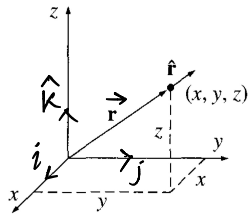
Lecture 2, 24/12/22

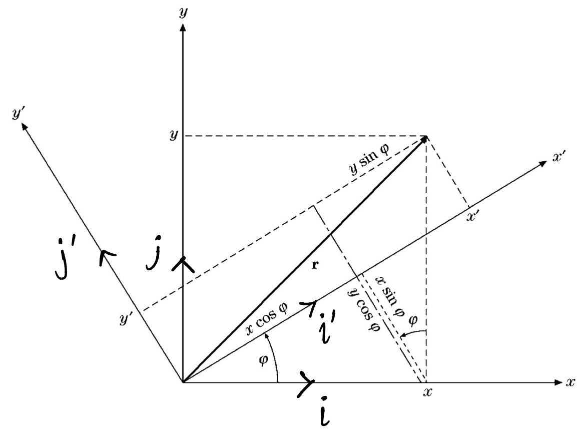
* Transformation of Vectors: Consider the position vector $\vec{r}$ of a point (x, y, z) in the Cartesian coordinate



$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Consider rotation of the coordinate about the z-axis by angle ϕ

In the new coordinate the position is (x', y', z') and the position vector



$$r' = x'\,\hat{i}' + y'\,\hat{j}' + z'\,\hat{k}'$$

From the diagram we find

$$\left.\begin{aligned} x' &= x\cos\varphi + y\sin\varphi, \\ y' &= -x\sin\varphi + y\cos\varphi. \\ z' &= z \end{aligned}\right\}$$

Any three component object (A_x, A_y, A_z) that transforms like the components of a position vector $\vec{r}$ is defined as a vector. In other words under the above coordinate transformation the components A_x, A_y & A_z transform as

$$A'_x = A_x \cos\varphi + A_y \sin\varphi,$$
$$A'_y = -A_x \sin\varphi + A_y \cos\varphi,$$
$$A'_z = A_z$$

It is better to write the above in matrix form

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$ *

* Magnitude of a vector remains invariant (unchanged) under rotation

$$A'_x = A_x \cos\phi + A_y \sin\phi, \quad A'_y = -A_x \sin\phi + A_y \cos\phi, \quad A'_z = A_z$$

Magnitude of the new vector $A' = \hat{\imath} A'_x + \hat{\jmath} A'_y + \hat{k} A'_z$ is given by

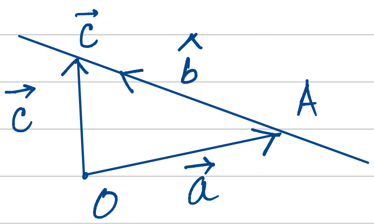
$$|A'| = \sqrt{(A'_x)^2 + (A'_y)^2 + (A'_z)^2}$$

= Substitute these equations

$$= \sqrt{A_x^2 + A_y^2 + A_z^2} \quad = |A|$$ Hence proved.

Some Applications:

Equation of a line: With reference to the the figure, we want to find the equation of the line going through the point A & R.



Position vector of point A is $\vec{a}$, w.r.to the point O

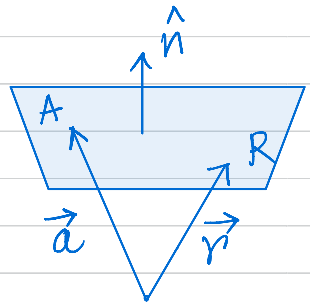
" " " " C is $\vec{c}$ " " " " "

Unit vector along the line is $\hat{b} = \dfrac{\vec{c}-\vec{a}}{|\vec{c}-\vec{a}|}$

For any scalar λ, the vector $\vec{a}+\lambda\hat{b}$ represents the position vector of a point on the line.

Position vector of a general point $\vec{r} = \vec{a} + \lambda\hat{b} = \vec{a} + \lambda \dfrac{\vec{c}-\vec{a}}{|\vec{c}-\vec{a}|}$

Equation of a plane: Can be expressed in many ways.



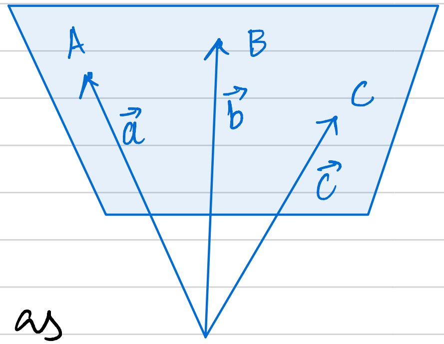
* With reference to the figure $\hat{n}$ is a unit vector normal to the plane.

A is any point on the plane with position vector $\vec{a}$.

If $\vec{r}$ is the position vector of any arbitrary point R on the plane then

$$(\vec{r}-\vec{a}).\hat{n} = 0 \Rightarrow \vec{r}.\hat{n} = \vec{a}.\hat{n}$$

represents the equation of the plane.



* W.r.to the figure, A, B & C are three points on a plane and they are not on a line. Their position vectors are $\vec{a}$, $\vec{b}$, & $\vec{c}$.

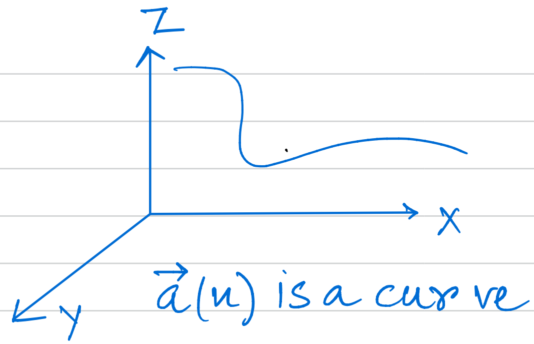
Position vector of any point on the plane can be written as

$$\vec{r} = \vec{a} + \lambda \frac{\vec{b}-\vec{a}}{|\vec{b}-\vec{a}|} + \mu \frac{\vec{c}-\vec{a}}{|\vec{c}-\vec{a}|}$$

this is then the equation of the plane.

Vector Calculus: Differentiation:

A vector $\vec{a}$ that is a function of scalar variable u



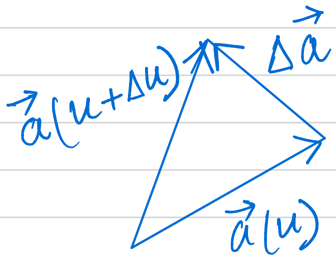
$\vec{a}(u)$ is a curve

$$\vec{a}(u) = \hat{i}\, a_x(u) + \hat{j}\, a_y(u) + \hat{k}\, a_z(u)$$

Here $a_x(u)$, $a_y(u)$ & $a_z(u)$ are scalar function of u, and represent the component of the vector $\vec{a}$ along the x, y and z coordinate axes. We assume that the variable u is continuous and so are the functions $a_x(u)$, $a_y(u)$, $a_z(u)$

A small change Δu in the variable u leads to a small change in the vector.

$$\Delta\vec{a} = \vec{a}(u+\Delta u) - \vec{a}(u)$$



Derivative of $\vec{a}(u)$ w.r.to u is then defined as

$$\frac{d\vec{a}}{du} = \lim_{\Delta u \to 0} \frac{\vec{a}(u+\Delta u) - \vec{a}(u)}{\Delta u}$$

$$= \hat{i} \lim_{\Delta u \to 0} \frac{a_x(u+\Delta u) - a_x(u)}{\Delta u} + \hat{j} \lim_{\Delta u \to 0} \frac{a_y(u+\Delta u) - a_y(u)}{\Delta u}$$

$$+ \hat{k} \lim_{\Delta u \to 0} \frac{a_z(u+\Delta u) - a_z(u)}{\Delta u}$$

Assuming the limits exist, we can write

$$\frac{d\vec{a}}{du} = \hat{\imath}\frac{da_x}{du} + \hat{\jmath}\frac{da_y}{du} + \hat{k}\frac{da_z}{du}$$

This means that, at least in the Cartesian coordinate we can do the derivative component by component

* Note that the derivative $d\vec{a}/du$ of a vector $\vec{a}(u)$ is a vector that is not in general parallel to $\vec{a}(u)$

Example! Position of a particle in a Cartesian coordinate is

$$\vec{r} = \hat{\imath}\,x(t) + \hat{\jmath}\,y(t) + \hat{k}\,z(t)$$

the velocity of the particle is given by

$$\vec{v}(t) = \hat{\imath}\frac{dx(t)}{dt} + \hat{\jmath}\frac{dy(t)}{dt} + \hat{k}\frac{dz(t)}{dt}$$

and the acceleration is

$$\vec{a}(t) = \hat{\imath}\frac{d^2x(t)}{dt^2} + \hat{\jmath}\frac{d^2y(t)}{dt^2} + \hat{k}\frac{d^2z(t)}{dt^2}$$

Suppose $\vec{r} = 2t^2\hat{\imath} + (3t-2)\hat{\jmath} + (3t^2-1)\hat{k}$

then $\vec{v}(t) = 4t\hat{i} + 3\hat{j} + 6t\hat{k}$

$\vec{a}(t) = 4\hat{i} + 6\hat{k}$

Composite Vectors:

$$\frac{d}{du}(\phi\vec{a}) = \phi\frac{d\vec{a}}{du} + \frac{d\phi}{du}\vec{a} \quad ; \quad \phi \text{ is a scalar}$$

$$\frac{d}{du}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{du} + \frac{d\vec{a}}{du}\cdot\vec{b}$$

$$\frac{d}{du}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{du} + \frac{d\vec{a}}{du}\times\vec{b}$$

{In this case the ordering is important.

* Differential of a Vector: In the previous example $\Delta\vec{a}$ is the change in $\vec{a}(u)$ due to change du in the variable u. In the limit $\Delta u \to 0$ the change $\Delta\vec{a}$ is infinitesimally small and is called the differential $d\vec{a}$

$$d\vec{a} = \frac{d\vec{a}}{du}\,du$$ → this is also a vector.

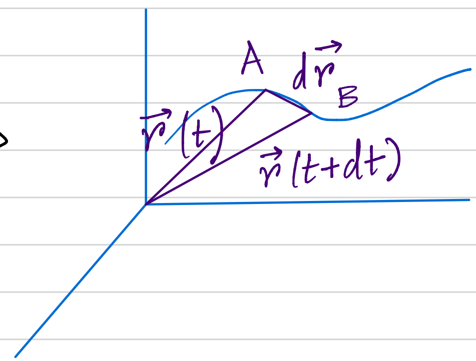
Example: If $d\vec{r}$ is infinitesimal change in position vector $\vec{r}$ then

$$d\vec{r} = \frac{d\vec{r}}{dt}\,dt = \vec{v}\,dt \quad ; \quad \vec{v}(t) \to \text{velocity}$$

Example!

Geometrical Meaning: Tangent Vector: Derivative of a vector is a vector that is tangent. Consider a particle that is moving in a Cartesian coordinate. The position vector of the particle at any time t is $\vec{r}(t)$

Position vector at t and $t+dt$ are $\vec{r}(t)$ and $\vec{r}(t+dt)$. Here the displacement vector $d\vec{r}$ is tangent to the curve at $\vec{r}(t)$. Hence the derivative $d\vec{r}/dt$ is tangent to the curve at $\vec{r}(t)$.



* Direction of the tangent vector $d\vec{r}/dt$ is independent of the parameter of the curve (in this case t)
* Magnitude of $d\vec{r}/dt$ depends on the parameter t.

Arc Length: Tangent vector can be used to calculate the length of a curve. In the above example, the distance between the nearby points A & B is

$$ds = |\vec{r}(t+dt) - \vec{r}(t)| = |d\vec{r}| + \mathcal{O}(|d\vec{r}|^2) + \cdots$$

{ using Taylor series expansion

Neglecting the terms of the order $|d\vec{r}|^2$

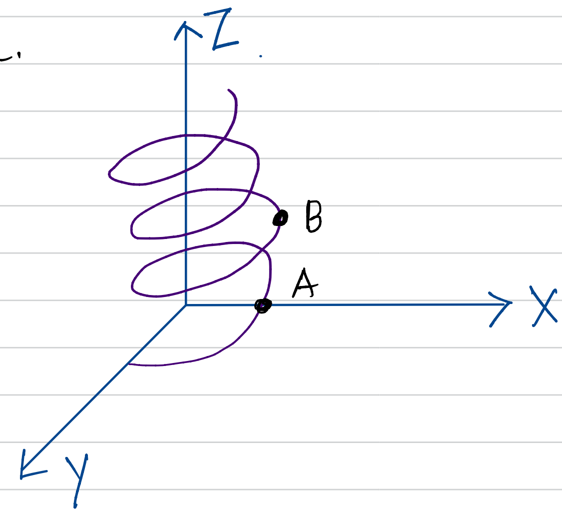
$$ds = |d\vec{r}| = \pm\left|\frac{d\vec{r}}{dt}dt\right| = \pm\left|\frac{d\vec{r}}{dt}\right|dt$$

I put + sign for distance measured in the direction of increasing t and − sign for the direction of decreasing t.

So the total length of the curve from time t_1 & t_2 is

$$s = \pm\int_{t_1}^{t_2}\left|\frac{d\vec{r}}{dt}\right|dt \quad \begin{cases} + \text{ sign for } t_2 > t_1 \\ - \text{ sign for } t_1 > t_2 \end{cases}$$

Example: The position vector of a particle in a Cartesian coordinate is given by $\vec{r} = \hat{i}\cos t + \hat{j}\sin t + \hat{k}\,t$ where t is the time.



The curve that the particle follows is a helix.

$$ds = \left|\frac{d\vec{r}}{dt}\right|dt$$

$$= |-\hat{i}\sin t + \hat{j}\cos t + \hat{k}|\,dt$$

$$= \sqrt{2}\,dt$$

So the length from time $t_1 = 0$ and $t_2 = t$ is $s = \int_0^{t_2}\sqrt{2}\,dt = \sqrt{2}\,t_2 = 2t$

For $t_1 = 0$ and $t_2 = 2\pi$ the particle makes one full rotation - shown by points A & B on the plot. Total arc length is given by $s = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$.