Countable & Uncountable Sets

→ regarding "size" of a set.

Def^n: Two sets A & B are said to be <u>equivalent</u> $(A \approx B)$ if $\exists$ a f^n $f: A \to B$ that is one-to-one and onto.

Notice that for sets A, B, C following holds:

1. $A \approx A$
2. If $A \approx B$, then $B \approx A$
3. If $A \approx B$ and $B \approx C$, then $A \approx C$.

The set of natural nos. $\mathbb{N} = \{1, 2, \dots\}$.

Any subset of $\mathbb{N}$ of the form $\{1, \dots, n\}$ is called a segment of $\mathbb{N}$, and $n \in \mathbb{N}$ is called no. of elements of the segment.

Two segments $\{1, \dots, m\}$ and $\{1, \dots, n\}$ are equiv. iff $m = n$.

A set that is equivalent to a segment is called a <u>finite set</u>. The empty set is also considered to be <u>finite</u> with zero elements. A set that is not finite is called an <u>infinite set</u>.

Def^n: A set A is called <u>countable</u> if it is equivalent to N, i.e., if $\exists$ a one-to-one correspondence of N with the elements of A.

Countable set A usually denoted as ~~$A = \{a_i\}_{i=1}^{n}$~~
$\underbrace{A = \{a_i\}_{i=1}^{n}}_{\text{enumeration of the set } A}$ for $n \in N$.

An infinite set that is not countable is called <u>uncountable set</u>.

<u>Thm:</u> Every infinite set contains a countable subset.

Proof: Let A be an infinite set; clearly $A \neq \phi$. Choose $a_1 \in A$, consider $A_1 = A \setminus \{a_1\} \neq \phi$ as A is infinite. Choose $a_2 \in A_1$, $A_2 = A \setminus \{a_1, a_2\} \neq \phi$ and so forth; $A_\infty = A \setminus \{a_1, a_2, \dots\}$ where $\{a_1, a_2, \dots\} \subset A$ and is countable as $\exists$ one-one and onto fn $f: \mathbb{N} \to \{a_1, \dots\}$.

A subset $S \subset \mathbb{N}$ is said to have a <u>least</u> (or a <u>first</u>) element if $\exists$ $k \in S$ s.t. $k \leq n$ for each $n \in S$. Note that ~~a~~ k is uniquely determined.

Two principles of the natural nos.:—

<u>The Well-ordering principle</u>: Every nonempty subset of $\mathbb{N}$ has a least element.

<u>The principle of Mathematical Induction</u>:

If a subset $S \subset \mathbb{N}$ ~~of N has a~~ satisfies the properties:

a) $1 \in S$ and b) $n+1 \in S$ whenever $n \in S$,

then $S = \mathbb{N}$.

<u>Thm</u>. Every subset of a countable set is either finite or else countable.

<u>Thm</u>: For an infinite set A, the following statements are equiv.

i) A is countable.

ii) $\exists$ a subset B of $\mathbb{N}$ and a f^n $f: B \to A$ that is onto.

iii) $\exists$ a f^n $g: A \to \mathbb{N}$ that is one-one.

<u>Thm</u>: Let $\{A_1, A_2, A_3, A_4, \ldots\}$ be a countable family of sets s.t. each A_i is a countable set. Then $A = \bigcup_{n=1}^{\infty} A_n$ is a countable set.

Proof: Let $A_n = \{a_1^n, a_2^n, \ldots\}$ for all $n \in \mathbb{N}$. $A = \bigcup_{n=1}^{\infty} A_n$. Consider $B = \{2^k \cdot 3^n : k, n \in \mathbb{N}\}$. Let $f: B \to A$ s.t. $f(2^k \cdot 3^n) = a_k^n$. Since f maps B onto A, A is countable set.