(v/x), 3–10
12	$\forall w\ (\forall x\ (\forall y\ ((x = y) \to ((y = w) \to (x = w)))))$	$\forall$I, 2–11

3. **Substitution of Equality**

$$\forall x\ (\forall y\ ((x = y) \to (r[z/x] = r[z/y]))\qquad \text{(EQsubs(r))}$$

where r is a term.

1	$\top$	
2	u $\quad \forall x\ (\forall y\ ((x = y) \to ((r[z/u] = r[z/x]) \to (r[z/u] = r[z/y]))))$	EQ2$[r[u/z] = r[z]]$
3	$\quad \forall y\ ((u = y) \to ((r[z/u] = r[z/u]) \to (r[z/u] = r[z/y]))$	$\forall$E(x/u), 2
4	$\quad v$ $\quad (u = v) \to ((r[z/u] = r[z/u]) \to (r[z/u] = r[z/v])$	$\forall$E(y/v), 3
5	$\quad\quad \forall x\ (x = x)$	EQ1
6	$\quad\quad r[z/u] = r[z/u]$	$\forall$E, 5
7	$\quad\quad\quad (u = v)$	
8	$\quad\quad\quad (r[z/u] = r[z/u]) \to (r[z/u] = r[z/v])$	$\to$E, 4, 7
9	$\quad\quad\quad (r[z/u] = r[z/v])$	$\to$E, 6, 8
10	$\quad\quad (u = v) \to (r[z/u] = r[z/v])$	$\to$I, 7–9
11	$\quad \forall y\ ((u = y) \to (r[z/u] = r[z/y])$	$\forall$I(v/y), 4–10
12	$\forall x\ (\forall y\ ((u = y) \to (r[z/x] = r[z/y]))$	$\forall$I(u/x), 2–11

4. **Example**

   Given a function f such that for some a, b, $f(a) = b$ and $f(b) = a$. Show that $f(f(a)) = a$.

1	$f(a) = b$	
2	$f(b) = a$	
3	$b = f(a)$	EQsymm, 1
4	$\forall x\ (\forall y\ (x = y) \to ((f(x) = a) \to (f(y) = a))$	EQ2[f(z)=a]
5	$\forall y\ (b = y) \to ((f(b) = a) \to (f(y) = a)$	$\forall$E(x/b), 4
6	$(b = f(a)) \to ((f(b) = a) \to (f(f(a)) = a)$	$\forall$E$(y/f(a))$, 5
7	$(f(b) = a) \to (f(f(a)) = a)$	$\to$E, 3, 6
8	$f(f(a)) = a$	$\to$E, 2, 7

1.2 Question 2

Determine the number of binary relations possible on A, whose cardinality is n, which are

1. **Reflexive**

   By definition of reflexivity, observe that the diagonal elements of the $A \times A$ matrix must be present in any reflexive binary relation. Therefore, the diagonal elements along with any subset from the remaining $n^2 - n$ elements is still a reflexive relation. So, the number of required relations is 2^{n^2-n}.
   .

2. **Symmetric**

   Consider a symmetric binary relation R and the matrix $A \times A$. By definition, elements of the form (a, a) can belong to a symmetric relation giving 2^n such relations. Let (a, b) and (b, a) be a symmetric pair of non-diagonal elements. There can only be two cases.

   (a) $(a, b) \in R$ and $(b, a) \in R$
   (b) $(a, b) \notin R$ and $(b, a) \notin R$

   Since there are $\frac{n^2-n}{2}$ unique symmetric pairs of non-diagonal elements, the number of relations containing non-diagonal elements is $2^{\frac{n^2-n}{2}}$. So, total number of required relations is $2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$.

3. **Asymmetric**

   Consider an asymmetric binary relation R and the matrix $A \times A$. By definition, elements of the form (a, a) cannot belong to an asymmetric relation. Let (a, b) and (b, a) be a symmetric pair of non-diagonal elements. There can only be three cases.

   (a) $(a, b) \in R$ and $(b, a) \notin R$
   (b) $(b, a) \in R$ and $(a, b) \notin R$
   (c) $(a, b) \notin R$ and $(b, a) \notin R$

   Since there are $\frac{n^2-n}{2}$ unique symmetric pairs of non-diagonal elements, the required number of relations is $3^{\frac{n^2-n}{2}}$.

4. **Antisymmetric**

   Consider the matrix $A \times A$. Using the same logic as above, considering non-diagonal elements the number of relations come out to be $3^{\frac{n^2-n}{2}}$. Furthermore, any subset of the diagonal elements will satisfy the antisymmetric property giving another 2^n valid relations. So, the required number of relations is $2^n \cdot 3^{\frac{n^2-n}{2}}$.

5. **Transitive**

   Counting transitive relations precisely is a challenging task. However, we can obtain good lower bounds and upper bounds. Any subset of the set of diagonal elements is an example transitive relation and thus, there are at least 2^n transitive relations (lower bound). A trivial upper bound is 2^{n^2}. To obtain a good upper bound, we focus on non-transitive relations. If we get a good lower bound for non-transitive relations, then total number of relations minus the lower bound gives a good upper bound for transitive relations. For example, irreflexive symmetric relations (except empty relation) are non-transitive relations, and there are at least $2^{\frac{n^2-n}{2}} - 1$ non-transitive relations. Therefore, a good upper bound for transitive relations is $2^{n^2} - 2^{\frac{n^2-n}{2}} + 1$.

1.3 Question 3

Check whether the relations given below are partial order or not.

1. **Problem:** Let Z be the set of all integers. Define a relation R on the set Z $\times$ Z by $(a, b)R(c, d)$ if and only if ad = bc, $\forall$ a, b, c, d ϵ Z. Prove or disprove: R is a partial-order relation.

Solution:

(a) Verify whether R is reflexive.
Consider some (a,b) ϵ N $\times$ N. Since ab = ba ϵ N,
(commutative) (substituting b in d, a in c) $\forall$ a,b,
R is reflexive.

(b) Verify whether R is anti-symmetric.
Assume, for some (a,b), (c, d) ϵ N $\times$ N, (a,b)R(c,d)
$\implies$ ad = bc — (1)
And (c,d)R(a,b)
$\implies$ cb = ad, which is the same as (1)
So, take a case where it does not hold and disprove.

Consider the elements (4,6) and (2,3)
Clearly, (4,6)R(2,3) and (2,3)R(4,6) but (2,3) $\neq$ (4,6)

(c) Verify whether R is transitive. This is not necessary as R is already proved to be not anti-symmetric.
Hence, R is NOT a partial order relation

2. **Problem:** A relation R is defined on the set N (set of natural numbers) by aRb if and only if a divides b, that is, R = $\{(a, b) \ \epsilon \ N \ \times \ N : a|b\}$. Prove or disprove: R is a partial-order relation.
Solution:

(a) Verify whether R is reflexive.
$\forall$ a ϵ N, a|a, making R reflexive.

(b) Verify whether R is anti-symmetric.
For some a,b ϵ Let aRb and bRa
$\implies$ a|b and b|a
$\implies$ a = b
$\therefore$ aRb and bRa $\implies$ a = b.
Hence, R is anti-symmetric by definition.

(c) Verify whether R is transitive.
Assume aRb and bRc for some a,b,c ϵ N.
$\implies$ a—b and b—c.
$\implies$ a—c.
$\therefore$ aRb and bRc $\implies$ aRc.
Hence, R is transitive by definition.

Since, R is reflexive, anti-symmetric and transitive, R is a partial order relation.