Generating Function

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

Properties

Let a, b, c be the numeric functions.

- If $b = \triangle a$, then

$$B(z) = \frac{1}{z}\left[A(z) - a_0\right] - A(z)$$

- If $c = \nabla a$, then

$$C(z) = (1 - z)A(z)$$

- If $c = a * b$, that is, c is the convolution of a and b, then

$$C(z) = A(z).B(z)$$

Generating Function

Problem: Consider the problem of determining c_r, the number of sequences of length r that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters.

Solution: Let $a_r =$ the number of sequences of length r that are made up from English letters $\{x, y, z\}$;
$b_r =$ the number of sequences of length r that are made up from Greek letters $\{\alpha, \beta\}$.
Then, we have,

$$\begin{aligned} a_r &= 3^r, r \geq 0 \\ b_r &= 2^r, r \geq 0 \end{aligned}$$

Then, for $c = a * b$, we have,

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} 3^i 2^{r-i}, r \geq 0.$$

Generating Function

Now,

$$C(z) = A(z).B(z), \tag{1}$$

where

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 3^r z^r = \sum_{r=0}^{\infty} (3z)^r = \frac{1}{1-3z} \tag{2}$$

$$B(z) = \sum_{r=0}^{\infty} b_r z^r = \sum_{r=0}^{\infty} 2^r z^r = \sum_{r=0}^{\infty} (2z)^r = \frac{1}{1-2z} \tag{3}$$

Hence,

$$\begin{aligned} C(z) &= A(z).B(z) \\ &= \frac{1}{1-3z}.\frac{1}{1-2z} = \frac{\alpha}{1-2z} + \frac{\beta}{1-3z}, \text{ say} \end{aligned}$$

Generating Function

Solving, we have, $\alpha = -2$ and $\beta = 3$. Thus,

$$\begin{aligned} C(z) &= \sum_{r=0}^{\infty} c_r z^r \\ &= -\frac{2}{1-2z} + \frac{3}{1-3z} \\ &= \sum_{r=0}^{\infty} [3.3^r - 2.2^r] z^r \\ &= \sum_{r=0}^{\infty} [3^{r+1} - 2^{r+1}] z^r \end{aligned}$$

As a result, we obtain:

$$c_r = 3^{r+1} - 2^{r+1},\ r \geq 0$$

Generating Function

Problem: Evaluate the sum

$$1 + 2 + 3 + \cdots + n$$

using the generating function.

Generating Function

Exercise: Evaluate the sum

$$1^2 + 2^2 + 3^2 + \cdots + n^2$$

using the generating function.