Subgroup

Definition (Left coset relation)

Let G be a group with subgroup H. The **left coset relation** on G with respect to H is the relation R with the property that ${}_{g_1}R_{g_2}$ iff $g_1^{-1} \cdot g_2 \in H, \forall g_1, g_2 \in G$.

Definition (Right coset relation)

Let G be a group with subgroup H. The **right coset relation** on G with respect to H is the relation R with the property that ${}_{g_1}R_{g_2}$ iff $g_1 \cdot g_2^{-1} \in H, \forall g_1, g_2 \in G$.

Subgroup

Theorem

The left (right) coset relation is an equivalence relation on a group G, and the equivalence classes are the left (right) cosets of G with respect to a subgroup H of G.

Normal Subgroup

Definition (Normal Subgroup)

A subgroup H of a group G is said to be a **normal subgroup** if the left coset partition induced by H is identical to the right coset partition induced by H.
Equivalently, H is normal if

$$g \cdot H = H \cdot g, \forall g \in G.$$

Theorem

*A subgroup H of a group G is **normal** if and only if*

$$g^{-1} \cdot H \cdot g \subseteq H, \forall g \in G.$$

*In other words, a subgroup H of a group G is **normal** if and only if*

$$g^{-1} \cdot h \cdot g \in H, \forall g \in G \, and \, h \in H.$$

Quotient group

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Theorem

If H is a normal subgroup of a group $\langle G, \cdot \rangle$, then the quotient structure $\langle G/H, \circ \rangle$ is a group, where $\circ$ is the composition of cosets defined by

$$[g] \circ [h] = [g \cdot h]$$

where $[g]$ denotes a left (right) coset of G relative to H and it is defined by $[g] = g \cdot H, \forall g \in G$, with respect to the left coset operation.

The group $\langle G/H, \circ \rangle$ is called the "quotient group" or "factor group" of G relative to the normal subgroup H.

Homomorphism

Definition (Homomorphism of semigroups)

Let $[S, \cdot]$ and $[T, *]$ be two semigroups. A mapping (function) $\theta : [S, \cdot] \rightarrow [T, *]$ is called a morphism (or homomorphism) of two semigroups $[S, \cdot]$ and $[T, *]$, if $\forall s_1, s_2 \in S$, $\theta(s_1 \cdot s_2) = \theta(s_1) * \theta(s_2)$.

Definition (Homomorphism of monoids)

Let $[S, \cdot, e_S]$ and $[T, *, e_T]$ be two monoids. A mapping (function) $\theta : [S, \cdot, e_S] \rightarrow [T, *, e_T]$ is called a morphism (or homomorphism), if the following conditions are met:

- (i) $\forall s_1, s_2 \in S$, $\theta(s_1 \cdot s_2) = \theta(s_1) * \theta(s_2)$.
- (ii) $\theta(e_S) = e_T$, where e_S and e_T denote the identity elements in the monoids $[S, \cdot, e_S]$ and $[T, *, e_T]$, respectively.

Homomorphism

Definition (Homomorphism of groups)

Let $[G, \cdot]$ and $[G', *]$ be two groups. A mapping (function) $\mu : [G, \cdot] \rightarrow [G', *]$ is called a morphism (or homomorphism), if the following conditions are met:

- (i) $\forall g, g' \in G$, $\mu(g \cdot g') = \mu(g) * \mu(g')$.
- (ii) $\mu(e_G) = e_{G'}$, where e_G and $e_{G'}$ denote the identity elements in the groups $[G, \cdot]$ and $[G', *]$, respectively.
- (iii) $[\mu(g)]^{-1} = \mu(g^{-1})$, $\forall g \in G$.

Homomorphism

Definition

Let g be a homomorphism from a structure $[X, \cdot]$ to another structure $[Y, *]$.

- If $g : X \to Y$ is onto (surjective), then g is called an **epimorphism**.
- If $g : X \to Y$ is one-one (injective), then g is called an **monomorphism**.
- If $g : X \to Y$ is one-one (injective) and onto (surjective) (that is, g is bijective), then g is called an **isomorphism**.
- If $g : X \to Y$ is called an **automorphism**, if $X = Y$ and g is a bijection.

Homomorphism

Theorem

Let $[G, \cdot]$ and $[G', *]$ be two groups. A mapping (function) $\mu : [G, \cdot] \rightarrow [G', *]$ is called a morphism (or homomorphism) of the groups $[G, \cdot]$ and $[G', *]$ if and only if

$$\mu(g \cdot g') = \mu(g) * \mu(g'), \forall g, g' \in G.$$

Homomorphism

Example

Let G be the group of non-zero real numbers under the multiplication operation. Determine whether the following functions are morphisms or not:

- (i) $\phi : G \to G$, where $\phi(x) = x^2$, for all $x \in G$.
- (ii) $\psi : G \to G$, where $\psi(x) = 2^x$, for all $x \in G$.